Fair Allocations in an Overlapping Generations Economy*

Ram Sewak Dubey† Francesco Ruscitti‡

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Abstract

This paper investigates fair (i.e., envy free and efficient) allocations in an overlapping generations economy without production and with two - period lived agents. We show that there exists a conflict between no-envy and efficiency when all generations have identical preferences. This conflict crucially depends on the size of the given young age consumption of the initial old generation relative to the young age consumption at the golden-rule. We then show that there exist non-stationary preferences for which such a conflict does not arise, regardless of the young age consumption of the initial old generation.

Keywords: `Efficiency, No-Envy, Fair allocations, Non-stationary preferences.`

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†Department of Economics, Feliciano School of Business, Montclair State University, Montclair, NJ 07043; E-mail: dubeyr@montclair.edu.

‡Department of Economics and Social Sciences, John Cabot University, Via della Lungara 233, 00165 Rome, Italy. Email: fruscitti@johncabot.edu

1 Introduction

Consider the following situation facing a social planner. At time $t = 1$, there are agents of different ages who can be broadly classified as young and old. The old agent is in the last period of her lifetime and has already consumed x_0 units of the consumption good when young. Each period, from now on into the future, the policy maker has to decide how to distribute a fixed amount of the given good between the young and old generations. What is an egalitarian allocation for this economy? Can such an egalitarian allocation be achieved efficiently? Given that the policy maker must take x_0 as given, what are, if any, the constraints on the set of feasible policies? The present paper is an attempt to address these questions.

In the literature on intertemporal social choice, two approaches have been followed to study efficient and equitable allocations. The first approach describes social preferences in terms of a social welfare quasi-order, a social welfare order, or a social welfare function defined on infinite utility streams, where these preferences are assumed to embody the notions of equity and efficiency.[1] Given an economy, these preferences can be used to choose among the set of feasible utility streams over time. The optimal allocations are those which generate utility streams that maximize the social preferences.[2]

An alternative approach consists in formalizing an economy and the set of its feasible allocations. The choice set is the set of allocations that satisfy the properties of equity and efficiency.[3]

The notion of efficiency used in both approaches involves some variant of the Pareto principle. The concept of equity which is considered varies widely, but the notions of equity employed in the first approach[4] can be translated into notions of equity that fit well in the second approach.

The concept of equity that we wish to study in this paper is known as *no-envy*, which was introduced by Foley (1967).[5] Unlike other notions of equity, in order to define the concept of no-envy it is necessary to know the set of feasible allocations of the underlying economy. This is because the property of an allocation being envy-free relies on each generation comparing its own *consumption bundle*

[1]Some of the classic contributions to this strand of literature are Koopmans (1960), Koopmans et al. (1964), Diamond (1965), Svensson (1980), Basu and Mitra (2003), Fleurbaey and Michel (2003), Zame (2007), Lauwers (2010). Recent contributions following this approach include Mitra (2005), Basu and Mitra (2007), Banerjee and Mitra (2010), Asheim et al. (2010), Zuber and Asheim (2012), Dubey and Mitra (2013), Asheim and Zuber (2014).

[2]For some recent contributions following this approach, see Mitra (2005), Basu and Mitra (2007), Banerjee and Mitra (2010), Asheim et al. (2010), Zuber and Asheim (2012), Dubey and Mitra (2013), Asheim and Zuber (2014) - who consider a model with variable population.

[3]For recent contributions following this approach, see Shinotsuka et al. (2007), Asheim et al. (2010), Issac and Piacquadio (2015), Piacquadio (2014).

[4]Some of the concepts of equity that have been discussed extensively are anonymity, which is a form of *procedural* equity, and the Hammond equity, strong equity, and Pigou-Dalton transfer principle, which are types of *consequentialist* equity.

[5]Other seminal papers on this equity notion are Kolm (1972), Varian (1974), and Yaari (1981).

to the other generations' consumption bundles. Our study is grounded in the second approach described above. Also, as opposed to other notions of equity, the utilities of different generations need not be comparable because each generation evaluates consumption bundles (delivered by a given allocation) using its own utility function.

Shinotsuka et al. (2007) have studied alternative notions of envy-free allocations in an overlapping generations economy without production and with two-period lived agents. We retain their framework, but we assume, in addition, that the population is stationary and that in each period there is a social endowment of one unit of a perishable consumption good. Such endowment is to be divided between the young and the old agent that are alive in that period. This simplifying assumption helps us bring the essential issues at hand into sharp focus. Also, unlike their focus on stationary allocations, we consider all type of allocations, stationary or not. In our opinion, this setting seems the most compelling framework to examine the notion of envy-free allocations in relation to efficiency. We then analyze the potential equity-efficiency tension the social planner is confronted with in determining the socially-optimal policy.

The first part of the paper deals with the case of *stationary* preferences. The potential conflict between efficiency and equity can be understood with a variety of specifications of the identical utility function. However, to demonstrate that this conflict is not due to pathological features of preferences, we deliberately work with "well-behaved" utility functions such that the economy has a unique golden-rule in the interior of the consumption set.

With the aid of the golden-rule tool, we find out that the clash between equity and efficiency is entirely determined by *history*. In other words, at the beginning of the planner's decision making process, x_0 (young age consumption of the initial old generation) is given and it turns out that the existence of such a conflict can be ascribed to the magnitude of x_0, in a sense we make precise in Theorem 1 below.

Indeed, our first result (Theorem 1) asserts that if x_0 is *high* relative to the golden-rule young-age consumption $\bar{x}$, then there is no allocation which is efficient and envy-free. If, instead, it is *low* relative to the golden-rule young-age consumption $\bar{x}$, then there always exists an efficient and envy-free allocation (referred to as a *fair* allocation).

This equity-efficiency trade-off may be explained intuitively as follows. If $x_0 > \bar{x}$, it is possible to secure golden-rule utility for all generations and provide a bit more utility to the initial old generation. On the other hand, any *constant* utility level for all generations higher than golden-rule utility is unfeasible. Then, in order to achieve equity one is forced to choose golden-rule utility for all generations; but this is clearly inefficient.

In the second part of the paper (section 4), we examine the case of non-stationary preferences. Intuitively, it would not be difficult to come up with economies that exhibit a conflict between equity and efficiency over larger subsets of possible histories. However, what is not obvious is whether, for every specification of (non-stationary) preferences, there are histories for which a conflict between equity and efficiency always arises. We construct preferences such that no conflict between equity and efficiency exists, regardless of how much the initial old generation consumes when young. It should be noted that since the social endowment is constant over time, the aforementioned result is obtained by letting different generations be endowed with different preferences. However, in order to highlight the strength of our result we need to retain the assumption that each agents' preferences are well-behaved (as in Theorem 1).

2 An Overlapping Generations Model

We consider a standard overlapping generations model in which consumers have a two-period lifetime and only one consumer is born in each period. Thus, consumer t is born at period t and is alive in periods t and $t+1$. Therefore, in any given period $t \in \mathbb{N}$[6] an old agent coexists with a young agent.[7]

For each generation $t \in \mathbb{M}$, we denote by x_t the young-age consumption and we let y_{t+1} denote old-age consumption. In each period $t \in \mathbb{N}$, one unit of a perishable good is exogenously available. This has to be divided between the young and old agents alive in period t. The choice as to how the single good is to be distributed to the young and the old agent in every period is made by the social planner at $t = 1$. Therefore, $x_0 \in [0, 1]$ is taken as given by the planner. In other words, it is exogenously given.

An *allocation* is a sequence $\langle x_t, y_{t+1} \rangle_{t=0}^{\infty}$ satisfying

$$\begin{aligned} &(x_t, y_{t+1}) \geqslant 0 \quad \text{for all } t \in \mathbb{M}, \\ &x_t + y_t \leqslant 1 \quad \text{for all } t \in \mathbb{N}, \text{ and } x_0 \text{ is given.} \end{aligned} \tag{1}$$

An allocation $\langle x_t, y_{t+1} \rangle_{t=0}^{\infty}$ is *stationary* if

$$(x_t, y_{t+1}) = (x_{t+1}, y_{t+2}) \text{ for all } t \in \mathbb{M}.$$

2.1 Equity and Efficiency

Even though x_0 is given from the perspective of policy making, the well-being of generation 0 can clearly be affected by decisions made at $t = 1$ (since such decision involve choosing y_1). Thus, the notions of both equity and efficiency should be formulated so that the well-being of generation 0 (and of all other generations) is taken into account.

Assume that preferences for consumption bundles of generation $t \in \mathbb{M}$ are defined on the consumption sets $X \equiv \mathbb{R}^2_+$ and are represented by a utility function u_t from X to $\mathbb{R}$.

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[6] $\mathbb{N}$ denotes the set of natural numbers, and $\mathbb{M} \equiv \mathbb{N} \cup \{0\}$.

[7] This is a standard overlapping generations model. The classic reference is Balasko and Shell (1980).

An allocation $\langle x_t, y_{t+1} \rangle_{t=0}^{\infty}$ is *envy-free* if for each $t \in \mathbb{M}$,

$$u_t(x_t, y_{t+1}) \geqslant u_t(x_s, y_{s+1}) \quad \text{for all } s \in \mathbb{M}.$$

An allocation $\langle x'_t, y'_{t+1} \rangle_{t=0}^{\infty}$ *dominates* another allocation $\langle x_t, y_{t+1} \rangle_{t=0}^{\infty}$ if

$$u_t(x'_t, y'_{t+1}) \geqslant u_t(x_t, y_{t+1}) \quad \text{for all } t \in \mathbb{M}, \tag{2}$$

and (2) holds with strict inequality for at least one $t \in \mathbb{M}$.

An allocation $\langle x_t, y_{t+1} \rangle_{t=0}^{\infty}$ is *efficient* if there is no allocation which dominates it. An allocation $\langle x_t, y_{t+1} \rangle_{t=0}^{\infty}$ is *fair* if it is both envy-free and efficient.

3 Fair Allocations with Stationary Preferences

In this section we examine the existence of fair allocations when agents have stationary preferences, i.e., there is a $u : X \to \mathbb{R}$ such that,

$$u_t = u \quad \text{for all } t \in \mathbb{M}.$$

In order to avoid that the equity-efficiency trade-off, established in Theorem 1 below, arises as a consequence of unusual specifications of preferences, in what follows we posit assumptions on u which guarantee that preferences are "well-behaved".

The utility function $u : X \to \mathbb{R}$ is assumed to satisfy:

A1. (Continuity) u is continuous on X.

A2. (Quasi-Concavity) u is quasi-concave on X.

A3. (Monotonicity) u is monotone on X, i.e., if $(x, y) \geqslant (x', y')$ then $u(x, y) \geqslant u(x', y')$, and strongly monotone on $\mathbb{R}^2_{++}$, i.e., if $(x, y) > (x', y')$, with $(x, y), (x', y') \in \mathbb{R}^2_{++}$, then $u(x, y) > u(x', y')$.

3.1 Golden Rule

Formally, a golden-rule is a pair $(\bar{x}, \bar{y})$ which solves the following problem

$$\begin{array}{ll} \max & u(x, y) \\ \text{subject to} & x + y \leqslant 1 \\ \text{and} & (x, y) \geqslant 0. \end{array} \tag{3}$$

It can be interpreted in the following way. For any exogenously given $x_0 \in [0, 1]$, consider the non-wasteful stationary allocation $\langle x_0, 1 - x_0 \rangle$ that yields stationary utility $u(x_0, 1 - x_0)$. Now, let x_0 vary between $[0, 1]$ and pick any value of x_0, say $\bar{x}_0$, that maximizes $u(x_0, 1 - x_0)$. That value is referred to as a golden-rule. Next, we will assume that u satisfies A4.

A4. There is a unique solution $(\bar{x}, \bar{y})$ to problem (3), and $(\bar{x}, \bar{y}) \gg 0$.

Of course, it follows from A3 and A4 that

$$\bar{x} + \bar{y} = 1. \tag{4}$$

Notice that by A4 there exists a unique golden rule in the interior of the consumption set. Define the sets

$$A = \{(x, y) \in [0, 1] \times [0, 1] : u(x, y) \geqslant u(\bar{x}, \bar{y})\},$$
$$B = \{(x, y) \in [0, 1] \times [0, 1] : u(x, y) > u(\bar{x}, \bar{y})\}.$$

It follows from the definition of golden-rule that

$$x + y > 1 \text{ for all } (x, y) \in B. \tag{5}$$

A useful property of the golden-rule can now be noted. To this end, we first define the following set. Given $\varepsilon > 0$, let

$$A(\varepsilon) = \{(x, y) \in [0, 1] \times [0, 1] : u(x, y) \geqslant u(\bar{x}, \bar{y}) + \varepsilon\}.$$

Lemma 1. *Given any $\varepsilon > 0$, there is $\delta > 0$ such that whenever $(x, y) \in A(\varepsilon)$,*

$$x + y \geqslant 1 + \delta \tag{6}$$

Proof. The set $A(\varepsilon)$ is non-empty (since $(1, 1) \in A(\varepsilon)$), bounded and closed (by A1). The function $h : A(\varepsilon) \to \mathbb{R}$, defined by $h(x, y) = x + y - 1$ for all $(x, y) \in A(\varepsilon)$, is continuous and positive on $A(\varepsilon)$ by (5). Therefore, it attains a minimum on $A(\varepsilon)$. Denoting this minimum value by δ, notice that (6) is satisfied and moreover $\delta > 0$ as desired. ∎

Now, let $(\hat{x}, \hat{y})$ be any vector that satisfies the following property:

$$(\hat{x}, \hat{y}) \in \mathbb{R}^2_{++}, \text{ and } \hat{x} + \hat{y} = 1, \text{ with } \hat{x} < \bar{x}.$$

Then, $\hat{y} > \bar{y}$. Let us define the sets

$$\widehat{A} = \{(x, y) \in [0, 1] \times [0, 1] : u(x, y) \geqslant u(\hat{x}, \hat{y})\},$$
$$\bar{A} = \{(x, y) \in \mathbb{R}^2_{++} : (x, y) \ll (\hat{x}, \hat{y})\}.$$

Clearly $\bar{A}$ is open and convex.

Lemma 2. *There exist p and q, with $p > q > 0$, such that*

$$px + qy \geqslant p\hat{x} + q\hat{y} \text{ for all} (x, y) \in \widehat{A}. \tag{7}$$

Proof. By A2, it's easy to see that $\widehat{A}$ is convex. Clearly, $(\hat{x}, \hat{y}) \in \widehat{A}$ and $(\hat{x}, \hat{y})$ is a boundary point of $\widehat{A}$. Hence, it follows from the finite dimensional supporting hyperplane theorem (see, e.g., Aliprantis and Border (2006, Theorem 7.36)) that there exists a nonzero (p, q) such that

$$px + qy \geqslant p\hat{x} + q\hat{y} \text{ for all } (x, y) \in \widehat{A}. \tag{8}$$

By construction of $(\hat{x}, \hat{y})$ and assumptions A2 and A3, if $\hat{y} < y < 1$ and $\hat{x} < x < 1$, $(\hat{x}, y)$ and $(x, \hat{y})$ both belong to the interior (in $\mathbb{R}^2$) of $\widehat{A}$. Thus, using Lemma 5.66 in Aliprantis and Border (2006), it follows from (8) that $q\,(y - \hat{y}) > 0$ and $p\,(x - \hat{x}) > 0$. The previous inequalities readily imply p and q are both positive. Thus, it remains to prove that $p > q$. To this end, notice that by assumption A4, $(\bar{x}, \bar{y})$ lies in the interior (in $\mathbb{R}^2$) of $\widehat{A}$. Hence, using (8) and invoking again Lemma 5.66 in Aliprantis and Border (2006) we get

$$p\bar{x} + q\bar{y} > p\hat{x} + q\hat{y}. \tag{9}$$

Given that $\bar{x} + \bar{y} = 1$ and $\hat{x} + \hat{y} = 1$, it follows from (9) that

$$p\,(\hat{y} - \bar{y}) > q\,(\hat{y} - \bar{y})\,.$$

Finally, recall that $\hat{y} > \bar{y}$, thus the preceding inequality immediately implies $p > q$. ∎

3.2 The Main Result

We are now in a position to state and prove the main result of the framework with stationary preferences.

Theorem 1. *(i) Suppose* $x_0 > \bar{x}$. *Then, there is no fair allocation.*

(ii) Suppose $0 < x_0 \leqslant \bar{x}$. *Then, the stationary allocation defined by*

$$(x_t, y_{t+1}) = (x_0, 1 - x_0) \text{ forall } t \in \mathbb{M},$$

is a fair allocation.

Proof. (i) Suppose, by way of obtaining a contradiction, that there is a fair allocation denoted by $\langle x_t, y_{t+1}\rangle_{t\in\mathbb{M}}$. Since it is envy-free, the utility evaluated at (x_t, y_{t+1}) must be constant for all $t \in \mathbb{M}$. Denote this constant utility level by ν. We claim that $\nu \leqslant u(\bar{x}, \bar{y})$. For, if $\nu > u(\bar{x}, \bar{y})$, then denoting $\nu - u(\bar{x}, \bar{y})$ by ε, we have that $\varepsilon > 0$. Consequently, by Lemma 1 there is $\delta > 0$ such that

$$x_t + y_{t+1} \geqslant 1 + \delta \geqslant x_t + y_t + \delta \text{ for all } t \in \mathbb{N}, \tag{10}$$

where we have used the fact that $x_t + y_t \leqslant 1$ for all $t \in \mathbb{N}$. Thus,

$$y_{t+1} \geqslant y_t + \delta \text{ for all } t \in \mathbb{N}. \tag{11}$$

Given (11), it is very easy to prove by induction that

$$y_t \geqslant y_1 + (t-1)\delta \text{ for all } t \in \mathbb{N}. \tag{12}$$

But (12) implies $y_t \to \infty$ as $t \to \infty$, which contradicts the fact that $y_t \leqslant 1$ for all $t \in \mathbb{N}$. This establishes the claim.

Consider now the sequence $\left\langle x'_t, y'_{t+1} \right\rangle_{t=0}^{\infty}$ defined by

$$x'_0 = x_0, (x'_t, y'_t) = (\bar{x}, \bar{y}) \text{ for all } t \in \mathbb{N}. \tag{13}$$

It is easy to check that $\left\langle x'_t, y'_{t+1} \right\rangle$ is an allocation. Further,

$$u\left(x'_t, y'_{t+1}\right) = u(\bar{x}, \bar{y}) \geqslant v = u(x_t, y_{t+1}) \text{ for all } t \in \mathbb{N}$$

and

$$u(x'_0, y'_1) = u(x_0, \bar{y}) > u(\bar{x}, \bar{y}) \geqslant v = u(x_0, y_1).$$

This establishes that $\langle x_t, y_{t+1} \rangle$ is inefficient, and we are done.

(ii) Clearly, $\langle x_t, y_{t+1} \rangle$ defined by

$$(x_t, y_{t+1}) = (x_0, 1 - x_0) \text{ for all } t \in \mathbb{M}, \tag{14}$$

is an allocation. It is also envy-free. It remains to show that it is efficient. Suppose, by way of contradiction, that there is an allocation $\left\langle x'_t, y'_{t+1} \right\rangle$ such that

$$u(x'_t, y'_{t+1}) \geqslant u(x_t, y_{t+1}) \text{ for all } t \in \mathbb{M}, \tag{15}$$

with strict inequality in (15) for some $t \in \mathbb{M}$. We need to consider two cases.

Case (a) $x_0 < \bar{x}$: By Lemma 2 and (15),

$$px'_t + qy'_{t+1} \geqslant px_0 + q(1 - x_0) \text{ for all } t \in \mathbb{M}. \tag{16}$$

Since $x'_0 = x_0$, (16) implies

$$(y'_1 - y_1) \geqslant 0, \tag{17}$$

and

$$(y'_{t+1} - y_{t+1}) \geqslant \left(\frac{p}{q}\right)(y'_t - y_t) \text{ for all } t \in \mathbb{N}. \tag{18}$$

Because there is strict inequality in (15) for some $t \in \mathbb{M}$, with $t + 1 \in \mathbb{N}$, $y'_{t+1} - y_{t+1} > 0$ must hold. Using (18), and proceeding by induction, we obtain

$$y'_{t+s} - y_{t+s} \geqslant \left(\frac{p}{q}\right)^{s-1} (y'_{t+1} - y_{t+1}), \text{ for all } s \in \mathbb{N}.$$

Since $y'_{t+1} > y_{t+1}$, $y_{t+s} = 1 - x_0$ for all $s \in \mathbb{N}$, and $\left(\frac{p}{q}\right) > 1$, we have $y'_{t+s} \to \infty$ as $s \to \infty$. This contradicts the fact that $y'_{t+s} \leqslant 1$ for all $s \in \mathbb{N}$.

Case (b) $x_0 = \bar{x}$: Firstly, note that $(x_t, y_{t+1}) = (\bar{x}, 1 - \bar{x})$ is indeed an allocation. Since this allocation is constant, clearly it is envy-free. We claim that such an allocation is also efficient. Indeed, suppose by way of obtaining a contradiction, that it is not efficient. Then, one can find an alternative allocation (x'_t, y'_{t+1}) such that

$$u\left(x'_t, y'_{t+1}\right) \geqslant u(\bar{x}, 1-\bar{x}) \text{ for all } t \in \mathbb{M} \text{ and} \tag{19}$$

$$u\left(x'_s, y'_{s+1}\right) > u(\bar{x}, 1-\bar{x}) \text{ for some } s \in \mathbb{M}. \tag{20}$$

We can assume, without any loss of generality, that $x'_t + y'_t = 1$ for all $t \in \mathbb{N}$. Therefore, it's easy to see that A4 and (20) imply $x'_s + y'_{s+1} > 1$. Thus, because $x_0 = \bar{x}$, by the preceding condition and (19) - (20), we have that

$$\left(x'_t, y'_{t+1}\right) \neq (\bar{x}, 1-\bar{x}), \text{ for all } t \geqslant s.$$

On the other hand, preferences are convex and $(\bar{x}, 1-\bar{x})$ is the unique solution to problem (3). Hence, $x'_t + y'_{t+1} > 1$ for all $t \geqslant s$. Since $x'_t + y'_t = 1$ for all $t \in \mathbb{N}$, $y'_{t+1} > y'_t$ for all $t \geqslant s$. Also, $y'_t \leqslant 1$ for all $t \in \mathbb{N}$. Thus, letting

$$\bar{s} = \sup\{y'_t\}_{t=s}^{\infty},$$

we have that $0 \leqslant \bar{s} \leqslant 1$ and

$$\{y'_t\}_{t=s}^{\infty} \to \bar{s}.$$

Similarly, $x'_t + y'_{t+1} > 1$ for all $t \geqslant s$ and $x'_t + y'_t = 1$ for all $t \in \mathbb{N}$ imply $x'_t > x'_{t+1}$ for all $t \geqslant s$. Also, $x'_t \geqslant 0$ for all $t \in \mathbb{N}$. Thus, letting

$$\bar{\alpha} = \inf\{x'_t\}_{t=s}^{\infty},$$

we have that $0 \leqslant \bar{\alpha} \leqslant 1$ and

$$\{x'_t\}_{t=s}^{\infty} \longrightarrow \bar{\alpha}.$$

Next, recall that $u\left(x'_t, y'_{t+1}\right) \geqslant u(\bar{x}, 1-\bar{x})$ for all $t \in \mathbb{N}$ (see (19) - (20)). Therefore, taking the limit in the above inequality and using A1 yields

$$u(\bar{\alpha}, \bar{s}) \geqslant u(\bar{x}, 1-\bar{x}). \tag{21}$$

We claim that $(\bar{\alpha}, \bar{s}) \neq (\bar{x}, 1-\bar{x})$. For, suppose by contradiction that $(\bar{\alpha}, \bar{s}) = (\bar{x}, 1-\bar{x})$. Then, it follows from above that

$$y'_s < y'_{s+1} < y'_{s+2} \cdots \leqslant 1-\bar{x},$$

and

$$x'_s > x'_{s+1} > x'_{s+2} \cdots \geqslant \bar{x},$$

which imply

$$x'_s > \bar{x}, \text{ and } y'_{s+1} < 1-\bar{x}. \tag{22}$$

Since $x'_0 = x_0 = \bar{x}$, (22) above immediately implies that $s > 0$. Hence, by (19) we get

$$y'_1 \geqslant 1-\bar{x} \Longrightarrow x'_1 \leqslant \bar{x} \Longrightarrow y'_2 \geqslant 1-\bar{x} \Longrightarrow x'_2 \leqslant \bar{x} \Longrightarrow y'_3 \geqslant 1-\bar{x} \cdots \Longrightarrow x'_s \leqslant \bar{x},$$

and $y'_{s+1} \geqslant 1-\bar{x}$, which contradicts (22) above. Since $\bar{\alpha} + \bar{s} = 1$ and $(\bar{\alpha}, \bar{s}) \neq (\bar{x}, 1-\bar{x})$, the previous inequality and assumption A4 imply $u(\bar{\alpha}, \bar{s}) < u(\bar{x}, 1-\bar{x})$, which contradicts (21) above.

∎

4 Fair Allocations with Non-Stationary Preferences

In this section we assume that preferences are non-stationary, in the sense that different generations are endowed with distinct preferences. We specify below the non-stationary preferences we have in mind.

$$u_t(x, y) = \begin{cases} \left(\frac{2^{t+2}}{1+2^{t+1}}\right) \cdot (x + y) & \text{for } y > (2^t - 0.5)\, x, \\ x + 2y & \text{otherwise.} \end{cases} \tag{23}$$

Notice that the utility function defined in (23) is continuous. In what follows, we show that there exists a fair allocation for all $x_0 \in [0, 1]$. Figure 1 below will be useful for understanding the geometric constructions underlying the following claims.

We split the proof into two cases: (a) $x_0 = 1$, and (b) $x_0 \in (0, 1)$. Note that in the case of $x_0 = 0$, the allocation in which every generation consumes $(0, 1)$ is clearly feasible and envy free. Moreover, it's easy to see that such allocation cannot be improved upon. Hence it is efficient as well.

(a) $x_0 = 1$: The existence of a fair allocation will be established through a series of claims, as follows.

Claim 1. *The allocation*

$$\left\langle x_t^*, y_{t+1}^* \right\rangle_{t=0}^{\infty} \equiv \left\langle \frac{1}{2^t}, 1 - \frac{1}{2^{t+1}} \right\rangle_{t=0}^{\infty} \tag{24}$$

is feasible, and

$$u_t\left(x_t^*, y_{t+1}^*\right) = 2$$

for all $t \in \mathbb{M}$.

Proof. Observe that

$$x_t^* + y_t^* = \frac{1}{2^t} + 1 - \frac{1}{2^t} = 1$$

for all $t \in \mathbb{N}$, which proves feasibility. Since

$$y_{t+1}^* = 1 - \frac{1}{2^{t+1}} = \left(2^t - 0.5\right)\left(\frac{1}{2^t}\right) = \left(2^t - 0.5\right) x_t^*,$$

for all $t \in \mathbb{M}$, the consumption bundle lies at the kink of the indifference curve for every generation. Therefore,

$$u_t\left(x_t^*, y_{t+1}^*\right) = x_t^* + 2y_{t+1}^* = \frac{1}{2^t} + 2\left(1 - \frac{1}{2^{t+1}}\right) = \frac{1}{2^t} + 2 - \frac{1}{2^t} = 2,$$

for all $t \in \mathbb{M}$. ∎

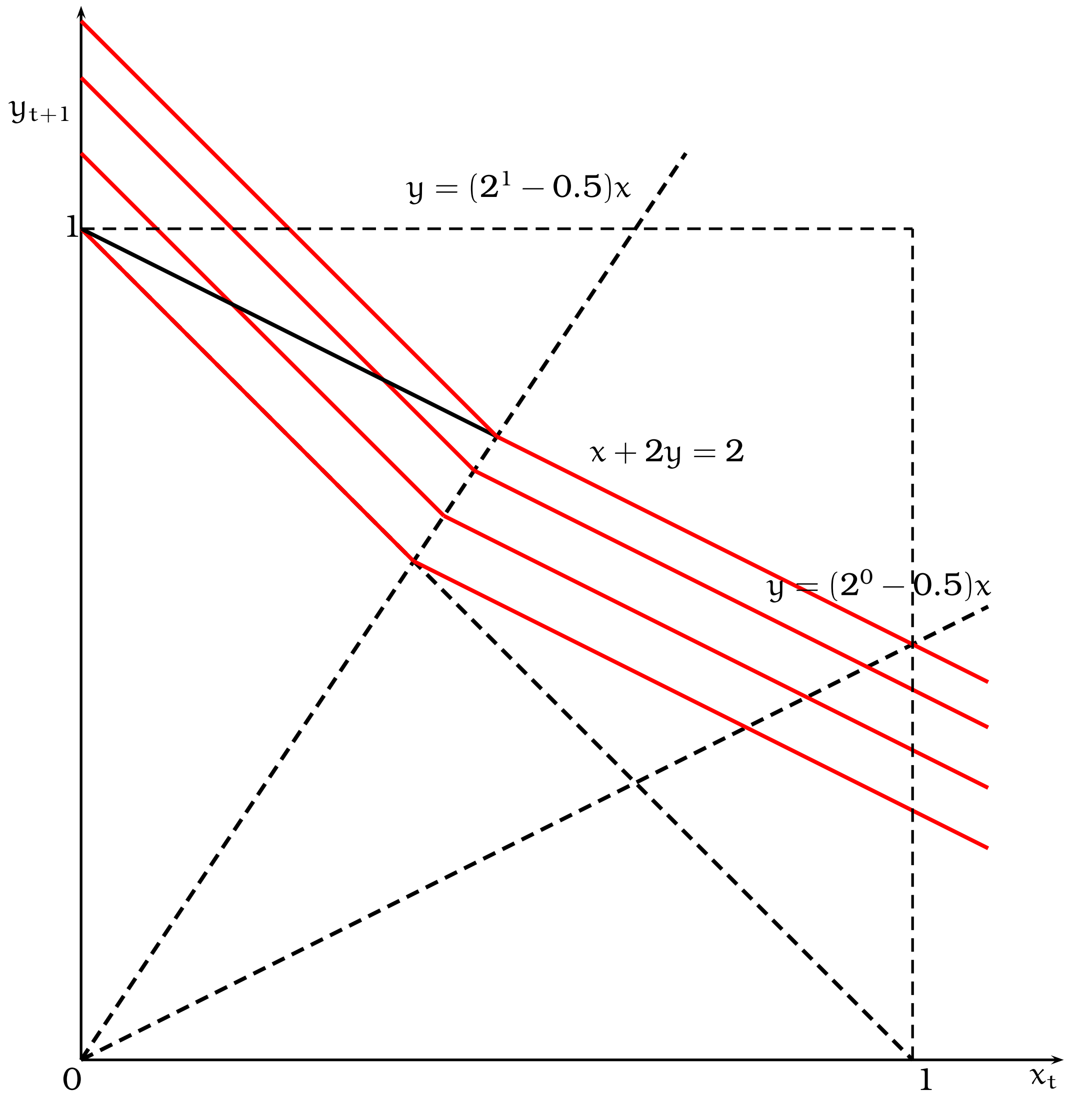

Figure 1: Indifference Curves (in red) for generation 1

Claim 2. *Generation* $t = S \in \mathbb{N}$ *is indifferent between its own consumption bundle* $\left(\frac{1}{2^S}, 1-\frac{1}{2^{S+1}}\right)$ *and the consumption bundle of all of its predecessors,* $S-1, \cdots, 0$.

Proof. We claim that he consumption bundle of generation $S-1, \cdots, 0$ satisfies the following condition

$$y_{t+1} < \left(2^S - 0.5\right) x_t, \text{ for } t = S-1, \cdots, 0.$$

To see this, note that

$$\frac{y_{t+1}}{x_t} = \frac{1-\frac{1}{2^{t+1}}}{\frac{1}{2^t}} = 2^t - 0.5 < 2^S - 0.5, \text{ for } t = S-1, \cdots, 0.$$

It follows from (23) that

$$u_S\left(x_t, y_{t+1}\right) = x_t + 2y_{t+1} = \frac{1}{2^t} + 2\left(1 - \frac{1}{2^{t+1}}\right) = \frac{1}{2^t} + 2 - \frac{1}{2^t} = 2,$$

for all $t = S-1, \cdots, 0$. Lastly, note that Claim 1 implies that the utility of generation S evaluated at its own consumption bundle is 2. ∎

Claim 3. *Generation* $t = S \in \mathbb{M}$ *strictly prefers its own consumption bundle* $\left(\frac{1}{2^S}, 1-\frac{1}{2^{S+1}}\right)$ *to the consumption bundle of all succeeding generations,* $S+1, \cdots$.

Proof. Observe that the consumption bundles of generation $S+1, \cdots$ satisfy the following property

$$y_{t+1} > \left(2^S - 0.5\right) x_t, \text{ for } t = S+1, \cdots.$$

Given $\left\langle \frac{1}{2^{S+s}}, 1-\frac{1}{2^{S+s+1}}\right\rangle$, with $s \geqslant 1$, it follows from (23) that

$$u_S(x^*_{S+s}, y^*_{S+s+1}) = \left(\frac{2^{S+2}}{1+2^{S+1}}\right)\left(\frac{1}{2^{S+s}} + 1 - \frac{1}{2^{S+s+1}}\right) = 2 \cdot \left(\frac{1+2^{t+s+1}}{2^s + 2^{t+s+1}}\right) < 2,$$

whereas

$$u_S(x^*_S, y^*_{S+1}) = 2,$$

as was to be proven. ∎

By claims 1, 2 and 3, we have shown that the allocation given by 24 is envy free. It remains to show that it is also efficient.

Claim 4. *The allocation given in (24) is efficient.*

Proof. Suppose, by way of obtaining a contradiction, that this allocation is not efficient. Then, there exists a different allocation

$$\left\{x'_t, y'_{t+1}\right\}_{t=0}^{\infty}, \text{ where } x'_0 = x_0 = 1,$$

that dominates $\left\{\frac{1}{2^t}, 1-\frac{1}{2^{t+1}}\right\}_{t=0}^{\infty}$, i.e., using Claim 1, for all $t \in \mathbb{M}$

$$u_t\left(x'_t, y'_{t+1}\right) \geqslant 2, \text{ and } u_s\left(x'_s, y'_{s+1}\right) > 2 \tag{25}$$

for some $s \in \mathbb{M}$. Let

$$\hat{s} = \min\left\{s \in \mathbb{M} : u_s\left(x'_s, y'_{s+1}\right) > 2\right\}.$$

We can assume, without any loss of generality, that

$$x'_t + y'_t = 1 \text{ for all } t \in \mathbb{N}.$$

Note that by the definition of $\hat{s}$, $x'_{\hat{s}} = \frac{1}{2^{\hat{s}}}$. Therefore, it's easy to see that for the second (strict) inequality in (25) to hold, it must be the case that

$$y'_{\hat{s}+1} > 1 - \frac{1}{2^{\hat{s}+1}}.$$

There are two cases to consider: $\hat{s} > 0$, and $\hat{s} = 0$.

Case 1): $\hat{s} > 0$. In this case,

$$x'_{\hat{s}} = \frac{1}{2^{\hat{s}}},\ y'_{\hat{s}+1} = 1 - \frac{1}{2^{\hat{s}+1}} + \delta, \text{for some } \delta > 0, \tag{26}$$

$$x'_{\hat{s}} + y'_{\hat{s}+1} = 1 + \frac{1}{2^{\hat{s}+1}} + \delta,$$

and $\left(x'_t, y'_{t+1}\right) = \left(\frac{1}{2^t}, 1-\frac{1}{2^{t+1}}\right)$ for all $0 \leqslant t < \hat{s}$. Then, it follows from (26), feasibility, and (25) that

$$\frac{y_{t+1}}{x_t} > 2^t - 0.5$$

for all $t \geqslant \hat{s}$. So, it follows from (23) that

$$u_t\left(x'_t, y'_{t+1}\right) = \left(\frac{2^{t+2}}{1+2^{t+1}}\right)\left(x'_t + y'_{t+1}\right)$$

for all $t \geqslant \hat{s}$. The latter condition, combined with (25) above implies

$$x'_t + y'_{t+1} \geqslant 1 + \frac{1}{2^{t+1}}, \tag{27}$$

for all $t \geqslant \hat{s}$. Since $\left\{x'_t, y'_{t+1}\right\}_{t=0}^{\infty}$ is an allocation, (27) readily implies

$$y'_{t+1} \geqslant y'_t + \frac{1}{2^{t+1}}$$

for all $t \geqslant \hat{s}$. Therefore, it is easy to show by induction that

$$y'_{\hat{s}+n} \geqslant y'_{\hat{s}+1} + \frac{1}{2^{\hat{s}+1}} - \frac{1}{2^{\hat{s}+n}} \text{ for all } n = 2, 3, \cdots . \tag{28}$$

Now, using (26), (28) can be rearranged as follows:

$$y'_{\hat{s}+n} \geqslant 1 + \delta - \frac{1}{2^{\hat{s}+n}} \text{ for all } n = 2, 3, \cdots . \tag{29}$$

Since $\frac{1}{2^{\hat{s}+n}} \to 0$ as $n \to \infty$, there exists a $N \in \mathbb{N}$ such that $\frac{1}{2^{\hat{s}+N}} < \delta$. But then for this N, we must have that

$$y'_{\hat{s}+N} \geqslant 1 + \delta - \frac{1}{2^{\hat{s}+N}} > 1 \tag{30}$$

which is a contradiction.

Case 2): $\hat{s} = 0$. The analysis of this case is basically identical to the one of case 1).

∎

(b) $x_0 \in (0, 1)$: Consider the following class of intervals

$$I_n \equiv \left[\frac{n+2}{2^{n+1}}, \frac{2n+2}{2^{n+1}}\right), \; n \in \mathbb{N}, \tag{31}$$

and notice that they form a partition of $(0, 1)$, i.e.,

$$I_n \cap I_m = \emptyset \text{ for } n \neq m; \text{ and } \bigcup_{n \in \mathbb{N}} I_n = (0, 1).$$

Given any $x_0 \in (0, 1)$, there exists a unique $T \in \mathbb{N}$ such that $x_0 \in I_T$, i.e.,

$$\frac{1}{2^{T+1}} \leqslant \frac{1}{T}\left(x_0 - \frac{1}{2^T}\right) < \frac{1}{2^T}. \tag{32}$$

T will be used to define the allocation we examine in claim 5.

Claim 5. *The allocation*

$$\langle \hat{x}_k, \hat{y}_{k+1} \rangle_{k=0}^{\infty} \equiv \left\langle \left(\frac{T-k}{T}\right) x_0 + \left(\frac{k}{T}\right) 2^{-T}, 1 - \left(\frac{T-k-1}{T}\right) x_0 - \left(\frac{k+1}{T}\right) 2^{-T} \right\rangle_{k=0}^{T-1}, \left\langle \frac{1}{2^k}, 1 - \frac{1}{2^{k+1}} \right\rangle_{k=T}^{\infty}. \tag{33}$$

is feasible.[8]

[8]For example, if $x_0 \in [\frac{3}{4}, 1)$, then the allocation (33) is given by

$$\left\langle \left(x_0, \frac{1}{2}\right), \left(\frac{1}{2^t}, 1 - \frac{1}{2^{t+1}}\right)_{t=1}^{\infty} \right\rangle,$$

and for $x_0 \in [\frac{1}{2}, \frac{3}{4})$, the allocation is

$$\left\langle \left(x_0, 1 - \frac{x_0}{2} - \frac{1}{8}\right), \left(\frac{x_0}{2} + \frac{1}{8}, 1 - \frac{1}{2^2}\right) \left(\frac{1}{2^t}, 1 - \frac{1}{2^{t+1}}\right)_{t=2}^{\infty} \right\rangle,$$

and so on.

Proof. It follows from (32) that for every $t = 0$ to $t = T - 1$ we have

$$\hat{x}_t = \left(\frac{T-t}{T}\right) x_0 + \left(\frac{t}{T}\right) 2^{-T} > 0.$$

Moreover,

$$\hat{x}_t - \hat{x}_{t+1} = \left(\frac{T-t}{T}\right) x_0 + \left(\frac{t}{T}\right) 2^{-T} - \left(\frac{T-t-1}{T}\right) x_0 - \left(\frac{t+1}{T}\right) 2^{-T} = \frac{1}{T}\left(x_0 - \frac{1}{2^T}\right) > 0.$$

Thus, $\langle \hat{x}_t \rangle$ is a decreasing sequence and $0 < \hat{x}_t < 1$ for all $t = 0, \cdots, T-1$. On the other hand, $0 < \hat{x}_t = \frac{1}{2^t} < 1$ for $t = T, \cdots$. Furthermore, $\hat{y}_t = 1 - \hat{x}_t$ for all $t \in \mathbb{N}$, which completes the proof of the claim. ∎

According to (33), the consumption bundles of generations T, $T+1$, $\cdots$ are the same as the consumption bundles of the corresponding generations defined in (24). We already know from Claim 2 and Claim 3 that generation $t \in \{T, T+1, \cdots\}$ is indifferent between its own consumption bundle and the consumption bundles of generations $\{T, T+1, \cdots, t-2, t-1\}$, and strictly prefers its own consumption bundle to the consumption bundle of all subsequent generations, $\{t+1, \cdots\}$. In what follows, first we will prove that generations T, $T+1$, $\cdots$ do not envy the consumption bundles of generations 0, $\cdots$, $T-1$. Once we accomplish this, we will have shown that generation $t \in \{T, T+1, \cdots\}$ does not envy consumption bundles of any other generations. Finally, in order to establish the no-envy property of the allocation given by (33), we will only need to prove that generations $s \in \{0, 1, \cdots, T-1\}$ are envy-free.

Claim 6. *Generation* T, $T+1$, $\cdots$ *strictly prefers its own consumption bundle to the consumption bundle of the generations* 0, $\cdots$, $T-1$.

Proof. For generation $s \in \{0, 1, \cdots, T-1\}$, the ratio $\frac{\hat{y}_{s+1}}{\hat{x}_s}$ attains its maximum at $s = T-1$ as $\hat{x}_{s+1} < \hat{x}_s$ for all s.[9] Further, using (32) for any $t \in \{T, T+1, \cdots\}$, it is easy to see that

$$\frac{\hat{y}_T}{\hat{x}_{T-1}} = \frac{1 - \frac{1}{2^T}}{\left(\frac{1}{T}\right)\left(x_0 - \frac{1}{2^T}\right) + \frac{1}{2^T}} < \frac{1 - \frac{1}{2^T}}{\frac{1}{2^T}} \leqslant \frac{\hat{y}_{t+1}}{\hat{x}_t}.$$

Therefore, using (23), one has that

$$u_t\left(\hat{x}_s, \hat{y}_{s+1}\right) = \hat{x}_s + 2\hat{y}_{s+1}.$$

Notice that

$$\hat{x}_s + 2 \cdot \hat{y}_{s+1} = 2 - x_0 + \frac{s+2}{T} \cdot \left(x_0 - \frac{1}{2^T}\right) \leqslant\ 2 - x_0 + \frac{T+1}{T} \cdot \left(x_0 - \frac{1}{2^T}\right).$$

[9]Note that $\frac{\hat{y}_{s+2}}{\hat{x}_{s+1}} = \frac{\hat{y}_{s+1} + \left(\frac{1}{T}\right)\left(x_0 - \frac{1}{2^T}\right)}{\hat{x}_s - \left(\frac{1}{T}\right)\left(x_0 - \frac{1}{2^T}\right)} > \frac{\hat{y}_{s+1} + \left(\frac{1}{T}\right)\left(x_0 - \frac{1}{2^T}\right)}{\hat{x}_s} > \frac{\hat{y}_{s+1}}{\hat{x}_s}$ for all $s \in \{0, 1, \cdots, T-1\}$.

On the other hand, it follows from (32) that

$$\begin{aligned}\hat{x}_s + 2 \cdot \hat{y}_{s+1} &= 2 - x_0 + \frac{s+2}{T} \cdot \left(x_0 - \frac{1}{2^T}\right) \leqslant \; 2 - x_0 + \frac{T+1}{T} \cdot \left(x_0 - \frac{1}{2^T}\right) \\ &= 2 - \frac{1}{2^T} + \frac{1}{T} \cdot \left(x_0 - \frac{1}{2^T}\right) < 2 - \frac{1}{2^T} + \frac{1}{T} \cdot \left(\frac{2T}{2^{T+1}}\right) \\ &= 2 - \frac{1}{2^T} + \frac{1}{2^T} = 2 = u_t\left(\hat{x}_t, \hat{y}_{t+1}\right).\end{aligned}$$

This establishes that agent $t \in \{T, T+1, \cdots\}$ attains higher utility from its own consumption bundle than utility from the consumption bundle of $s \in \{0, 1, \cdots, T-1\}$. ∎

Next, Claim 7 establishes a fact which will be instrumental in computating the utility of generation $s \in \{0, 1, \cdots, T-1\}$.

Claim 7.

$$\frac{\hat{y}_{s+1}}{\hat{x}_s} > 2^s - 0.5,$$

for every $s \in \{0, 1, \cdots, T-1\}$.

Proof. It is easy to see from (33) that

$$\hat{x}_{s+1} = \hat{x}_s - \frac{1}{T}\left(x_0 - \frac{1}{2^T}\right).$$

Furthermore,

$$\begin{aligned}\left(\frac{1}{2^s} - \hat{x}_s\right) - \left(\frac{1}{2^{s+1}} - \hat{x}_{s+1}\right) &= \left(\frac{1}{2^s} - \frac{1}{2^{s+1}}\right) - \left(\hat{x}_s - \hat{x}_{s+1}\right) \\ &= \frac{1}{2^{s+1}} - \frac{1}{T}\left(x_0 - \frac{1}{2^T}\right) > \frac{1}{2^{s+1}} - \frac{1}{2^T} \geqslant 0.\end{aligned}$$

Thus, $\left(\frac{1}{2^s} - \hat{x}_s\right)$ is decreasing in s and attains its minimum at $s = T-1$. Define $x_s^* = \frac{1}{2^s}$, $y_{s+1}^* = 1 - \frac{1}{2^{s+1}}$, and observe that

$$\begin{aligned}x_s^* - \hat{x}_s = \frac{1}{2^s} - \hat{x}_s &= \frac{1}{2^s} - x_0 + \frac{s}{T}\left(x_0 - \frac{1}{2^T}\right) \\ &\geqslant \frac{1}{2^{T-1}} - x_0 + \frac{T-1}{T}\left(x_0 - \frac{1}{2^T}\right), \\ &= \frac{1}{2^{T-1}} - x_0 + \left(x_0 - \frac{1}{2^T}\right) - \frac{1}{T}\left(x_0 - \frac{1}{2^T}\right) \\ &= \frac{1}{2^{T-1}} - \frac{1}{2^T} - \frac{1}{T}\left(x_0 - \frac{1}{2^T}\right) = \frac{1}{2^T} - \frac{1}{T}\left(x_0 - \frac{1}{2^T}\right) > 0,\end{aligned}$$

where the last strict inequality follows from (32). Moreover, it is easy to see from (33) that $x_T^* = \frac{1}{2^T}$. Given that

$$x_{s+1}^* + y_{s+1}^* = \hat{x}_{s+1} + \hat{y}_{s+1},$$

we get $\hat{y}_{s+1} \geqslant y_{s+1}^*$ for all $0 \leqslant s \leqslant T-1$. Then, it follows from (24) that

$$\frac{\hat{y}_{s+1}}{\hat{x}_s} \geqslant \frac{y_{s+1}^*}{\hat{x}_s} > \frac{y_{s+1}^*}{x_s^*} = \frac{1-\frac{1}{2^{s+1}}}{\frac{1}{2^s}} = 2^s - 0.5.$$

∎

In order to prove that generation $s \in \{0, 1, \cdots, T-1\}$ is envy free, we will proceed in three steps. In the first step, we establish that generation $s \in \{0, 1, \cdots, T-1\}$ prefers its own consumption bundle to the consumption bundles of generations T, $T+1$, $\cdots$. In the second step, we show that generation $s \in \{0, 1, \cdots, T-1\}$ is indifferent between its own consumption bundle and the consumption bundles of generations $\{s+1, \cdots, T-1\}$. In the third step, we prove that generation $s \in \{0, 1, \cdots, T-1\}$ prefers its own consumption bundle to the consumption bundles of generations $\{0, 1, \cdots, s-1\}$.

Claim 8. *Generation* $s \in \{0, 1, T-1\}$ *prefers its own consumption bundle to the consumption bundle of generations* $t \in \{T, \cdots\}$.

Proof. Observe that by Claim 7, (23) and (33)

$$\begin{aligned} u_s(\hat{x}_s,\ \hat{y}_{s+1}) - u_s(\hat{x}_t, \hat{y}_{t+1}) &= \left(\frac{2^{s+2}}{1+2^{s+1}}\right)(\hat{x}_s + \hat{y}_{s+1}) - \left(\frac{2^{s+2}}{1+2^{s+1}}\right)(\hat{x}_t + \hat{y}_{t+1}) \\ &= \left(\frac{2^{s+2}}{1+2^{s+1}}\right)\left(1 + \frac{1}{T}\left(x_0 - \frac{1}{2^T}\right) - 1 - \frac{1}{2^{t+1}}\right) \\ &\geqslant \left(\frac{2^{s+2}}{1+2^{s+1}}\right)\left(\frac{1}{2^{T+1}} - \frac{1}{2^{t+1}}\right) \ \geqslant\ 0, \end{aligned}$$

where the weak inequality in the third line follows from (32) and the fact that $t \geqslant T$. ∎

Claim 9. *Generation* $s \in \{0, 1, T-2\}$ *is indifferent between its own consumption bundle and the consumption bundle of generation* $t \in \{s+1, \cdots, T-1\}$.

Proof. Notice that

$$\hat{x}_s + \hat{y}_{s+1} = \hat{x}_t + \hat{y}_{t+1} = 1 + \frac{1}{T}\left(x_0 - \frac{1}{2^T}\right).$$

Therefore,

$$\begin{aligned} u_s(\hat{x}_s,\ \hat{y}_{s+1}) - u_s(\hat{x}_t,\ \hat{y}_{t+1}) &= \left(\frac{2^{s+2}}{1+2^{s+1}}\right)(\hat{x}_s + \hat{y}_{s+1}) - \left(\frac{2^{s+2}}{1+2^{s+1}}\right)(\hat{x}_t + \hat{y}_{t+1}) \\ &= \frac{2^{s+2}}{1+2^{s+1}}\left[1 + \frac{1}{T}\left(x_0 - \frac{1}{2^T}\right) - 1 - \frac{1}{T}\left(x_0 - \frac{1}{2^T}\right)\right] = 0. \end{aligned}$$

Thus, no agent $s \in \{0, T-2\}$ envies the allocation of its successors $\{s+1, \cdots, T-1\}$. ∎

Claim 10. *Generation* $s \in \{1, \cdots, T-1\}$ *weakly prefers its own consumption bundle to the consumption bundle of generation* $t \in \{0, \cdots, s-1\}$.

Proof. As observed in claim 9, the allocation of each of these generations is such that

$$\hat{x}_s + \hat{y}_{s+1} = 1 + \frac{1}{T}\left(x_0 - \frac{1}{2^T}\right). \tag{34}$$

We need to consider two cases.

(a) $\frac{\hat{y}_{t+1}}{\hat{x}_t} > (2^s - 0.5)$: In this case, the consumption bundles of generations s and t are such that

$$u_s(\hat{x}_s,\ \hat{y}_{s+1}) = \left(\frac{2^{s+2}}{1+2^{s+1}}\right)(\hat{x}_s + \hat{y}_{s+1}),$$

and

$$u_s(\hat{x}_t,\ \hat{y}_{t+1}) = \left(\frac{2^{s+2}}{1+2^{s+1}}\right)(\hat{x}_t + \hat{y}_{t+1}).$$

But then, we know from (34) that

$$u_s(\hat{x}_s,\ \hat{y}_{s+1}) - u_s(\hat{x}_t,\ \hat{y}_{t+1}) = 0,$$

i.e., generation s is indifferent between its own consumption bundle and that of generation t.

(b) $\frac{\hat{y}_{t+1}}{\hat{x}_t} \leqslant (2^s - 0.5)$: In this case,

$$u_s(\hat{x}_t,\ \hat{y}_{t+1}) = \hat{x}_t + 2\hat{y}_{t+1}.$$

Then,

$$\begin{aligned}
u_s(\hat{x}_s,\ \hat{y}_{s+1}) - u_s(\hat{x}_t,\ \hat{y}_{t+1}) &= \left(\frac{2^{s+2}}{1+2^{s+1}}\right)(\hat{x}_s + \hat{y}_{s+1}) - (\hat{x}_t + 2\hat{y}_{t+1}) \\
&= \left(\frac{2^{s+2}}{1+2^{s+1}}\right)(\hat{x}_t + \hat{y}_{t+1}) - (\hat{x}_t + 2\hat{y}_{t+1}) \\
&= \left(\frac{2^{s+1}-1}{2^{s+1}+1}\right)\hat{x}_t - \left(\frac{2}{2^{s+1}+1}\right)\hat{y}_{t+1} \\
&= \left(\frac{2^{s+1}-1}{2^{s+1}+1}\right)\hat{x}_t\left[1 - \left(\frac{2}{2^{s+1}-1}\right)\left(\frac{\hat{y}_{t+1}}{\hat{x}_t}\right)\right] \\
&\geqslant \left(\frac{2^{s+1}-1}{2^{s+1}+1}\right)\hat{x}_t\left[1 - \left(\frac{2}{2^{s+1}-1}\right)(2^s - 0.5)\right] = 0,
\end{aligned}$$

where in the last inequality we have used the fact that $\frac{\hat{y}_{t+1}}{\hat{x}_t} \leqslant (2^s - 0.5)$. Thus, agent s weakly its prefers own consumption bundle to the consumption bundles of all such predecessors.

▮

Claims 8, 9, and 10 establish that generation $s \in \{0, \cdots, T-1\}$ weakly prefers its own consumption bundle to the consumption bundle of other generations. Therefore, allocation defined in (33) is envy-free. It remains to show that it is also efficient. Before we take up this task, it is worth noting that the utility values attained across this allocation are as follows. For generation $s \in \{0, 1, \cdots, T-1\}$, using (32) we get

$$u_s(x_s, y_{s+1}) = \left(\frac{2^{s+2}}{1+2^{s+1}}\right)\left(1 + \frac{1}{T}\left(x_0 - \frac{1}{2^T}\right)\right) < \left(\frac{2^{s+2}}{1+2^{s+1}}\right)\left(1 + \frac{1}{2^T}\right) \leqslant 2, \tag{35}$$

and for generation $t \in \{T, T+1, \cdots\}$, we have that

$$u_t(x_t, y_{t+1}) = 2. \tag{36}$$

Claim 11. *The allocation given by (33) is efficient.*

Proof. Suppose, by way of obtaining a contradiction, that this allocation is not efficient. Then, there exists a different allocation,

$$\left\{x'_t, y'_{t+1}\right\}_{t=0}^{\infty} \text{ where } x'_0 = x_0 \in (0, 1),$$

that Pareto-dominates it, i.e.,

(a) for all agents $s \in \{0, 1, \cdots, T-1\}$,

$$u_s\left(x'_s, y'_{s+1}\right) \geqslant \left(\frac{2^{s+2}}{1+2^{s+1}}\right)\left(1 + \frac{1}{T}\left(x_0 - \frac{1}{2^T}\right)\right);$$ [10]

(b) given (36), for all agents $t \in \{T, T+1, \cdots\}$

$$u_t\left(x'_t, y'_{t+1}\right) \geqslant 2;$$

(c) at least one agent is strictly better off.

Mimicking the approach in the proof of Claim 4, let $\hat{s}$ be the first generation which is strictly better off. There are two possible cases to consider.

(i) $\hat{s} < T$, i.e.,

$$u_{\hat{s}}\left(x'_{\hat{s}}, y'_{\hat{s}+1}\right) > \left(\frac{2^{\hat{s}+2}}{1+2^{\hat{s}+1}}\right)\left(1 + \frac{1}{T}\left(x_0 - \frac{1}{2^T}\right)\right),$$

(ii) $\hat{s} \geqslant T$, i.e.,

$$u_{\hat{s}}\left(x'_{\hat{s}}, y'_{\hat{s}+1}\right) > 2.$$

[10]The right hand side member of the above inequality follows from (35).

We can assume, without any loss of generality, that

$$x'_t + y'_t = 1 \text{ for all } t \in \mathbb{N}.$$

We consider case (i) first. By definition of $\hat{s}$, we know that

$$x'_{\hat{s}} = \left(\frac{T-\hat{s}}{T}\right) x_0 + \left(\frac{\hat{s}}{T}\right) 2^{-T},$$

and

$$y'_{\hat{s}+1} = 1 - \left(\frac{T-\hat{s}-1}{T}\right) x_0 - \left(\frac{\hat{s}+1}{T}\right) 2^{-T} + \delta, \text{ for some } \delta > 0.$$

Therefore, it is easy to see that for all $s \in \{\hat{s}+1, \cdots, T-1\}$,

$$u_s\left(x'_s, y'_{s+1}\right) = \left(\frac{2^{s+2}}{1+2^{s+1}}\right)\left(x'_s + y'_{s+1}\right).$$

In order for

$$u_s\left(x'_s, y'_{s+1}\right) \geqslant \left(\frac{2^{s+2}}{1+2^{s+1}}\right)\left(1 + \frac{1}{T}\left(x_0 - \frac{1}{2^T}\right)\right),$$

to hold for all $\hat{s} < s < T$, we must have

$$x'_s + y'_{s+1} \geqslant 1 + \frac{1}{T}\left(x_0 - \frac{1}{2^T}\right),$$

which is equivalent to

$$x'_{s+1} \leqslant \left(\frac{T-s-1}{T}\right) x_0 + \left(\frac{s+1}{T}\right) 2^{-T} - \delta.$$

This leads to $x'_T \leqslant \frac{1}{2^T} - \delta$. Since

$$u_t\left(x'_t, y'_{t+1}\right) = \left(\frac{2^{t+2}}{1+2^{t+1}}\right)\left(x'_t + y'_{t+1}\right) \geqslant 2$$

for all $t \geqslant T$, we must have that

$$x'_t + y'_{t+1} \geqslant 1 + \frac{1}{2^{t+1}}, \text{ or } y'_{t+1} \geqslant y'_t + \frac{1}{2^{t+1}},$$

for all $t \geqslant T$. Note that

$$y'_{T+1} \geqslant 1 - \frac{1}{2^T} + \delta + \frac{1}{2^{T+1}} = 1 - \frac{1}{2^{T+1}} + \delta. \tag{37}$$

Next, we show by induction that

$$y'_{T+n} \geqslant y'_{T+1} + \frac{1}{2^{T+1}} - \frac{1}{2^{T+n}} \tag{38}$$

for all $n = 2, 3, \cdots$. Clearly (38) is true for $n = 2$ since

$$y'_{T+2} \geqslant y'_{T+1} + \frac{1}{2^{T+2}} = y'_{T+1} + \frac{1}{2^{T+1}} - \frac{1}{2^{T+2}}.$$

Now, assume that (38) is true for n. We must show that it holds true also for $n+1$. To see this, notice that

$$y'_{T+n} \geqslant y'_{T+1} + \frac{1}{2^{T+1}} - \frac{1}{2^{T+n}}$$

and

$$y'_{T+n+1} \geqslant y'_{T+n} + \frac{1}{2^{T+n+1}}.$$

Adding the two inequalities yields

$$y'_{T+n+1} \geqslant y'_{T+n} + \frac{1}{2^{T+n+1}} \geqslant y'_{T+1} + \frac{1}{2^{T+1}} - \frac{1}{2^{T+n}} + \frac{1}{2^{T+n+1}} = y'_{T+1} + \frac{1}{2^{T+1}} - \frac{1}{2^{T+n+1}}.$$

This establishes (38). Given (37), the latter can be rearranged as follows:

$$y'_{T+n} \geqslant 1 + \delta - \frac{1}{2^{T+n}} \text{ for all } n = 2, 3, \cdots .$$

As $\frac{1}{2^{T+n}} \to 0$ as $n \to \infty$, there exists a $N \in \mathbb{N}$ such that $\frac{1}{2^{T+N}} < \delta$. But then, for this N we must have that

$$y'_{T+N} \geqslant 1 + \delta - \frac{1}{2^{T+N}} > 1$$

which delivers the desired contradiction.

Case (ii): In this case the proof is basically identical to the argument set forth above to deal with case 1) in Claim 4. ∎

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