Non-linear Pricing on Information Structure with Application in Platform Design

Kai Hao Yang*

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Abstract

We study an optimal pricing problem for an intermediary through which transactions between a monopoly and the consumers take place and consumers receive information about the commodity. The intermediary can provide information to the consumers and charge the monopoly accordingly. We characterize the optimal menus and show that a menu consisting of *(garbled) upper censorship* that displays negative targeting feature is optimal and that surplus reduces comparing to a benchmark where the monopoly has control of the information technology.

KEYWORDS: Monopolistic pricing, platform design, screening, targeting, non-linear pricing, information design.

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1 Introduction

When making purchasing decisions, consumer's information about the commodity plays a central role, which in turns affects profits of a producer or a seller that can be generated

*Department of Economics, University of Chicago, khyang@uchicago.edu. I am indebted to Ben Brooks, Emir Kamenica, Elliot Lipnowski, Roger Myerson Doron Ravid and Phil Reny for valuable comments and suggestions.

by the sale. How well a consumer is informed about the quality of commodities and what kinds of information does a consumer posses greatly affects how much profit a producer can make by selling these commodities. Moreover, due to rapid development of information technology in the past few decades, the channels and platforms through which consumers receive information about commodities have been widely expanded. However, under many real-world contexts, the channel and platforms through which consumers receive information are neither owned by the producer nor the consumers themselves. Instead, it is often the case that a centralized third party owns the technology and the platform so that it has the ability to provide various information to the consumers of different commodities. For instance, Online platforms such as eBay or Amazon have a well-publicized website on which the commodities are presented to the consumers and different information about the commodities are provided to the consumers via photos, certificates, and descriptions of the product. Or, television broadcasters such as QVC have well-known broadcasting channels through which the sellers can present their products on the programs and during which the information about the commodity can be presented. Likewise, advertising agencies help sellers design advertisements and use them to publicize the commodities and inform the potential consumers. In these scenarios, the third party who owns the technology or platform for information provision are often not involved directly in the trade between the consumers and the producers, nor does it have direct interests in the profit that the producers can make (e.g. The third party often do not own shares of the seller's firm). Instead, such third party only has interests in monetary payments made by the producers. In other words, the owner of the information technology or platform often does not care about the outcome of the trade, rather, it wishes to "sell" such technology or platform—by designing different outlays and contents that contains different information about the commodity and charge prices accordingly—to the producers who seek to maximize profit by selling a commodity to consumers. In the era of vast development of informational technology and computational abilities, it is then of great importance to understand the incentives for such platform or information technology owners and how would such centralized informational intermediary, together with the flexibility of the information they can convey, affect the market. In this article, our main goal is to understand the incentive for such informational intermediary.

Specifically, we examine the optimal way for such intermediary to price and "sell" to the producers its technology or platform and the ways to convey information to the consumers. Furthermore, we investigate the impact on welfare of such information provision process that is concentrated to an intermediary which is separated from the producers and consumers.

To fix ideas, consider a simple parameterized model that illustrates why information conveyed to the consumer is crucial in determining seller's profit and why selling such information technology would be profitable for a intermediary. Suppose that a seller is trying to sell an indivisible good to a buyer. The buyer's valuation of the good is $v \in [0, \bar{v}]$, for some $\bar{v} \geq 3$. Consider a seller with cost $c \in [0, 1]$ who does not know the exact value of v but only that v is drawn from a uniform distribution on $[0, \bar{v}]$. If the buyer knows exactly the value of v, then the seller will choose a posted price that solves $\max_{v \in [0,\bar{v}]}(v - c)(1 - v/\bar{v})$, which gives profit of $((\bar{v} - c)/2)^2$. On the other hand, if the buyer knows nothing but the fact that the valuation is drawn from a uniform distribution, any seller with cost less than or equal to the expectation of v, $\bar{v}/2$ will set of price at $\bar{v}/2$ and the seller with cost greater than $\bar{v}/2$ will not sell at all. As such, the seller's profit is $((\bar{v} - 2c)/2)^+$. Notice that with different costs, the seller would prefer different information that the buyer has. For sellers whose costs are low, they will prefer the buyer to be not informed than to be fully informed about v, whereas for sellers whose costs are high, they will prefer otherwise. Therefore, as an intermediary who has the technology to provide the buyer different information, it could then benefit from setting up a menu of different information that is going to be provided to the buyer and charge the seller with different prices.

As a preview of our main result, a corollary of our analysis (Theorem 1) suggests that if the sellers are uniformly distributed in terms of their costs, the following menu will be optimal: There is a continuum of items in this menu, each of them is indexed by a cutoff $k \in [0, 2]$. For each item indexed by k, the intermediary provides information to the buyer so that the buyer perfectly learns about v whenever it is below k and learns nothing else whenever it is above k and charges a price $a + b(1 - k^2/4)$, where $a, b > 0$ are constants that depend only on $\bar{v}$. More generally, we show that an *upper censorship menu*—menu consisting of items that perfectly inform the buyer when the value is below a certain cutoff and nothing else when it is above the cutoff with a price that depend on the level of the cutoff—is optimal under an assumption on the distribution that requires the *information rent* of seller to be relatively small comparing to that of the buyer under full information, to which uniform distributions introduced above is an example. Furthermore, we also show that in general (Theorem 2), without additional assumption on the distribution of valuation and costs, a *garbled upper censorship menu* that resembles upper censorship menus, except that information the buyer gets when the valuation is below the cutoff is possibly garbled and that the largest cutoff in the menu becomes lower. Such optimal menus reflects two interesting features. First, it involves non-linear pricing and contains infinitely many items in the menu. As it is the *information structure* that is sold by the intermediary, a constant per-unit price cannot be well-defined. Furthermore, even if we can index the items in this menu by a one dimensional variable k, the price depends on k in a non-linear way in general. Second, any element in this optimal menu displays a *negative targeting* feature that has been discussed and documented in the advertising literature (see, for example Pancras & Sudhir (2007) and Blake, Nosko & Tadelis (2013)). That is, in order to implement the upper (or garbled upper) censorship, the intermediary will provide detailed information to the buyer when his value is low and inform him perfectly (or give relatively more information) while the information provided to high-value buyer is relatively coarse. This optimal menu prescribes the intermediary to adopt (garbled) upper censorship that display negative targeting and the scope of targeting is determined by the cutoff of each items. For items with higher cutoff, the buyer receives more information in the sense of Blackwell order and the scope of targeting is larger.

The rest of this paper is structured as follows: In the next section, we summarize the related literature and mark the connections and differences between the literature and our paper. In section 3, we present the model and some preliminary analyses. In section 4, we provide characterizations of optimal menus under the baseline model. Section 5 includes an extension in which we allow the intermediary to also contract on publicity of the advertisement. Section 6 concludes.

2 Related Literature

This paper is related to several branches of literature in interplay between monopolistic pricing and information structure, selling information, and Bayesian persuasion. In the monopolistic pricing literature, Lewis & Sappington (1994) also examines how the change in consumer's information affects a monopolist's profit. They show that for a given monopolist with a constant production cost, the buyer having either full information or no information is optimal for the monopolist. Our model distinct from theirs in two major aspects. First, we examine how a third party would price information structures for a monopolist to purchase, instead of examining optimal information structure from a monopolist's perspective directly. Second, although the setting of Lewis & Sappington (1994) is close to a benchmark of our model in which the monopolist can choose information structure that the buyer has directly, we maintain an assumption that the commodity is indivisible so that buyers have 0-1 demand while the consumer's demand in their model can be more general. On the other hand, Lewis and Sappington (1994) restrict the information structures to vary within a one-dimensional family by assuming a particular disclosure rule, whereas our model allows full flexibility of the choice of information structure. Johnson & Myatt (2006) also studies how change in the distribution of buyers' valuation, which is equivalent to the information that buyer has in our model, affects a monopolist's profit. In particular, they show that when the distributions are ordered by the rotational ordering, the monopolist will prefer two extremes of the order. Using the language of buyers' information, this result is similar to Lewis & Sappington (1994) in that it implies that under a particular one dimensional (and hence, totally-ordered) family of information structures, a monopolist will either prefer the most informative one or the least informative one. Again, our model differs from theirs in that we focus on a third party's optimal menu and that we allow complete flexibility in providing information structures. Recent developments, on the other hand, have adopted flexible information structures. Bergemann, Brooks & Morris (2015) characterizes all the possible surplus division that can arise by giving the monopolist different information about the buyer's valuation. Roesler & Szentesz (2017) examines the buyer-optimal information structure when facing a monopoly.

There are several works that also study a problem of pricing information. Bergemann & Bonatti (2011) studies a pricing problem of a data provider who can provide information about the match value for a seller whose profit from trade depends on the match value of each consumer and the amount of investments the seller makes in each consumer. Although having a similar title, the model Horner & Skrzypacz (2016) is about disclosing an agent's private information to a decision maker who can choose whether to hire the agent to make the decision. The sell of information in their model is endogenous in the sense that it is through transfers that induces to agent to take proper tests to reveal their information in a selected equilibrium in their dynamic setting. Bergemann, Bonatti and Smolin (2018) studies an optimal menu for a data provider to sell experiments to a decision maker who has a private estimate about the state. Our model differs from the theirs in that our intermediary sells information to affect the information of the *buyer* to the *seller*, which affects the seller's value function on different information indirectly, whereas the model in Bergemann, Bonattil and Smolin (2018) focuses on selling information structure to a decision maker whose value function depends on the information she purchases directly.

Furthermore, our screening framework of selling information structure is analogous to standard monopolistic screening problems as in Mussa & Rosen (1978), Myerson (1981) and Maskin & Riely (1984). The screening problem in our model is more complicated in that it is essentially a mixture of adverse selection and moral hazard problem from the intermediary's perspective. Also, our assumption about the intermediary's ability to commit to a menu and the characterization of information structure follows from the the Bayesian persuasion literature, as Kamenica & Gentzkow (2011), Gentzkow & Kamenica (2016).

3 Model

There is a buyer (he), a seller (she) and an information intermediary (it). The seller is selling an indivisible object to the buyer. The buyer has quasi-linear preference with valuation $v \in [0, \bar{v}]$, for some $\bar{v} \in \mathbb{R}_+$. The buyer does not know about his valuation *á priori*. Rather, he has to learn about his valuation, which follows a common prior F, through the information provided by the intermediary. More precisely, the intermediary has the technology to design (and commit to) *information structures* in order to inform the buyer. After privately learning about the valuation, the seller then interacts with the buyer by designing selling mechanisms to maximize profit. The seller has private information about her constant marginal cost of production $c \in [0, \bar{c}]$, for some $\bar{c} \in \mathbb{R}_+$. This private cost is drawn from a common prior G. As such, the intermediary can "sell" the information technology to the seller by posting a menu of information structures and the associated payments. We assume that the intermediary has no direct interest in the allocation of the object but only revenue. Moreover, we also assume that the seller becomes visible to the buyer *only if* she uses the intermediary's technology. That is, if the seller does not buy from any item in the menu that the intermediary provides, she will then not be able to interact with the buyer and thus will always receive zero profit. To sum up, the timing of the model is described as below:

1. Nature draws valuation $v \sim F$ and cost $c \sim G$.
2. The intermediary posts a menu of information structures and payments to the seller.
3. The seller chooses whether and what item to buy from the menu posted by the intermediary.
4. Based on the selected item (if any), the seller pays the payment to the intermediary and the intermediary implements the information structure.
5. The buyer receives private signals from the information structure implemented by the intermediary (if any) and update beliefs about his valuation.
6. The seller (if possible) then designs selling mechanism to sell the object. When the buyer is indifferent, he breaks ties in favor of seller.[1]

Since the buyer has quasi-linear preference, the interim expected value is the only payoff relevant statistic for a given information structure. As such, the marginal distribution of the interim expected value conveys all the payoff-relevant aspect of a given information structure.

[1]Notice that from a subgame-perfection perspective, such tie-breaking rule is a result rather than an assumption, since only breaking tie in favor of the seller can the equilibrium exists.

As in Gentzkow & Kamenica (2016),[2] we may represent the information structures by the collection of CDFs of which the prior F is a mean preserving spread. That is, the collection of information structures can be represented by the set

$$\mathcal{H}_F := \left\{ H : [0, \bar{v}] \to [0,1] \middle| \int_0^x H(z)dz \leq \int_0^x F(z)dz, \, \forall x \in [0,\bar{v}] \right. \\ \left. \int_0^{\bar{v}} H(x)dx = \bar{v} - \mathbb{E}_F[v], \, H \text{ is a CDF} \right\}.$$

On the other hand, by quasi-linearity of the buyer's preference again, given any information structure $H \in \mathcal{H}_F$, since the interim expected value follows the distribution H, it is well-known that posted price mechanisms always achieves the maximal profit for the seller with any cost $c \in [0, \bar{c}]$. Thus, given any information structure $H \in \mathcal{H}_F$, when the buyer is provided (private) information according to H, the seller's profit maximization problem can be reduced to[3]

$$\max_{x \in [0,\bar{v}]} (x - c)(1 - H(x^-)).$$

Finally, by the standard revelation principle arguments, it is without loss to restrict the intermediary to post direct menus that are incentive compatible and individually rational: Menus that ask the seller to report her private cost c and assign an information structure $H^c \in \mathcal{H}_F$, an amount of payment $t(c) \in \mathbb{R}$ and a *publicizing policy*—decisions of whether to make the buyer aware of the seller, denoted by $\alpha(c) \in \{0,1\}$—to each report in a way that the seller will always participate and report truthfully. Formally, the intermediary posts incentive compatible and individually rational direct menu that takes form of $(H^c, t(c), \alpha(c))_{c \in [0,\bar{c}]}$ such that $H^c \in \mathcal{H}_F$, $t(c) \in \mathbb{R}$ and $\alpha(c) \in \{0,1\}$ for all $c \in [0,\bar{c}]$ and that

$$\alpha(c) \cdot \max_{x \in [0,\bar{v}]} (x - c)(1 - H^c(x^-)) - t(c) \geq \max \left\{ \alpha(c') \cdot \max_{x \in [0,\bar{v}]} (x - c)(1 - H^{c'}(x^-)) - t(c'), 0 \right\}.$$

for all $c, c' \in [0, \bar{c}]$, to maximize expected revenue

$$\mathbb{E}_G[t(c)] = \int_0^{\bar{c}} t(c) G(dc).$$

[2]Similar characterizations appear in many recent developments in the literature of mechanism and information design, see for instance Neeman (2003), Bergenamm and Pesendorfer (2007), Shi (2012), Roseler & Szentes (2017), Kolotilin et al. (2017), Bergemann, Brooks and Morris (2017a), Du (2017), Dworczak and Martini (2017), Brooks and Du (2018).

[3]As conventional, $H(x^-) := \lim_{\delta \downarrow 0} H(x - \delta)$ is the left-limit of H at x. The left limit is taken since the buyer breaks ties in favor of the seller and thus always buys when the price is equal to his expeted value.

Before proceeding in characterizing the incentive compatible and individually rational menus, we first observe that given a menu $(H^c, t(c), \alpha(c))_{c\in[0,\bar{c}]}$, for a seller with cost c, when the implemented information structure is $H^{c'}$ and the publicizing decision is $\alpha(c')$, the seller has profit

$$\Pi(c,c') := \alpha(c') \cdot \max_{x\in[0,\bar{c}]}(x-c)(1-H^{c'}(x^-))$$

and thus, if a seller with cost c reports her cost to be c', the net profit is given by

$$V(c,c') := \Pi(c,c') - t(c'),$$

which resembles the payoff functions as in standard screening problem (see, for example, Mussa & Rosen (1978), Myerson (1981), Maskin & Riley (1984)) with agents that have quasi-linear preference. However, in this model, a complication arises as the function Π is endogenous—it is derived from an optimization problem of the seller and depends on the information structure through a seller's optimal pricing strategy. In other words, from the intermediary's perspective, our model is in fact a mixture of screening—inducing the seller to report truthfully and moral hazard—inducing the seller to set prices in a desirable way. Nevertheless, as the object is indivisible, the cost of production affects seller's profit in an affine fashion. This gives the problem enough of structure so that the standard envelope characterization can be modified to accommodate our setting. This is given by the following Lemma.

Lemma 1. *Suppose that a menu $(H^c, t(c), \alpha(c))_{c\in[0,\bar{c}]}$ is incentive compatible and individually rational. Then for any selection*

$$x(c,c') \in \alpha(c') \cdot \operatorname*{argmax}_{x\in[0,\bar{v}]}(x-c)(1-H^{c'}(x^-)),$$

1. $t(c) = t(\bar{c}) + \alpha(c)\cdot \max_{x\in[0,\bar{v}]}(x-c)(1-H^c(x^-)) - \int_c^{\bar{c}} \alpha(z)(1-H^z(x(z,z)^-))dz$.

2. $t(\bar{c}) \leq 0$.

3. *The function* $c \mapsto \alpha(c)(1-H^c(x(c,c)^-))$ *is nonincreasing.*

4. $\int_c^{c'}[\alpha(z)(1-H^z(x(z,z)^-)) - \alpha(c')(1-H^{c'}(x(z,c')^-)))]dz \geq 0$ *for any* $c', c \in [0,\bar{c}]$.[4]

[4]As conventional, for $a, b\in\mathbb{R}$, $b<a$ and any measurable function f, we define

$$\int_a^b f(x)dx := -\int_b^a f(x)dx.$$

Conversely, suppose that for a menu $(H^c, t(c), \alpha(c))_{c\in[0,\bar{c}]}$*, there exists a selection*

$$x(c, c') \in \operatorname*{argmax}_{x\in[0,\bar{v}]} \alpha(c')(x - c)(1 - H^{c'}(x^-))$$

satisfying:

1. $t(c) = \alpha(c) \cdot \max_{x\in[0,\bar{v}]}(x - c)(1 - H^c(x^-)) - \int_c^{\bar{c}} \alpha(z)(1 - H^z(x(z, z)^-))dz.$
2. $\int_c^{c'} [\alpha(z)(1 - H^z(x(z, z)^-)) - \alpha(c')(1 - H^{c'}(x(z, c')^-)))]dz \geq 0$ *for any* $c', c \in [0, \bar{c}]$.

Then $(H^c, t(c), \alpha(c))_{c\in[0,\bar{c}]}$ *is incentive compatible and individually rational.*

The proof of Lemma 1 can be found in the Appendix. Lemma 1 is similar to standard envelope characterization of inventive compatibility, where the induced probability of trade for a given information structure and publicizing decision is, $\alpha(c')(1 - H^{c'}(x(c, c')^-))$, is analogous to the role of "allocation" in standard problems. However, since the intermediary is facing a mixture problem rather than a one-dimensional screening problem, local incentive constraints will not be sufficient for global incentive compatibility in general, even with monotonicity of the (on path) probability of trade, as the possibility of double deviations complicates the incentive constraints. As a result, the characterization in Lemma 1 involves a family of inequalities that rules out all incentives to misreport, with or without double deviations, rather than a compact monotonicity condition as in standard screening problems.

By Lemma 1, for any incentive compatible menu $(H^c, t(c), \alpha(c))_{c\in[0,\bar{c}]}$, by Fubini's theorem, expected revenue can then be written as:

$$t(\bar{c}) + \int_0^{\bar{c}} \alpha(c) \left(\max_{x\in[0,\bar{v}]} (x - c)(1 - H^c(x^-)) - (1 - H^c(x(c, c)^-))\frac{G(c)}{g(c)} \right) G(dc),$$

where $x(c, c)$ is a selection of $\operatorname{argmax}_{x\in[0,\bar{v}]}(x - c)(1 - H^c(x^-))$. Lemma 1 and individual rationality then implies that optimal menus can be found by solving the problem:

$$\sup_{\{H^c\}\subset\mathcal{H}_F,\, \alpha:[0,\bar{c}]\to\{0,1\}} \int_0^{\bar{c}} \alpha(c) \left(\max_{x\in[0,\bar{v}]} (x - c)(1 - H^c(x^-)) - (1 - H^c(x(c, c)^-))\frac{G(c)}{g(c)} \right) G(dc)$$

$$\text{s.t.} \int_c^{c'} [\alpha(z)(1 - H^z(x(z, z)^-)) - \alpha(c')(1 - H^{c'}(x(z, c')^-)]dz \geq 0,\ \forall c, c' \in [0, \bar{c}], \tag{1}$$

where $x(z, c')$ is the largest element in $\operatorname{argmax}_{x\in[0,\bar{v}]}(x - z)(1 - H^{c'}(x^-))$ for all $z, c' \in [0, \bar{c}]$.

In the next section, we aim to solve the problem defined in (1) and derive the optimal menu for the intermediary.

4 Optimal Menu for the Intermediary

In this section, we will find an optimal menu that solves (1) and discuss the implications. To facilitate the derivation and stress the main intuition, we maintain regularity assumptions in this section. Specifically, we assume that both F and G are absolutely continuous and admit densities f and g, respectively. Furthermore, the *virtual valuation* under the prior F, $\phi(v) := v - \frac{1-F(v)}{f(v)}$, is strictly increasing and the *virtual cost* given by G, $\psi(c) := \min\{c + \frac{G(c)}{g(c)}, \bar{v}\}$ is also strictly increasing whenever $\psi(c) < \bar{v}$.[5]

As in standard screening problems, if we can find a family of information structures $\{H^c\}$ and a publicizing policy α that maximizes the integrand of the objective function in (1) pointwisely for all $c \in [0, \bar{c}]$ and find a transfer $t : [0, \bar{c}] \to \mathbb{R}$ such that the menu $(H^c, t(c), \alpha(c))_{c \in [0,\bar{c}]}$ is incentive compatible and individually rational, then the problem is solved. However, due to the double deviation concerns noted above, such approach might not be valid, as the pointwise maximization solution might not be incentive compatible. In this case, a more subtle approach will be needed. Specifically, as the main reason for the failure of pointwise maximization approach is that local incentive constraints with monotonicity are not sufficient—due to the possibility of double deviations, we need to keep track of all the global incentive constraints and the double deviation constraints when solving the problem. In what follows, we will use the duality approach to characterize the solution.[6] Below, we will first solve the problem for the intermediary under an additional assumption on the distributions F and G so that pointwise maximization approach is valid and characterize the solution, and then we will solve the problem generally by using duality approach.

4.1 Optimality of Upper Censorship Menu

Below, we first show that under an additional assumption about the distributions F and G, the solution for the intermediary's problem takes a simple form—an *upper censorship menu*

[5] The regularity assumption can be relaxed and the solution we provide below will not be affected qualitatively. Detailed discussions can be found in the Online Appendix.

[6] Using duality approach to solve mechanism design problems when local incentive constraints are not sufficient aligns with the recent development in the literature, see Bergemann, Brooks and Morris (2017b), where the approach is implicitly applied and Carroll & Segel (2017) with explicit application.

maximizes the intermediary's revenue. An upper censorship menu has a simple structure: For each reported cost c, the intermediary will provide an information structure so that whenever the buyer's value is below a certain cutoff, he learns exactly his value whereas when the buyer's value is above the cutoff, he learns nothing else than the value being above the cutoff. We will show that under an assumption that requires the seller's virtual cost not to be too high relative to the buyer's virtual valuation under full information, an upper censorship menu with cutoffs given by the virtual cost, $\psi(c)$, is optimal.

Specifically, we assume that for any $c \in [0, \bar{c}]$, $\phi(\psi(c)) \leq c$. That is, the virtual cost of seller with cost c is *below* the optimal monopolist price for this seller when the buyer has full information and define an *upper censorship menu* as follows:

Definition 1. Let $\widetilde{\psi} : [0, \bar{c}] \to [0, \bar{v}]$ be a nondecreasing function. Fix any $c \in [0, \bar{c}]$. An information structure $H \in \mathcal{H}_F$ is an *upper censorship* with cutoff $\widetilde{\psi}(c)$ if

$$H(x) = \begin{cases} F(x), & \text{if } x \in [0, \widetilde{\psi}(c)) \\ F(\widetilde{\psi}(c)), & \text{if } x \in [\widetilde{\psi}(c), \mathbb{E}_F[v|v > \widetilde{\psi}(c)]) \\ 1, & \text{if } x \in [\mathbb{E}_F[v|v > \widetilde{\psi}(c)], \bar{v}]. \end{cases}, \forall x \in [0, \bar{v}].$$

Moreover, we say that a menu $(H^c, t(c), \alpha(c))_{c \in [0, \bar{c}]}$ is an *upper censorship menu* with cutoff $\widetilde{\psi}$ if for all $c \in [0, \bar{c}]$, H^c is an upper censorship with cutoff $\widetilde{\psi}(c)$ whenever $\alpha(c) = 1$.

Our first result can then be formally stated as follows:

Theorem 1. *Suppose that $\phi(\psi(c)) \leq c$ for all $c \in [0, \bar{c}]$. For each $c \in [0, \bar{c}]$, let H_u^c be an upper censorship with cutoff $\psi(c)$, let $\alpha_u(c) = 1$ and let*

$$t_u(c) := (v(c) - c)(1 - F(\psi(c))) - \int_c^{\bar{c}} (1 - F(\psi(z)))dz,$$

where $v(c) := \mathbb{E}_F[v|v > \psi(c)]$. Then $(H_u^c, t_u(c), \alpha_u(c))_{c \in [0, \bar{c}]}$ is incentive compatible, individually rational and maximizes the intermediary's revenue among all the inventive compatible and individually rational menus.

Formal proof of Theorem 1 can be found in the Appendix. We provide a graphical illustration below. First notice that the set of available information structures for the intermediary can be represented by a family of convex functions the are majorized by the function

$x \mapsto \int_x^{\bar{v}}(1-F(z))dz$ and majorizes the function $x \mapsto (\mathbb{E}_F[v]-x)^+$, and share the same values at the end points, as illustrated in Figure 1, in which we plot the integral, from x to $\bar{v}$, as a function of x, of the prior F as the blue curve, the degenerate distribution F_0 that puts probability 1 on $\mathbb{E}_F[v]$ has the green curve and a generic $H \in \mathcal{H}_F$ as the red curve. Second, consider any $H \in \mathcal{H}_F$, fix a $c \in [0,\bar{c}]$ and consider a seller with cost $\psi(c)$. For simplicity, assume that H is continuous and that $\text{argmax}_{x\in[0,\bar{v}]}(x-\psi(c))(1-H(x))$ is a singleton and is denoted by x^*. Then the buyer's surplus is $\int_{x^*}^{\bar{v}}(1-H(x))dx$ and the seller's profit is $(x^*-\psi(c))(1-H(x^*))$. These two quantities can be easily represented on the graph introduced above, as illustrated in Figure 2. In general, for any information structure $H \in \mathcal{H}_F$ and any $x^* \in \text{argmax}_{x\in[0,\bar{v}]}(x-\psi(c))(1-H(x^-))$, the buyer's surplus is exactly the value of the convex function associated with H at x^* and the seller's profit is the difference between the height of the intersection of the vertical line $x=\psi(c)$ and the tangent line segment that is tangent to the convex function associated with H at x^* and the height of buyer's surplus.[7]

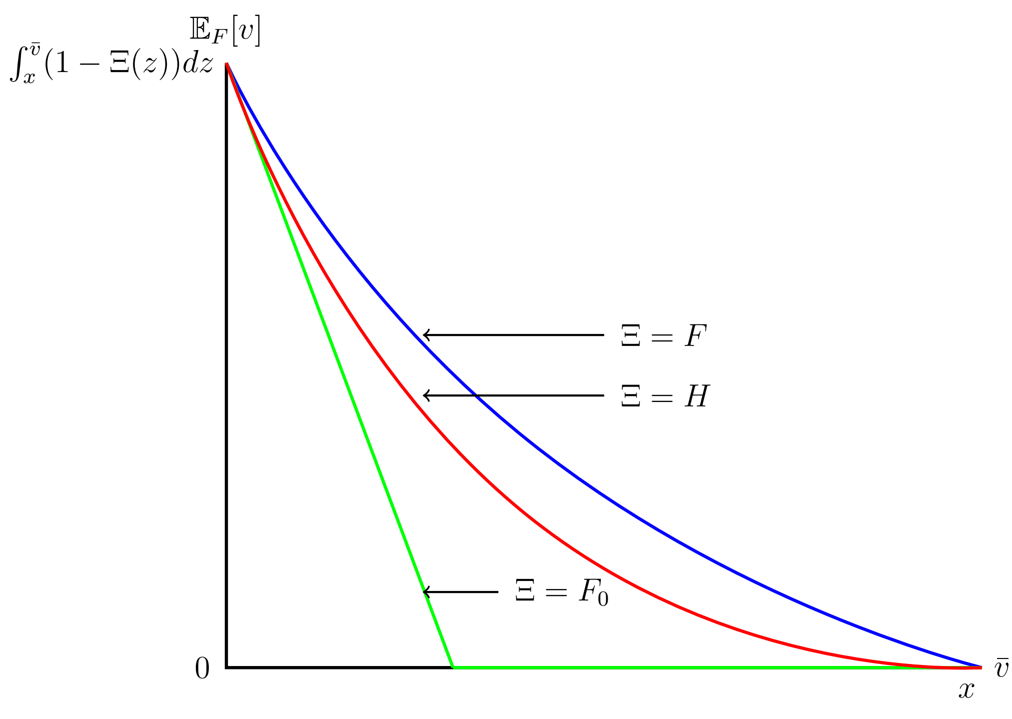

Figure 1: Feasible Set $\mathcal{H}_F$.

With this graphical representation, it is then convenient to solve the intermediary's prob-

[7]We thank Doron Ravid for suggesting this graphical representation.

lem (1) by the following procedure. Consider the integrad of the objective in (1) for any fixed $H \in \mathcal{H}_F$. Take any $x_H(c) \in \operatorname{argmax}_{x\in[0,\bar{v}]}(x-c)(1-H(x^-))$, we then notice that:

$$\begin{aligned} &\max_{x\in[0,\bar{v}]}(x-c)(1-H(x^-)) - (1-H(x_H(c)^-))\frac{G(c)}{g(c)} \\ \leq &\max_{x\in[0,\bar{v}]}(x-\psi(c))(1-H(x^-)). \end{aligned} \tag{2}$$

That is, for any $c \in [0,\bar{c}]$, whenever $\alpha(c) = 1$, the integrand of the objective in (1) is bounded from above by the maximized profit of a hypothetical seller with cost $\psi(c)$ instead of c. As such, if we can find a family of information structures $\{H^c\}_{c\in[0,\bar{c}]}$ that maximizes the optimal profit of this hypothetical seller, $\max_{x\in[0,\bar{v}]}(x-\psi(c))(1-H^c(x^-))$, for all $c \in [0,\bar{c}]$, and gives a nonnegative value, together with a transfer $t : [0,\bar{c}] \to \mathbb{R}$ such that the menu $(H^c, t(c), \alpha(c))_{c\in[0,\bar{c}]}$ with $\alpha \equiv 1$, is incentive compatible and individually rational, then it must be a solution of the intermediary's problem (1), as it attains an upper bound of its relaxed problem.

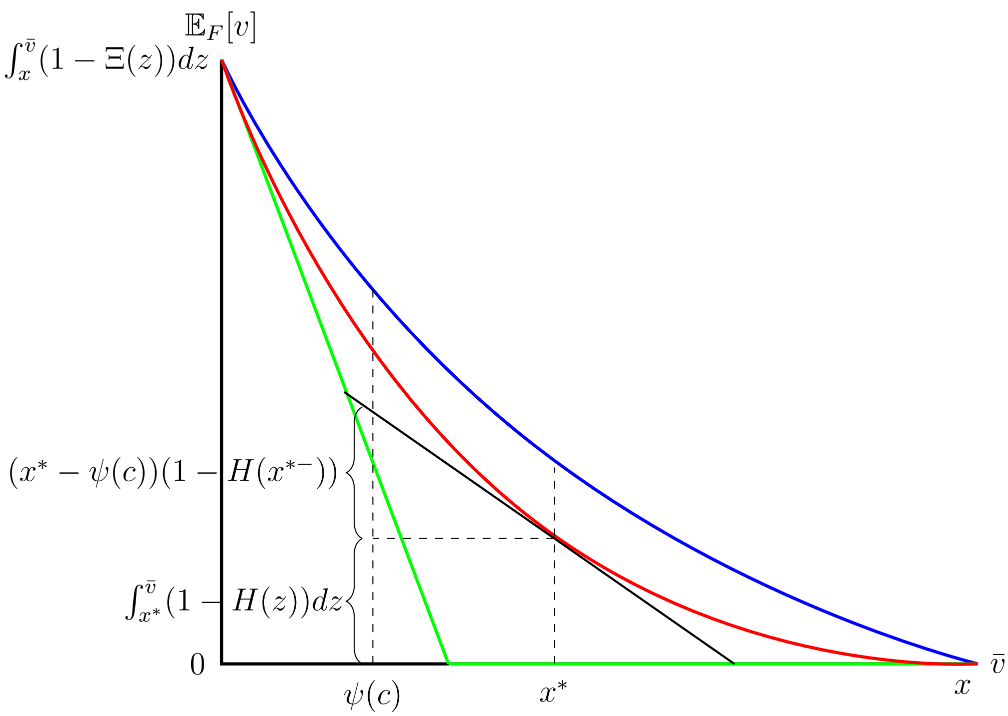

Figure 2: Representing Seller's Profit.

We will now argue that the upper censorship menu with cutoff $\psi(c)$ indeed satisfies the criteria above. First, fix any $c \in [0,\bar{c}]$ and suppose that a seller's cost is $\psi(c)$. Take any information structure $H \in \mathcal{H}_F$ and any $x^* \in \text{argmax}_{x\in[0,\bar{v}]}(x-\psi(c))(1-H(x^-))$. Consider an alternative information structure that garbles H: Inform the buyer nothing but whether his expected value given by H is above the original price x^*, as illustrated by the transition from the doted curve to the red curve in Figure 3. Under this information structure, every buyer value that would have bought at x^* under the information structure H will still buy at price x^*. However, all the buyers will then have valuation $\mathbb{E}_H[v|v > x^*]$ conditional on buying, meaning that the seller can set a higher price to extract surplus. In fact, the optimal price for the seller under this garbled information structure is exactly $\mathbb{E}_H[v|v > x^*]$, which allows the seller to extract all the expected surplus under the original information structure H. As such, to maximize the seller's optimal profit, it is without loss to search across information structures that has only two realizations. However, across those information structures, the largest surplus that can be extracted is $\int_{\psi(c)}^{\bar{v}}(1-F(x))dx \geq 0$, the total expected surplus when the seller's cost is $\psi(c)$. This is achieved by an information structure that discloses whether the buyer's value is above the cutoff $\psi(c)$, as illustrated by the green curve in Figure 3. Although the information structure that discloses whether the buyer's value is above a cutoff

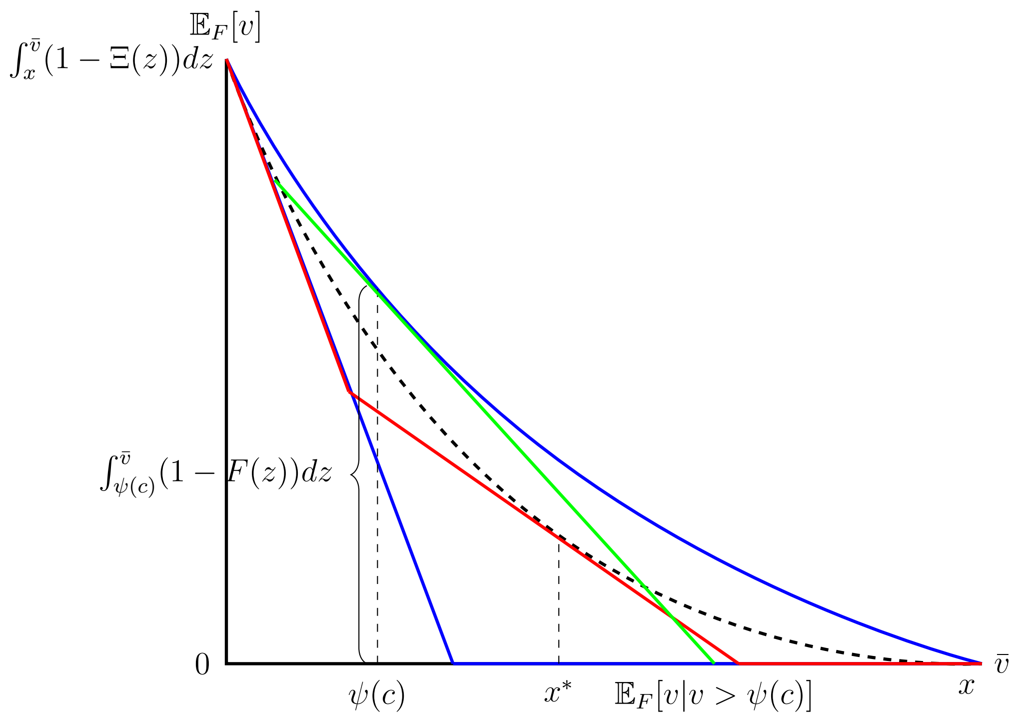

Figure 3: Upper-bound on Revenue.

$\psi(c)$ maximizes optimal profit of the hypothetical seller, $\max_{x\in[0,\bar{v}]}(x-\psi(c))(1-H^c(x^-))$, pointwisely, it may fail to be incentive compatible due to two reasons. First, the seller with costs c may not be willing to set the same optimal price as the hypothetical seller with cost $\psi(c)$. Second, the seller may have incentive to misreport. The intermediary may address the first issue by further giving the buyer all the information about his valuation whenever his value is below the cutoff $\psi(c)$ so that the information structure becomes an upper censorship with cutoff $\psi(c)$. Due to the assumption $\phi(\psi(c)) \leq c$, the seller with cost c would never optimally set prices at any point below $\psi(c)$, as $\psi(c)$ is below the optimal monopolist price under full information and the monopolist's profit function is single-peaked due to regularity. Therefore, for the seller with cost c, her optimal monopoly price under this information structure must coincide with the hypothetical seller with cost $\psi(c)$, namely $v(c)$. This shows that the upper censorship is able to incentivize the seller to set prices in a desirable way so that the pointwise upper bound in (2) can be attained. Finally, for incentive compatibility, using the assumption $\phi(\psi(c)) \leq c$ again, it can be verified that the inequality given by Lemma 1 is satisfied for all c, c'. Together, this shows that an upper censorship menu with cutoff ψ is optimal.

To better understand the implication of Theorem 1, we may consider a benchmark in which the seller has full control of the technology to provide information for the buyer to learn about his valuation. In this case, previous arguments imply that the seller with cost $c \in [0, \bar{c}]$ would prefer an upper censorship with cutoff c, as such information structure concentrates all the posterior expected value that are profitable for the seller (i.e. whenever $v > c$) to a singleton $\mathbb{E}[v|v > c]$ so that the seller can extract all the expected surplus by setting a price at $\mathbb{E}[v|v > c]$ and obtain profit $(\mathbb{E}[v|v > c] - c)(1 - F(c))$.

Being implicit in the statement of Theorem 1, under this optimal upper censorship, a seller with any cost $c \in [0, \bar{c}]$ would not only be willing to report truthfully and then face a buyer with distribution of posterior given by H_u^c, but would also optimally set a price at the highest possible posterior valuation $v(c)$. As such, for seller with cost $c \in [0, \bar{c}]$, profit from selling the object to the buyer is $(v(c) - c)(1 - F(\psi(c)))$. By comparing the intermediary-optimal information structure and the seller-optimal information structure, we can then see that the upper censorship is in fact the seller-optimal information structure had her cost c

been replaced by her virtual cost $\psi(c)$. Intuitively, the optimal upper censorship given in Theorem 1 is chosen so that the seller with cost c is treated as if she has the technology to provide information to the buyers and has cost $\psi(c)$ and then *internalizes* all the information rents for having private cost when making pricing decisions. By providing such information structure, together with a properly designed transfer rule to extract profits from the seller, the menu $(H_u^c, t_u(c), \alpha_u(c))_{c\in[0,\bar{c}]}$ will then be optimal. In other words, the upper censorship menu given in Theorem 1 grants the technology of providing information to the seller but with an adjustment of cost to incorporate information rents and then extract the sellers surplus via a transfer that is paid up-front.

Although the solution to the intermediary's revenue maximization problem given above is in terms of direct menu, there is a more interpretable menu that implements this solution. Consider a menu that contains a continuum of items, each of these items contains two components: an information structure that will be given to the buyer if selected and a "price" for such information structure. These items are indexed by a one-dimensional parameter $k \in [0, \psi(\bar{c})]$. For each k, the information structure $H^k \in \mathcal{H}_F$ is an upper censorship with cutoff k so that the buyer learns precisely his valuation when it is below k and nothing else when it is above k. On the other hand, the price for this information structure is[8]

$$t_k = (\mathbb{E}_F[v|v > k] - \psi^{-1}(k))(1 - F(k)) - \int_{\psi^{-1}(k)}^{\bar{c}} (1 - F(\psi(z)))dz.$$

Together, the menu $(H^k, t_k)_{k\in[0,\psi(\bar{c})]}$ implements the upper censorship menu given in Theorem 1, as a seller with cost $c \in [0, \bar{c}]$ will self-select herself to purchase the item indexed by $k = \psi(c)$ and yields the same outcome. Notice that when F and G are specialized to uniform distributions on $[0, \bar{v}]$ and $[0, 1]$, respectively, for some $\bar{v} \geq 3$, the menu $(H^k, t_k)_{k\in[0,\psi(\bar{c})]}$ is exactly the one presented in the introduction.

In addition to being easy to interpret, such upper censorship menu has another strength—it is in fact easy to implement in many real world contexts. An upper censorship can be understood as an information structure that gives the consumer detailed information about the commodity up to a certain level, beyond which the consumer will only have a

[8]Notice that under the assumption $\phi(\psi(c)) \leq c$ for all $c \in [0, \bar{c}]$, $\psi(c) < \bar{v}$ for all $c \in [0, \bar{c}]$ and hence, by regularity, $\psi^{-1} : [0, \bar{v}] \to [0, \bar{c}]$ is well defined.

coarse estimate. Now consider an example of selling objects whose contents are unclear *á priori* on an Online platform such as Amazon—for instance, textbooks. A commonly used disclosure policy under this context is to provide "previews" of these objects. For example, using the "Look Inside Program", Amazon provides book previews on the website and often allow the consumers to read selected pages and the table of contents of a book. Since such previews are limited, a consumer cannot learn about the object completely but only receive partial information about his valuation for the object. Moreover, in these contexts, it is relatively easy for the consumer to learn that the object has low value to him whereas only a coarse estimation can be formed when the value is higher. For instance, a consumer who is looking for graduate-level real analysis textbooks will learn precisely that a textbook for undergraduate real analysis is not suitable for him by simply looking at the table of content. On the other hand, even after previewing the table of content and several pages of a well-written, self-contained graduate-level real analysis book, such consumer can still not figure out the precise value of this book to him but rather has a coarse estimation. The nature of such contexts allow the intermediary to implement upper censorship information structure in a simple way: By determining the size or time frame of the preview, it effectively pins down a cutoff of an upper censorship—the larger/longer the preview the higher the cutoff. According to the discussions above, once an upper censorship is determined, the price for such information structure can then be computed.

4.2 Optimal Menu with General Distributions

Although the assumption $\phi(\psi(c)) \leq c$ guarantees optimality of a simple upper censorship menu, this assumption may not be satisfied for many reasonable applications. Qualitatively, it requires the information rent of the seller to be sufficiently small comparing to the information rent of the buyer under the prior, which, under regularity, is equivalent to requiring that the seller's virtual cost $\psi(c)$ is always below her optimal monopoly price when facing a fully-informed buyer. However, there are reasonable scenarios in which the seller's information rent is considerably high relative to the buyer's information rent under full information, such as cases when the prior distribution has low variances. To complete the analysis, we will now solve for the optimal menu without any assumptions on the distributions F and G

other than regularity.

As a preview of the result, the optimal menu in this general case has a similar structure. Specifically, recall that with $\phi(\psi(c)) \leq c$ for all $c \in [0, \bar{c}]$, the upper censorship menu with cutoff ψ has several critical features: 1) It publicizes the seller's object regardless of the reported cost $c \in [0, \bar{c}]$. 2) Given any reported $c \in [0, \bar{c}]$, it fully informs the buyer when his valuation is below the cutoff $\psi(c)$ and nothing else when his valuation is above the cutoff. 3) For seller with any cost $c \in [0, \bar{c}]$, the optimal monopoly price under this upper censorship is the buyer's expected value given that it is above the cutoff $\psi(c)$ and 4) truthfully reporting is optimal for the seller. The solution we characterize in this general environment displays similar features. In particular, for 1), it also publicizes the seller's object for all the reports except for a right tail on the interval $[0, \bar{c}]$. For 2) it also gives no further information to the buyer when his valuation is above a certain cutoff. However, the cutoff level may not be $\psi(c)$ and when the buyer's valuation is below the cutoff, the information that he receives might be garbled rather than his true value. Furthermore, both feature 3) and feature 4) are preserved.

More formally, the solution in the general environment has the following properties:

Definition 2. Let $\widetilde{\psi} : [0, \bar{c}] \to [0, \bar{v}]$ be a nondecreasing function. Fix any $c \in [0, \bar{c}]$. An information structure $H \in \mathcal{H}_F$ is a *garbled upper censorship* with cutoff $\widetilde{\psi}(c)$ if

$$H(x) = \begin{cases} F(\widetilde{\psi}(c)), & \text{if } x \in [\widetilde{\psi}(c), \mathbb{E}_F[v|v > \widetilde{\psi}(c)]) \\ 1, & \text{if } x \in [\mathbb{E}_F[v|v > \widetilde{\psi}(c)], \bar{v}]. \end{cases}, \forall x \in [\widetilde{\psi}(c), \bar{v}]$$

and

$$\int_0^{\widetilde{\psi}(c)} H(x)dx = \int_0^{\widetilde{\psi}(c)} F(x)dx.$$

Furthermore, a garbled upper censorship with cutoff $\widetilde{\psi}(c)$ is said to be *responsive* if

$$\mathbb{E}_F[v|v > \widetilde{\psi}(c)] \in \underset{x \in [0,\bar{v}]}{\operatorname{argmax}} (x - c)(1 - H(x^-)).$$

Also, a menu $(H^c, t(c), \alpha(c))_{c \in [0,\bar{c}]}$ is called a *responsive garbled upper censorship menu* with cutoff $\widetilde{\psi}$ if H^c is a responsive garbled upper censorship with cutoff $\widetilde{\psi}(c)$ for all $c \in [0, \bar{c}]$ whenever $\alpha(c) = 1$.

Using this formal definition, the above description for the optimal menu means that a responsive upper censorship menu with cutoff $\widetilde{\psi}$ is optimal, for some nondecreasing function $\widetilde{\psi} : [0, \bar{c}] \to [0, \bar{v}]$. Indeed, as Lemma 2 in the Appendix shows, one of the responsive garbled upper censorship menus must be optimal for the intermediary. It then remains to describe the cutoff function and the publicizing policy that gives an optimal menu. Before stating the formal result, we first examine how the intermediary's problem (1) depend on the choice of cutoff and how to understand the incentive constraints and responsiveness.

First, notice that by Lemma 1, for any incentive compatible menu $(H^c, t(c), \alpha(c))_{c\in[0,\bar{c}]}$, the function $c \mapsto \alpha(c)(1 - H^c(x(c, c)^-))$ must be nonincreasing. As such, whenever $\alpha(c) = 0$ for some $c \in [0, \bar{c})$, it must be that $\alpha(c')(1-H^{c'}(x(c', c'))) = 0$ for all $c' \in [c, \bar{c}]$. Therefore, (1) can be rewritten as choosing an upper bound of the reported costs to which the intermediary will publicize their product. That is, (1) can be written as:

$$\begin{aligned}
\sup_{\{H^c\}\subset\mathcal{H}_F,\, c_\alpha\in[0,\bar{c}]} & \int_0^{c_\alpha} \left(\max_{x\in[0,\bar{v}]} (x - c)(1 - H^c(x^-)) - (1 - H^c(x(c, c)^-))\frac{G(c)}{g(c)} \right) G(dc) \\
\text{s.t. } & \int_c^{c'} [H^{c'}(x(z, c')^-) - H^z(x(z, z)^-))]dz \geq 0,\ \forall c, c' \in [0, c_\alpha], \\
& H^c(x(c, c)^-) < 1,\ \forall c \in [0, c_\alpha).
\end{aligned} \tag{3}$$

As a result, the intermediary's problem can be solved separately: First, fix a $c_\alpha \in [0, \bar{c}]$ and choose a cutoff function $\widetilde{\psi}$ such that there exists a family of garbled upper censorship $\{H^c\}_{c\in[0,c_\alpha]}$ with cutoffs $\widetilde{\psi}$ that maximizes the objective of (3) subject to being responsive and incentive compatible. Then select an optimal c_α.

To understand the incentive compatibility and responsiveness constraints, in Figure 4 below, we fix a nondecreasing function $\widetilde{\psi}$ and plot the CDF of an upper censorship with cutoff $\widetilde{\psi}(c) \in (0, \bar{v})$ as the blue curve, where the jump point is given by $\widetilde{v}(c) := \mathbb{E}_F[v|v > \widetilde{\psi}(c)]$. This upper censorship is clearly a garbled upper censorship. For it to be responsive, a seller with cost c has to be willing to set the price at $\widetilde{v}(c)$. One way to understand this constraint is through the following graphical approach. Let $\widetilde{\pi}(c) := (\widetilde{v}(c) - c)(1 - F(\widetilde{\psi}(c)))$ be the seller's profit when setting a price at $\widetilde{v}(c)$. For the price $\widetilde{v}(c)$ to be optimal under the upper censorship menu, it has to be that

$$(x-c)(1-F(x)) \leq \widetilde{\pi}(c) \iff F(x) \geq \left(1-\frac{\widetilde{\pi}(c)}{(x-c)^{+}}\right)^{+}, \forall x \in [0, \widetilde{\psi}(c)].$$

That is, the CDF F has to be above the CDF of a *Pareto distribution* with parameters $\widetilde{\pi}(c)$ and c for all $x \in [0, \widetilde{\psi}(c)]$, as illustrated by the green curve in Figure 4. If, on the other hand, the Pareto CDF is above F, as illustrated by the red curve in Figure 4, then setting a price at $\widetilde{v}(c)$ would not be optimal under the upper censorship for the seller with cost c. In fact, the graphs of Pareto distributions can be regarded as *iso-profit curves* for the seller and the direction of increment is toward the button-right corner. Responsiveness is then equivalent to requiring that the graph of the CDF of an information structure must be always above the graph of the Pareto CDF with parameters $\widetilde{\pi}$ and c on $[0, \widetilde{\psi}(c)]$.

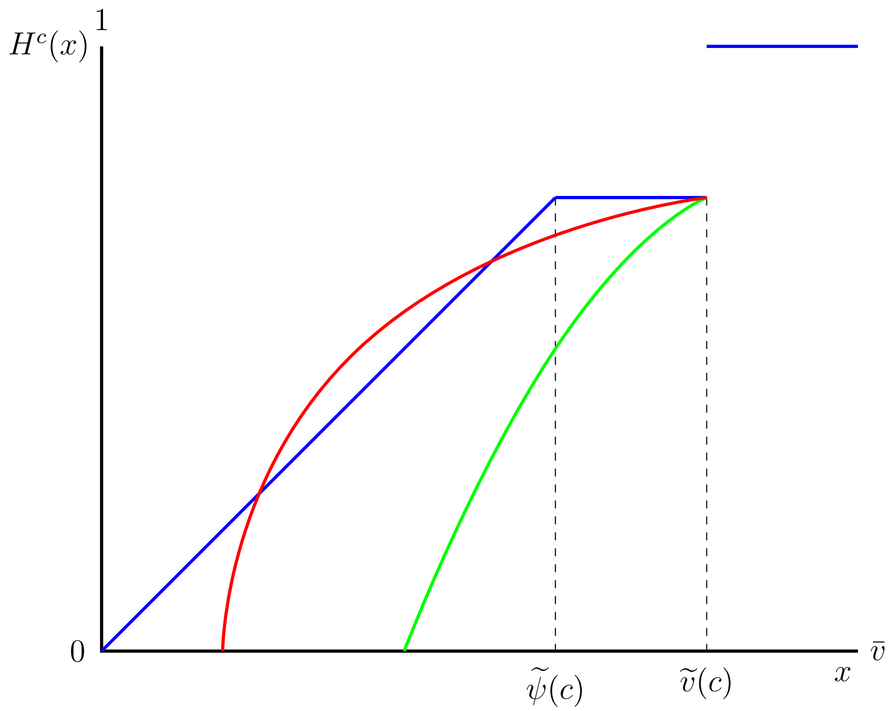

Figure 4: Upper Censorship with Cutoff $\widetilde{\psi}(c)$.

With the observation above, we then know that for any information structure H to be a responsive garbled upper censorship with cutoff $\widetilde{\psi}(c) \in [0, \bar{v}]$, it must be that: 1) The graph of H is always above the graph of the Pareto CDF with parameters $\widetilde{\pi}(c)$ and c. 2) The conditional CDF $F(x|x \leq \widetilde{\psi}(c))$ is a mean preserving spread of $H(x|x \leq \widetilde{\psi}(c))$. As such, a necessary condition for a garbled upper censorship menu $(H^c, t(c), \alpha(c))_{c\in[0,\bar{c}]}$ with cutoff $\widetilde{\psi}$ to be responsive and incentive compatible is that whenerver $\alpha(c) = 1$, the Pareto CDF $(1 - \widetilde{\pi}(c)/(x - c)^+)^+$ satisfies the second-order stochastic dominance constraint at $x = P(\widetilde{\psi}(c), c)$, where, due to regularity, this Pareto CDF crosses with F at at most two points other than zero and $P(\widetilde{\psi}(c), c)$ denotes the largest crossing point. That is: For all $c \in [0, \bar{c}]$ such that $\alpha(c) = 1$,

$$\int_0^{P(\widetilde{\psi}(c),c)} \left(1 - \frac{\widetilde{\pi}(c)}{(x-c)^+}\right)^+ dx \le \int_0^{P(\widetilde{\psi}(c),c)} F(x)dx. \tag{4}$$

Since if not, for the garbled upper censorship H to be responsive, as observed above, it must be that $H(x) \ge (1 - \widetilde{\pi}(c)/(x - c)^+)^+$ for all $x \in [0, \widetilde{\psi}(c)]$ and thus

$$\int_0^{\widetilde{\psi}(c)} H(x)dx \ge \int_0^{\widetilde{\psi}(c)} \left(1 - \frac{\widetilde{\pi}(c)}{(x-c)^+}\right)^+ dx > \int_0^{\widetilde{\psi}(c)} F(x)dx,$$

so that F cannot be a mean preserving spread of H.

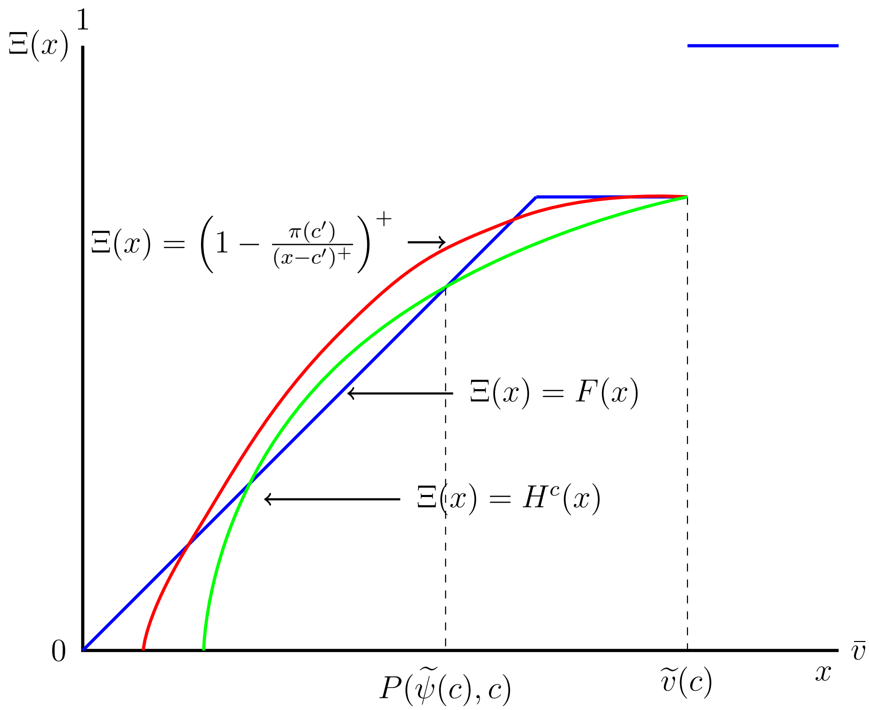

Figure 5: Necessary Condition for Responsiveness.

However, (4) is not sufficient for a garbled upper censorship menu $(H^c, t(c), \mathbf{1}_{[0,c_\alpha]}(c))_{c\in[0,\bar{c}]}$ with cutoff $\widetilde{\psi}$ to be responsive and incentive compatible at the same time, due to the possibility of double deviations. To see this, fix any $c_\alpha \in [0, \bar{c}]$, suppose that H^z is responsive for all $z \in [0, c_\alpha]$ and that (4) holds with equality for some $c \in (0, c_\alpha]$. Then the only possible way for H^c to be responsive is that $H^c(x) = (1 - \widetilde{\pi}(c)/(x-c)^+)^+$ for all $x \in [0, P(\widetilde{\psi}(c), c)]$. As such, for any $c' \in [0, c)$, the associated Pareto distribution (i.e. the iso-profit curve for the seller with cost c' at the price $\widetilde{v}(c)$, $(1 - \widetilde{\pi}(c')/(x-c')^+)^+$) is always above H^c, as illustrated in Figure 5, where the red curve represents the iso-profit curve $(1 - \widetilde{\pi}(c')/(x-c')^+)^+$ and the green curve represents such H^c. This implies that whenever the seller's cost is below c, she would optimally set a price at $\widetilde{\pi}(c) + c$, which gives $H^c((\widetilde{\pi}(c) + c)^-) = 0$. Consequently, if the seller's cost is c' and she misreports to be of cost c, the deviation gain is

$$\int_{c'}^{c} [F(\widetilde{\psi}(z)) - H^c((\widetilde{\pi}(c) + c)^-)]dz = \int_{c'}^{c} F(\widetilde{\psi}(z))dz > 0$$

and thus there exits no transfers $t : [0, \bar{c}] \to \mathbb{R}$ such that the menu $(H^c, t(c), \mathbf{1}_{[0,c_\alpha]}(c))_{c \in [0,\bar{v}]}$ is incentive compatible.

This problem occurs because (4) only accounts for the incentives for a truth-telling seller to set prices correctly so that the garbled upper censorship H^c could be responsive, but fail to account the misreporting seller to set prices in a desirable way and thus creates incentives for misreporting. The following menu, however, accounts for both by adjusting the cost for which we plot the iso-profit curves. Fix any nondecreasing function $\widetilde{\psi} : [0, \bar{c}] \to [0, \bar{v}]$ and fix $c_\alpha \in [0, \bar{c}]$ For each $c \in [0, c_\alpha]$, let

$$\widetilde{k}(c) := c - \frac{1}{F(\widetilde{\psi}(c))} \int_0^c F(\widetilde{\psi}(z))dz$$

and construct the Pareto iso-profit curve for seller with cost $\widetilde{k}(c)$ when she sets a price at $\widetilde{v}(c)$. That is, the CDF $(1 - \widetilde{\pi}(\widetilde{k}(c))/(x - \widetilde{k}(c))^+)^+$. Again, by regularity of F, this iso-profit curve crosses with F at at most two points other than zero, as illustrated by the green curve in Figure 6. Now if

$$\int_0^{P(\widetilde{\psi}(c), \widetilde{k}(c))} \left(1 - \frac{\widetilde{\pi}(\widetilde{k}(c))}{(x - \widetilde{k}(c))^+}\right)^+ dx = \int_0^{P(\widetilde{\psi}(c), \widetilde{k}(c))} F(x)dx,$$

we may then take $H^c(x) = (1 - \widetilde{\pi}(\widetilde{k}(c)))/(x - \widetilde{k}(c))^+)^+$ for all $x \in [0, P(\widetilde{\psi}(c), \widetilde{k}(c)))$ and $H^c(x) = F(x)$ for any $x \in [P(\widetilde{\psi}(c), \widetilde{k}(c)), \widetilde{\psi}(c)]$. Then clearly, as the iso-profit curve of setting price at $\widetilde{v}(c)$ for the seller with cost $c > \widetilde{k}(c)$ is below H^c, H^c is responsive, Furthermore, since the iso-profit curve crosses with F at two points other than zero, F is indeed a mean preserving spread of H^c. Finally, notice that for seller with any cost $c' \in (\widetilde{k}(c), c]$, optimal monopoly price under H^c is $\widetilde{v}(c)$, whereas for seller with cost $c' \in [0, \widetilde{k}(c))$, trade occurs with probability one given the optimal monopoly price under H^c. Therefore, by construction of $\widetilde{k}$, for any $c' \in [0, c]$, the deviation gain from misreporting to be of cost c is

$$\int_{c'}^{c} F(\widetilde{\psi}(z))dz - (c - \max\{c', \widetilde{k}(c)\})F(\widetilde{\psi}(c)) \leq \int_{0}^{c} F(\widetilde{\psi}(z))dz - (c - \widetilde{k}(c))F(\widetilde{\psi}(c)) = 0.$$

On the other hand, for any cost $c' \in (c, c_\alpha]$, setting price at $\widetilde{v}(c)$ is always optimal and therefore deviation gain is

$$\int_{c}^{c'} [F(\widetilde{\psi}(c)) - F(\widetilde{\psi}(z))]dz \leq 0,$$

which then ensures that there is no incentive for any other costs c' to misreport to be of cost c.

In fact, even if

$$\int_{0}^{P(\widetilde{\psi}(c),\widetilde{k}(c))} \left(1 - \frac{\widetilde{\pi}(\widetilde{k}(c))}{(x - \widetilde{k}(c))^+}\right)^+ dx < \int_{0}^{P(\widetilde{\psi}(c),\widetilde{k}(c))} F(x)dx,$$

an "ironing" procedure as illustrated by the red curve in Figure 6 can be applied and yields another feasible information structure H^c that is responsive and gives non-positive deviation gain for any cost c' to misreport to be of cost c.[9] Together, as long as

$$\int_{0}^{P(\widetilde{\psi}(c),\widetilde{k}(c))} \left(1 - \frac{\widetilde{\pi}(\widetilde{k}(c))}{(x - \widetilde{k}(c))^+}\right)^+ dx \leq \int_{0}^{P(\widetilde{\psi}(c),\widetilde{k}(c))} F(x)dx,$$

for all $c \in [0, c_\alpha]$, we can construct an incentive compatible, individually rational and responsive menu $(H^c, t(c), \mathbf{1}_{[0,c_\alpha]}(c))_{c\in[0,\bar{c}]}$ with cutoff $\widetilde{\psi}$.

In fact, by selecting a proper cutoff function $\widetilde{\psi}$ and a proper upper bound c_α, the menu constructed above will be optimal for the intermediary. To construct such function. We first notice that as c increases,

$$\int_{0}^{P(\widetilde{\psi}(c),\widetilde{k}(c))} \left(1 - \frac{\widetilde{\pi}(\widetilde{k}(c))}{(x - \widetilde{k}(c))^+}\right)^+ dx - \int_{0}^{P(\widetilde{\psi}(c),\widetilde{k}(c))} F(x)dx$$

[9]Formally, we find the convex hull of the minimum between the integral of prior F and such Pareto distribution and take the smallest sub-differential pointwisely. See the last two steps of the proof of Theorem 2 in the Appendix.

also increases. As such, if we define

$$c^* := \sup \left\{ c \in [0, \bar{c}] \middle| \int_0^{P(\psi(c),k(c))} \left(1 - \frac{\pi(k(c))}{(x - k(c))^+}\right)^+ dx \le \int_0^{P(\psi(c),k(c))} F(x)dx \right\},$$

where

$$k(c) := c - \frac{1}{F(\psi(c))} \int_0^c F(\psi(z))dz$$

and

$$\pi(c) := (v(c) - c)(1 - F(\psi(c))),$$

for all $c \in [0, \bar{c}]$, then

$$\int_0^{P(\psi(c),k(c))} \left(1 - \frac{\pi(k(c))}{(x - k(c))^+}\right)^+ dx \le \int_0^{P(\psi(c),k(c))} F(x)dx$$

if and only if $c \in [0, c^*]$. Now let $\psi^*(c) := \min\{\psi(c), \psi(c^*)\}$, and define

$$\hat{c} := \sup \left\{ c \in [0, \bar{c}] | \mathbb{E}_F[v|v > \psi(c^*)] \ge \psi(c) \right\},$$

the cutoff ψ^* and the publicizing policy $\mathbf{1}_{[0,\hat{c}]}$ are then in fact optimal, which is stated in the following Theorem.

Theorem 2. *For each $c \in [0, \bar{c}]$, let*

$$\alpha_{gu}(c) := \mathbf{1}_{[0,\hat{c}]}(c)$$

and let

$$t_{gu}(c) := (v^*(c) - c)(1 - F(\psi^*(c))) - \int_c^{\bar{c}} (1 - F(\psi^*(z)))dz,$$

where $v^*(c) := \mathbb{E}_F[v|v > \psi^*(c)]$. Then there exists a family of information structures $\{H^c_{gu}\}_{c\in[0,\bar{c}]} \subset \mathcal{H}_F$ such that $(H^c_{gu}, t_{gu}(c), \alpha_{gu}(c))_{c\in[0,\bar{c}]}$ is an incentive compatible, individually rational and responsive garbled upper censorship menu with cutoff ψ^* that maximizes the intermediary's revenue among all the inventive compatible and individually rational menus.

Formal proof of Theorem 2 can be found in the Appendix. We sketch the steps and stress the intuition here. As noted above, the main difficulty of the proof is the possibility of double deviation and hence local incentive constraint fails to imply global incentive constraints. We address this problem by examine the dual. Specifically, we first characterize the incentive

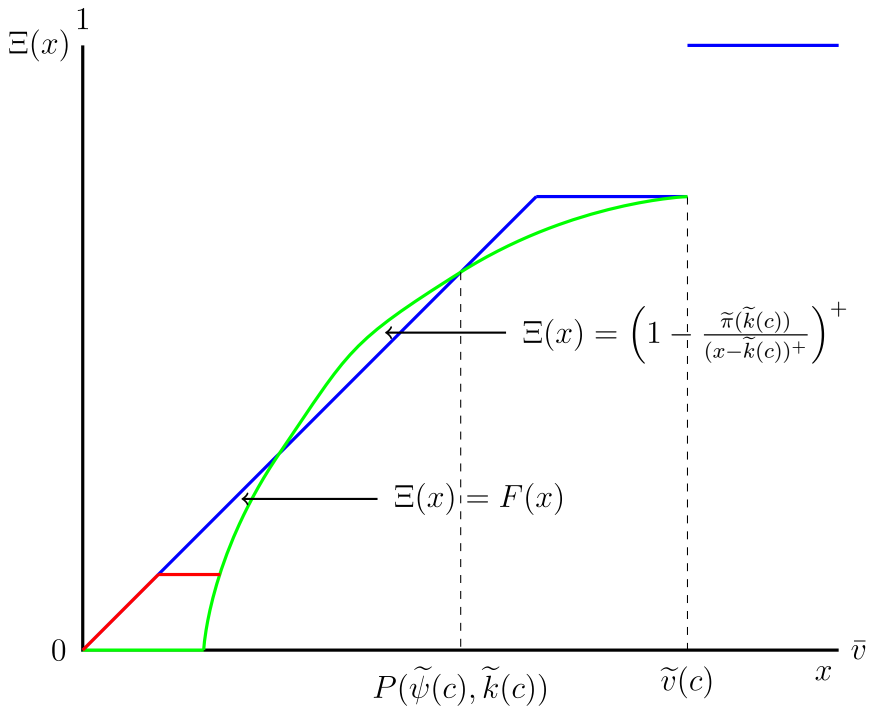

Figure 6: Optimal Responsive Garbled upper Censorship.

compatible, individually rational and responsive garbled upper censorship menus for any fixed upper bound c_α by a family of inequalities and further identify the critical ones that must be binding under any optimal menu. As such, for each fixed c_α, the intermediary's problem (3) then becomes a constraint maximization problem, to which we can write a dual problem. Then, we find the Lagrange multipliers under which the garbled upper censorship constructed above, together with the associated transfer induced by Lemma 1, is a solution of the dual problem and is incentive compatible and individually rational as well. By weak duality, we are then ensured that such menu is indeed optimal for a fixed c_α. Finally, we note that the optimal information structure and transfer does not depend on the selected c_α. As such, it then suffices to show that setting $c_\alpha = \hat{c}$ is indeed optimal.

Qualitatively, the optimal garbled upper censorship $\{H^c_{gu}\}_{c\in[0,\bar{c}]}$ is based on the Pareto distribution with parameters $\pi(k(c))$ and $k(c)$, possibly with some "ironing" procedures to ensure mean-preserving spread property. Under a particular garbled upper censorship H^c_{gu}, different costs of the seller are grouped into two classes, one that will optimally set prices at $v^*(c)$ as the truthfully-reporting type, while the others will set a price that gives zero

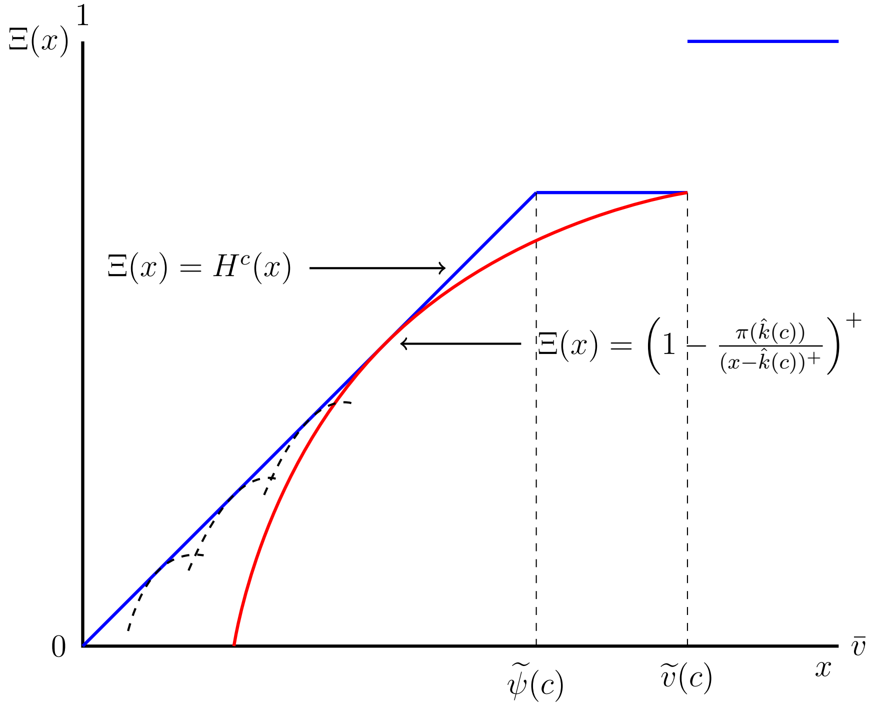

Figure 7: Optimality of Pareto-shape Garbling.

probability of trade. The reason for using this Pareto-shape garbling can be seen below: Fix any nondecreasing function $\widetilde{\psi} : [0, \bar{c}] \to [0, \bar{v}]$ and consider any incentive compatible, individually rational and responsive garbled upper censorship menu $(H^c, t(c), \mathbf{1}_{[0,c_\alpha]}(c))_{c\in[0,\bar{c}]}$ with cutoff $\widetilde{\psi}$. Recall that for any $c \in [0, c_\alpha]$, a family of Pareto-shaped distributions represent the iso-profit curves for each costs $c' \in [0, c]$ and must be everywhere below a CDF. As illustrated in Figure 7, where we take $H^c \equiv F$ on $[0, \widetilde{\psi}(c)]$,the distribution H^c must be above the upper envelope of a family of Pareto distributions. Indeed, each dotted curve in Figure 7 represents a Pareto distribution induced by a particular cost and the tangent point represents the optimal monopoly price for the seller with a given cost. Furthermore, notice that since $\widetilde{v}(c) > \widetilde{\psi}(c)$, there exists some $\hat{k}(c) \in [0, c)$ such that for all sellers with cost $c' \in [\hat{k}(c), c]$, setting price at $\widetilde{v}(c)$ as the truthfully-reporting type c does is optimal. Whenever such cutoff type $\hat{k}(c)$ is positive, some mis-reporting types would set prices differently and thus monotonocity of $\widetilde{\psi}$ would not be sufficient for global incentive compatibility. Such cutoff type $\hat{k}(c)$ must induce a Pareto distribution that crosses $(\widetilde{v}(c), F(\widetilde{\psi}(c)))$ and tangents to the distribution H^c at some lower points, as illustrated by the red curve in Figure 7.

Clearly, since $H^c \in \mathcal{H}_F$, the Pareto distribution induced by $\hat{k}(c)$ must satisfy the second order stochastic dominance constraint but with

$$\int_0^{P(\widetilde{\psi}(c),\hat{k}(c))} \left(1 - \frac{\widetilde{\pi}(\hat{k}(c))}{(x-\hat{k}(c))^+}\right)^+ dx < \int_0^{P(\widetilde{\psi}(c),\hat{k}(c))} F(x)dx. \tag{5}$$

Therefore, we may reduce the cutoff cost $\hat{k}(c)$ so that (5) holds with equality. Notice that by reducing $\hat{k}(c)$ locally, the first order effect on deviation gain for any type $c' \in [0, \hat{k}(c)]$ is

$$H^c(x(\hat{k}(c), c)) - F(\widetilde{\psi}(c)) < 0$$

and thus incentive compatibility can still be preserved by reducing the cutoff $\hat{k}(c)$. As a result, any incentive compatible, individually rational and responsive garbled upper censorship menu $(H^c, t(c), \mathbf{1}_{[0,c_\alpha](c)})_{c\in[0,\bar{c}]}$ induces another Pareto-shaped garbled upper censorship menu with the same cutoff $\widetilde{\psi}$. Moreover, since under any responsive menu with cutoff $\widetilde{\psi}$, the objective for the intermediary in (1) is

$$\int_0^{c_\alpha} (\widetilde{v}(c) - \psi(c))(1 - F(\widetilde{\psi}(c)))G(dc),$$

which only depends on the cutoff $\widetilde{\psi}$ and the upper bound of publicizing c_α. This illustrates the reason why Pareto-shaped garbling as described above can be optimal.[10]

Another way to understand the optimal garbled upper censorship menu in Theorem 2 is through the cutoff function ψ^*. In Figure 8, we plot the virtual cost ψ (blue curve) and the cutoff function ψ^* (red curve). Notice that if we drop the incentive constraints for truthfully-reporting costs and consider only the pricing constraint for responsiveness. In this case, (4) is in fact both necessary and sufficient for existence of a responsive garbled upper censorship with cutoff $\widetilde{\psi}$. This relation pins down a largest possible cutoff function below ψ that can be supported by a responsive garbled upper censorship menu, denoted by $\widehat{\psi}$ and illustrated by the green curve in Figure 8. Since the objective for the intermediary

[10]When examining the dual problem, there is a more convenient way to establish optimality of Pareto-shaped garbling by observing that the objective in the dual problem is a convex functional on family of uniformly bounded increasing functions, which ensures that one of the extreme points must be optimal. Under a our characterization, such extreme points correspond to Pareto-shape garbling. See detailed discussions in the Appendix.

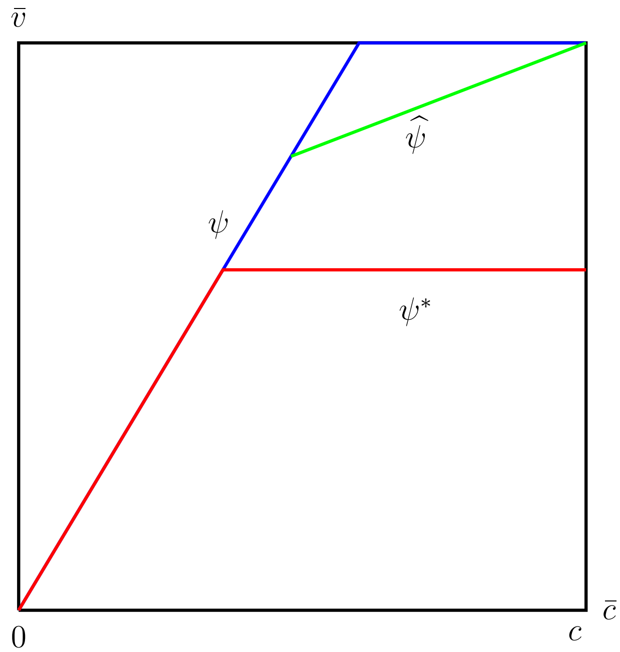

Figure 8: Cutoff Functions.

depends only on the cutoff function under responsive garbled upper censorship menus, the cutoff function $\widehat{\psi}$ and a proper publicizing policy would then be optimal when ignoring the truthfully-reporting constraints. To incorporate the truthfully-reporting constraints, we must further reduce $\widehat{\psi}$ to rule out mis-reporting incentives. Since the definition of function k gives binding incentive constraints for the lowest type 0, it is the "cheapest" way, in the sense that least adjustment for $\hat{\psi}$ is needed, to accommodate truthfully-reporting constraints among all the Pareto-shape garblings. This then gives the cutoff function ψ^*. Notice that at c^*, the Pareto iso-profit curve $(1-\pi(k(c^*))/(x-k(c^*))^+)^+$ agrees with the distribution $H_{gu}^{c^*}$ on $[0, \psi(c^*)]$, meaning that the constraint for responsiveness is binding at c^*. Furthermore, for any fixed $c_\alpha \in [c^*, \bar{c}]$, under the menu $(H_{gu}^c, t_{gu}(c), \mathbf{1}_{[0,c_\alpha]}(c))_{c\in[0,\bar{c}]}$, type 0 is indifferent between truthfully reporting and reporting to be of any cost $c \in [c^*, c_\alpha]$. However, since there is no distortion at the bottom under the menu $(H_{gu}^c, t_{gu}(c), \mathbf{1}_{[0,c_\alpha]}(c))_{c\in[0,\bar{c}]}$, the intermediary cannot further improve by relaxing the incentive constraint for type 0 in order to gain more for type c^*, which suggests optimality of the garbled upper censorship with cutoff ψ^* for each fixed c_α. Finally, since for any c_α, the expected revenue of the intermediary under the menu $(H^c_{gu}, t_{gu}(c), \mathbf{1}_{[0,c_\alpha]}(c))_{c\in[0,\bar{c}]}$ is

$$\int_0^{c_\alpha} (\widetilde{v}(c) - \psi(c))(1 - F(\widetilde{\psi}(c)))G(dc)$$

and is maximized by setting $c_\alpha = \hat{c}$, the menu $(H^c_{gu}, t_{gu}(c), \alpha_{gu}(c))_{c\in[0,\bar{c}]}$ is indeed optimal.

Notice that from the payoff perspective, the only difference between the optimal upper censorship menu in Theorem 1 and the optimal garbled upper censorship menu in Theorem 2 is when the seller's realized cost c is above the threshold c^*. It is straightforward to show that when $\phi(\psi(c)) \leq c$ for all $c \in [0, \bar{c}]$, $c^* = \hat{c} = \bar{c}$ and therefore $\psi \equiv \psi^*$ and $\alpha_{gu} \equiv 1$ on $[0, \bar{c}]$, meaning that the two menus give the same revenue when the condition in Theorem 1 is satisfied. On the other hand, when $c^* < \bar{c}$, the optimal garbled upper censorship menu in Theorem 2 will give all the sellers with cost between c^* and $\hat{c}$ the same information structure at the same price, which reflects a *bundling* property of this optimal menu and rule out all the sellers with costs above $\hat{c}$. The intuition is that, when the virtual costs of the seller is too high, granting information rents to these sellers will be too costly since there is not enough of leverage to create profit from the sell via manipulating the buyer's information. The intermediary would then have to sacrifice the additional gains from trade for high-virtual cost sellers to avoid paying too much information rent.

4.3 Welfare Analysis and Comparative Statics

We end this section by providing some welfare analysis and comparative statics. For simplicity, we focus on the case when the condition of Theorem 1 holds. Analyses with general regular distributions have qualitatively similar implications and can be found in the Online Appendix. To begin with, assume that F and G satisfy the condition $\phi(\psi(c)) \leq c$ for all $c \in [0, \bar{c}]$ so that the upper censorship menu with cutoff ψ is optimal. By Theorem 1, the total surplus generated by the sell is

$$(v(c) - c)(1 - F(\psi(c))) = \int_{\psi(c)}^{\bar{v}} (1 - F(x))dx + (\psi(c) - c)(1 - F(\psi(c))).$$

With the probability of trade being $(1 - F(\psi(c)))$, for all $c \in [0, \bar{c}]$. Comparing this with the benchmark case in which the seller has the technology to provide information to the buyer, where the optimal information structure, as noted above, gives total surplus generated by trade

$$(\mathbb{E}_F[v|v > c] - c)(1 - F(c)) = \int_c^{\bar{v}} (1 - F(x))dx$$

and a probability of trade $1 - F(c)$, we have the following observation:

Proposition 1 (Welfare Comparison)**.** *Suppose that $\phi(\psi(c)) \leq c$. Then the expected total surplus generated by trade and the probability of efficient trade are larger when the seller has control of the information technology than when the intermediary has control of the information technology.*

In brief, Proposition 1 shows that when the seller does not have the technology to provide information to the buyer directly but has to do so by interacting with a intermediary who has this technology, since the seller has private information about production cost, the ownership of such information technology matters. Indeed, when the seller has to buy such information technology from the intermediary, due to the presence of incomplete information, the seller would demand information rent from the intermediary and total surplus will be reduced since the intermediary has to provide information structures so that the seller would be willing to internalize her information rent when making pricing decisions.

Finally, we examine how the shifts of the distribution of valuation and the distribution of production cost affects the intermediary's revenue and the total surplus generated by trade. Specifically, take any two pairs of distributions F_1, G_1 and F_2, G_2 such that $\phi_i(\psi_j(c)) \leq c$, for all $c \in [0, \bar{c}]$, all $i, j \in \{1, 2\}$, where ϕ_i, ψ_i are the induced virtual value and virtual cost of F_i, G_i, respectively, for all $i \in \{1, 2\}$. The previous observation then gives us the following comparative statics analysis:

Proposition 2 (Comparative Statics)**.**

1. *Suppose that F_1 first order stochastic dominates F_2. That is, $F_1 \leq F_2$. Then the total surplus, the intermediary's revenue and seller's expected net profit under (F_1, G_i) are larger than those under (F_2, G_i), $i \in \{1, 2\}$.*

2. *Suppose that F_1 is a mean preserving spread of F_2. Then intermediary revenue under (F_1, G_i) is larger than that under (F_2, G_i), $i \in \{1, 2\}$.*

3. *Suppose that G_2 dominates G_1 in the hazard rate order. That is $g_1/G_1 \geq g_2/G_2$. Then the total surplus, the intermediary's revenue and the seller's expected revenue are larger under (F_i, G_1) than those under (F_i, G_2), $i \in \{1, 2\}$.*

To summarize, when the buyer's value becomes higher, in the sense of first order stochastic dominance, it becomes easier for the intermediary to generate trade surplus by providing proper information to the buyer and therefore total surplus, the intermediary's revenue and the seller's net profit all increase. When the distribution of valuation becomes more spread-out, in the sense of mean preserving spread, the informational tools for the intermediary becomes more flexible and therefore revenue increases. Finally, when the seller's cost shifts in hazard rate order, causing a reduction of information rent and the costs, in the sense of first order stochastic dominance, the seller retains less information rent and the distortion on information structure for her to internalize pricing decision reduces. These two factors jointly increase total surplus and intermediary's revenue as well. Furthermore, although the reduction of information rent and reduction of production has opposite effects on the seller's net profit, Proposition 2 shows that the gain in total surplus offsets the loss of information rent of the seller and hence also increases seller's expected net profit.

5 Extension: Contracting Publicity

In the baseline model analyzed above, we assumed that the buyer becomes aware of the seller's object only through the intermediary's technology so that the seller gets zero profit if she does not buy from any of the items in the menu that the intermediary offers. This often occurs in situations where the seller does not have significant publicity on the market and the intermediary owns a platform that is well known to the buyer, on which the seller's object can be presented. For instance, when a seller wants to sell an object on Online platforms such as Amazon, or when a seller wants to advertise her object through television programs such as QVC. However, in such scenarios, it is reasonable to argue that the intermediary—owning a platform that can not only provide information, but can also publicize the seller's object—can also screen the seller on the *extensive margin* by controlling how public it can make the seller's object be to the buyer, in addition to the *intensive margin* by providing different information to the buyer, as modeled above. In this section, we introduce this extra leverage to the intermediary. Specifically, we consider two extensions of the baseline model provided above. In the first extension, we allow the intermediary's menu to contain not only information structures that will be use to inform the buyer but also the level of publicity the seller's object is going to receive, which we refer as *non-discriminatory publicizing*. In the second extension, we further allow the intermediary to publicize the seller's object discriminatorily by "targeting" different buyers as well, which we refer as *discriminatory publicizing*.

5.1 Non-discriminatory Publicizing

For non-discriminatory publicizing, we allow the intermediary to also control the publicity of the seller's product to all the buyers. As such, in the associated direct menu, the publicizing policy α becomes a function with range on the interval $[0,1]$, where $\alpha(c)$ stands for the probability that the buyer will be aware of the seller's object when the reported cost is c. Comparing to the baseline model, the only difference is that α can now take any values in $[0,1]$. Thus, except that α has different range, characterization of incentive compatible and individually rational menus is the same as Lemma 1 and hence the revenue maximization problem (1) will be modified as:

$$\sup_{\{H^c\}\subset\mathcal{H}_F,\,\alpha:[0,\bar{c}]\to[0,1]} \int_0^{\bar{c}} \alpha(c)\left(\max_{x\in[0,\bar{v}]}(x-c)(1-H^c(x^-))-(1-H^c(x(c,c)^-))\frac{G(c)}{g(c)}\right)G(dc)$$

$$\text{s.t.} \int_c^{c'} [\alpha(z)(1-H^z(x(z,z)^-))-\alpha(c')(1-H^{c'}(x(z,c')^-)]dz \geq 0,\ \forall c,c'\in[0,\bar{c}], \tag{6}$$

It is rather straightforward to see that under the assumption of Theorem 1, $\phi(\psi(c))\leq c$ for all $c\in[0,\bar{c}]$, the upper censorship menu with cutoff ψ as in Theorem 1, together with an *always-publicizing* policy $\alpha_u\equiv 1$ maintains to be optimal. To see this, analogous to (2), we have that for any information structure $H\in\mathcal{H}_F$, any selection $x_H(c)\in\operatorname{argmax}_{x\in[0,\bar{v}]}(x-c)(1-H(x^-))$ and any publicizing policy α,

$$\begin{aligned}&\alpha(c)\left[\max_{x\in[0,\bar{v}]}(x-c)(1-H(x^-))-(1-H(x_H(c)^-))\frac{G(c)}{g(c)}\right]\\ \leq &\max_{x\in[0,\bar{v}]}(x-\psi(c))(1-H(x^-)).\end{aligned} \tag{7}$$

As the upper censorship menu maximizes the optimal profit for the hypothetical seller with cost $\psi(c)$, by (7), the revenue given by the upper censorship menu with cutoff ψ and publicizing policy $\alpha_u \equiv 1$ attains an upper bound of the seller's revenue. As this menu is incentive compatible and individually rational by arguments above and by Lemma 1, it also solves the intermediary's problem. As such, whenever the upper censorship menu with cutoff ψ is incentive compatible and individually rational, fully-publicizing upper censorship menu with cutoff ψ is optimal.

On the other hand, when we do not have the assumption $\phi(\psi(c)) \leq c$ for all $c \in [0, \bar{c}]$, the fully-publicized upper censorship menu with cutoff ψ will not be incentive compatible, as noted in the previous section. To understand the optimal menu in the general case, we first notice that optimality of responsive garbled upper censorship menus established by Lemma 2 is still valid when we allow the publicizing policy to take interior values. Furthermore, similar to the baseline model, by incentive compatibility, the set of seller's costs under which probability of sell is zero must be an interval $[c_\alpha, \bar{c}]$ for some $c_\alpha \in [0, \bar{c}]$. Thus, given any publicizing policy $\alpha : [0, \bar{c}] \to [0, 1]$, under any responsive garbled upper censorship with cutoff $\widetilde{\psi}$, the intermediary's expected revenue can be written as:

$$\int_0^{c_\alpha} \alpha(c)(\widetilde{v}(c) - \psi(c))(1 - F(\widetilde{\psi}(c)))G(dc). \tag{8}$$

Therefore, as in the proof of Theorem 2, the intermediary's problem can now be represented by a constraint maximization problem. In fact, the Lagrange multipliers constructed in the proof of Theorem 2 also warrants the garbled upper censorship menu $(H^c_{gu}, t_{gu}(c), \mathbf{1}_{[0,\hat{c}]}(c))_{c \in [0,\bar{c}]}$. Using the same argument as in the proof of Theorem 2, we can then conclude that the menu $(H^c_{gu}, t_{gu}(c), \mathbf{1}_{[0,\hat{c}]}(c))_{c \in [0,\bar{c}]}$ is still optimal and the intermediary's maximized revenue is the same as the baseline model where it cannot control the level of publicity.

5.2 Discriminatory Publicizing

On the other hand, for discriminatory publicizing, we allow the intermediary to not only control the publicity, but also decide which consumers are going to be aware of the object. For the ease of exposition, we interpret the baseline model to be with a continuum of consumers, whose distribution of valuation is F and each consumer does not know their valuation but knows the distribution for the whole economy. With this interpretation, discriminatory publicizing can be thought of as a technology that allows the intermediary to track down a consumer's true value and decides to whom to show a particular commodity—which has become possible thanks to the computational technology and availability of consumers' data including browsing histories and cookies. More formally, this means that the intermediary can select any arbitrary subset of consumers A to be aware of the object while all the other consumers remains ignorant. A compact way to model such selection is through the prior distribution F. Notice that when a consumer is not informed about the existence of the object, from the seller's perspective, it is equivalent to say that such consumer has zero value. As such, a discriminatory publicizing policy can be described by a (measurable) targeting set $A \subseteq [0, \bar{v}]$. Indeed, let μ_F be the probability measure associated with the CDF F. If the intermediary targets the group of consumers A, the prior distribution of valuations effectively becomes:

$$\mu^A = \mu_F|_A + \delta_{\{0\}} \cdot \mu_F([0, \bar{v}] \backslash A).$$

That is, the consumers who are not in the target set A are all treated as having zero valuation. For any targeting set A, let F^A be the CDF associated with μ^A. By the same arguments as in the baseline model, together with an additional requirement that whoever is not targeted cannot receive any information, the collection of information structures can then be characterized by the set

$$\mathcal{H}_A := \{H \in \mathcal{H}_{F^A} | H(0) = F^A(0)\}.$$

As such, for any $c \in [0, \bar{c}]$, an item of a direct menu consists of three components: A targeting set $A^c \in \mathcal{B}([0, \bar{v}])$; an information structure $H^c \in \mathcal{H}_{A^c}$, and a transfer $t(c) \in \mathbb{R}$ and the intermediary's problem is to find and incentive compatible and individually rational direct menu $(A^c, H^c, t(c))_{c \in [0, \bar{c}]}$ to maximize revenue.

As in the baseline model, incentive compatible and individually rational menus can be characterized by the envelope arguments, which allows us to determine the transfer t up to a constant once the targeting sets $\{A^c\}$ and the information structures $\{H^c\}$ are specified. Therefore, by Fubini's theorem, the intermediary's revenue maximization problem can be written as:

$$\sup_{\{A^c\}\subset\mathcal{B}([0,\bar{v}]),\,\{H^c\}\subset\mathcal{H}_{A^c}} \int_0^{\bar{c}} \left(\max_{x\in[0,\bar{v}]} (x-c)(1-H^c(x^-)) - (1-H^c(x(c,c)^-))\frac{G(c)}{g(c)} \right) G(dc)$$

$$\text{s.t. } \int_c^{c'} [(1-H^z(x(z,z)^-)) - (1-H^{c'}(x(z,c')^-)]dz \geq 0,\ \forall c,c' \in [0,\bar{c}]. \tag{9}$$

With the additional leverage on the publicizing policy, it is clear that the intermediary can do better.[11] Below, we argue that the intermediary's maximized revenue is exactly the expected revenue given by the upper censorship menu with cutoff ψ, even when the assumption $\phi(\psi(c)) \leq c$ for all $c \in [0,\bar{c}]$ fails. Indeed, first notice that since for any targeted set $A \in \mathcal{B}([0,\bar{v}])$, we must have

$$\int_0^x F^A(z)dz \leq \int_0^x F(z)dz, \forall x \in [0,\bar{v}].$$

Using the same arguments as in the proof of Theorem 1, a pointwise upper bound of the integrand of the objective in (9) is $(v(c)-\psi(c))(1-F(\psi(c)))$ and therefore

$$R^* := \int_0^{\bar{c}} (v(c)-\psi(c))(1-F(\psi(c)))G(dc)$$

is an upper bound on the intermediary's revenue. It then suffices to show that there exists an incentive compatible, individually rational menu $(A^c, H^c, t(c))_{c\in[0,\bar{c}]}$ such that the intermediary's expected menu attains R^*.

To see this, for any $c \in [0,\bar{c}]$, take $A^c_* := [\psi(c),\bar{v})$ and let

$$H^c_*(x) := \begin{cases} F(\psi(c)), & \text{if } x \in [0, v(c)) \\ 1, & \text{if } x \in [v(c),\bar{v}] \end{cases}.$$

Then $H^c \in \mathcal{H}_{A^c}$ for all $c \in [0,\bar{c}]$. Moreover, any seller with costs $c' \in [0, v(c)]$, including $c' = c$, will optimally set the price at $v(c)$ and any seller with cost $c' \in (v(c),\bar{c}]$ will not sell the object. Together, this implies that $H^c(x(c,c)^-) = F(\psi(c))$ for all $c \in [0,\bar{c}]$ and that

$$\int_c^{c'} [(1-H^z_*(x(z,z)^-)) - (1-H^{c'}_*(x(z,c')^-)]dz \geq 0,$$

[11] To see this, notice that if one take $A^c = [0,\bar{v}]$, such targeting set will then correspond to the publicizing policy $\alpha(c) = 1$ in the baseline model, whereas $\alpha(c) = 0$ can also be replicated by taking $A^c = \emptyset$.

for all $c, c' \in [0, \bar{c}]$. As such, there exists a transfer t^* such that the menu $(A^c_*, H^c_*, t^*(c))_{c\in[0,\bar{c}]}$ is incentive compatible and individually rational, with expected revenue

$$\int_0^{\bar{c}} t^*(c)G(dc) = R^*.$$

The solution constructed above can be implemented by a rather simple menu as follows. Consider a menu contains a continuum of items, each of them is indexed by $k \in [0, \min\{\psi(\bar{c}), \bar{v}\}]$. For each k, the targeted set is $[k, \bar{v}]$. That is, only the consumers with valuation above k can see the object. Furthermore, all these consumers receive the same signal, so that their information about their valuation will be nothing but being above k. Finally, the price for this item is

$$t_k = (\mathbb{E}_F[v|v \geq k] - \psi^{-1}(k))(1 - F(k)) - \int_{\psi^{-1}(k)}^{\bar{c}} (1 - F(\psi(z)))dz.$$

In other words, such menu exhibits *positive targeting* on the extensive margin and *no targeting* on the intensive margin. Comparing to the optimal menu in the baseline model that exhibits *negative targeting* on the intensive margin, items in this optimal menu also provides very coarse information to the buyers whose valuation is above the cutoff. However, instead of providing very detailed information to the buyers with lower valuation, when discriminatory publicizing is feasible, the intermediary will exclude these buyers by not targeting at them at all, which leads to positive targeting.

6 Conclusion

We studied an optimal pricing problem for an informational intermediary to sell information structures that informs the consumers of a monopolist. We found that the optimal menu displays an upper censorship feature consisting of information structures that discloses all—or a partially garbled, in general—information to the buyer when the valuation is low and no information when the valuation is high, which displays a negative targeting feature. Such menu contains a continuum of items that prescribes the range of consumer values that will be given full (or partially garbled, in general) information and prices for each information structure. A welfare analysis suggests that total surplus generated by the sell reduces comparing to a benchmark where the monopolist can disclose information themselves, due to the presence of incomplete information of production cost. We also show that both the intermediary's revenue and the monopolist's net profit increases when either the distribution of valuation shifts up in the sense of first order stochastic dominance or when the distribution of costs shifts up in the sense of hazard rate dominance. Moreover, the intermediary's revenue increases when the distribution of valuation becomes more dispersed. Finally, we extend our result to a situation when the intermediary can also contract on how well the monopolist is going to be perceived by the consumers. If the publicizing policy is non-discriminatory, the results in the baseline model carries over. On the other hand, if the intermediary is allowed to publicize the seller's commodity discriminatorily, then a menu that consists of *positive targeting* on the extensive margin, and provide no information to those who are targeted is shown to be optimal. Throughout the analyses, the outside option of the seller has been assumed to be zero, meaning that the seller must sell the commodity through the intermediary. Different specification of outside options, for instance, facing a buyer who has no information, could be analogously considered. This could be a topic for future research.

Although our model is a revenue maximizing problem of an informational intermediary who owns the technology to publicize a commodity and provide information to the consumers, the techniques that we developed in this framework—the characterization of inventive compatible menu, the pointwise maximization approach and the duality approach that solves for optimal menu—are applicable to a broader class of setting where the intermediary might have different objectives and the outside options of the buyer of information structures might be different, including regulatory policies on information disclosure of new products (e.g. financial products or drugs) that a government can implement to improve social welfare. These can also be topics for future studies.

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Appendix

A. Proof of Lemma 1.

Proof of Lemma 1. For necessity, consider any incentive compatible and individually rational menu $(H^c, t(c), \alpha(c))_{c\in[0,\bar{c}]}$. Let

$$\Pi(c,c') := \max_{x\in[0,\bar{v}]} \alpha(c')(x-c)(1-H^{c'}(x^-))$$

be the seller's expected profit under the information structure $H^{c'} \in \mathcal{H}_F$, publicizing policy $\alpha(c') \in \{0,1\}$ and cost c. By the envelope theorem (Milgrom & Segal, 2002), since the function

$$c \mapsto \alpha(c')(x-c)(1-H^{c'}(x^-))$$

is absolutely continuous with value uniformly bounded by $-\bar{c}$ and $\bar{v}$ for any fixed $x \in [0,\bar{v}]$, $c' \in [0,\bar{c}]$, $\Pi(\cdot, c')$ is absolutely continuous for all $c' \in [0,\bar{c}]$ and its derivative exists and equals to

$$\Pi_1(c,c') = -\alpha(c')(1-H^{c'}(x(c,c')^-)) \tag{10}$$

for any selection $x(c,c') \in \operatorname{argmax}_{x\in[0,\bar{v}]} \alpha(c')(x-c)(1-H^{c'}(x^-))$, for (Lebesgue) almost all $c \in [0,\bar{c}]$.

Now let

$$V(c,c') := \Pi(c,c') - t(c')$$

be the seller's profit net of transfer if the cost is c and the (mis)report c'. Incentive compatibility then implies

$$V^*(c) := V(c,c) = \max_{c'\in[0,\bar{c}]} V(c,c').$$

Since $\Pi(\cdot,c')$ is absolutely continuous and uniformly bounded by $-\bar{c}$ and $\bar{v}$, by the envelope theorem again,

$$V^*(c) = V(\bar{c}) - \int_c^{\bar{c}} \Pi_1(z,z)dz = V(\bar{c}) + \int_c^{\bar{c}} (1-H^z(x(z,z)^-))dz.$$

Rearranging, we have:

$$t(c) = t(\bar{c}) + \alpha(c)\cdot \max_{x\in[0,\bar{v}]}(x-c)(1-H^c(x^-)) - \int_c^{\bar{c}} (1-H^z(x(z,z)^-))dz,$$

which established assertion 1.

Furthermore, since V^* is nonincreasing, individual rationality implies that $-t(\bar{c}) = V^*(\bar{c}) \geq 0$, which established assertion 2.

In addition, by assertion 1, for any $c, c' \in [0, \bar{c}]$,

$$
\begin{aligned}
&\int_c^{c'} [\alpha(z)(1 - H^z(x(z,z)^-)) - \alpha(c')(1 - H^{c'}(x(z,c')))]dz \\
=&\int_c^{c'} \alpha(z)(1 - H^z(x(z,z)^-))dz - \int_c^{c'} \Pi_1(z,c')dz \\
=&\int_c^{c'} \alpha(z)(1 - H^z(x(z,z)^-))dz - (\Pi(c,c') - \Pi(c',c')) \qquad (11) \\
=&\int_c^{\bar{c}} \alpha(z)(1 - H^z(x(z,z)^-))dz - (\Pi(c,c') - \Pi(c',c')) - \int_{c'}^{\bar{c}} \alpha(z)(1 - H^z(x(z,z)^-))dz \\
=&V(c,c) - V(c,c') \\
\geq&0,
\end{aligned}
$$

where the first equality follows from (10), the second equality follows from the fundamental theorem of calculus and the last equality follows from assertion 1. This establishes assertion 3.

Finally, notice that for all $c' \in [0, \bar{c}]$, $\Pi(\cdot, c')$ is a pointwise supremun of a family of affine functions ans this is convex. Therefore, V^* is also convex as it is a pointwise supremum of a family of convex functions. Together, its subgradient $-(1 - H^c(x(c,c)^-)$ is nondecreasing in c. This proves assertion 4.

Conversely, take any menu $(H^c, t(c), \alpha(c))_{c \in [0,\bar{c}]}$ and any selection $x(c,c') \in \operatorname{argmax}_{x \in [0,\bar{c}]} \alpha(c')(x - c)(1 - H^{c'}(x^-))$ that satisfy conditions 1 and 2. Again, let $\Pi(c,c') := \max_{x \in [0,\bar{v}]} \alpha(c')(x - c)(1 - H^{c'}(x^-))$ and let $V(c,c') := \Pi(c,c') - t(c')$. By condition 1 and 2, (10) and (11), for any $c, c' \in [0, \bar{c}]$,

$$
V(c,c) - V(c,c') = \int_c^{c'} [\alpha(z)(1 - H^z(x(z,z)^-)) - \alpha(c')(1 - H^{c'}(x(z,c')^-)]dz \geq 0,
$$

where the inequality follows from condition 2. This completes the proof. ∎

B. Proofs of Main Results

B1. Proof of Theorem 1

Proof of Theorem 1. We first construct an upper bound of the intermediary's revenue given by (1), denoted by R^* and then show that the proposed upper censorship menu $(H_u^c, t_u(c), \alpha_u(c))_{c \in [0,\bar{c}]}$ is incentive compatible, individually rational and attains the upper bound R^*. First recall that given any incentive compatible and individually menu $(H^c, t(c), \alpha(c))_{c \in [0,\bar{c}]}$. Lemma 1 gives the intermediary's revenue as:

$$
t(\bar{c}) + \int_0^{\bar{c}} \alpha(c) \left((x(c,c) - c)(1 - H^c(x(c,c)^-)) - (1 - H^c(x(c,c)^-)) \frac{G(c)}{g(c)} \right) G(dc),
$$

where for each $c \in [0, \bar{c}]$, $x(c, c)$ is the largest selection of $\text{argmax}_{x\in[0,\bar{v}]}(x - c)(1 - H^c(x^-))$. Then for each $c \in [0, \bar{c}]$,

$$\alpha(c)\left((x(c,c) - c)(1 - H^c(x(c,c)^-)) - (1 - H^c(x(c,c)^-))\frac{G(c)}{g(c)}\right) \le \max_{x\in[0,\bar{v}]}(x - \psi(c))(1 - H^c(x^-)).$$

and thus for any incentive compatible and individually rational menu $(H^c, t(c), \alpha(c))_{c\in[0,\bar{c}]}$, the intermediary's revenue is bounded from above by

$$\int_0^{\bar{c}} \max_{x\in[0,\bar{v}]}(x - \psi(c))(1 - H^c(x^-))G(dc).$$

On the other hand, since $H^c \in \mathcal{H}_F$ for all $c \in [0, \bar{c}]$, for any $\tilde{x}(c, c) \in \text{argmax}_{x\in[0,\bar{v}]}(x - \psi(c))(1 - H^c(x^-))$,

$$\begin{aligned}
&\max_{x\in[0,\bar{v}]}(x - \psi(c))(1 - H^c(x^-)) \\
\le&(\tilde{x}(c,c) - \psi(c))(1 - H^c(\tilde{x}(c,c)^-)) + \int_{\tilde{x}(c,c)}^{\bar{v}}(1 - H^c(z))dz \\
\le&\int_{\psi(c)}^{\bar{v}}(1 - H^c(z))dz \\
\le&\int_{\psi(c)}^{\bar{v}}(1 - F(z))dz \\
=&(\mathbb{E}_F[v|v > \psi(c)] - \psi(c))(1 - F(\psi(c))),
\end{aligned}$$

where second inequality follows from monotonicity of H^c, the third inequality follows from the fact that F is a mean-preserving spread of H^c and the last equality follows from integration by parts. Together, we have that for any incentive compatible and individually rational menu $(H^c, t(c), \alpha(c))_{c\in[0,\bar{c}]}$,

$$\begin{aligned}
R^* :=& \int_0^{\bar{c}}(v(c) - \psi(c))(1 - F(\psi(c)))G(dc) \\
\ge& \int_0^{\bar{c}} \max_{x\in[0,\bar{v}]}(x - \psi(c))(1 - H^c(x^-))G(dc). \\
\ge& \int_0^{\bar{c}} \alpha(c)\left((x(c,c) - c)(1 - H^c(x(c,c)^-)) - (1 - H^c(x(c,c)^-))\frac{G(c)}{g(c)}\right)G(dc)
\end{aligned}$$

and therefore the intermediary's revenue given by any incentive compatible and individually rational menu $(H^c, t(c), \alpha(c))_{c\in[0,\bar{c}]}$ must be no greater than R^*.

Now notice that under the upper censorship menu $(H_u^c, t_u(c), \alpha_u(c))_{c\in[0,\bar{c}]}$, if each truthful-reporting seller whose cost is $c \in [0, \bar{c}]$ sets price optimally at $v(c)$, then for all $c \in [0, \bar{c}]$,

$$\max_{x\in[0,\bar{c}]}(x - c)(1 - H_u^c(x^-)) = (v(c) - \psi(c))(1 - F(\psi(c))). \tag{12}$$

Therefore, it suffices to show that (12) holds for the upper censorship menu $(H_u^c, t_u(c), \alpha(c))_{c\in[0,\bar{c}]}$ and that this menu is incentive compatible and individually rational, as this would imply that the menu $(H_u^c, t_u(c), \alpha_u(c))_{c\in[0,\bar{c}]}$ is feasible and attains the upper bound R^* of problem (1).

Indeed, first notice that $\phi(\psi(c)) \le c$ for all $c \in [0,\bar{c}]$ is equivalent to:

$$\psi(c) \le x_F(c),\ \forall c \in [0,\bar{c}],$$

where $x_F(c)$ is the unique element of $\operatorname{argmax}_{x\in[0,\bar{v}]}(x-c)(1-F(x))$. Take and fix any $c \in [0,\bar{c}]$, under the upper censorship H_u^c, for a seller with cost c, setting price at any $x \in [0,\psi(c)]$ gives profit

$$(x-c)(1-F(x)) \le (\psi(c)-c)(1-F(\psi(c))),$$

since the function $x \mapsto (x-c)(1-F(x))$ is single-peaked by regularity and since $x \le \psi(c) \le x_F(c)$. Furthermore, setting any prices in $[\psi(c), v(c))$ must be worse than setting price at $v(c)$ since H_u^c is a constant on $(\psi(c), v(c))$. Finally, for any $x \in (v(c),\bar{v}]$, the seller gets zero profit by setting a price at x. Together, for the truthfully-reporting seller with cost c, setting price at $v(c)$ is indeed optimal.

Moreover, for any $c, c' \in [0,\bar{c}]$, if $c' \le c$,

$$\begin{aligned}&\int_{c'}^{c}[(1-H_u^c(x(z,z)^-)) - (1-H_u^c(x(z,c)^-))]dz\\ =&\int_{\tilde{c}}^{c}[F(\psi(c)) - F(\psi(z))]dz + \int_0^{\tilde{c}}[F(x_F(z)) - F(\psi(z))]dz \ge 0,\end{aligned}$$

for some $\tilde{c} \in [0,c]$, where the inequality follows from monotonocity of ψ and that $x_F(c) \ge \psi(c)$ for all $c \in [0,\bar{c}]$. On the other hand, if $c' \in (c, v(c))$, then for any $z \in [c,c']$, $H_u^c(x(z,c)^-) = F(\psi(c)) \le F(\psi(z))$ by construction of H_c^u and by monotonocity of ψ. As such,

$$\int_{c'}^{c}[(1-H_u^c(x(z,z)^-)) - (1-H_u^c(x(z,c)^-))]dz = \int_{c}^{c'}[F(\psi(z)) - F(\psi(c))]dz \ge 0.$$

Finally, if $c' > v(c)$, optimal price under H_u^c gives zero profit and thus deviation gain must be negative. Together with Lemma 1, the upper censorship menu $(H_u^c, t_u(c), \alpha_u(c))_{c\in[0,\bar{c}]}$ is indeed incentive compatible and individually rational. This completes the proof. ■

B2. Proof of Theorem 2

Before stating the formal proof of Theorem 2, we first outline each step and sketch the structure of the proof. First, we will show that to maximize the expected revenue by choosing among all the possible incentive compatible and individually rational menu, it is without loss to restrict attention to incentive compatible, individually rational and responsive garbled upper censorship menus. This will be done by Lemma 2. Second, we develop a characterization for incentive compatible, individually rational and responsive garbled upper censorship menus that allows us to represent them by a family of inequalities so that the intermediary's problem can then be expressed as a constraint optimization problem. This will be done by Lemma 3 and Lemma 4. Next, we will then identify a class of critical constraints for the constraint optimization problem just obtained and show that for any feasible choice in the constraint optimization, two types of constraints must be met with equality. This is the content of Lemma 5. Finally, we will use the critical constraints to write down the dual of the constraint optimization problem and find the proper Lagrange multipliers so that the proposed menu indeed solves the dual problem, which effectively closes the duality gap and establishes optimality.

Proof of Theorem 2.

Step 1: We first show that it is without loss to restrict attention to the family of incentive compatible, individually rational and responsive upper censorship menus. This is implied by the following Lemma.

Lemma 2. *For any incentive compatible and individually ration menu $(H^c, t(c), \alpha(c))_{c \in [0,\bar{c}]}$, there exists a nondecreasing function $\widetilde{\psi} : [0, \bar{c}] \to [0, \bar{v}]$ and an incentive compatible and individually rational menu $(\widehat{H}^c, \widehat{t}(c), \widehat{\alpha}(c))_{c \in [0,\bar{c}]}$ such that $\widehat{H}^c$ is a responsive garbled upper-censorship with cutoff $\widetilde{\psi}(c)$ for all $c \in [0, \bar{c}]$ such that $\widehat{\alpha}(c) = 1$ and that*

$$\int_0^{\bar{c}} t(c)G(dc) \leq \int_0^{\bar{c}} \widehat{t}(c)G(dc).$$

Proof. Let $(H^c, t(c), \alpha(c))_{c \in [0,\bar{c}]}$ be an incentive compatible and individually rational menu. For each $c, z \in [0, \bar{c}]$, denote $x(z, c)$ by the largest element of

$$\underset{x \in [0,\bar{v}]}{\operatorname{argmax}} \, \alpha(c)(x - z)(1 - H^c(x^-))$$

and let $x(c) := x(c, c)$. Notice that by Lemma 1, the function $c \mapsto \alpha(c)(1 - H^c(x(c)^-))$ must be nonincreasing. Therefore, the set $\{c \in [0, \bar{c}] | \alpha(c)(1 - H^c(x(c)^-)) = 0\}$ must be an interval with supremum $\bar{c}$ and infimum $c_0 \in [0, \bar{c}]$. Notice that this implies $\alpha(c) = 1$ and $H^c(x(c)^-) < 1$ for all $c \in [0, c_0]$.

Thus, by Lemma 1 and Fubini's theorem,

$$\begin{aligned}\int_0^{\bar{c}} t(c)G(dc) &= t(\bar{c}) + \int_0^{c_0} (x(c)-\psi(c))(1-H^c(x(c)^-))G(dc)\\ &\leq \int_0^{c_0} (x(c)-\psi(c))(1-H^c(x(c)^-))G(dc). \qquad (13)\end{aligned}$$

For any $c \in [0, c_0]$, let $\widetilde{\psi}(c) := F^{-1}(H^c(x(c)^-))$. Notice first that by Lemma 1, $\widetilde{\psi}$ is nondecreasing. Now define an information structure $\widehat{H}^c$ as follows:

$$\widehat{H}^c(x) := \begin{cases} H^c(x), & \text{if } x \in [0, \underline{v}(c)) \\ F(\widetilde{\psi}(c)), & \text{if } x \in [\underline{v}(c), \widetilde{v}(c)) \\ 1, & \text{if } x \in [\widetilde{v}(c), \bar{v}] \end{cases},$$

where $\underline{v}(c)$ is uniquely determined by the equation

$$\int_0^{\widetilde{\psi}(c)} F(x)dx = \int_0^{\underline{v}(c)} H^c(x)dx + (\widetilde{\psi}(c) - \underline{v}(c))F(\widetilde{\psi}(c))$$

and

$$\widetilde{v}(c) := \mathbb{E}_F[v|v > \widetilde{\psi}(c)].$$

We claim that $\widetilde{v}(c)$ is the largest optimal price for a seller with cost c under the information structure $\widehat{H}^c$. That is,

$$\widetilde{v}(c) = \max\left\{\underset{x\in[0,\bar{v}]}{\operatorname{argmax}}(x-c)(1-\widehat{H}^c(x^-))\right\}.$$

Indeed, any $x \in (\underline{v}(c), \widetilde{v}(c))$ cannot be optimal since the function is strictly increasing on $(\underline{v}(c), \widetilde{v}(c))$. Also, any $x \in (\widetilde{v}(c), \bar{v}]$ cannot be optimal either since it gives zero profit to the seller. Finally, for any $x \in [0, \underline{v}(c)]$,

$$\begin{aligned}(x-c)(1-\widehat{H}^c(x^-)) &= (x-c)(1-H^c(x^-))\\ &\leq (x(c)-c)(1-H(x(c)^-))\\ &\leq (\mathbb{E}_{H^c}[v|v > x(c)] - c)(1-H(x(c)^-))\\ &\leq (\mathbb{E}_F[v|v > \widetilde{\psi}(c)] - c)(1-F(\widetilde{\psi}(c))),\end{aligned}$$

where the first equality follows from the fact that $x \in [0, \underline{v}(c)]$, the first inequality follows from optimality of $x(c)$ under H^c and the last inequality follows from the construction of $\widetilde{\psi}$ do that $F(\widetilde{\psi}(c)) = H^c(x(c)^-)$ and that $\mathbb{E}_{H^c}[v|v > x(c)] \leq \mathbb{E}_F[v|v > \widetilde{\psi}(c)]$, which follows from $F(\widetilde{\psi}(c)) = H^c(x(c)^-)$ and the fact that $H^c \in \mathcal{H}_F$.

For any $c \in [c_0, \bar{c}]$, select any $H \in \mathcal{H}_F$ and let $\widehat{H}^c := H$. We will now verify that there exists some transfer $\widehat{t} : [0, \bar{c}] \to \mathbb{R}$ such that the menu $(\widehat{H}^c, \widehat{t}(c), \mathbf{1}_{[0,c_0]}(c))_{c\in[0,\bar{c}]}$ is incentive compatible. By Lemma 1, it suffices to show that for any $c', c \in [0, c_0]$ with $c' < c$,

$$\int_{c'}^{c} [\widehat{H}^c(\widehat{x}(z,c)^-) - \widehat{H}^z(\widehat{x}(z,z)^-)]dz \geq 0,$$

where $\widehat{x}(z,c)$ is the largest element of

$$\operatorname*{argmax}_{x\in[0,\bar{v}]}(x - z)(1 - \widehat{H}^c(x^-)).$$

Take and fix any $c', c \in [0, c_0]$ with $c' < c$. Notice that as argued above, since $\widetilde{v}(c)$ is the largest optimal price for a seller with cost c under $\widehat{H}^c$, $\widehat{H}^c(\widehat{x}(c,c)^-) = F(\widetilde{\psi}(c)) = H^c(x(c))$ for all $c \in [0, c_0]$. On the other hand, notice that by construction of $\widehat{H}^c$, for any $z \in [c', c]$, either $\widehat{x}(z,c) = \widetilde{v}(c)$ or $\widehat{x}(z,c) = x(z,c) \in [0, \underline{v}(c))$. In addition, since the original menu $(H^c, t(c), \alpha(c))_{c\in[0,\bar{c}]}$ is incentive compatible, Lemma 1 implies that

$$\int_{c'}^{c} [H^c(x(z,c)^-) - H^z(x(z,z)^-)]dz \geq 0.$$

Together,

$$\begin{aligned}
\int_{c'}^{c} [\widehat{H}^c(\widehat{x}(z,c)^-) - \widehat{H}^z(\widehat{x}(z,z)^-)]dz &= \int_{c'}^{c} [\widehat{H}^c(\widehat{x}(z,c)^-) - F(\widetilde{\psi}(z))]dz \\
&\geq \int_{c'}^{c} [H^c(x(z,c)^-) - F(\widetilde{\psi}(z))]dz \\
&= \int_{c'}^{c} [H^c(x(z,c)^-) - H^z(x(z,z)^-)]dz \\
&\geq 0,
\end{aligned}$$

where the two equalities follows from the constructions of $\widehat{H}^z$ and $\widetilde{\psi}$ so that $\widehat{H}^z(\widehat{x}(z,z)^-) = F(\widetilde{\psi}(z)) = H^z(x(z,z)^-)$, the first inequality follows from the observation that $\widehat{H}^c(\widetilde{v}(c)^-) = F(\widetilde{\psi}(c)) \geq H^c(x)$ for any $x \in [0, \underline{v}(c))$.

Finally, since as noted above, $F(\widetilde{\psi}(c)) = H^c(x(c)^-)$ and $\mathbb{E}_{H^c}[v|v > x(c)] \leq \mathbb{E}_F[v|v > \widetilde{\psi}(c)]$, for any $c \in [0, c_0]$, we have:

$$\begin{aligned}
(x(c) - \psi(c))(1 - H(x(c)^-)) &\leq (\mathbb{E}_{H^c}[v|v > x(c)] - \psi(c))(1 - H(x(c)^-)) \\
&\leq (\mathbb{E}_F[v|v > \widetilde{\psi}(c)] - \psi(c))(1 - F(\widetilde{\psi}(c))) \qquad (14)\\
&= (\widetilde{v}(c) - \psi(c))(1 - F(\widetilde{\psi}(c))).
\end{aligned}$$

Together, by Lemma 1, let

$$\widehat{t}(c) := \mathbf{1}_{[0,c_0]}(c)\left[(\widetilde{v}(c)-c)(1-F(\widetilde{\psi}(c))) - \int_c^{\bar{c}}(1-F(\widetilde{\psi}(z)))dz\right].$$

Then $(\widehat{H}^c, \widehat{t}(c), \mathbf{1}_{[0,c_0]}(c))_{c\in[0,\bar{c}]}$ is an incentive compatible and individually rational menu, $\widehat{H}^c$ is a responsive garbled upper-censorship with cutoff $\widetilde{\psi}$ and

$$\begin{aligned}\int_0^{c_0}\widehat{t}(c)G(dc) &= \int_0^{\bar{c}}(\widetilde{v}(c)-\psi(c))(1-F(\widetilde{\psi}(c)))G(dc)\\ &\geq \int_0^{\bar{c}}(x(c)-\psi(c))(1-H(x(c)^-))G(dc)\\ &\geq \int_0^{\bar{c}}t(c)G(dc),\end{aligned}$$

where the last two inequalities follow from (13) and (14). This completes the proof. ∎

Step 2: We now characterize the family of incentive compatible, individually rational and responsive garbled upper censorship menus by a collection of inequalities. This is the result of Lemma 4. To show Lemma 4, we first need Lemma 3 to simplify the procedure.

Lemma 3. *Fix any $\tilde{c}\in[0,\bar{c}]$. Given any functions $p:[0,\tilde{c}]\to[0,\bar{v}]$ and $q:[0,\bar{c}]\to[0,1]$. Then*

$$(p(c)-c)(1-q(c)) \geq (p(c')-c)(1-q(c')), \forall c,c'\in[0,\tilde{c}] \tag{15}$$

if and only if

1. *$(p(c)-c)(1-q(c)) = (p(\tilde{c})-\tilde{c})(1-q(\tilde{c})) + \int_c^{\tilde{c}}(1-q(z))dz$ for all $c\in[0,\tilde{c}]$.*

2. *q is nondecreasing.*

Proof. For necessity, consider and pair of functions p,q satisfying (15). Let $\pi(c,c') := (p(c')-c)(1-q(c'))$, for any $c,c'\in[0,\tilde{c}]$. Notice that for each $c'\in[0,\tilde{c}]$, the function $\pi(\cdot,c')$ is absolutely continuous and uniformly bounded by $-\tilde{c}$ and $\bar{v}$. By the envelope theorem, the function

$$\pi^*(c) := \pi(c,c) = \max_{c'\in[0,\bar{c}]}\pi(c,c')$$

is also absolutely continuous and the derivative exists and equals to $-(1-q(c))$ for almost all $c\in[0,\tilde{c}]$. Thus,

$$(p(c)-c)(1-q(c)) = \pi^*(c) = \pi^*(\tilde{c}) + \int_c^{\tilde{c}}(1-q(z))dz,$$

which establishes assertion 1. Moreover, since $\pi(\cdot, c')$ is an affine function, π^* is a pointwise supremum of a family of affine functions and thus is convex. As a result, its derivative, $-(1-q)$, is nondecreasing.

For sufficiency, consider a pair of functions p, q satisfying 1 and 2. Then, for any $c, c' \in [0, \tilde{c}]$,

$$\begin{aligned}
&(p(c)-c)(1-q(c)) - (p(c')-c)(1-q(c')) \\
=&(p(c)-c)(1-q(c)) - (p(c')-c')(1-q(c')) - (c'-c)(1-q(c')) \\
=&\int_c^{c'} (1-q(z))dz - (c'-c)(1-q(c')) \\
=&\int_c^{c'} (q(c')-q(z))dz \\
\geq&0,
\end{aligned}$$

where the second equality follows from 1 and the inequality follows from 2. ■

Lemma 4. *For any $c_\alpha \in [0, \bar{c}]$, any nondecreasing function $\widetilde{\psi} : [0, c_\alpha] \to [0, \bar{v}]$, let $\widetilde{v}(c) := \mathbb{E}_F[v | v > \widetilde{\psi}(c)]$, $\widetilde{\pi}(c) := (\widetilde{v}(c) - c)(1 - F(\widetilde{\psi}(c))), \forall c \in [0, \bar{c}]$. There exists $t : [0, \bar{c}] \to \mathbb{R}$, such that $(H^c, t(c), \mathbf{1}_{[0,c_\alpha](c)})_{c \in [0,\bar{c}]}$ is an incentive compatible, individually rational and responsive upper censorship menu with cutoff $\widetilde{\psi}$ if and only if there exists $q : [0, c_\alpha]^2 \to [0, 1]$ such that:*

1. *$H^c(\widetilde{\psi}(c)^-) = F(\widetilde{\psi}(c))$, $\int_0^x [H^c(z) - F(z)]dz \geq 0$, for all $x \in [0, \widetilde{\psi}(c)]$, with equality at $x = \widetilde{\psi}(c)$, for all $c \in [0, c_\alpha]$.*
2. *$\int_{c'}^c [q(z|c) - F(\widetilde{\psi}(z))] \geq 0$, for all $c, c' \in [0, c_\alpha]$ with $c' \leq c$.*
3. *For any $c \in [0, c_\alpha]$, $x \in [0, \widetilde{\psi}(c)]$, $H^c(x^-) \geq \max_{c' \in [0,c]} [1 - (\widetilde{\pi}(c) + \int_{c'}^c (1 - q(z|c)dz))/(x - c')^+]^+$ with equality if and only if $(x - c')(1 - q(c'|z)) = \widetilde{\pi}(c) + \int_{c'}^c (1 - q(z|c))dz$ for some $c' \in$ $\mathrm{argmax}_{c' \in [0,c]} [1 - (\widetilde{\pi}(c) + \int_{c'}^c (1 - q(z|c)dz))/(x - c')^+]^+$.*
4. *$q(\cdot|c)$ is nondecreasing and $q(c|c) = F(\widetilde{\psi}(c))$ for all $c \in [0, c_\alpha]$.*

Proof. For necessity, consider any incentive compatible, individually rational and responsive upper censorship menu $(H^c, t(c), \mathbf{1}_{[0,c_\alpha]}(c))_{c \in [0,\bar{c}]}$ with cutoff $\widetilde{\psi}$. Since H^c is an upper censorship, assertion 1 is clearly satisfied.

Since H^c is responsive, for any $c \in [0, c_\alpha]$,

$$(x - c)(1 - H^c(x^-)) \leq (\widetilde{v}(c) - c)(1 - F(\widetilde{\psi}(c))) = \widetilde{\pi}(c), \forall x \in [0, \bar{v}]$$

and $H^c(x^-) = 1$ for all $x \in (\widetilde{v}(c), \bar{v}]$. Therefore,

$$\widetilde{v}(c) \in \text{argmax}(x-c)(1-H^c(x^-)), \forall c \in [0, c_\alpha].$$

On the other hand, for any $c', c \in [0, c_\alpha]$ with $c' \leq c$, take any selection

$$x(c', c) \in \underset{x\in[0,\bar{v}]}{\text{argmax}}(x-c')(H^c(x^-))$$

and let

$$\begin{aligned} p(c'|c) &:= x(c', c) \\ q(c'|c) &= H^c(x(c', c)^-). \end{aligned}$$

Since $(H^c, t(c), \mathbf{1}_{[0,c_\alpha]}(c))_{c\in[0,\bar{c}]}$ is incentive compatible, Lemma 1 then ensures that

$$\int_{c'}^{c} [q(z|c) - F(\widetilde{\psi}(z))]dz = \int_{c'}^{c} [H^c(x(z,c)) - F(\widetilde{\psi}(z)))]dz \geq 0,$$

which establishes assertion 2.

Now notice that as shown above, $q(c|c) = F(\widetilde{\psi}(c))$ and $(p(c|c) - c)(1 - q(c|c)) = \widetilde{\pi}(c)$ for all $c \in [0, c_\alpha]$. Since $x(c', c) \in \text{argmax}_{x\in[0,c_\alpha]}(x - c')(H^c(x^-))$, for any such c, c' we have

$$(x-c')(1-H^c(x^-)) \leq (p(c'|c) - c')(1-q(c'|c)), \forall x \in [0, \bar{v}]. \tag{16}$$

Rearranging, we have:

$$H^c(x^-) \geq 1 - \frac{(p(c'|c) - c')(1-q(c'|c))}{(x-c')^+}, \forall x \in [0, \bar{v}]. \tag{17}$$

Notice that the right hand side of (17) does not depend on c', we thus have

$$H^c(x^-) \geq \max_{c'\in[0,c]} \left[1 - \frac{(p(c'|c) - c')(1-q(c'|c))}{(x-c')^+}\right]^+, \forall x \in [0, \widetilde{\psi}(c)], \forall c \in [0, c_\alpha]. \tag{18}$$

Moreover, by $x(c', c) \in \text{argmax}_{x\in[0,\bar{v}]}(x - c')(H^c(x^-))$ again, for any $c', c'' \in [0, c]$ and for any $c \in [0, c_\alpha]$,

$$(p(c'|c) - c')(1-q(c'|c)) \geq (p(c''|c) - c')(1-q(c''|c)). \tag{19}$$

By Lemma 3, $\hat{q}(\cdot|c)$ is nondecreasing, which establishes assertion 4.

Furthermore, also by Lemma 3,

$$(p(c'|c)-c')(1-q(c'|c)) = (p(c|c)-c)(1-q(c|c)) + \int_{c'}^{c} (1-q(z|c))dz = \widetilde{\pi}(c) + \int_{c'}^{c} (1-q(z|c))dz, \tag{20}$$

for any $c \in [0, c_\alpha]$ and any $c' \in [0, c]$. Combining (18) and (20), we obtain

$$H^c(x^-) \leq \max_{c' \in [0,x]} \left[1 - \frac{\widetilde{\pi}(c) + \int_{c'}^{c}(1 - \hat{q}(z|c)}{(x - c')^+}\right]^+, \forall x \in [0, \bar{v}].$$

Moreover, by (20), for any $c \in [0, c_\alpha]$, any $x \in [0, \widetilde{\psi}(c)]$,

$$(x - c')(1 - q(c'|c)) = \widetilde{\pi}(c) + \int_{c'}^{c}(1 - q(z|c))dz$$

for some $c' \in \text{argmax}_{c' \in [0,c]}[1 - (\widetilde{\pi}(c) + \int_{c'}^{c}(1 - q(z|c)dz))/(x - c')]^+$, if and only if $x = x(c', c) = p(c'|c)$ and

$$H^c(x^-) = \max_{c' \in [0,c]} \left[1 - \frac{(p(c'|c) - c')(1 - q(c'|c))}{(x - c')^+}\right]^+,$$

which, together with (19), establishes assertion 3.

Conversely, for sufficiency, take any $q : [0, c_\alpha]^2 \to [0, 1]$ and $\{H^c\}_{c \in [0, c_\alpha]}$ satisfying conditions 1, 2, 3 and 4. Notice that if H^c is not an upper censorship, then for each $c \in [0, c_\alpha]$, $x \in [0, \bar{v}]$, let

$$\widehat{H}^c(x) := \begin{cases} H^c(x), & \text{if } x \in [0, \widetilde{\psi}(c)) \\ F(\widetilde{\psi}(c)), & \text{if } x \in [\widetilde{\psi}(c), \widetilde{v}(c)) \\ 1, & \text{if } x \in [\widetilde{v}(c), \bar{v}] \end{cases}.$$

By assertion 1, $\widehat{H}^c \in \mathcal{H}_F$ and is an upper censorship with cutoff $\widetilde{\psi}(c)$ for all $c \in [0, c_\alpha]$ and $\widehat{H}^c \equiv H^c$ on $[0, \widetilde{\psi}]$. Thus suffices to take H^c as an upper censorship and verify that there exists $t : [0, \bar{c}] \to \mathbb{R}$ such that $(H^c, t(c), \mathbf{1}_{[0,c_\alpha]}(c))_{c \in [0,\bar{c}]}$ is an incentive compatible, individually rational and responsive menu.

For responsiveness, notice that by assertion 3, for any $x \in [0, \widetilde{\psi}(c)]$,

$$H^c(x^-) \geq \max_{c' \in [0,c]} \left[1 - \frac{\widetilde{\pi}(c) + \int_{c'}^{c}(1 - q(z|c))dz}{(x - c')^+}\right]^+ \geq \left[1 - \frac{\widetilde{\pi}(c)}{(x - c)^+}\right]^+.$$

Rearranging, we have:

$$(x - c)(1 - H^c(x^-)) \leq \widetilde{\pi}(c) = (\widetilde{v}(c) - c)(1 - H^c(\widetilde{v}(c)^-)).$$

On the other hand, any $x \in (\widetilde{v}(c), \bar{v}]$ gives zero profit under $\widehat{H}^c$. Thus,

$$\widetilde{v}(c) \in \underset{x \in [0,\bar{v}]}{\text{argmax}}(x - c)(1 - H^c(x^-)), \forall c \in [0, c_\alpha].$$

Therefore, $\{H^c\}_{c \in [0,c_\alpha]}$ is indeed responsive.

For incentive compatibility and individual rationality, by Lemma 3, since $q(\cdot|c)$ is increasing and $q(c|c) = F(\widetilde{\psi}(c))$ for all $c \in [0, c_\alpha]$, there exists $p : [0, c_\alpha]^2 \to [0, \bar{v}]$ such that

$$(p(c'|c) - c')(1 - q(c'|c)) = \widetilde{\pi}(c) + \int_{c'}^{c}(1 - q(z|c))dz, \forall c' \in [0, c],\ c \in [0, c_\alpha]$$

and that

$$(p(c'|c) - c')(1 - q(c'|c)) \geq (p(c''|c) - c')(1 - q(c''|c)), \forall c', c'' \in [0, c],\ c \in [0, c_\alpha].$$

As such, for any $c \in [0, c_\alpha]$, $x \in [0, \widetilde{\psi}(c)]$ and any $c' \in [0, c]$, whenever

$$H^c(x^-) = \max_{c' \in [0,c]} \left[1 - \frac{\widetilde{\pi}(c) + \int_{c'}^{c}(1 - q(z|c))dz}{(x - c')^+}\right]^+, \tag{21}$$

by assertion 3 and (3), there exists c' such that $H^c(x) = q(c'|c)$

$$(x - c')(1 - H^c(x^-)) \geq \widetilde{\pi}(c) + \int_{c''}^{c}(1 - q(z|c)) = (p(c''|c) - c')(1 - q(c''|c)), \forall c'' \in [0, c],$$

for some $c' \in [0, c_\alpha]$. Assertion 3 then implies that whenever (21) holds for some $x \in [0, \widetilde{\psi}(c)]$, $c' \in [0, c_\alpha]$

$$(x - c')(1 - H^c(x^-)) \geq (y - c')(1 - H^c(y^-))$$

for any y such that (21) holds for some (possibly distinct) $c'' \in [0, c]$.

On the other hand, for any $x \in [0, \widetilde{\psi}(c)]$ such that (21) does not hold, there must be some $c' \in [0, c_\alpha]$ such that

$$(x - c')(1 - H^c(x^-)) < \widetilde{\pi}(c) + \int_{c'}^{c}(1 - q(z|c))dz$$

and therefore

$$(x - c')(1 - H^c(x^-)) < (y - c')(1 - H^c(y^-))$$

for $y = p(c'|c)$. Together, it must be that $\operatorname{argmax}_{x \in [0, \widetilde{\psi}(c)]}(x - c')(1 - H^c(x^-))$ is exactly the set where (21) holds for some $c' \in [0, c_\alpha]$. Thus, we may take a selection $x(c', c) \in \operatorname{argmax}_{x \in [0, \widetilde{\psi}(c)]}(x - c')(1 - H^c(x^-))$ such that $H^c(x(c', c)^-) = q(c'|c)$ for all $c \in [0, c_\alpha]$, $c' \in [0, c]$. Then, by Lemma 1 and assertion 2, there exists $t : [0, \bar{c}] \to \mathbb{R}$ such that $(H^c, t(c), \mathbf{1}_{[0,c_\alpha]}(c))_{c \in [0,\bar{c}]}$ is incentive compatible and individually rational. ■

Step 3: Now, we identify two critical constraints for the intermediary's problem. Lemma 5, together with Lemma 4, show that for any incentive compatible, individually rational and responsive garbled upper censorship menu, two families of equality constraints must be satisfied, which will later be used in forming the dual problem.

Lemma 5. *For any $c_\alpha \in [0,\bar{c}]$, any nondecreasing function $\widetilde{\psi} : [0,c_\alpha] \to [0,\bar{v}]$, any $\{H^c\}_{c\in[0,\bar{c}]} \subset \mathcal{H}_F$ and any $q : [0,c_\alpha]^2 \to [0,1]$ satisfying conditions 1-4 in Lemma 4, there exists some $\hat{q} : [0,c_\alpha]^2 \to [0,1]$ such that $\hat{q}(c|c) = F(\widetilde{\psi}(c))$, $\hat{q}(\cdot|c)$ is increasing and*

$$\int_0^{\widetilde{\psi}(c)} [\Gamma^c_{\hat{q},\widetilde{\psi}}(x) - F(x)]dx = 0$$
$$\int_0^c [\hat{q}(z|c) - F(\widetilde{\psi}(z))]dz = 0,$$

where

$$\Gamma^c_{q,\widetilde{\psi}}(x) := \begin{cases} q(0|c), & \text{if } x \in [0, \underline{x}_q(c)) \\ \max_{c'\in[0,c_\alpha]} \left[1 - \frac{\widetilde{\pi}(c) + \int_{c'}^c (1-q(z|c))dz}{(x-c')^+}\right]^+, & \text{if } x \in [\underline{x}_q(c), \bar{x}_q(c)) \\ F(\widetilde{\psi}(c)), & \text{if } x \in [\bar{x}_q(c), \widetilde{\psi}(c)] \end{cases},$$

for any $q : [0,c_\alpha]^2 \to [0,1]$ with $q(\cdot|c)$ being nondecreasing , any $c \in [0,c_\alpha]$ and any $x \in [0,\widetilde{\psi}(c)]$, with

$$\underline{x}_q(c) := \frac{1}{(1-q(0|c))} \left[\widetilde{\pi}(c) + \int_0^c (1-q(z|c))dz\right] + c$$
$$\bar{x}_q(c) := \frac{1}{(1-q(\bar{z}_q(c)^-|c))} \left[\widetilde{\pi}(c) + \int_{\bar{z}_q(c)}^c (1-q(z|c))dz\right] + c$$
$$\bar{z}_q(c) := \inf\{z \in [0,c] | q(z|c) = F(\widetilde{\psi}(c))\}$$

Proof. Take any $\widetilde{\psi}$, $\{H^c\}_{c\in[0,c_\alpha]}$ and q satisfying conditions 1-4 in Lemma 4. Fix any $c \in [0,c_\alpha]$. By Lemma 4, we have

$$\int_0^{\widetilde{\psi}(c)} [H^c(x) - F(x)]dx = 0.$$

Moreover, since $\widetilde{v}(c) > \widetilde{\psi}$, we must have $\bar{z}_q(c) < c$ and $F(\widetilde{\psi}(c)) = F(\widetilde{\psi}(c)) > q(\bar{z}_q(c)^-|c)$.

Consider first the case where

$$\int_0^c [q(z|c) - F(\widetilde{\psi}(z))] > 0,$$

first define $\underline{z}_q(c)$ as the unique z that solves to the equation

$$\int_z^c q(t|c)dt = \int_0^c F(\widetilde{\psi}(t))dx.$$

For this $\underline{z}_q(c)$, consider the following construction: Define

$$\hat{q}(z|c) := \begin{cases} 0, & \text{if } z \in [0, \underline{z}_q(c)) \\ q(z|c), & \text{if } z \in [\underline{z}_q(c), c] \end{cases}.$$

Notice that we then have

$$\int_0^c [\hat{q}(z|c) - F(\widetilde{\psi}(z))]dz = 0.$$

If, furthermore,

$$\int_0^{\underline{x}_{\hat{q}}(c)} \left[H^c(x) - \left(1 - \frac{\widetilde{\pi}(c) + \int_0^c (1-\hat{q}(z|c))dz}{x}\right)^+ \right] dx$$
$$\leq \int_{\bar{x}_q(c)}^{\widetilde{v}(c)} \left[F(\widetilde{\psi}(c)) - \left(1 - \frac{\widetilde{\pi}(c) + \int_{\bar{z}_{\hat{q}}(c)}^c (1-\hat{q}(z|c))dz}{(x - \bar{z}_q(c))^+}\right)^+ \right] dx,$$

then there exists some $q_0 \in [q(\bar{z}_q(c)^-|c), F(\widetilde{\psi}(c))]$ such that

$$\int_0^{\underline{x}_{\hat{q}}(c)} \left[H^c(x) - \left(1 - \frac{\widetilde{\pi}(c) + \int_0^c (1-\hat{q}(z|c))dz}{x}\right)^+ \right] dx$$
$$= (\widetilde{v}(c) - \hat{x})F(\widetilde{\psi}(c)) - \int_{\bar{x}_q(c)}^{\widetilde{v}(c)} \left(1 - \frac{\widetilde{\pi}(c) + \int_{\bar{z}_{\hat{q}}(c)}^c (1-\hat{q}(z|c))dz}{(x - \bar{z}_{\hat{q}}(c))^+}\right)^+ dx,$$

where

$$\hat{x} := -\frac{1}{1-q_0}\left[\widetilde{\pi}(c) + \int_{\bar{z}_{\hat{q}}}^c (1-\hat{q}(z|c))dz\right] + c.$$

Thus, by possibly redefining $\hat{q}(\bar{z}_{\hat{q}}|c)$ as q_0, we then have

$$\int_0^{\widetilde{\psi}(c)} \Gamma^c_{\hat{q},\widetilde{\psi}}(x)dx = \int_0^{\widetilde{\psi}(c)} H^c(x)dx = \int_0^{\widetilde{\psi}(c)} F(x)dx.$$

Furthermore, since the the point $\bar{z}_q(c)$ has Lebesgue measure zero, we still have

$$\int_0^c [\hat{q}(z|c) - F(\widetilde{\psi}(z))]dz = 0.$$

On the other hand, if

$$\int_0^{\underline{x}_{\hat{q}}(c)} \left[H^c(x) - \left(1 - \frac{\widetilde{\pi}(c) + \int_0^c (1-\hat{q}(z|c))dz}{x}\right)^+ \right] dx$$
$$> \int_{\bar{x}_q(c)}^{\widetilde{v}(c)} \left[F(\widetilde{\psi}(c)) - \left(1 - \frac{\widetilde{\pi}(c) + \int_{\bar{z}_{\hat{q}}(c)}^c (1-\hat{q}(z|c))dz}{(x - \bar{z}_q(c))^+}\right)^+ \right] dx,$$

we construct a similar procedure for any $x \in [\underline{x}_q(c), \bar{x}_q(c)]$. Specifically, since the mapping

$$c' \mapsto \left[1 - \frac{\widetilde{\pi}(c) + \int_{c'}^c (1-q(z|c))dz}{(x-c')^+}\right]^+$$

is continuous on $[0, c]$, its set of maximizers admits a continuous selection

$$c_q(x) \in \operatorname*{argmax}_{c' \in [0,c]} \left[1 - \frac{\widetilde{\pi}(c) + \int_{c'}^c (1-q(z|c))dz}{(x-c')^+}\right]^+.$$

Observe that for any $x \in [\underline{x}_q(c), \bar{x}_q(c)]$, $c_q(x) \geq \underline{z}_q(c)$ and therefore there exists $q_x \geq 0$ such that

$$\int_{c_q(x)}^{c} [q(z|c) - F(\widetilde{\psi}(z))]dz + \int_0^{c_q(x)} [q_x - F(\widetilde{\psi}(z))]dz = 0$$

Now define

$$q_x(z|c) := \begin{cases} q_x, & \text{if } z \in [0, c_q(x)) \\ q(z|c), & \text{if } z \in [c_q(x), c] \end{cases}$$

and let

$$\hat{x}(x) := \frac{1}{1-q_x}\left[\widetilde{\pi}(c) + \int_{c_q(x)}^{c} (1 - q_x(z|c))dz\right].$$

Then, as long as

$$\int_0^{\hat{x}(\bar{x}_q(c))} \left[H^c(x) - \left(1 - \frac{\widetilde{\pi}(c) + \int_{\bar{z}_q(c)}^c (1 - q_x(z|c))dz}{(x - \bar{z}_q(c))^+}\right)^+\right] dx$$
$$\leq \int_{\bar{x}_q(c)}^{\widetilde{v}(c)} \left[F(\widetilde{\psi}(c)) - \left(1 - \frac{\widetilde{\pi}(c) + \int_{\bar{z}_{\hat{q}}(c)}^c (1 - q_x(z|c))dz}{(x - \bar{z}_q(c))^+}\right)^+\right] dx,$$

by the intermediate value theorem, since the selection c_q is continuous and $c_q(\bar{x}_q(c)) = \bar{z}_q(c)$ by construction, there must be some $x^* \in [\underline{x}_q(c), \bar{x}_q(c)]$ such that

$$\int_0^{\hat{x}(x^*)} \left[H^c(x) - \left(1 - \frac{\widetilde{\pi}(c) + \int_{c_q(x^*)}^c (1 - q_x(z|c))dz}{(x - c_q(x^*))^+}\right)^+\right] dx$$
$$= \int_{\bar{x}_q(c)}^{\widetilde{v}(c)} \left[F(\widetilde{\psi}(c)) - \left(1 - \frac{\widetilde{\pi}(c) + \int_{\bar{z}_{\hat{q}}(c)}^c (1 - q_x(z|c))dz}{(x - \bar{z}_q(c))^+}\right)^+\right] dx.$$

We then have

$$\int_0^{\widetilde{\psi}(c)} \Gamma^c_{q_{x^*}, \widetilde{\psi}}(x)dx = \int_0^{\widetilde{\psi}(c)} H^c(x)dx = \int_0^{\widetilde{\psi}(c)} F(x)dx.$$

Therefore, by defining $\hat{q}(\cdot|c) \equiv q_{x^*}(\cdot|c)$, we then have the desired $\hat{q}$.

Finally, if

$$\int_0^{\hat{x}(\bar{x}_q c)} \left[H^c(x) - \left(1 - \frac{\widetilde{\pi}(c) + \int_{\bar{z}_q(c)}^c (1 - q_x(z|c))dz}{(x - \bar{z}_q(c))^+}\right)^+\right] dx$$
$$< \int_{\bar{x}_q(c)}^{\widetilde{v}(c)} \left[F(\widetilde{\psi}(c)) - \left(1 - \frac{\widetilde{\pi}(c) + \int_{\bar{z}_{\hat{q}}(c)}^c (1 - q_x(z|c))dz}{(x - \bar{z}_q(c))^+}\right)^+\right] dx,$$

then there exists $x^* \in [0, \bar{x}_q(c)]$ such that

$$\int_0^{x^*} \max\left\{q_{\bar{x}_q(c)}, \left(1 - \frac{\widetilde{\pi}(c) + \int_{\bar{z}_q(c)}^c (1 - q(z|c))dz}{(x - \bar{z}_q(c))^+}\right)^+\right\} dx + (\widetilde{v}(c) - x^*)F(\widetilde{\psi}(c)) = \int_0^{\widetilde{\psi}(c)} H^c(x)dx.$$

Then, by defining

$$\hat{q}(z|c) := \begin{cases} q_{\bar{x}_q}, & \text{if } z \in [0, \bar{z}_q(c)] \\ F(\widetilde{\psi}(c)), & \text{if } z \in (\bar{z}_q(c), c] \end{cases},$$

we then have

$$\int_0^{\widetilde{\psi}(c)} \Gamma^c_{\hat{q},\widetilde{\psi}}(x)dx = \int_0^{\widetilde{\psi}(c)} H^c(x)dx = \int_0^{\widetilde{\psi}(c)} F(x)dx,$$

as desired.

Finally, if, on the other hand,

$$\int_0^c [q(z|c) - F(\widetilde{\psi}(z))]dz = 0,$$

then it must be that $q(0|c) = 0$ since by Lemma 4,

$$\int_{c'}^c [q(z|c) - F(\widetilde{\psi}(z))]dz \geq 0$$

and $F(\widetilde{\psi}(0)) = 0$. Thus, by selecting a proper $\tilde{q} \in (0, q(\underline{z}_q(c)^+|c))$ and letting $\tilde{q}(z|c) := \tilde{q}$ for all $z \in [0, \underline{z}_q c]$ and $\tilde{q}(z|c) := q(z|c)$ otherwise, we will have

$$\tilde{H}^c(x) \geq \max_{c' \in [0, c_\alpha]} \left[1 - \frac{\widetilde{\pi}(c) + \int_{c'}^c (1 - \tilde{q}(z|c))dz}{(x - c')^+}\right]^+,$$
$$\int_0^{\widetilde{\psi}(c)} \tilde{H}(c)dx = \int_0^{\widetilde{\psi}(c)} H^c(x)dx = \int_0^{\widetilde{\psi}(c)} F(x)dx$$

and

$$\int_0^c [\hat{q}(z|c) - F(\widetilde{\psi}(z))]dz > 0,$$

where

$$\tilde{H}^c(x) := \begin{cases} \tilde{q}, & \text{if } x \in [0, \tilde{x}) \\ H^c(x), & \text{if } x \in [\tilde{x}, \widetilde{\psi(c)}] \end{cases},$$

and

$$\tilde{x} := \frac{1}{1 - \tilde{q}} \left[\widetilde{\pi}(c) + \int_{\underline{z}_q(c)}^c (1 - \tilde{q}(z|c))dz\right].$$

Since the previous arguments hinge only on the property that

$$\int_0^{\widetilde{\psi}(c)} H^c(x)dx = \int_0^{\widetilde{\psi}(c)} F(x)dx$$

and that $H^c(x) \geq \max_{c' \in [0, c_\alpha]}[1 - (\widetilde{\pi}(c) + \int_{c'}^c (1 - q(z|c))dz)/(x - c')^+]^+$ for all $x \in [0, \widetilde{\psi}(c)]$, by replacing H^c with $\tilde{H}^c$, q with $\tilde{q}$ and repeating the procedures, we can then find the desired $\hat{q}$. ■

Step 4: We now show that for any fixed $c_\alpha \in [0, \bar{c}]$, the menu induced by the family of information structure $\{H^c_{gu}\}_{c\in[0,c_\alpha]} \subset \mathcal{H}_F$ and the publicizing policy $\mathbf{1}_{[0,c_\alpha]}$, with H^c_{gu} being a responsive upper censorship with cutoff $\psi^*(c)$, is incentive compatible and individually rational and maximized the intermediary's revenue among all incentive compatible, individually rational and responsive garbled upper censorship menu with the same publicizing policy. To this end, first notice that when $c_\alpha \in [0, c^*]$, since $\psi^*(c) = \psi(c)$, the proof of Theorem 1 then ensures the desired property. Below we discuss the case when $c_\alpha \in (c^*, \bar{c}]$

By using the characterizations above, we can form a dual for the original problem with a fixed c_α. Fix any Borel measures μ, ν on the measurable space $[0, c_\alpha]$ (endowed with the Borel algebra). Let

$$
\begin{aligned}
D(\mu,\nu) := \sup_{\widetilde{\psi},q} \Bigg[&\int_0^{c_\alpha} (\widetilde{v}(c) - \psi(c))(1 - F(\widetilde{\psi}(c)))G(dc) - \int_0^{c_\alpha} \left(\int_0^{\bar{v}} (1 - \Gamma^c_{q,\widetilde{\psi}}(x)) - (1 - F(x)) \right) dx\mu(dc) \\
&- \int_0^{c_\alpha} \left(\int_0^c [q(z|c) - F(\widetilde{\psi}(z))]dz \right) \nu(dc) \Bigg],
\end{aligned} \tag{22}
$$

where the supremum is taken over all nondecreasing function $\widetilde{\psi} : [0, c_\alpha] \to [0, \bar{v}]$ and all $q : [0, c_\alpha]^2 \to [0, 1]$ such that $q(\cdot|c)$ is nondecreasing and $q(c|c) = F(\widetilde{\psi}(c))$, for all $c \in [0, c_\alpha]$.

Notice that for any incentive compatible, individually rational and responsive garbled upper censorship menu $(H^c, t(c), \mathbf{1}_{[0,c_\alpha]}(c))_{c\in[0,\bar{c}]}$ with cutoff $\widetilde{\psi}$, the expected revenue for the intermediary is

$$
R(\widetilde{\psi}) := \int_0^{c_\alpha} (\widetilde{v}(c) - \psi(c))(1 - F(\widetilde{\psi}(c)))G(dc).
$$

By Lemma 4 and Lemma 5, for any such menu, there exists $q_{\widetilde{\psi},H} : [0, c_\alpha]^2 \to [0, c_\alpha]$ such that $q_{\widetilde{\psi},H}(\cdot|c)$ is nondecreasing, $q_{\widetilde{\psi},H}(c|c) = F(\widetilde{\psi}(c))$,

$$
\begin{aligned}
&\int_0^{\bar{v}} [(1 - \Gamma^c_{q_{\widetilde{\psi},H},\widetilde{\psi}}(x)) - (1 - F(x))] = 0, \\
&\int_0^c [q_{\widetilde{\psi},H}(z|c) - F(\widetilde{\psi}(z))]dz = 0,
\end{aligned}
$$

for all $c \in [0, c_\alpha]$.

As such, for any such menu, for any Borel measures μ, ν on $[0, c_\alpha]$,

$$
\begin{aligned}
R(\widetilde{\psi}) =& \int_0^{c_\alpha} (\widetilde{v}(c) - \psi(c))(1 - F(\widetilde{\psi}(c)))G(dc). \\
=& \int_0^{c_\alpha} (\widetilde{v}(c) - \psi(c))(1 - F(\widetilde{\psi}(c)))G(dc) - \int_0^{c_\alpha} \left(\int_0^{\bar{v}} (1 - \Gamma^c_{q_{\widetilde{\psi},H}, \widetilde{\psi}}(x)) - (1 - F(x)) \right) dx \mu(dc) \\
& - \int_0^{c_\alpha} \left(\int_0^{c} [q_{\widetilde{\psi},H}(z|c) - F(\widetilde{\psi}(z))] dz \right) \nu(dc) \\
\leq & \sup_{\widetilde{\psi}, q} \Bigg[\int_0^{c_\alpha} (\widetilde{v}(c) - \psi(c))(1 - F(\widetilde{\psi}(c)))G(dc) - \int_0^{c_\alpha} \left(\int_0^{\bar{v}} (1 - \Gamma^c_{q, \widetilde{\psi}}(x)) - (1 - F(x)) \right) dx \mu(dc) \\
& - \int_0^{c_\alpha} \left(\int_0^{c} [q(z|c) - F(\widetilde{\psi}(z))] dz \right) \nu(dc) \Bigg] \\
=& D(\mu, \nu),
\end{aligned}
$$

and therefore, if R^* is the supremum of the expected revenue among all possible incentive compatible, individually rational and responsive garbled upper censorship menus with publicizing policy $\mathbf{1}_{[0,c_\alpha]}$, we have

$$R^* \leq D(\mu, \nu).$$

It then suffices to show that there exists Borel measures μ^*, ν^* and a function $q^* : [0, c_\alpha]^2 \to [0, 1]$ with $q^*(\cdot|c)$ being nondecreasing and $q^*(c|c) = F(\psi^*(c))$ such that (ψ^*, q^*) solves the dual problem (22), that

$$
\begin{aligned}
& \int_0^{c_\alpha} \left(\int_0^{\bar{v}} [(1 - \Gamma^c_{q^*, \psi^*}(x)) - (1 - F(x)) dx \right) \mu^*(dc) = 0, \\
& \int_0^{c_\alpha} \left(\int_0^{c} [q^*(z|c) - F(\psi^*(z))] dz \right) \nu^*(dc) = 0
\end{aligned}
$$

for all $c \in [0, c_\alpha]$ and that there exists an incentive compatible, individually rational and responsive garbled upper censorship menu with cutoff ψ^* and publicizing policy $\mathbf{1}_{[0,c_\alpha]}$, as this would imply:

$$R^* \leq D(\mu^*, \nu^*) = R(\psi^*) \leq R^*.$$

Together, we will then have

$$R^* = R(\psi^*),$$

as desired.

Indeed, for any Borel measures μ, ν on $[0, c_\alpha]$, (22) can be written as:

$$
\begin{aligned}
& \sup_{\widetilde{\psi}} \Bigg[\int_0^{c_\alpha} (\widetilde{v}(c) - \psi(c))(1 - F(\widetilde{\psi}(c)))G(dc) \\
& + \sup_{q} \left(\int_0^{c_\alpha} \left(\int_0^{\bar{v}} [\Gamma^c_{q, \widetilde{\psi}}(x) - F(x)] dx \right) \mu(dc) - \int_0^{c_\alpha} \left(\int_0^{c} [q(z|c) - F(\widetilde{\psi}(z))] dz \right) \nu(dc) \right) \Bigg].
\end{aligned}
$$

Notice that for any fixed $\widetilde{\psi}$, the functional

$$q \mapsto \Gamma^c_{q,\widetilde{\psi}}$$

is convex, as it is essentially a pointwise supremum of a family of affine functionals. Also, since for each $c \in [0, c_\alpha]$, the collection of nondecreasing functions $q(\cdot|c) : [0, c] \to [0, \widetilde{\psi}(c)]$, $\mathcal{Q}(c)$, is convex, for each $c \in [0, c_\alpha]$, and any fixed nondecreasing function $\widetilde{\psi} : [0, c_\alpha] \to [0, \bar{v}]$, the maximization problem

$$\max_{q(\cdot|c)\in\mathcal{Q}(c)} \int_0^{\bar{v}} [\Gamma^c_{q,\widetilde{\psi}}(x) - F(\widetilde{\psi})]dx - \int_0^c [q(z|c) - F(\widetilde{\psi}(z))]dz$$

has a solution and one of the extreme points of $\mathcal{Q}(c)$, which take form of $q(z|c) \in \{0, F(\widetilde{\psi}(c))\}$ for all $z \in [0, c]$, attains the maximum. Therefore, (22) can be reduced to choosing cutoff points of the extreme points of $\mathcal{Q}(c)$, denoted as $k(c)$, instead of choosing among all nondecreasing functions for each $c \in [0, c_\alpha]$. That is:

$$\begin{aligned}
D(\mu, \nu) = &\sup_{\widetilde{\psi}} \Bigg[\int_0^{c_\alpha} (\widetilde{v}(c) - \psi(c))(1 - F(\widetilde{\psi}(c)))G(dc) \\
&+ \sup_{k:[0,c_\alpha]\to[0,c_\alpha],\, k(c)\le c} \Bigg(\int_0^{c_\alpha} \left(\int_0^{\bar{v}} [\Gamma^c_{\mathbf{1}\{z\ge k(c)\},\widetilde{\psi}}(x) - F(x)]dx \right) \mu(dc) \\
&- \int_0^{c_\alpha} \left(\int_0^c [F(\widetilde{\psi}(c))\mathbf{1}\{z \ge k(c)\} - F(\widetilde{\psi}(z))]dz \right) \nu(dc) \Bigg) \Bigg].
\end{aligned}$$

Notice that by definition of $\Gamma^c_{\mathbf{1}\{z\ge k(c)\},\widetilde{\psi}}$, fix any $\widetilde{\psi}$ and k,

$$\int_0^{\bar{v}} [\Gamma^c_{\mathbf{1}\{z\ge k(c)\},\widetilde{\psi}}(x) - F(x)]dx \ge \int_0^{P(\widetilde{\psi}(c),k(c))} [\gamma(x, \widetilde{\psi}(c), k(c)) - F(x)]dx,$$

where

$$\gamma(x, \widetilde{\psi}(c), k(c)) := \left(1 - \frac{(\widetilde{v}(c) - k(c))(1 - F(\widetilde{\psi}(c)))}{(x - k(c))^+} \right)^+, \forall x \in [0, \widetilde{\psi}(c)]$$

and

$$P(\widetilde{\psi}(c), k(c)) := \begin{cases} \max \Phi^{-1}_{k(c)}((\widetilde{v}(c) - k(c))(1 - F(\widetilde{\psi}(c)))), & \text{if } \Phi^{-1}_{k(c)}((\widetilde{v}(c) - k(c))(1 - F(\widetilde{\psi}(c)))) \neq \emptyset \\ 0, & \text{if } \Phi^{-1}_{k(c)}((\widetilde{v}(c) - k(c))(1 - F(\widetilde{\psi}(c)))) = \emptyset \end{cases},$$

with $\Phi_z(x) := (x - z)(1 - F(x))$ being the profit function of a seller with a cost $z \in [0, c_\alpha]$.

Now fix Borel measures μ, ν on $[0, c_\alpha]$ that are absolutely continuous with respect to the

Lebesgue measure with densities m, n, respectively, and consider an auxiliary problem:

$$
\begin{aligned}
D'(\mu,\nu) := \sup_{\widetilde{\psi},k} \Bigg[\int_0^{c_\alpha} \Bigg(& (\widetilde{v}(c)-\psi(c))(1-F(\widetilde{\psi}(c)))g(c) \\
& - \left(\int_0^{P(\widetilde{\psi}(c),k(c))} [(1-\gamma(x,\widetilde{\psi}(c),k(c))) - (1-F(x))]dx \right) m(c) \\
& - \left((c-k(c))F(\widetilde{\psi}(c)) - \int_0^c F(\widetilde{\psi}(z))dz \right) n(c) \Bigg) dc \Bigg].
\end{aligned} \tag{23}
$$

Notice that for any fixed $\widetilde{\psi}$, (23) is a (concave) pointwise maximization problem of choosing $k(c) \leq c$, whereas for any fixed k, (23) is a (concave) variational problem.[12] As such, for ψ^*, k^* to be optimal under the Borel measures μ, ν, it is equivalent to that $k^*(c)$ solves the pointwise first order condition given ψ^*

$$
\begin{aligned}
& m(c) \int_0^{P(\psi^*(c),k^*(c))} \gamma_3(x,\psi^*(c),k^*(c))dx = -F(\psi^*(c))n(c) \\
\iff & m(c) = -\frac{F(\psi^*(c))}{\int_0^{P(\psi^*(c),k^*(c))} \gamma_3(x,\psi^*(c),k^*(c))dx} n(c),
\end{aligned} \tag{24}
$$

whenever $P(\psi^*(c), k^*(c)) > 0$ and that ψ^* solves the Euler-Largrange equation given k^*:

$$
n(c) = \frac{d}{dc}\left[(\psi(c)-\psi^*(c))g(c) + m(c)\int_0^{P(\psi^*(c),k^*(c))} \gamma_2(x,\psi^*(c),k^*(c))dx - (c-k^*(c))n(c) \right], \tag{25}
$$

for (Lebesgue) almost all $c \in [0, c_\alpha]$.

Substituting (24) into (25) and integrating both sides with the fact that $\psi^*(0) = 0$, we then have:

$$
N(c) := \int_0^c n(z)dz = (\psi(c)-\psi^*(c))g(c) + \Omega^*(c)n(c), \tag{26}
$$

where

$$
\Omega^*(c) := -\frac{\int_0^{P(\psi^*(c),k^*(c))} \gamma_2(x,\psi^*(c),k^*(c))dx F(\psi^*(c))}{\int_0^{P(\psi^*(c),k^*(c))} \gamma_3(x,\psi^*(c),k^*(c))dx f(\psi^*(c))} - (c-k^*(c))
$$

CLAIM 1: Ω^* is strictly decreasing on $[c^*, c_\alpha]$ and $\Omega^*(c) \geq 0$ for all $c \in [c^*, c_\alpha]$.

---

[12] This can be seen by observing that $(\widetilde{v}(c)-\psi(c))(1-F(\widetilde{\psi}(c))) = \int_{\widetilde{\psi}(c)}^{\bar{v}}(1-F(x))dx - (\psi(c)-\widetilde{\psi}(c))(1-F(\widetilde{\psi}(c)))$, $(\widetilde{v}(c)-k(c))(1-F(\widetilde{\psi}(c))) = \int_{\widetilde{\psi}(c)}^{\bar{v}}(1-F(x))dx + (\widetilde{\psi}(c)-k(c))(1-F(\widetilde{\psi}(c)))$ and by letting $\eta(c) := \frac{1}{F(\widetilde{\psi}(c))}\int_0^c F(\widetilde{\psi}(z))dz$.

Notice that (26) is a first order linear differential equation and therefore the initial value problem with $N(c^*) = 0$ has a unique solution

$$N^*(c) = \zeta(c)\left(\int_{c^*}^{c} \zeta(z)\frac{(\psi(z) - \psi(c^*))}{g(z)}dz\right),$$

where

$$\zeta(c) := \exp\left(\int_0^c \frac{1}{\Omega^*(c)}\right), \forall c \in [0, c_\alpha].$$

It can be verified that N^* is increasing and therefore is indeed a CDF of a Borel measure with density n^*. As such, let

$$m^*(c) := -\frac{F(\psi^*(c))}{\int_0^{P(\psi^*(c),k^*(c))} \gamma_3(x, \psi^*(c), k^*(c))dx} n^*(c), \forall c \in [0, c_\alpha],$$

and let μ^*, ν^* be the Borel measures induced by m^* and n^*. Notice that $\text{supp}(\mu^*) = \text{supp}(\nu^*) = [c^*, c_\alpha]$ and that ψ^*, k^* solves the auxiliary problem (23). Moreover, by construction, for any $c \in [c^*, c_\alpha]$,

$$\int_0^{\bar{v}} [\Gamma^c_{\mathbf{1}\{z \geq k^*(c)\}, \psi^*}(x) - F(x)]dx = 0 = \int_0^{P(\psi^*(c),k^*(c))} [\gamma(x, \psi^*(c), k(c)) - F(x)]dx$$

and therefore,

$$D'(\mu^*, \nu^*) = D(\mu^*, \nu^*).$$

Furthermore, let $q^*(z|c) := \mathbf{1}\{z \geq k^*(c)\}$. Then (ψ^*, q^*) indeed solves the dual problem (22) and

$$\int_0^{c_\alpha} \left(\int_0^{\bar{v}} [(1 - \Gamma^c_{q^*,\psi^*}(x)) - (1 - F(x))]dx\right) \mu^*(dc) = 0,$$
$$\int_0^{c_\alpha} \left(\int_0^c [q^*(z|c) - F(\psi^*(z))]dz\right) \nu^*(dc) = 0$$

for all $c \in [0, c_\alpha]$.

Finally, for any $c \in [0, c_\alpha]$, any $x \in [0, \bar{v}]$, let

$$\widehat{H}^c(x) := \begin{cases} \left[1 - \frac{(v^*(c) - k^*(c))(1 - F(\psi^*(\psi(c)))}{(x - k^*(c))^+}\right]^+, & \text{if } x \in [0, P(\psi^*(c), k^*(c))) \\ F(x), & \text{if } x \in [P(\psi^*(c), k^*(c)), \psi^*(c)) \\ F(\psi^*(c)), & \text{if } x \in [\psi^*(c), v^*(c)) \\ 1, & \text{if } x \in [v^*(c), \bar{v}] \end{cases},$$

where $v^*(c) := \mathbb{E}_F[v|v \geq \psi^*(c)]$. On the other hand, for any $H \in \mathcal{H}_F$ denote the integral of H by $I_H(x) := \int_x^{\bar{v}} (1 - H(z))dz$ for all $x \in [0, \bar{v}]$. Now let

$$I^c_{gu}(x) := \text{conv}\Big(\min\{I_F(x), I_{\widehat{H}^c}(x)\}\Big).$$

By construction, for all $c \in [0, c_\alpha]$ I^c_{gu} is convex and thus its subdifferential, $\partial I^c_{gu}(x)$, is nonempty for all $x \in [0, \bar{v}]$. Finally, let

$$H^c_{gu}(x) := \inf \partial I^c_{gu}(x), \, \forall x \in [0, \bar{v}], c \in [0, c_\alpha].$$

It can be verified that $H^c_{gu} \in \mathcal{H}_F$ for all $c \in [0, c_\alpha]$ and that $\{H^c_{gu}\}_{c \in [0, c_\alpha]}$ satisfies the sufficient conditions 1 and 2 in Lemma 1. Then by Lemma 4 and Lemma 1, there exists a transfer $t_{gu} : [0, c_\alpha] \to \mathbb{R}$ such that $(H^c_{gu}, t_{gu}(c), \mathbf{1}_{[0,c_\alpha]}(c))_{c \in [0, c_\alpha]}$ is indeed an incentive compatible, individually rational and responsive upper censorship menu with cutoff ψ^*, as desired.

Step 5: Finally, it remains to find c_α that maximizes

$$\int_0^{c_\alpha} (v^*(c) - \psi(c))(1 - F(\psi^*(c)))G(dc).$$

By definition of $\hat{c}$ and by monotonocity of ψ, and thus of ψ^*,

$$(v^*(c) - \psi(c))(1 - F(\psi^*(c))) \geq 0 \iff c \leq \hat{c}.$$

Since G has full support, the function

$$c_\alpha \to \int_0^{c_\alpha} (v^*(c) - \psi(c))(1 - F(\psi^*(c)))G(dc)$$

is indeed maximized at $\hat{c}$. This completes the proof. ■

C. Proofs for Welfare Analysis and Comparative Statics

Proof of Proposition 1. Notice that for each $c \in [0, \bar{c}]$, probability of efficient trade is the probability of the event that trade occurs when the buyer's value is greater than the seller's cost. Since $\psi(c) > c$ for all $c \in [0, \bar{c}]$,

$$\int_0^{\bar{c}} (1 - F(c))G(dc) > \int_0^{\bar{c}} (1 - F(\psi(c)))G(dc),$$

which implies that the probability of efficient trade is larger when the seller has control of the information technology.

On the other hand, since ψ is increasing and $\psi(c) > c$ for all $c \in [0, \bar{c}]$,

$$\begin{aligned}
&\int_{\psi(c)}^{\bar{v}} (1 - F(x))dx + (\psi(c) - c)(1 - F(\psi(c))) \\
&< \int_{\psi(c)}^{\bar{v}} (1 - F(x))dx + \int_{c}^{\psi(c)} (1 - F(x))dx \\
&= \int_{c}^{\bar{v}} (1 - F(x))dx,
\end{aligned}$$

for all $c \in [0, \bar{c}]$. Thus,

$$\int_0^{\bar{c}} (v(c) - c)(1 - F(\psi(c)))G(dc) < \int_0^{\bar{c}} (\mathbb{E}_F[v|v > c] - c)(1 - F(c))G(dc).$$

This completes the proof. ■

Proof of Proposition 2. For 1., notice that the intermediary's revenue is given by

$$\int_0^{\bar{c}} (v(c) - \psi(c))(1 - F(\psi(c)))G(dc),$$

and total surplus is

$$\int_0^{\bar{c}} (v(c) - c)(1 - F(\psi(c)))G(dc),$$

and the seller's expected net profit is

$$\int_0^{\bar{c}} \left(\int_c^{\bar{c}} (1 - F(\psi(z)))dz \right) G(dc)$$

for any distributions F, G satisfying $\phi(\psi(c)) \le c$ for all $c \in [0, \bar{c}]$. As such for any $i \in \{1, 2\}$,

$$\begin{aligned}
&(v_1(c) - \psi_i(c))(1 - F_1(\psi_i(c))) = \int_{\psi_i(c)}^{\bar{v}} (1 - F_1(x))dx \\
&\ge \int_{\psi_i(c)}^{\bar{v}} (1 - F_2(x))dx = (v_1(c) - \psi_i(c))(1 - F_1(\psi_i(c))),
\end{aligned}$$

and

$$\begin{aligned}
(v_1(c) - c)(1 - F_1(\psi_i(c))) &= \int_{\psi_i(c)}^{\bar{v}} (1 - F_1(x))dx + (\psi_i(c) - c)(1 - F_1(\psi_i(c))) \\
\ge (v_1(c) - c)(1 - F_2(\psi_i(c))) &= \int_{\psi_i(c)}^{\bar{v}} (1 - F_2(x))dx + (\psi_i(c) - c)(1 - F_2(\psi_i(c))),
\end{aligned}$$

and also

$$\int_c^{\bar{c}} (1 - F_1(\psi_i(z)))dz \ge \int_c^{\bar{c}} (1 - F_2(\psi_i(z)))dz,$$

for all $c \in [0, \bar{c}]$ and therefore

$$\int_0^{\bar{c}} (v_1(c) - \psi_i(c))(1 - F_1(\psi_i(c)))G_i(dc) \ge \int_0^{\bar{c}} (v_2(c) - \psi_i(c))(1 - F_2(\psi_i(c)))G_i(dc)$$

and

$$\int_0^{\bar{c}} (v_1(c) - c)(1 - F_1(\psi_i(c)))G(dc) \geq \int_0^{\bar{c}} (v_2(c) - c)(1 - F_2(\psi_i(c)))G(dc),$$

and also

$$\int_0^{\bar{c}} \left(\int_c^{\bar{c}} (1 - F_1(\psi_i(z)))dz \right) G(dc) \geq \int_0^{\bar{c}} \left(\int_c^{\bar{c}} (1 - F_2(\psi_i(z)))dz \right) G(dc),$$

for any $i \in \{1, 2\}$.

For 2., notice that by using integration by parts, for all $c \in [0, \bar{c}]$, $i, j \in \{1, 2\}$,

$$\int_{\psi_i(c)}^{\bar{v}} (1 - F_j(x))dx = \int_0^{\bar{v}} \mathbf{1}\{x \geq \psi_i(c)\}(1 - F_j(x))dx = \int_0^{\bar{v}} (x - \psi_i(c))^+ F_j(dx).$$

Therefore, since the function $x \mapsto (x - \psi(c))^+$ is convex and since F_1 is a mean preserving spread of F_2, for any $i \in \{1, 2\}$,

$$\begin{aligned}
&\int_0^{\bar{c}} (v_1(c) - \psi_i(c))(1 - F_1(\psi_i(c)))G_i(dc) \\
&= \int_0^{\bar{c}} \left(\int_0^{\bar{v}} (x - \psi_i(c))^+ F_1(dx) \right) \\
&\geq \int_0^{\bar{c}} \left(\int_0^{\bar{v}} (x - \psi_i(c))^+ F_1(dx) \right) \\
&= \int_0^{\bar{c}} (v_2(c) - \psi_i(c))(1 - F_2(\psi_i(c)))G_i(dc)
\end{aligned}$$

For 3., first notice that the hazard rate dominance implies that $\psi_1 \leq \psi_2$ and that

$$G_1(c) = \exp\left(- \int_c^{\bar{c}} \frac{1}{\psi_1(z)} dz \right) \geq \exp\left(- \int_c^{\bar{c}} \frac{1}{\psi_2(z)} dz \right) = G_2(c).$$

That is, G_2 first order stochastic dominates G_1. As such, for each $i \in \{1, 2\}$,

$$\begin{aligned}
&\int_0^{\bar{c}} (v_1(c) - \psi_1(c))(1 - F_i(\psi_1(c)))G_1(dc) \\
&= \int_0^{\bar{c}} \left(\int_{\psi_1(c)}^{\bar{v}} (1 - F_i(x))dx \right) G_1(dc) \\
&\geq \int_0^{\bar{c}} \left(\int_{\psi_2(c)}^{\bar{v}} (1 - F_i(x))dx \right) G_1(dc) \\
&\geq \int_0^{\bar{c}} \left(\int_{\psi_2(c)}^{\bar{v}} (1 - F_i(x))dx \right) G_2(dc) \\
&= \int_0^{\bar{c}} (v_2(c) - \psi_2(c))(1 - F_i(\psi_1(c)))G_2(dc),
\end{aligned}$$

where the first inequality follows from $\psi_1 \leq \psi_2$ and the second inequality follows from the fact that G_2 first order stochastic dominates G_1 and that ψ_2 is increasing. Similarly, for each $i \in \{1,2\}$,

$$\begin{aligned}
&\int_0^{\bar{c}} (v_1(c) - c)(1 - F_i(\psi_1(c)))G_1(dc) \\
&= \int_0^{\bar{c}} \left(\int_{\psi_1(c)}^{\bar{v}} (1 - F_i(x))dx + (\psi_1(c) - c)(1 - F(\psi_1(c))) \right) G_1(dc) \\
&\geq \int_0^{\bar{c}} \left(\int_{\psi_2(c)}^{\bar{v}} (1 - F_i(x))dx + (\psi_2(c) - c)(1 - F(\psi_2(c))) \right) G_1(dc) \\
&\geq \int_0^{\bar{c}} \left(\int_{\psi_2(c)}^{\bar{v}} (1 - F_i(x))dx + (\psi_2(c) - c)(1 - F(\psi_2(c))) \right) G_1(dc) \\
&= \int_0^{\bar{c}} (v_2(c) - c)(1 - F_i(\psi_2(c)))G_2(dc).
\end{aligned}$$

Finally, for the same reasons,

$$\int_0^{\bar{c}} \left(\int_c^{\bar{c}} (1 - F_i(\psi_1(z)))dz \right) G_1(dc) \geq \int_0^{\bar{c}} \left(\int_c^{\bar{c}} (1 - F_i(\psi_2(z)))dz \right) G_2(dc)$$

This completes that proof. ■

D. Proof of Claim

Proof of Claim 1. Take a sequence of strictly increasing and differentiable functions ψ_n such that $\{\psi_n\} \to \psi^*$ pointwisely. For each $n \in \mathbb{N}$, let

$$\begin{aligned}
k_n(c) &:= c - \frac{1}{F(\psi_n(c))} \int_0^c F(\psi_n(z))dz \\
\pi_n(c) &:= \int_{\psi_n(c)}^{\bar{v}} (1 - F(x))dx + (\psi_n(c) - k_n(c))(1 - F(\psi_n(c))),
\end{aligned}$$

and let

$$\begin{aligned}
k^*(c) &:= c - \frac{1}{F(\psi^*(c))} \int_0^c F(\psi^*(z))dz \\
\pi^*(c) &:= \int_{\psi^*(c)}^{\bar{v}} (1 - F(x))dx + (\psi^*(c) - k^*(c))(1 - F(\psi^*(c))),
\end{aligned}$$

for all $c \in [0, \bar{c}]$. Notice that since $\{\psi_n\} \to \psi^*$ pointwisely, by the dominated convergence theorem, $\{\pi_n\} \to \{\pi^*\}$ pointwisely as well and $(\psi_n(c) - k_n(c)) \geq 0$ for all $c \in [0, \bar{c}]$ for n large enough. As such, for n large enough,

$$\pi_n'(c) = -(\psi_n(c) - k_n(c))(1 - F(\psi_n(c))) - \frac{(1 - F(\psi_n(c)))}{F(\psi_n(c))^2} \int_0^c F(\psi_n(z))dz < 0, \forall c \in [0, \bar{c}]$$

This then implies that for n large enough,

$$\frac{d}{dc}\left[\int_0^{P(\psi_n(c),k_n(c))}[(1-\gamma(x,\psi_n(c),k_n(c))-(1-F(x))]dx\right]\leq 0, \forall c\in[0,\bar{c}].$$

Therefore,

$$\begin{aligned}
0 \geq& \frac{d}{dc}\left[\int_0^{P(\psi_n(c),k_n(c))}[(1-\gamma(x,\psi_n(c),k_n(c)))-(1-F(x))]dx\right]\\
=& f(\psi_n(c))\psi_n'(c)\Bigg[-\frac{1}{f(\psi_n(c))}\int_0^{P(\psi_n(c),k_n(c))}\gamma_2(x,\psi_n(c),k_n(c))dx\\
&-\int_0^{P(\psi_n(c),k_n(c))}\gamma_3(x,\psi_n(c),k_n(c))dx\frac{\int_0^c F(\psi_n(z))dz}{F(\psi_n(c))^2}\Bigg]\\
=& f(\psi_n(c))\psi_n'(c)\left[-\frac{\int_0^{P(\psi_n(c),k_n(c))}\gamma_2(x,\psi_n(c),k_n(c))dxF(\psi_n(c))}{\int_0^{P(\psi_n(c),k_n(c))}\gamma_3(x,\psi_n(c),k_n(c))dxf(\psi_n(c))}-(c-k_n(c))\right]\\
&\times\frac{\int_0^{P(\psi_n(c),k_n(c))}\gamma_3(x,\psi_n(c),k_n(c))dx}{F(\psi_n(c))},
\end{aligned}$$

for all $c\in[0,\bar{c}]$, $n\in\mathbb{N}$. Furthermore, direct calculation shows that

$$\frac{\int_0^{P(\psi_n(c),k_n(c))}\gamma_3(x,\psi_n(c),k_n(c))dx}{F(\psi_n(c))}<0, \forall c\in[0,\bar{c}],\ n\in\mathbb{N}.$$

Together, since $f(\psi_n(c))\psi_n'(c)>0$ for all $c\in[0,\bar{c}]$ and $n\in\mathbb{N}$, for n large enough,

$$\Omega_n(c):=-\frac{\int_0^{P(\psi_n(c),k_n(c))}\gamma_2(x,\psi_n(c),k_n(c))dxF(\psi_n(c))}{\int_0^{P(\psi_n(c),k_n(c))}\gamma_3(x,\psi_n(c),k_n(c))dxf(\psi_n(c))}-(c-k_n(c))\geq 0,$$

for all $c\in[0,\bar{c}]$. Finally, by the dominance convergence theorem, $\{\Omega_n\}\to\Omega^*$ pointwisely and therefore

$$\Omega^*(c)\geq 0, \forall c\in[0,\bar{c}].$$

Direct computation then shows that Ω^* is strictly decreasing on $[c^*,\bar{c}]$. This completes the proof. ■