NO TRADE AND YES TRADE THEOREMS FOR HETEROGENEOUS PRIORS

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ABSTRACT. First we show that even under non-common priors the classical no trade theorem obtains. However, speculative trade becomes mutually acceptable, if traders put at least slight probability on the trading partner being irrational. Our model, thus, provides a generalization of the result of Neeman (1996) for the case of heterogeneous priors. We also derive bounds on disagreements in the case of heterogeneous priors and p-common beliefs.

1. INTRODUCTION

One commonly hears it said that 'if there is a common prior among agents, then no trade is possible between them', which is indeed true. It is, however, also not any less common to hear the (erroneous) inverse to that statement, that is, a claim that 'if priors are not common, then an unbounded volume of trade may ensue'. The goal of this paper is two-fold: first we show that even if priors are not common (and there are arbitrarily large disagreements among agents about the expected value of a random variable), a corresponding version of the no-trade theorem obtains. Second, we extend to the case of heterogeneous priors a result due to Neeman (1996b) that shows the necessity of the existence of slight irrationality of each player for trade to occur. Specifically, we demonstrate that when agents do not share a common prior, for trade to occur it is sufficient that each agent i believes that with some arbitrarily small probability *other agents* are irrational, while ascribing probability one that himself he is perfectly lucid.

The surprising No Trade theorem of Milgrom and Stokey (1982), which states that under the assumption of common priors and ex-ante Pareto efficient allocations the arrival of private information will not induce further trade if the acceptability of a proposed trade is common knowledge among traders, has for several decades been a much-discussed counter-intuitively negative result of the theoretical literature. Under its usual interpretation as a 'no speculation' result, it has been perplexing because it stands in stark

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contrast to the immense volume of speculative trade observed daily in security markets.

The No Trade theorem is usually presented as building upon two other surprising results, the No Disagreement theorem and the No Betting theorem. These theorems essentially state that if agents have common priors, they will never take opposite sides of any commonly acceptable proposed bet even after receiving private information. Moreover, the no betting result is bi-directional, i.e., a common prior precludes betting while heterogeneous priors imply the existence of agreeable bets (i.e. that both agents would *agree to disagree* about the expected value of a random variable). It is this that leads to the common but erroneous argument that the No Trade theorem is also dependent on the common prior assumption. Indeed, the counter-intuitive quality of the No Trade theorem contrasted with observed trade volumes is sometimes adduced as an argument against assuming common priors in 'the real world'. In this view, dropping the common prior assumption puts the theory back in harmony with the empirical existence of speculative trade.

We show here that, in fact, the No Trade result is independent of the question of whether or not priors are common; it follows solely from the combination of ex-ante Pareto efficiency and common knowledge of an agreed trade. One needs to weaken common knowledge of rationality to restore the possibility of mutually agreed trading.

This insight is not entirely new. Neeman (1996b) presents a model with common priors in which speculative trade is possible between traders with only common p-belief of rationality.[1] However, the model in that paper must assume that each trader ascribes some positive probability to *himself* being irrational at some state, making its interpretation difficult. It would seem much more natural to postulate that each trader accepts a trade based on a belief that others are irrationally acquiescing to the terms of the trade, while being certain that he is immaculately rational.

Neeman (1996b), noting the problematic aspect of all agents ascribing a non-zero probability of self-irrationality, conjectures that with heterogeneous priors trade can occur with each trader assigning positive probability only to the other trader being irrational, leaving himself perfectly rational. We show here that this is indeed true, and that furthermore it suffices for one trader to ascribe positive probability, as small as desired, of the other trader being irrational at some state for trade to occur.

[1] For a related result see also Dow et al (1990)

2. PRELIMINARIES

Let I be a pair of players. Let Ω be an uncountable compact metric state space, endowed with a certain topology τ. Denote by Σ the Borel σ-field of Ω generated by topology τ. Assume that the corresponding space of measures $\Delta(\Omega)$ is endowed with the topology of weak convergence.

For each i, the collection $\Pi^i \equiv \{\Pi^i(\omega) \mid \omega \in \Omega\}$ is a partition of Ω. We assume that this partition is finite for all i. $\mathbf{\Pi} = (\Pi^i)_{i\in I}$ is called a partition profile of the set of players $i \in I$. Denote by $\mathcal{F}^i$ the σ-field generated by Π^i, so that $\mathcal{F}^i$ consists of all unions of elements of Π^i. It follows that $\mathcal{F}^i \subset \Sigma$.

The meet of $\mathbf{\Pi}$, denoted $\wedge\mathbf{\Pi}$, is the partition that is the finest among the partitions that are simultaneously coarser than all the partitions Π^i. We will denote by $\mathbf{\Pi}(\omega)$ the element of $\wedge\mathbf{\Pi}$ containing ω. $\mathbf{\Pi}$ is called connected when $\wedge\mathbf{\Pi} = \{\Omega\}$.

For each $i \in I$ and $\omega \in \Omega$, let $t_i(\omega)$ be a probability measure on Ω, such that:

(a) $t_i(\omega)(\Pi_i(\omega)) = 1$;
(b) for each $\omega' \in \Pi^i(\omega)$, $t_i(\omega') = t_i(\omega)$.

The function $t_i : \Omega \to \Delta(\Omega)$ is a type function and $t_i(\omega)$ is a type of i at $\omega \in \Omega$. Throughout the paper we assume that $t_i(\omega)$ is a continuous function and each $i \in I$ has a regular non-atomic Borel probability measure $t_i(\omega)$ on Ω at every state $\omega \in \Omega$. A type space corresponds to the tuple

$$\mathcal{T} = \{I, \Omega, (\Pi_i, t_i)_{i\in I}\}$$

A probability measure $\mu^i \in \Delta(\Omega)$ is a prior for player i if for every event $E \in \Sigma$, $\mu^i(E) = \int_\Omega t^i(\omega)(\cdot)(E)d\mu^i(\cdot)$. In other words, μ^i is a prior if i's types $t^i(\omega)$ are the posteriors of μ^i conditional on i's information function t^i. A probability measure $\mu \in \Delta(\Omega)$ is a common prior if it is a prior for each $i \in I$. We will not assume the existence of common priors here unless it is explicitly stated that one exists.

For a measurable random variable f over Ω, the prior expectation of a player i with respect to a prior μ^i will be denoted $E_{\mu^i} f = \int_\Omega f(\cdot)d\mu^i(\cdot)$. The posterior expectation of f by player i at a state ω will be denoted

$$E_i(f \mid \Pi^i(\omega)) = \int_\Omega f(\cdot)dt^i(\omega)(\cdot). \tag{1}$$

The expected value of an event H will be understood to be the expected value of the standard characteristic function 1^H which is defined as:

$$1^H(\omega) = \begin{cases} 1 & \text{if } \omega \in H \\ 0 & \text{if } \omega \notin H \end{cases}$$

Given the type space $\mathcal{T}$, denote the set of all priors of player i by $W^i(\mathcal{T})$, or simply by W^i when $\mathcal{T}$ is understood.[2] In general, W^i is a set of probability distributions, not a single element; it is, in fact, a closed and convex set (Samet (1998), Heifetz (2006)).

To articulate players' mutual beliefs and knowledge we proceed following the generalization of the concept of common knowledge due to Monderer and Samet (1989). In particular, suppose $\mathcal{T}$ is a type profile and that for each i, μ^i is a prior for player i. Then player i p-*believes* event E at ω if $\mu^i(E \mid \pi^i(\omega)) \geq p$. The event that '$i$ p-believes E', denoted $B_i^p(E)$ is

$$B_i^p(E) := \{\omega \mid \mu^i(E \mid \Pi^i(\omega)) \geq p\}. \tag{2}$$

An event E is called (heterogenously)[3] evident p-belief if for each $i \in I$

$$E \subseteq B_i^p(E). \tag{3}$$

An event E is (heterogenously) common p-belief at ω if there exists an evident p-belief event $F \ni \omega$ such that for all $i \in I$,

$$F \subseteq B_i^p(E). \tag{4}$$

It is straightforward to show that for every $0 \leq p \leq 1$, player i, and $E \in \Sigma$, one has $B_i^p(E) \in \mathcal{F}^i$, i.e. B_i^p is measurable with respect to $\mathcal{F}^i$. containing ω.

Finally, we note that when $p = 1$, the above definition corresponds to 'knowledge' rather than 'belief'.

3. Common Priors: No Betting and No Trade

3.1. No Disagreements and No Agreeable Bets.

The main characterisation of the existence of common priors in the literature is based on the concept of agreeable bets.

Definition 1. Given an n-player type space $\mathcal{T}$, an n-tuple of random variables $\{f^1, \dots, f_n\}$ is a *bet* if $\sum_{i=1}^n f^i = 0$. ♦

Definition 2. A bet $\{f^1, \dots, f_n\}$ is an *agreeable bet* if it is common knowledge that $E_i f^i(\omega) > 0$ for all players i. ♦

[2] Strictly speaking, the set of priors of a player i depends solely on i's type function t^i, not on the full type profile $\mathcal{T}$. However, since we are studying connections between sets of priors of different players, we will find it more convenient to write $W^i(\mathcal{T})$.

[3] We use the term heterogenously here to refer to the possibility that the players may not share a common prior.

In the special case of a two-player type space Definition 2 implies that we may consider a random variable f to be an agreeable bet if it is common knowledge that $E_1 f > 0 > E_2 f$ (by working with the pair $\{f, -f\}$).

We can now state the Aumann No Disagreement Theorem: *A compact type space has a common prior if and only if there does not exist an agreeable bet.*

The most accessible proof of this result is in Heifetz (2006). The seminal work on no disagreement is the famous paper of Aumann (1976), which proved the result in one direction (a common prior implies no disagreement) in the special case of finite type spaces and bets that are restricted to characteristic functions over events. This was extended to bets that are any random variable over a finite type space with a common prior by Sebenius and Geanakoplos (1983). The converse direction was independently proved by Morris (1995) and Samet (1998) for finite type spaces and by Feinberg (2000) for compact type spaces.

Weakening the condition that the posterior values players assign to events are common knowledge, Monderer and Samet (1989) extend the no disagreement theorem by showing that if there is a common prior and common p-belief regarding the posteriors players assign to an event then those posterior values cannot differ by more than $2(1-p)$. Following this, Neeman (1996a) showed that this upper bound on disagreement can be reduced to $1-p$ (but no further).

3.2. **The Exchange Economy and No Trade.**

Based on Milgrom and Stokey (1982), we consider an economy with a set of players $I = \{1, 2\}$, operating in an environment of uncertainty represented by a finite type space $\mathcal{T}$. For simplicity of exposition, the model here is restricted to a one-dimensional commodity space. Let $u^i : \mathbb{R} \to \mathbb{R}$ be player i's von-Neumann-Morgenstern utility function. Allocations are presumed to be state-contingent, hence let $e^i(\omega)$ denote player i's (initial) allocation at state ω, i.e., $e^i : \Omega \to \mathbb{R}$, and denote by $e = \{e^1, e^2\}$ the total (initial) allocation.

Definition 3. A *trade* $B = \{(B^1, B^2)\}$ between the players is a state-contingent commodity transfer, i.e., a pair of random variables $B^i : \Omega \to \mathbb{R}$ satisfying the constraint

$$B^1(\omega) + B^2(\omega) = 0.$$

for all $\omega \in \Omega$. A trade is *feasible* if for all i and all $\omega \in \Omega$

$$e^i(\omega) + B^i(\omega) \geq 0$$

with strict inequality for some i and some ω. ♦

Definition 4. Given a pair of priors μ^1, μ^2 of players 1 and 2 respectively, an allocation $e = \{e^1, e^2\}$ is *ex-ante Pareto optimal* if there does not exist a feasible trade $B = (B^1, B^2)$ such that

$$E_{\mu^i}(u^i(e^i + B^i)) \geq E_{\mu^i}(u^i(e^i))$$

for all i, with strict inequality for some i. ♦

Milgrom and Stokey (1982), building on the results of Aumann (1976), proved what has come to be called the No Trade theorem: in a knowledge structure with a common prior and *ex ant*e Pareto efficiency, rational agents can never agree to a commonly known and mutually acceptable feasible trade in the interim period after they receive private information and calculate their posterior expected values.

Note that the statement of the theorem goes in only one direction: a common prior and common knowledge of acceptance of a trade nullifies a possibility of trade occurring. The no disagreements theorem, as extended to no betting, is bi-directional: if there is a common prior there is no betting, and if there is no common prior then there exists a mutually agreeable bet.

It is tempting to try to conclude that no trade can similarly be extended to its converse, that is, that if there is no common prior then traders will able in the interim period to conduct a mutually agreeable trade. We will show in Section 5 that this is not true: even when there is no common prior, the traders will still fail to agree to trade in the interim period. No betting is *not* equivalent to no trade. For trade to occur, more is required than heterogeneous priors alone. We show in Section 6 that if at least one agent believes in the possibility that the other trader may be irrational, then heterogeneous priors can induce mutually acceptable trade.

4. Bounded Disagreements

Given the negative result of the previous Section, a natural question that arises is what happens when the assumptions of common priors and common knowledge of a particular event (namely, the event of positive expected values on the part of all players) are weakened. This is what we will do in the remaining of this paper.

The established measure of how far players are from common knowledge of an event is the concept of common p belief, introduced by Monderer and Samet (1989). We present in Section 4.2 a way to measure how far a type space is from having a common prior. In Section 4.3 we put these together common p-belief and our measure of the distance from common prior to study bounds on possible disagreements between players when both the common prior and the common knowledge assumptions are weakened.

4.1. **Background Assumptions.**

We assume throughout a fixed two-player type space τ with each the partition Π^i for each player i further assumed to be finite, such that τ is connected, i.e., $\Pi^1 \wedge \Pi^2 = \{\Omega\}$. This enables us to equate statements regarding common knowledge of an event with statements regarding mutual knowledge of the event at every state in Ω, simplifying the presentation. The generalisation to non-connected spaces, which can always be decomposed to connected subspaces, is straight-forward.

Given a distribution $\mu \in \Delta(\Omega)$ and an event A, if $A \cap \mathrm{supp}(\mu) = \emptyset$, then $E_\mu(f \mid A)$ in (1) is undefined. It turns out, however, that for our purposes in this paper, we may conveniently adopt the convention that

$$E_\mu(f \mid A) = \begin{cases} \frac{1}{\mu(A)} \int_{\omega \in A} f(\omega) d\mu & \text{if } \mu(A) \neq 0 \\ 0 & \text{if } \mu(A) = 0 \end{cases} \tag{5}$$

This convention will be assumed throughout the rest of this paper.

4.2. **Prior Distance.**

The main idea here is straightforward. Each player has a compact and closed set of priors. When these sets intersect – intuitively speaking, when there is 'zero distance' between them – there is a common prior. Thus, if they are disjoint, we will seek points in player 1's set of priors and points in player 2's set of priors, such that the distance between these points is the 'smallest distance between the sets of priors'. Since we are trying to find bounds on disagreements, the 'distance' needs to measure how 'far apart' distributions are with regard to conditional probabilities, given an event. To capture notions of closeness of priors, we need a topology.

Given $\varphi^1, \varphi^2 \in \Delta(\Omega)$, a random variable f, and an event $A \in \Sigma$, define d' by:

$$d'(\varphi^1, \varphi^2, A, f) := \left| E_{\varphi^1}(f \mid A) - E_{\varphi^2}(f \mid A) \right|. \tag{6}$$

In words, d' is the absolute difference between the expectations assigned by φ^1 and φ^2 to f conditional on A.

Denote by B the collection of measurable random variables f over Ω satisfying the property that $0 \leq f(\omega) \leq 1$ for all $\omega \in \Omega$, and then use d' to define

$$d''(\varphi^1, \varphi^2, A) := \sup_{f \in B} d'(\varphi^1, \varphi^2, A, f). \tag{7}$$

Restricting the random variables to the closed and bounded set B ensures that the supremum in Equation (7) is a well-defined real number.

Finally, define

$$d(\varphi^1, \varphi^2) := \sup_{A \in \Sigma} d''(\varphi^1, \varphi^2, A). \tag{8}$$

Lemma 1, in the appendix, shows that d in fact defines a metric over $\Delta(\Omega)$.[4]

Intuitively, d measures the 'worst case' of disagreement between agents holding beliefs φ^1 and φ^2 (over all normalised random variables). By the normalisation assumption, $0 \leq d(\varphi^1, \varphi^2) \leq 1$ for any pair of distributions φ^1 and $\varphi^2 \in \Delta(\Omega)$.

Note also the following: suppose that there is an event A such that $\varphi^i(A) = 0$ and $\varphi^j(A) \neq 0$. Then, using Equation (5) and setting f to be $f(\omega) = 1$ for all $\omega \in A$, we have $E_{\varphi^i}(f \mid A) = 0$ while $E_{\varphi^j}(f \mid A) = 1$, so that $d''(\varphi^1, \varphi^2, A) = 1$, and therefore $d(\varphi^1, \varphi^2) = 1$. This makes perfect intuitive sense: if $\varphi^i(A) = 0$ and $\varphi^j(A) \neq 0$, then i and j have the most 'violent disagreement' possible – one regards the expected value of any f given A to be a well-defined real number, the other regards it as undefined, because he regards A as impossible. Hence φ^1 and φ^2 should being maximally distant under d, which is exactly the intuitive interpretation we give to $d(\varphi^1, \varphi^2) = 1$.

Definition 5. Given a type space $\mathcal{T}$, and the related sets of priors W^1 and W^2 of players 1 and 2 respectively, the value

$$\delta = \inf_{\varphi \in W^1, \varphi' \in W^2} d(\varphi, \varphi') \tag{9}$$

will be called the *prior distance of* $\mathcal{T}$. A pair of priors $\mu^1 \in W^1$ and $\mu^2 \in W^2$ such that $d(\mu^1, \mu^2) = \delta$ are a pair of δ-*nearest priors of* $\mathcal{T}$. ⧫

Definition 5 intuitively measures the distance between the pair of sets of priors of players 1 and 2, W^1 and W^2, by identifying the 'closest points' between the two sets according to the metric d. If the sets have non-empty intersection, there is a common prior (hence no disagreement) and the prior distance is zero. If the prior distance δ is non-zero, the sets of priors are non-intersecting, there is no common prior (hence there is disagreement), and δ can be interpreted as providing a measure of possible posterior disagreement.

4.3. **Almost-Almost: Almost Common Knowledge with Almost Common Priors.**

Monderer and Samet (1989) consider models with common priors but 'almost common knowledge', via common p-beliefs. With the notion of

[4] This metric induces a topology on the space of measures $\Delta(\Omega)$, and the resulting topology is finer than the usual weak* topology.

nearest priors, we can relax both the assumption of common priors and the assumption of common knowledge, to yield an 'almost-almost' result, with nearest priors serving as a proxy for 'almost common priors', and common p-beliefs as 'almost common knowledge', bounding disagreements regarding posteriors.

Let $\mathcal{T}$ be a connected type space. Suppose that $\mathcal{T}$ has δ-prior distance, with (μ^1, μ^2) a pair of δ-nearest priors. Fix a random variable g with $0 \leq g(\omega) \leq 1$ for all ω.

Next, define functions f^i, for the players $i = 1, 2$, by

$$f^i(\omega) := \mu^i(g \mid \Pi^i(\omega));$$

i.e., $f^i(\omega) = f^i(\Pi^i(\omega))$ is i's posterior probability of g, given $\Pi^i(\omega)$. Let η_i be two numbers in the interval $[0, 1]$, satisfying $\eta_1 \geq \eta_2$. Consider the event

$$C = \{\omega \mid f^1(\omega) \geq \eta_1\} \cap \{\omega \mid f^2(\omega) \leq \eta_2\}.$$

In words, C is the event that player 1's posterior probability of g is greater than or equal to η_1 and player 2's posterior probability of g is less than or equal to η_2.

Proposition 1. *Suppose that $\mathcal{T}$ has δ-prior distance and that there is common p-belief at a state ω^* in the event*

$$C = \{\omega \mid f^1(\omega) \geq \eta_1\} \cap \{\omega \mid f^2(\omega) \leq \eta_2\}.$$

Then $|\eta_1 - \eta_2| \leq 1 - p(1 - \delta)$.

Proposition 1 immediately leads to the following theorem:

Theorem 1. *Suppose that $\mathcal{T}$ has δ-prior distance and that the posteriors of an event H are common p-belief at some $\omega \in \Omega$. Then those posterior beliefs cannot differ by more than $1 - p(1 - \delta)$.*

The above result highlights how common p-belief and δ-nearest priors work together to bound disagreements, beyond the common prior and common knowledge cases.

When $\delta = 0$, i.e., there is a common prior, Theorem 1 recapitulates the result of the main theorem of Neeman (1996a) (for a pair of players), and when both $p = 1$ and $\delta = 0$, i.e., there is both a common prior and common knowledge, we recapitulate the No Disagreements Theorem of Aumann (1976).

5. Heterogeneous Priors and No Trade

5.1. **Preliminaries.**

We work with the following model of rational trade. Let A denote the set of actions available to the traders in I, namely

$$A = \{\text{'buy'}, \text{'sell'}, \text{'refrain'}\}.$$

For each observed partition $\Pi^i(\omega)$, offered trade $B(\omega)$, and suggested trade price q trader i strategy prescribes an action in A, we denote as $s^i(\omega, B, q)$ player i's strategy function. The space of all strategy functions is denoted as S.

In choosing his strategy, trader i takes into account his utility function u^i for money. Given that a trade occurs if and only if one trader is willing to sell and the other to buy, we can derive a 'utility function from trade' for each trader i, labelled $v^i(s^i(\omega), s^j(\omega), B, q)$ as

$$v^i(s(\omega), B, q) = \begin{cases} u^i(e^i(\omega) + B(\omega) - q) & \text{if } s^i(\omega) = \text{'buy'}, s^j(\omega) = \text{'sell'} \\ u^i(e^i(\omega) - B(\omega) + q) & \text{if } s^i(\omega) = \text{'sell'}, s^j(\omega) = \text{'buy'} \\ u^i(e^i(\omega)) & \text{otherwise.} \end{cases}$$

Trader i is *rational* at $\omega \in \Omega$ with respect to a proposed trade B and price q if $\omega \in R^i(a, B, q)$, where $R^i(s, B, q)$ is defined as

$$\begin{aligned} &R^i(s, B, q) \\ &:= \left\{\omega \mid s^i(\omega, B, q) \in \arg\max\nolimits_{s^i \in S} E_i(v^i(s^i(\omega), s^j(\omega), B, q \mid \Pi^i(\omega)))\right\}. \end{aligned}$$

Hence, as in games with incomplete information, rationality is defined interactively, as being trader i's best response to s^j.

Assuming w.l.o.g. that trader 1 is the buyer and trader 2 is the seller, let $T(s, B, q)$ denote the event at which both traders are rational and trade is beneficial to both parties:

$$\begin{aligned} &T(s, B, q) \\ &:= \left\{\omega \in R^1 \cap R^2 \mid \begin{array}{l} \text{'buy'} = \arg\max E(v^1(s(\omega), B, q) \mid \Pi^1(\omega)) \\ \text{'sell'} = \arg\max E(v^2(s(\omega), B, q) \mid \Pi^2(\omega)) \end{array}\right\} \end{aligned}$$

5.2. **No Trade Even When Priors Are Not Common.**

It is common error to believe that no trade results follow immediately from no-betting results and that therefore one must assume a common prior in order to derive a no-trade result. This is not true; in contrast to betting, no-trade holds even *without* the assumption of a common prior if one is working with a model of ex-ante Pareto efficiency. This is shown in the next result.

Theorem 2. *For a pair of players, let u^1 and u^2 respectively be a pair of (not necessarily risk-neutral nor increasing) utility functions, μ^1 and μ^2 respectively be a pair of priors (allowing for heterogeneous priors), and e^1*

and e^2 respectively be a pair of allocations that are ex-ante Pareto-efficient with respect to μ^1 and μ^2.

Suppose that there exists a trade B, a price q, and a pair of strategies a^1, a^2 such that there is common knowledge of rationality and common knowledge of a positive probability of trade. Then neither player can strictly prefer trade to no-trade.

It may be instructive to compare the no trade result in Theorem 2 with the No Betting Theorem. Superficially, it may seem that since heterogeneous priors guarantee the existence of an agreeable bet, that should carry over to an agreed trade, because a proposed trade could be constructed out of a proposed bet. The difference between the two results comes down to the assumption of ex ante Pareto efficiency in the no-trade result, which plays no role in the no-betting result.

The essence of the original proof of Aumann (1976) that 'common priors imply no agreement' is rather similar to the heart of the proof of Theorem 2, namely any common knowledge disagreement that holds in each partition element can be aggregated to a disagreement over the entire state space, contradicting the assumption of a common prior. If there are heterogenous priors, one could run exactly the same argument. The result would be non-equal ex ante expected values, one per each player, but that would not lead to any contradiction, since heterogeneous priors can accommodate ex ante disagreement in the full state space. In contrast, in the trade model, any mutually accepted interim trade (under common knowledge) is translated under aggregation into a mutually accepted ex ante trade, violating the ex ante Pareto efficiency assumption even when there are heterogeneous priors.

6. Heterogeneous Priors and Trade

As Theorem 2 indicates, for trade to be feasible it is insufficient to weaken the common prior assumption. In fact, one needs also to weaken the assumption of common knowledge of rationality; that is, there needs to be a 'suspicion of irrationality' in the sense that each trader believes he is rational but suspects that the other trader is not.

Making this precise requires adopting a bounded rationality approach. There are several models of bounded rationality in the literature. We will make use here of the minimal bounded rationality assumption necessary for trade to occur in the context of heterogeneous priors. Specifically, we let common knowledge of rationality be weakened to common p-belief in rationality for p arbitrarily close to 1, where furthermore each agent believes himself to be rational with probability 1.

6.1. **Common p-Belief in Rationality and Trade.**

Neeman (1996b) presents a model of common p-belief in rationality, in the context of common priors, that enables trade to occur with unbounded volumes of trade. For that model to work, each trader must assume that there is a non-zero probability that *he himself* may be irrational. There are interpretational difficulties in such an approach; noting this, in the concluding remarks of that paper, Neeman (1996b) states that 'it might be argued that using different priors alleviates some of the difficulties with the interpretation of the model, in particular regarding the way we chose to model irrationality.'

We show here in detail in Theorem 3 that this is indeed true: in a model of *heterogeneous* priors, common p-belief in rationality enables trade to occur under conditions of 'suspicion of irrationality', in which each trader asserts zero probability that he is irrational but ascribes positive probability that the other trader is irrational. In addition, although the volume of trade may be unlimited, by measuring how far the traders are from a common prior using prior distance, Theorem 3 provides limits on the range of prices under which trade may occur.

The next two definitions are straightforward generalisations of definitions in Neeman (1996b) to the case of heterogeneous priors, that is, we suppose that the players respectively have priors μ^i, μ^j, where we allow for $\mu^i \neq \mu^j$.[5]

Definition 6. A pair of strategies $s^1(\omega, B, q)$ and $s^2(\omega, B, q)$ is a *heterogeneous* $(1-\rho^1, 1-\rho^2)$*-rationality Nash equilibrium* if for at all i, $\mu^i(R^j(s)) \geq 1 - \rho^i$. ♦

In words, each trader i ascribes probability at least $1 - \rho^i$ of trader j rationally optimising with respect to trader i's actions. This leaves scope for each trader to believe that his or her strategy is rational with probability 1.

Definition 7 is taken directly from Neeman (1996b).

Definition 7. Let $0 < p < 1$. Then two partitions Π^1 and Π^2 are p-*overlapping* if for each trader i there exists an index set K^i and a non-empty set $\pi^i = \bigcup_{k \in K^i} \Pi^i_k$ such that the following two conditions hold:

(1) $\mu^i(\pi^j \mid \Pi^i_k) \geq p$ for all $\Pi^i_k \subseteq \pi^i$; and
(2) for any two non-empty index subsets $K^{1'} \subseteq K^1$ and $K^{2'} \subseteq K^2$, $\mu^i(\pi^{1'} \Delta \pi^{2'}) > 0$ for at least one i, where $\pi^{i'} = \bigcup_{k \in K^{i'}} \Pi^i_k$ and Δ denotes symmetric difference. ♦

The definition of p-overlapping partitions is quite technical but it essentially provides a necessary and sufficient condition for the existence of a

[5] When $\mu^i = \mu^j$, the definitions in Neeman (1996b) are recaptured.

common p-belief event, namely $\pi^1 \cap \pi^2$, that is not common knowledge, nor are any of it sub-events common knowledge either.

Theorem 3. *Let $\mathcal{T}$ be a type space with δ prior distance, with $\delta > 0$. Let $\rho^1, \rho^2 \geq 0$, with $\rho^i > 0$ for at least one i, and let $0 < p < 1$. Then the following two conditions are equivalent:*

(I) *The information structures are p-overlapping.*
(II) *There exists a proposed trade B, a price q, and strategies $s^1(\omega), s^2(\omega)$ such that:*
  (a) *the strategies $s^1(\omega), s^2(\omega)$ form a $(1-\rho^1, 1-\rho^2)$-rationality Nash equilibrium with respect to the trade (B, q).*
  (b) *$T(s, B, q)$ is heterogeneous common p-belief at some $\omega \in \Omega$.*

Furthermore, the range of the price q at which trade may occur is limited by $y^1 > q > y^2$, where $y^1 - y^2 \leq (1 - p(1-\delta))\|B\|_\infty$.

Proof. See Appendix.

As Theorem 3 indicates, for trade to occur it suffices for one trader to ascribe positive probability to the other trader being irrational (not necessarily both traders), while each trader ascribes zero probability to himself being irrational.

6.2. Common p-Belief in Rationality: The General Case.

In this subsection we allow the functions u_i to incorporate agents' risk aversion. We parametrise utility functions by two positive parameters M, ψ. Specifically we consider the set of strictly increasing, concave, and differentiable utility functions

$$U_{M,\psi} = \{u : [-M, M] \to \mathbb{R} \mid \; u'(M) \geq \psi\}$$

i.e. allowing for a general preference representation. It turns out that the previous theorem extends easily to this more general case, provided that agents are not too risk averse, in which case one can find a non-empty range of the prices allowing for the trade to happen.

Theorem 4. *Let $\mathcal{T}$ be a type space with δ prior distance, with $\delta > 0$. Let $\rho^1, \rho^2 \geq 0$, with $\rho^i > 0$ for at least one i, and $p < 1$. Let there be also given some positive ψ and M. Then the following two conditions are equivalent:*

(I) *There exists a $\sigma > 0$ such that there exists a trade B, a price q, and strategies $s^1(\omega), s^2(\omega)$ such that for all traders with utility functions $u \in U_{M,\psi}$ that satisfy $\sup_{|x|<M} |u''(x)| < \sigma$ (i.e. sufficiently risk tolerant traders):*
  (a) *the strategies $s^1(\omega), s^2(\omega)$ form a $(1-\rho^1, 1-\rho^2)$-rationality Nash equilibrium with respect to the trade (B, q).*

(b) *$T(s, B, q)$ is heterogeneous common p-belief at some $\omega \in \Omega$.*

(II) *The information structures are p-overlapping.*

Furthermore, the range of the price q at which trade may occur is limited by $y^1 - C^1(\mu^1, u^1) > q > y^2 + C^2(\mu^2, u^2)$ with $y^1 - y^2 \leq (1 - p(1 - \delta))\|B\|_\infty - (C^1(\mu^1, u^1) + C^2(\mu^2, u^2))$.

Hence, compared to the result of Theorem 3, the more risk averse are the agents (i.e. the higher is $C^1(\mu^1, u^1) + C^2(\mu^2, u^2)$), the smaller is the range of prices under which trade will happen. In other words, risk aversion can be expected to have a dampening effect on trade by limiting the range of prices at which traders will agree to conduct trades.

7. Concluding Remarks

In this paper we have shown that on the one hand, the no trade theorem obtains even if players do not share a common prior; one the other hand a small departure from the common knowledge of rationality is sufficient to restore the positive trade result. This departure can be arbitrarily small, i.e. in common p-belief of rationality p can approach 1 arbitrarily close (provided that information structures remain to be p-overlapping) and moreover each agent can believe himself to be perfectly rational.

Thus the results of our paper shed light on the connection of the no trade theorem to empirical evidence regarding the positive relationship between volumes of trade and traders' over-confidence (see, e.g., Grinblatt and Keloharju (2009)). The no trade theorem is valid within a highly idealised theoretical world of rationality and common knowledge of rationality, but in a nearby 'neighbourhood' of this world it fails. In a less idealised and unrealistic world, a more realistic trade theorem obtains.

8. Appendix – Proofs

Lemma 1. *d is a metric.*

Proof. By definition, $d(\varphi^1, \varphi^2) \geq 0$ for all $\varphi^1, \varphi^2 \in \Delta(\Omega)$, and it is almost immediately clear that equality holds if and only if $\varphi^1 = \varphi^2$.

Similarly, that $d(\varphi^1, \varphi^2) = d(\varphi^2, \varphi^1)$ follows from the definition.

For the triangle inequality, suppose $\varphi^1, \varphi^2, \varphi^3 \in \Delta(\Omega)$. Temporarily fix $A \in \Sigma$ and $f \in B$. We initially want to establish

$$d'(\varphi^1, \varphi^3, A, f) \leq d'(\varphi^1, \varphi^2, A, f) + d'(\varphi^2, \varphi^3, A, f), \tag{10}$$

which holds true in all possible cases, as we detail:

(a) If $A \cap \text{supp}(\varphi^1) = \emptyset$ and $A \cap \text{supp}(\varphi^3) = \emptyset$: $d'(\varphi^1, \varphi^3, A, f) = 0$, and Equation (10) is automatically satisfied.

(b) If $A \cap \text{supp}(\varphi^1) = \emptyset$, $A \cap \text{supp}(\varphi^2) = \emptyset$ and $A \cap \text{supp}(\varphi^3) \neq \emptyset$: $d'(\varphi^1, \varphi^3, A, f) = d'(\varphi^2, \varphi^3, A, f) = \left|E_{\varphi^3}(f|A)\right|$ and $d'(\varphi^1, \varphi^2, A, f) = 0$. Then $d'(\varphi^1, \varphi^3, A, f) = d'(\varphi^1, \varphi^2, A, f) + d'(\varphi^2, \varphi^3, A, f)$.

(c) If $A \cap \text{supp}(\varphi^1) = \emptyset$, $A \cap \text{supp}(\varphi^2) \neq \emptyset$ and $A \cap \text{supp}(\varphi^3) \neq \emptyset$: $d'(\varphi^2, \varphi^3, A, f) = \left|E_{\varphi^2}(f|A) - E_{\varphi^3}(f|A)\right|$, $d'(\varphi^1, \varphi^3, A, f) = \left|E_{\varphi^3}(f|A)\right|$, and $d'(\varphi^1, \varphi^2, A, f) = \left|E_{\varphi^2}(f|A)\right|$. Since $E_{\varphi^3}(f|A) = E_{\varphi^2}(f|A) - E_{\varphi^2}(f|A) - E_{\varphi^3}(f|A)$, it follows that $d'(\varphi^1, \varphi^3, A, f) \leq d'(\varphi^1, \varphi^2, A, f) - d'(\varphi^2, \varphi^3, A, f)$.

(d) If $A \cap \text{supp}(\varphi^1) \neq \emptyset$, $A \cap \text{supp}(\varphi^2) = \emptyset$ and $A \cap \text{supp}(\varphi^3) = \emptyset$: $d'(\varphi^1, \varphi^3, A, f) = d'(\varphi^1, \varphi^2, A, f) = \left|E_{\varphi^1}(f|A)\right|$, and $d'(\varphi^2, \varphi^3, A, f) = 0$, so (10) is satisfied, since $\left|E_{\varphi^1}(f|A)\right| = \left|E_{\varphi^1}(f|A)\right| + 0$.

(e) If $A \cap \text{supp}(\varphi^1) \neq \emptyset$, $A \cap \text{supp}(\varphi^2) = \emptyset$, and $A \cap \text{supp}(\varphi^3) \neq \emptyset$: $d'(\varphi^1, \varphi^3, A, f) = \left|E_{\varphi^1}(f|A) - E_{\varphi^3}(f|A)\right|$, $d'(\varphi^1, \varphi^2, A, f) = \left|E_{\varphi^1}(f|A)\right|$, and $d'(\varphi^2, \varphi^3, A, f) = \left|E_{\varphi^3}(f|A)\right|$. Since
$$\left|E_{\varphi^1}(f|A) - E_{\varphi^3}(f|A)\right| \leq \left|E_{\varphi^1}(f|A)\right| + \left|E_{\varphi^3}(f|A)\right|,$$
(10) is satisfied.

(f) If $A \cap \text{supp}(\varphi^1) \neq \emptyset$, $A \cap \text{supp}(\varphi^2) \neq \emptyset$ and $A \cap \text{supp}(\varphi^3) = \emptyset$: $d'(\varphi^1, \varphi^3, A, f) = \left|E_{\varphi^1}(f|A)\right|$, $d'(\varphi^1, \varphi^2, A, f) = \left|E_{\varphi^1}(f|A) - E_{\varphi^2}(f|A)\right|$, and $d'(\varphi^2, \varphi^3, A, f) = \left|E_{\varphi^2}(f|A)\right|$. Since $E_{\varphi^1}(f|A) = E_{\varphi^1}(f|A) - E_{\varphi^2}(f|A) + E_{\varphi^2}(f|A)$, it follows that $\left|E_{\varphi^1}(f|A)\right| \leq \left|E_{\varphi^1}(f|A) - E_{\varphi^2}(f|A)\right| + \left|E_{\varphi^2}(f|A)\right|$, so that (10) is satisfied.

(g) If $A \cap \text{supp}(\varphi^1) \neq \emptyset$, $A \cap \text{supp}(\varphi^2) \neq \emptyset$ and $A \cap \text{supp}(\varphi^3) \neq \emptyset$: Since $E_{\varphi^1}(f|A) - E_{\varphi^3}(f|A) = E_{\varphi^1}(f|A) - E_{\varphi^2}(f|A) + E_{\varphi^2}(f|A) - E_{\varphi^3}(f|A)$,
$$\left|E_{\varphi^1}(f|A) - E_{\varphi^3}(f|A)\right| \leq \left|E_{\varphi^1}(f|A) - E_{\varphi^2}(f|A)\right| + \left|E_{\varphi^2}(f|A) - E_{\varphi^3}(f|A)\right|$$
which in this case is inequality (10).

Given Equation (10), by definition of d'', $d'(\varphi^1, \varphi^3, A, f) \leq d''(\varphi^1, \varphi^2, A) + d''(\varphi^2, \varphi^3, A)$, and therefore $d''(\varphi^1, \varphi^3, A) = \sup_{f \in B} d'(\varphi^1, \varphi^3, A, f) \leq d''(\varphi^1, \varphi^2, A) + d''(\varphi^2, \varphi^3, A)$. Similar reasoning, with respect to d'' and varying $A \in \Sigma$, establishes that $d(\varphi^1, \varphi^3) \leq d(\varphi^1, \varphi^2) + d(\varphi^2, \varphi^3)$. ∎

Proof of Proposition 1. If $p = 0$, the conclusion is trivially true, so we can assume that $p > 0$.

By assumption, C is common p-belief at ω^*. Then, by Equation (4), there exists an evident p-belief event $E \ni \omega^*$ such that $E \subseteq B_i^p(C)$, for all $i \in I$. Define $\pi^1 = B_1^p(E) = \{\omega \in \Omega \mid \mu^1(E \mid \Pi^1(\omega)) \geq p\}$, $\pi^2 = B_2^p(E) = \{\omega \in \Omega \mid \mu^2(E \mid \Pi^2(\omega)) \geq p\}$, and let $\pi = \pi^1 \cap \pi^2$. The fact that C is common p-belief at ω^* guarantees that π is not empty and that $\mu^1(\pi) > 0$, and $\mu^2(\pi) > 0$.

Because E is evident p-belief, by Equation (3)

(11) $E \subseteq B_1^p(E)$, and $E \subseteq B_2^p(E)$, hence $E \subseteq B_1^p(E) \cap B_2^p(E)$,

and

$$E \subseteq B_1^p(C) \text{ and } E \subseteq B_2^p(C). \tag{12}$$

Applying monotonicity to Equation (11) yields, for $i = 1, 2$, that

$$B_i^p(E) \subseteq B_i^p(B_1^p(E) \cap B_2^p(E)),$$

or

$$\pi^i \subseteq B_i^p(\pi). \tag{13}$$

Applying $B_i^p B_i^p = B_i^p$ to Equation (12) yields, for $i = 1, 2$, that $B_i^p(E) \subseteq B_i^p(B_i^p(C))$, or

$$\pi^i \subseteq B_i^p(C). \tag{14}$$

Because B_1^p is measurable with respect to $\mathcal{F}^1$, π^1 is a union of disjoint sets $\{A_k \in \Pi^1\}$. By Equation (13), $\pi^1 \subseteq B_1^p(\pi)$, and by definition $B_1^p(\pi) = \{\omega \in \Omega \mid \mu^1(\pi \mid \Pi^1(\omega)) \geq p\}$, hence $\mu^1(\pi \mid A_k) \geq p$ for all k. This in turn implies that $\mu^1(\pi \mid \pi^1) \geq p$. But since $\pi \subseteq \pi^1$, we have that

$$\mu^1(\pi^2 \mid \pi^1) \geq p. \tag{15}$$

An exactly similar argument leads to $\mu^2(\pi^1 \mid \pi^2) \geq p$.

We also have that $\mu^1(g \mid \pi^1) = \eta_1$. Otherwise, there exists an $\omega' \in \pi^1$ such that $\mu^1(g \mid \Pi^1(\omega')) \neq \eta_1$, which would mean that $\mu^1(C \mid \Pi^1(\omega')) = 0$, contradicting Equation (14). A similar argument establishes $\mu^2(g \mid \pi^2) = \eta_2$.

For any function h satisfying $h \geq 0$, $\mu^1(h \cdot 1_{\pi^1}) \geq \mu^1(h \cdot 1_{\pi^1} \cdot 1_{\pi^2})$, hence

$$\frac{\int_{\omega\in\Omega} h(\omega) 1_{\pi^1}(\omega) d\mu^1}{\mu^1(\pi^1)} \geq \frac{\mu^1(\pi^2 \cap \pi^1)}{\mu^1(\pi^1)} \cdot \frac{\int_{\omega\in\Omega} h(\omega) 1_{\pi^1}(\omega) 1_{\pi^2}(\omega) d\mu^1}{\mu^1(\pi^1 \cap \pi^2)}.$$

This is the same as saying $\mu^1(h \mid \pi^1) \geq \mu^1(\pi^2 \mid \pi^1)\, \mu^1(h \mid \pi)$. Applying Equation (15) yields

$$\mu^1(h \mid \pi^1) \geq p\mu^1(h|\pi). \tag{16}$$

Substituting g for h in Equation (16) gives

$$\eta_1 \geq p\mu^1(g \mid \pi). \tag{17}$$

Substituting, instead, $1 - g$ for h in Equation (16) yields

$$\eta_1 \leq p\mu^1(g \mid \pi) + (1 - p). \tag{18}$$

We arrive at $p\mu^1(g \mid \pi) \leq \eta_1 \leq p\mu^1(g \mid \pi) + (1-p)$. An entirely symmetric argument gives $p\mu^2(g \mid \pi) \leq \eta_2 \leq p\mu^2(g \mid \pi) + (1-p)$. Since μ^1 and μ^2 are δ-separated priors,

$$|p\mu^1(g \mid \pi) - p\mu^2(g \mid \pi)| = |p(\mu^1(g \mid \pi) - \mu^2(g \mid \pi))| \leq p\delta,$$

and we conclude that $|\eta_1 - \eta_2| \leq 1 - p + p\delta$. ∎

Proof of Theorem 1. The conditional probability of an event H occurring is the conditional probability of the characteristic function 1_H, hence this is a special case of Proposition 1. ∎

Proof of Theorem 2. For trade to occur, there must exist an event

$$T = \{\omega \mid s^i(\omega, B, q) = \text{“buy”}, s^j(\omega, B, q) = \text{“sell”}\}$$

that both traders regard as a positive probability event, where trader i buys B at price q, and trader j sells it, i.e., the associated trade is

$$t^i(\omega) = (B(\omega) - q) \cdot 1_T,\ t^j(\omega) = -(B(\omega) - q) \cdot 1_T.$$

The assumption of the common knowledge of (interim) rationality of the players implies that for any state ω in the connected state space Ω,

$$E_i(v^i(e^i(\omega) + t^i(\omega)) \mid \Pi^i(\omega)) \geq E_i(v^i(e^i(\omega)) \mid \Pi^i(\omega)) \tag{19}$$

and

$$E_j(v^j(e^j(\omega) + t^j(\omega)) \mid \Pi^j(\omega)) \geq E_j(v^j(e^j(\omega)) \mid \Pi^j(\omega)). \tag{20}$$

In words, Equation (19) states that at each of his interim partition elements, trader i has positive expectation of beneficial trade; Equation (20) says the same with regards to trader j. We may now aggregate over all the partition elements of trader i to the full state space Ω. By the properties of the expectation operator this yields, for trader i according to his prior:

$$E_{\mu^i}(v^i(e^i + t^i)) \geq E_{\mu^i}(v^i(e^i)).$$

Doing the same for trader j yields

$$E_{\mu^j}(v^j(e^j + t^j)) \geq E_{\mu^j}(v^j(e^j)).$$

Hence if even one of the traders strictly benefits from trade at event T, we deduce that the initial allocations e^i and e^j are not ex-ante Pareto-optimal initial allocations, a contradiction. ∎

Proposition 2. *An event C is heterogeneous common p-belief at $\omega \in \Omega$ if and only if for each $i \in I$ there exists an index set K^i and a non-empty set $\pi^i = \bigcup_{k \in K^i} \Pi^i_k$ such that $\omega \in \pi^i$ and both of the following conditions hold:*

(A1) $\mu^i(\bigcap_{h \in I} \pi^h \mid \Pi^i_k) \geq p$*, for all* $i \in I$ *and* $\Pi^i_k \subseteq \pi^i$

(A2) $\mu^i(C \mid \Pi^i_k) \geq p$, *for all* $i \in I$ *and* $\Pi^i_k \subseteq \pi^i$.

Proof . ($\Rightarrow$) Suppose that C is a heterogeneous common p-belief at ω. Then by definition there exists a heterogeneous evident p-belief event E such that $\omega \in E$ and $E \subseteq \bigcap_{i \in I} B^i_p(C)$. Let $i \in I$ be chosen arbitrarily. Define

$$\pi^i = \bigcup_{\mu^i(\pi^i_k \cap E) > 0} \Pi^i_k.$$

By definition, it must be the case that $E \subseteq \pi^i$ for all i, hence $E \subseteq \bigcap_{h \in I} \pi^h$. We also have that for all $\Pi^i_k \subseteq \pi^i$, because $\Pi^i_k \cap E \neq \emptyset$, there exists $\omega' \in E$ such that $\Pi^i_k = \Pi^i(\omega')$. Hence, for all $i \in I$ and all $\Pi^i_k \subseteq \pi^i$

$$\mu^i\left(\bigcap_{h \in I} \pi^h \mid \Pi^i_k\right) \geq \mu^i(E \mid \Pi^i_k) \geq p.$$

By the same reasoning, $\mu^i(C \mid \Pi^i_k) \geq p$ for all $\Pi^i_k \subseteq \pi^i$.

($\Leftarrow$) Let

$$E = \bigcap_{h \in I} \pi^h.$$

Since $\omega \in \pi^i$ for all i, $\omega \in E$. To show that C is heterogeneous common p-belief it suffices to show that E is heterogeneous evident p-belief and that $E \subseteq \bigcap_{i \in I} B^i_p(C)$. Suppose that $\omega' \in E$. By definition $\Pi^i(\omega') = \Pi^i_k$ for some $\Pi^i_k \subseteq \pi^i$. Since $\mu^i(\bigcap_{h \in I} \pi^h \mid \Pi^i_k) \geq p$ for every $i \in I$ and $E = \bigcap_{h \in I} \pi^h$, we immediately have

$$\mu^i(E \mid \Pi^i(\omega')) = \mu^i(E \mid \Pi^i_k) \geq p.$$

Since $\mu^i(C \mid \Pi^i_k) \geq p$ for every $i \in I$, we deduce that

$$\mu^i(C \mid \Pi^i(\omega')) = \mu^i(C \mid \Pi^i_k) \geq p.$$

∎

Proof of Theorem 3.

(I) implies (II).

Since $\delta > 0$, there is no common prior, hence there exists a bet $\overline{B}$, i.e., w.l.o.g., $\overline{B}$ satisfies the property that for each state ω, $E_1(\overline{B} \mid \Pi_1(\omega)) > 0 > E_2(\overline{B} \mid \Pi_2(\omega))$ meaning there is common knowledge that in the interim stage player 1 will be willing to buy the bet and player 2 will be willing to take the opposite side and sell. That is not enough to fashion a trade, as can be seen from Theorem 2. One needs to construct from $\overline{B}$ a trade that includes 'some irrationality'.

By the assumption of p-overlapping, we may identify $\pi^1 = \bigcup_{k\in K^1} \Pi^1_k$ and $\pi^2 = \bigcup_{k\in K^2} \Pi^2_k$ that satisfy (1) and (2) of Definition 7. From this, there is common p-belief in $\pi^1 \cap \pi^2$.

For each $i = 1, 2$ chose a subset of Ω denoted as F^j such that $F^j \subseteq \pi^i \setminus \pi^j$, $F^j \subsetneq \Pi^i(\omega)$ for any $\Pi^i(\omega)$, $\mu^i(F^j) \leq \rho^i$ and $\mu^i(F^i) = 0$.

Now, since $\overline{B}$ is an acceptable bet and the number of partitions is finite for both players, there exist y^1 and y^2 such that for each state ω, $E_1(\overline{B} \mid \Pi_1(\omega)) > y^1 > 0 > y^2 > E_2(\overline{B} \mid \Pi_2(\omega))$. Let $0 < q < \min(|y^1|, |y^2|)$. We use this to define a trade B as follows: for $\omega \in F^1 \subseteq \pi^2 \setminus \pi^1$, let $B(\omega) = \overline{B}(\omega) - c^1$, where c^1 is a constant sufficiently large to make $E_1(B \mid \Pi^1(\omega)) < q$. Similarly, for $\omega \in F^2 \subseteq \pi^1 \setminus \pi^2$, let $B(\omega) = \overline{B}(\omega) + c^2$, where c^2 is a constant sufficiently large to make $E_2(B \mid \Pi^2(\omega)) > -q$. For all other states ω, let $B(\omega) = \overline{B}(\omega)$.

We construct a pair of strategies s^1 and s^2 as follows:

$$s^1(\omega, B, q) = \begin{cases} \textit{`buy'} & \text{if } \Pi^1(\omega) \cap (\pi^1 \cup F^1) \\ \textit{`refrain'} & \text{otherwise;} \end{cases}$$

$$s^2(\omega, B, q) = \begin{cases} \textit{`sell'} & \text{if } \Pi^2(\omega) \cap (\pi^2 \cup F^2) \\ \textit{`refrain'} & \text{otherwise,} \end{cases}$$

Note that because each player i puts ex ante probability $\mu_i(F^i) = 0$, so he essentially plays the strategy

$$s^i(\omega, B, q) = \begin{cases} \textit{`buy / sell'} & \text{if } \Pi^i(\omega) \cap \pi^i \\ \textit{`refrain'} & \text{otherwise;} \end{cases}$$

which from the perspective of player i is a best reply to player j strategy

$$s^j(\omega, B, q) = \begin{cases} \textit{`sell / buy'} & \text{if } \Pi^j(\omega) \cap (\pi^j \cup F^j) \\ \textit{`refrain'} & \text{otherwise,} \end{cases}$$

We now show that indeed the above strategy, rational from player's own perspective, the best reply for each player i to s^j.

Consider without loss of generality player 1:

- If $\Pi^1(\omega) \subseteq \pi^1 \cap \pi^2$, then at any state in $\Pi^1(\omega)$ 'buy' is optimal for trader 1, because his conditional expectation is y^1, trade is offered at a price $q < y^1$ and it takes place with probability 1.
- For $\omega \notin \pi^1 \cup \pi^2$, trade does not occur.
- For $\omega \in \pi^2 \setminus \pi^1$, trader 1 believes that trader 2 is willing to sell. Furthermore trader 1 believes that trader 2 believes that trader 1's strategy calls on him to buy at states ω such that $\Pi^1(\omega) \cap F^1 \neq \emptyset$ (as player 2 believes that player 1's expectation at such an ω, $E(B \mid$

$\Pi^1(\omega) \cap \{\omega \mid s^2(\omega, B, q) = \text{'sell'}\})$, is less than q) and refrain otherwise.

- For $\omega \in \pi^1 \setminus \pi^2$ 'buy' is still optimal for trader 1, because by construction trader 1's expected value conditional on $\Pi^1(\omega)$ and on the event of trade (which happens on π^2 and F^2, even if trading is irrational on F^2 for player 2) for is still greater than or equal to y^1.

Finally the result follows because $\mathbf{\Pi}$ is connected.

A symmetric argument shows that the rational strategy of player 2 is a best reply to "irrational" strategy of player 1.

By construction, $\mu^1(F^2) \leq \rho^1$ and $\mu^2(F^1) \leq \rho^2$ (with $\rho^i > 0$ for at least one i) hence $s(\omega, B, q)$ is a $(1-\rho^1, 1-\rho^2)$-rationality NE equilibrium.

That $T(s, B, q)$ is common-p belief at any $\omega \in \pi^1 \cap \pi^2$ follows because it satisfies (A2) and π^1 and π^2 satisfy (A1).

Finally, Proposition 1 implies that $y^1 - y^2 \leq (1 - p(1-\delta))\|B\|_\infty$.

(II) implies (I).

Suppose that there exists a proposed trade B, a price q, and strategies s^1, s^2 such that $T(s, B, q)$ is heterogeneous common p-belief at $\omega \in \Omega$. By Proposition 2 there exist two sets π^1 and π^2 satisfying Condition (A1) and (A2) of that proposition. Condition (A1) implies Condition (1) of Definition 7.

Suppose next that Condition (2) fails to obtain. Then there exist nonempty index sets $K^{1'} \subseteq K^1$ and $K^{2'} \subseteq K^2$ such that both $\mu^1(\pi^{1'} \Delta \pi^{2'}) = 0$ and $\mu^2(\pi^{1'} \Delta \pi^{2'}) = 0$. But for any $\omega \in T(s, B, q)$, 'buy' is a unique optimising action on $\Pi^1(\omega)$, and since trader 1 is rational in $T(s, B, q)$, it follows that $s^1(\omega') = \text{'buy'}$ for all $\omega' \in \Pi^1(\omega)$. A similar statement holds for $s^2(\omega') = \text{'sell'}$ for all $\omega' \in \Pi^2(\omega)$. Hence, if both $\mu^1(\pi^{1'} \Delta \pi^{2'}) = 0$ and $\mu^2(\pi^{1'} \Delta \pi^{2'}) = 0$, if follows that for all $\omega \in \pi^{1'} \Delta \pi^{2'}$, $\Pi^1(\omega) \in \pi^{1'}$ and 'buy' is trader 1's optimising action, and $\Pi^2(\omega) \in \pi^{2'}$ and 'sell' is trader 2's optimising action. Therefore at any $\omega \in \pi^{1'} \Delta \pi^{2'}$ there is common knowledge of strictly improving trade among rational traders, contradicting Theorem 2. ∎

Proof of Theorem 4. The proof of Theorem 4 generally follows similar steps to those of the corresponding result of Neeman (1996b), including the lemma on page 94 of that paper (that lemma does not depend at all on the existence or non-existence of a common prior among different agents).

(II) implies (I). Let sets F^1 and F^2 be generated as in the proof of Theorem 3 and let trade B satisfy all the conditions specified in the statement and proof of Theorem 3. As before, since $\delta > 0$ there exists a bet $\overline{B}$ such that $E_1(\overline{B} \mid \Pi(\omega)) = y^1$ for all ω and $E_2(\overline{B} \mid \Pi(\omega)) = y^2$ for all ω, with $v^1 > v^2$ w.l.o.g. Define $\hat{C}^i(\mu^i, u^i, \Pi^i_k(\omega))$ as given by

$$E_i(u^i(e^i(\omega) + (-1)^{i-1}(B - \hat{C}^i(\mu^i, u^i, \Pi^i_k(\omega))) \mid \Pi^i_k(\omega)) \\ = E_i(u^i(e^i(\omega)) \mid \Pi^i_k(\omega))$$

for all $\Pi^i_k(\omega)) \in \pi^i$ and also define

$$C^i(\mu^i, u^i) := \min_{\Pi^i_k(\omega) \in \pi^i} \hat{C}^i(\mu^i, u^i, \Pi^i_k(\omega)).$$

Consider a price in the interval $(y^2 + C^2(\mu^2, u^2), y^1 - C^1(\mu^1, u^1))$. An argument similar to the proof of Theorem 3 shows that s^1 and s^2 constitutes a $(1 - \rho)$-rationality Nash equilibrium, if the agents are sufficiently risk tolerant. Similarly to the proof of Theorem 3, within π^1 agent 1 is rational and accepting B at price q is the optimal action (his expectation of B is smaller than q and in addition q also incorporates a risk premium for him). On F^1 agent 1 is irrational as 'buy' is sub-optimal for him. On $\Omega \setminus (\pi^1 \cup F^1)$ refraining from trade is the optimal action.

The same argument holds symmetrically for agent 2, who is rational everywhere except on F^2 (an event on which he himself again puts zero probability). Hence s^1 and s^2 constitute a $(1-\rho)$-rationality Nash equilibrium. In addition, $T(s, B, q)$ is heterogeneous common p-belief at any $\omega \in \pi^1 \cap \pi^2$, by the same reasoning as in the proof of Theorem 3.

(I) implies (II). The proof is similar to the proof of Theorem 3. ∎

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