Incentivizing Team Production by Indivisible Prizes

Electoral Competition under Proportional Representation*

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Abstract

This paper formulates proportional representation in a parliamental election as a multi-prize contest among political parties. In particular, we analyze the performance of commonly-used list rule, and investigate what the optimal list rule is when candidates differ in their abilities to contribute. We show that, in order to maximize the aggregated effort exerted by the party candidates, each party should assign the highest ability candidates to the middle of the list, while the top priority rankings and low priority rankings should be assigned to lower ability candidates under the optimal list rule. Then, we turn to the optimal mechanism. When individual effort cost function is not too convex and the complemantarities of individual efforts are not too strong, we show that the optimal monotonic mechanism is the optimal list rule. Additional interesting observations are that under the same conditions, (i) if the optimal list rule gives the highest ability candidate rank k, then the optimal (nonmonotonic) rule also selects her as the winner if and only if the party wins k seats or more, and (ii) the optimal rule selects the lowest ability candidate to the parliament if only one seat is won, unless the party is very small.

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1 Introduction

There are countries of which parliament seats are allocated proportionally by the number of votes each party collected. Each party announces a list of candidates with a priority order (a list rule), who compete as a team with other party candidates. Obvious questions that come up in our mind are: Is a list rule a desirable way to allocate prizes to the team members' efforts to collect votes for the party? If so, what is the best way for a party to order heterogeneous ability candidates in its list rule?

Crutzen and Sahuguet (2017) and Crutzen, Flamand, and Sahuguet (2017) are the first to analyze the incentive structure of the list rule in a parliamental election, or more generally in a contest between teams that compete for multiple indivisible prizes. They set up a multi-prize contest model with a CES team-effort aggregator function, and compare different electoral systems and different intra-team prize allocation rules. Their main analyses are on ex ante symmetric two party case with homogeneous candidates. In this paper, we extend this basic model, allowing for heterogeneous abilities of candidates in order to see the performance of the list rule and how a party leader should allocate heterogenous candidates on the priority list. More generally, this paper analyzes competition by the parties by employing a party-optimal rule that maximize the party's winning probability given other parties' effort levels.

We first show that there is an electoral equilibrium in multi-party proportional representation with list rules for any priority list for each party. Then, we analyze the equilibrium system of equations. A party that exert more aggregate effort than another party in equilibrium achieves a higher winning probability, having higher expected number of winning seats in the parliament. We also show that a rule change to improve a party's aggregate effort increases the party's winning probability, if each party's aggregate effort is stable in equilibrium (although it is not necessary). This justifies that a party's behavior to choose its assignment rule to maximize its aggregate effort given other parties' aggregate efforts as given. With this result, we concentrate on each party's selection of optimal assignment rule including list rules.

We analyze the party-optimal rule for each party. Imposing a natural single-crossigness assumption on the probabilities of number of winning seats, we characterize party-optimal rules. When individual effort cost function is not too convex and the complementarities of individual efforts are not too strong, we show that the optimal mechanism is deterministic: i.e., when any k seats are won by a party, then the party assigns the seats to a certain size-k subset of candidates. We provide an algorithm to calculate the optimal mechanism in this case (weaker complementarity and not too convex cost function). The optimal rule assigns the highest ability candidate to about probability one half to get a parliamnet seat. This is to encourage highest ability person to exert more effort for the party. Note that the optimal (effort-maximizing) rule is not necessarily monotonic— even if a candidate could go to the parliament in the case that the party gets k seats, it does not mean that the same candidate can go to the parliament in the case of $k+1$ seats are won by the party. If the expected number of winning seats exceeds one, the lowest ability candidate may go to the parliament when only one seat is won in the realized state, exerting no effort. Then, we impose a natural monotonicity requirement on the mechanism (if a candidate goes to the parliament when k seats are won, then she will go to the paliament when more than k seats are won). This monotonicity requirement assures that all candidates exert effort totheir parties. We show that the optimal monotonic rule is the optimal list rule, which assigns the highest ability candidate in the middle of the list. The lowest ability candidates are assigned to the top or the bottom of the optimal list rule which maximizes the aggregate effort exerted by all candidates. When individual cost function is rather convex and the complementarities of individual efforts are strong, the probabilities of candidates' going to the parliament are ranked by their abilities.

To be completed.

1.1 Related Literature

(*)**To be completed.** Include references in the above two papers.

2 The Model

There are J parties (teams) that are competing for n parliament seats (indivisible prizes). Each party j has n candidates who differ in their abilities (effectiveness) in contributing to her party by making effort.

Candidate i in party j has ability a_{ij} and she decides how much effort e_{ij} to contribute to her party j. Party j's winning number of seats is a random variable through a Tullock-style contest among J parties based on the ratios of parties' efforts E_js. In our basic model, we assume that seat allocation is determined through "winning probability" of each party j:

$$p_j = \frac{E_j}{E_1 + ... + E_J},$$

and we assume that p_j solely explain the number of seats party j wins as a random variable. mbut later this assumption will be weakened to first-order stochastic dominance with single-crossingness when we analyze effort-maximizing intra-seat allocation rule. Each party's effort aggregator function is assumed to be a CES function

$$E_j = \left(\sum_{i=1}^{n} a_{ij} e_{ij}^{1-\sigma} \right)^{\frac{1}{1-\sigma}},$$

where $a_{ij} > 0$ represents member ij's ability in making effort e_{ij} for all $i = 1, ..., n$ and party $j = 1, 2, ..., J$. Each candidate i's individual effort cost is specified as $\frac{1}{\beta} e_{ij}$. The following relationship holds regarding candidate i's effort and party j's aggregate effort:

$$\frac{\partial E_j}{\partial e_{hj}} = a_{hj} \left(\sum_{i=1}^{n} a_{ij} e_{hj}^{1-\sigma} \right)^{\frac{\sigma}{1-\sigma}} e_{hj}^{-\sigma} = a_{hj} E_j^{\sigma} e_{hj}^{-\sigma}$$

Each party decides how it allocates winning number of seats among its candidates. One rule that is often used in a parliament election is a list rule: party announces the priority ordering of its candidates, and depending on the number of seats it wins, the highest priority candidates go to the parliament. In the basic model, we analyze this list rule, then later we investigate what the optimal rules are.

3 A Party's Probability of Winning k Seats

Let parties' winning probability vector be $p = (p_1, ..., p_J)$, and consider the probability of party $j = 1$'s winning k_1 seats. Let $P_j = (P_j^k)_{k=0}^n \in \Delta^{n+1}$ be probability distribution of party j's number of winning seats: i.e., P_j^k is the probability of party j's winning k seats with $\sum_{k=0}^n P_j^k = 1$. Assuming that seat allocation is determined by i.i.d., we have

$$
\begin{aligned}
P_1^k &= VC(n, k_1)p_1^{k_1} \\
&\quad \times \left[\sum_{k_2=0}^{n-k_1} C(n-k_1, k_2)p_2^{k_2} \sum_{k_3=0}^{n-k_1-k_2} C(n-k_1-k_2, k_3)p_3^{k_3} \times ... \right. \\
&\qquad \left. \times \sum_{k_{J-1}=0}^{n-k_1-...-k_{r-2}} C(n-k_1-...-k_{J-2}, k_{J-1})p_{J-1}^{k_{J-1}} p_J^{n-k_1-...-k_{J-2}-k_{J-1}} \right]
\end{aligned}
$$

First note that $(p_i + p_j)^k = \sum_{\ell=0}^k C(k, \ell)p_i^\ell p_j^{k-\ell}$ for any k. Setting $k = n - k_1 - ... - k_{J-2}$, we have

$$
\sum_{k_{J-1}=0}^{n-k_1-\cdots-k_{J-2}} C(n-k_1-...-k_{J-2}, k_{J-1})p_{J-1}^{k_{J-1}} p_J^{n-k_1-...-k_{J-2}-k_{J-1}} = (p_{J-1} + p_J)^{n-k_1-...-k_{J-2}}
$$

Repeatedly applying this, we have

$$
\begin{aligned}
P_1^{k_1} &= C(n, k_1)p_1^{k_1} (p_2 + ... + p_J)^{n-k_1} \\
&= C(n, k_1)p_1^{k_1} (1 - p_1)^{n-k_1}
\end{aligned}
$$

Thus, the probability of party j's winning k seats is:

$$
P_j^k = C(n, k)p_j^k (1 - p_j)^{n-k} .
$$

Note that

$$
\begin{aligned}
\frac{dP_j^k}{dp_j} &= C(n, k)p_j^k (1 - p_j)^{n-k} \left(\frac{k}{p_j} - \frac{n-k}{1-p_j} \right) \\
&= C(n, k)p_j^{k-1} (1 - p_j)^{n-k-1} (k - np_j)
\end{aligned}
$$

That is, the probability of party j's winning k seats decreases with an increase in p_j for $k < k^* = \lfloor np_j \rfloor + 1$, and increases for $k \geq k^*$.

We will employ a slightly more general setup than i.i.d. random winning seat assumption. We let $P_j = (P_j^k)_{k=0}^n \in \Delta^{n+1}$ be functions of p_j only: $P_j^k = P_j^k(p_j)$ for all $k = 0, ..., n$. This probability distribution should naturally satisfy the first order stochastic dominance property:

First-Order Stochastic Dominance (FOSD). For all $j = 1, ..., J$, all $p_j \in (0, 1)$, and all $m = 1, ..., n$, $\sum_{k=m}^n \frac{\partial P^k}{\partial p_j} > 0$ holds.

In addition to this, we may impose the following plausible condition that is satisfied in i.i.d. case ($k^*(p_j) = \lfloor np_j \rfloor + 1$).

Single-Crossingness on Winning Probabilities. For all $j = 1, .., J$, and all $p_j \in (0, 1)$, $P_j = P(p_j) = (P^k(p_j))_{k=0}^n \in \Delta^{n+1}$ satisfies the following condition: there is $k^*(p_j) \in \{1, ..., n-1\}$ such that (i) $\frac{\partial P^k}{\partial p_j} \leq 0$ for all $k = 0, ..., k^*(p_j) - 1$, and (ii) $\frac{\partial P^k}{\partial p_j} > 0$ for all $k = k^*(p_j), ..., n$.

Under this Single-Crossingness condition, we will analyze each party's choice of rules of assigning seats to its candidates, including list rules.

4 General Seat Allocation Rule and List Rule

A list rule is a simple and commonly used rule in proportional representation parliament elections in many coutries. Party j's candidates' names are listed with priority order, and if party j wins k seats then the top k candidates on the list go to the parliament. That is, the mth candidate on the list will go to the parliament with probability $\sum_{k=m}^n P_j^k(p_j)$.

We can analyze each party's effort-maximizing rules by using a more general framework. A **general (stochastic) seat allocation rule** is a list of functions $(q^k)_{k=1}^n$ such that $q^k : \mathcal{S}(k) \to [0, 1]$ such that $\mathcal{S}(k) \equiv \{S \subseteq N_j : |S| = k\}$ and $\sum_{S \in \mathcal{S}(k)} q(S) = 1$ for all $k = 1, ..., n$. A general seat allocation rule assigns probabilities to which subset of k candidates go to the parliament when k seats are won in the election.

When a general allocation rule is used, the member i of team j has the following benefit function

$$B_{ij} = V \sum_{k=1}^{n} \sum_{S \in \mathcal{S}_i(k)} q(S) P^k(p_j),$$

where $\mathcal{S}_i(k) = \{S \in \mathcal{S}_i(k) : i \in S\}$.

Let $(l_1, l_2, ..., l_n)$ be the list of priority ordering of candidates' names for a list rule. Then, for each k, let $S_k = \{l_1, ..., l_k\} \in \mathcal{S}(k)$ and let $q^k(S_k) = 1$ and $q^k(S) = 0$ for all $S \in \mathcal{S}(k) \backslash \{S_k\}$, and all $k = 1, ..., n$. Thus, any list rule can be represented as a general seat assignment rule.

Taking the derivative of B_{ij} with respect to e_{ij}, we obtain,

$$\begin{aligned}
\frac{\partial B_{ij}}{\partial e_{ij}} &= V \sum_{k=1}^{n} \sum_{S \in \mathcal{S}_i(k)} q(S) \frac{dP^k}{dp_j} \frac{E_{-j}}{(E_{-j} + E_j)^2} \frac{\partial E_j}{\partial e_{ij}} \\
&= V \sum_{k=1}^{n} \sum_{S \in \mathcal{S}_i(k)} q(S) \frac{dP^k}{dp_j} (1 - p_j) \frac{p_j}{E_j} \frac{\partial E_j}{\partial e_{ij}} \\
&= \frac{V}{e_{ij}} \frac{e_{ij}}{E_j} \frac{\partial E_j}{\partial e_{ij}} \sum_{k=1}^{n} \sum_{S \in \mathcal{S}_i(k)} q(S) \frac{dP^k}{dp_j} (1 - p_j) p_j \\
&= \frac{a_{ij} V}{e_{ij}} \left(\frac{e_{ij}}{E_j}\right)^{1-\sigma} \sum_{k=1}^{n} \sum_{S \in \mathcal{S}_i(k)} q(S) \frac{dP^k}{dp_j} (1 - p_j) p_j
\end{aligned}$$

Thus, the first order condition assuming an interior solution is

$$\frac{\partial B_{ij}}{\partial e_{ij}} - e_{ij}^{\beta-1} = \frac{a_{ij} V}{e_{ij}} \left(\frac{e_{ij}}{E_j}\right)^{1-\sigma} \sum_{k=1}^{n} \left(\sum_{S \in \mathcal{S}_i(k)} q(S) \right) \frac{dP^k}{dp_j} (1 - p_j) p_j - e_{ij}^{\beta-1} = 0$$

or

$$a_{ij} V \left(\frac{1}{E_j}\right)^{1-\sigma} \sum_{k=1}^{n} \left(\sum_{S \in \mathcal{S}_i(k)} q(S) \right) \frac{dP^k}{dp_j} (1 - p_j) p_j = e_{ij}^{\sigma+\beta-1}$$

or

$$e_{ij} = \left[a_{ij} V \left(\frac{1}{E_j}\right)^{1-\sigma} \sum_{k=1}^{n} \left(\sum_{S \in \mathcal{S}_i(k)} q(S) \right) \frac{dP^k}{dp_j} (1 - p_j) p_j \right]^{\frac{1}{\sigma+\beta-1}}$$

Note that this solution makes sense only when $\sum_{k=1}^{n} \left(\sum_{S \in \mathcal{S}_i(k)} q(S) \right) \frac{dP^k}{dp_j} (1 - p_j) p_j > 0$. Otherwise, $e_{ij} = 0$

must hold. Thus, formally, we can write

$$e_{ij} = \left[a_{ij} V \left(\frac{1}{E_j} \right)^{1-\sigma} \max \left\{ \sum_{k=1}^{n} \left(\sum_{S \in \mathcal{S}_i(k)} q(S) \right) \frac{dP^k}{dp_j} \left(1 - p_j\right) p_j, 0 \right\} \right]^{\frac{1}{\sigma+\beta-1}}$$

$$\begin{aligned}
E_j &= \left(\sum_{h=1}^{n} a_{hj} e_{hj}^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \\
&= \left(\sum_{i=1}^{n} a_{ij} \left[a_{ij} V \left(\frac{1}{E_j} \right)^{1-\sigma} \max \left\{ \sum_{k=1}^{n} \left(\sum_{S \in \mathcal{S}_i(k)} q(S) \right) \frac{dP^k}{dp_j} \left(1 - p_j\right) p_j, 0 \right\} \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{1}{1-\sigma}} \\
&= E_j^{\frac{-(1-\sigma)}{\sigma+\beta-1}} V^{\frac{1}{\sigma+\beta-1}} \left(\sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}} \left[\max \left\{ \sum_{k=1}^{n} \left(\sum_{S \in \mathcal{S}_i(k)} q(S) \right) \frac{dP^k}{dp_j} \left(1 - p_j\right) p_j, 0 \right\} \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{1}{1-\sigma}}
\end{aligned}$$

or

$$E_j^{\frac{\beta}{\sigma+\beta-1}} = V^{\frac{1}{\sigma+\beta-1}} \left(\sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}} \left[\max \left\{ \sum_{k=1}^{n} \left(\sum_{S \in \mathcal{S}_i(k)} q^k(S) \right) \frac{dP^k}{dp_j} \left(1 - p_j\right) p_j, 0 \right\} \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{1}{1-\sigma}}$$

or

$$\begin{aligned}
E_j &= \left\{ V \left(\sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}} \left[\max \left\{ \sum_{k=1}^{n} \left(\sum_{S \in \mathcal{S}_i(k)} q^k(S) \right) \frac{dP^k}{dp_j} \left(1 - p_j\right) p_j, 0 \right\} \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{\sigma+\beta-1}{1-\sigma}} \right\}^{\frac{1}{\beta}} \\
&= \left\{ V \left(\sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}} \left[\max \left\{ \sum_{k=1}^{n} r_i^k \mu^k(p_j), 0 \right\} \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{\sigma+\beta-1}{1-\sigma}} \right\}^{\frac{1}{\beta}},
\end{aligned}$$

where $r_i^k = \sum_{S \in \mathcal{S}_i(k)} q^k(S) \in [0, 1]$ *is the probability of candidate* i *goes to the parliament when party* j *wins* k *seats, and* $\mu^k(p_j) = \frac{dP^k}{dp_j}\left(1 - p_j\right) p_j$ *which denotes the impact of increasing* p_j *(or aggregate effort* E_j*) on the probability of winning* k *seats.* Since when k seats are won, k party j candidates need to go to the parliament, $\sum_{i=1}^{n} r_i^k = k$ must hold for all $k = 1, ..., n$. To summarize the analysis, we have the following Lemma.

Lemma 1. Suppose that party j uses generalized assignment rule $q = (q^k)_{k=1}^n$. Then, candidate i makes a positive effort if and only if $\sum_{k=1}^n r_i^k \mu^k(p_j) > 0$. The resulting equilibrium aggregate effort by party j is

$$E_j = \left\{ V \left(\sum_{i=1}^n a_{ij}^{\frac{\beta}{\sigma+\beta-1}} \left[\max\left\{ \sum_{k=1}^n r_i^k \mu^k(p_j), 0 \right\} \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{\sigma+\beta-1}{1-\sigma}} \right\}^{\frac{1}{\beta}}$$

5 Equilibrium Analysis

Here, we will investigate how competition by the parties work. In particular, what each party tries to maximize when parties are competing for the number of seats in the parliament. Since we assume that $(P^k(p_j))_{k=1}^n$ is first-order stochastically dominated by $(P^k(p_j'))_{k=1}^n$ for $p_j' > p_j$, each party j should try to maximize p_j. Since $p_j = \frac{E_j}{E_1+...+E_J}$, it seems to make sense for party j to choose rule $q = (q^k)_{k=1}^n$ in order to maximize E_j given E_{-j}. However, $p = (p_1, ..., p_j, ..., p_J)$ is actually determined in the interactions with other parties in equilibrium, and it is important to check our intuitive approach makes sense.[1]

In this section, we start with existence of equilibrium. It is easy to observe that E_j depends only on p_j — nothing else ($E_j = E_j(p_j)$). Thus, we can use the following fixed-point mapping to prove the existence of equilibrium. Let $p = (p_1, ..., p_J)$ and

$$f_j(p) = \frac{E_j(p_j)}{\sum_{k=1}^J E_k(p_k)}$$

for all $j = 1, ..., J$. Then $f(p) = (f_1(p), ..., f_J(p))$ is a fixed point mapping from simplex $\Delta^J \equiv \left\{ p \in \mathbb{R}_+^J : \sum_{k=1}^J p_k = 1 \right\}$ to itself, which is a continuous function. By Brouwer's fixed point theorem, there exists a fixed point $p^* = f(p^*)$.

Theorem 1 (Existence Theorem). There exists an equilibrium for any profile of list rule $a = (a_j)_{j=1}^J$, where $a_j = (a_{j1}, ..., a_{jn})$.

Let $\alpha_{ij} = a_{ij}^{\frac{\beta}{\sigma+\beta-1}}$. The following result is an immededate consequence of equilibrium condition.

[1] In the general equilibrium framework (in trade theory), we see cases of transfer paradox and immiserizing growth occurring.

Proposition 1 (Winning Probability Ranking). In every equilibrium, the winning probabilities of parties j and h satisfy the following:

$$p_j \gtreqless p_h \Longleftrightarrow \sum_{i=1}^{n} \alpha_{ij} \left[\max \left\{ \sum_{k=1}^{n} r_i^k \mu^k(p_j), 0 \right\} \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \gtreqless \sum_{i=1}^{n} \alpha_{ih} \left[\max \left\{ \sum_{k=1}^{n} r_i^k \mu^k(p_h), 0 \right\} \right]^{\frac{1-\sigma}{\sigma+\beta-1}}$$

In order to see how a party's rule choice affects equilibrium probability of winning, we analyze how equilibrium probability distribution responds to an increase in a party member's ability. Then, party j's aggregated effort is written as

$$E_j = \left\{ V \left(\sum_{i=1}^{n} \alpha_{ij} \left[\max \left\{ \sum_{k=1}^{n} r_i^k \mu^k(p_j), 0 \right\} \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{\sigma+\beta-1}{1-\sigma}} \right\}^{\frac{1}{\beta}},$$

An equilibrium is described by the following system of equations:[2]

$$\begin{pmatrix} p_1 \\ \vdots \\ p_j \\ \vdots \\ p_J \end{pmatrix} = \begin{pmatrix} \frac{E_1(p_1)}{E_1(p_1)+E_{-1}(p_{-1})} \\ \vdots \\ \frac{E_j(p_j)}{E_j(p_j)+E_{-j}(p_{-j})} \\ \vdots \\ \frac{E_J(p_J)}{E_J(p_J)+E_{-J}(p_{-J})} \end{pmatrix}$$

We will consider a comparative static exercise of increasing α_{ij}. We drop the last equation from the system

[2] This analysis is valid for any prize allocation rule and for any functional form of the effort aggregator function.

since $\sum_{j=1}^{J} p_j = 1$. Totally differentiating the system, we obtain

$$\begin{pmatrix} dp_1 \\ \vdots \\ dp_j \\ \vdots \\ dp_{J-1} \end{pmatrix} = \begin{pmatrix} \frac{\frac{\partial E_1}{\partial p_1}}{E} - \frac{E_1 \frac{\partial E_1}{\partial p_1}}{E^2} & \cdots & -\frac{E_1 \frac{\partial E_j}{\partial p_j}}{E^2} & \cdots & -\frac{E_1 \frac{\partial E_{J-1}}{\partial p_{J-1}}}{E^2} \\ \vdots & \ddots & \vdots & & \vdots \\ -\frac{E_j \frac{\partial E_1}{\partial p_1}}{E^2} & \cdots & \frac{\frac{\partial E_j}{\partial p_j}}{E} - \frac{E_j \frac{\partial E_j}{\partial p_j}}{E^2} & \cdots & -\frac{E_j \frac{\partial E_{J-1}}{\partial p_{J-1}}}{E^2} \\ \vdots & & \vdots & \ddots & \vdots \\ -\frac{E_{J-1} \frac{\partial E_1}{\partial p_1}}{E^2} & \cdots & -\frac{E_{J-1} \frac{\partial E_j}{\partial p_j}}{E^2} & \cdots & \frac{\frac{\partial E_{J-1}}{\partial p_{J-1}}}{E} - \frac{E_{J-1} \frac{\partial E_{J-1}}{\partial p_{J-1}}}{E^2} \end{pmatrix} \begin{pmatrix} dp_1 \\ \vdots \\ dp_j \\ \vdots \\ dp_{J-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial E_j}{\partial \alpha_{ij}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} d\alpha_{ij}$$

Since $\frac{\frac{\partial E_j}{\partial p_j}}{E} - \frac{E_j \frac{\partial E_j}{\partial p_j}}{E^2} = \frac{1}{E}\frac{\partial E_j}{\partial p_j} - \frac{p_j}{E}\frac{\partial E_j}{\partial p_j}$, we have

$$\begin{pmatrix} 1 - \frac{1}{E}\frac{\partial E_1}{\partial p_1} + \frac{p_1}{E}\frac{\partial E_1}{\partial p_1} & \cdots & \frac{p_1}{E}\frac{\partial E_j}{\partial p_j} & \cdots & \frac{p_1}{E}\frac{\partial E_{J-1}}{\partial p_{r-1}} \\ \vdots & \ddots & \vdots & & \vdots \\ \frac{p_j}{E}\frac{\partial E_1}{\partial p_1} & \cdots & 1 - \frac{1}{E}\frac{\partial E_j}{\partial p_j} + \frac{p_j}{E}\frac{\partial E_j}{\partial p_j} & \cdots & \frac{p_j}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}} \\ \vdots & & \vdots & \ddots & \vdots \\ \frac{p_{J-1}}{E}\frac{\partial E_1}{\partial p_1} & \cdots & \frac{p_{J-1}}{E}\frac{\partial E_j}{\partial p_j} & \cdots & 1 - \frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}} + \frac{p_{J-1}}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}} \end{pmatrix} \begin{pmatrix} dp_1 \\ \vdots \\ dp_j \\ \vdots \\ dp_{J-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial E_j}{\partial \alpha_{mj}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} d\alpha_{mj}$$

Let $\eta_i = \frac{1}{E}\frac{\partial E_j}{\partial p_j} = \frac{p_j}{E_j}\frac{\partial E_j}{\partial p_j}$ be party i's aggregated effort elasticity of its probability of winning. We can prove the following Proposition.

Lemma 2. Suppose that candidate i's ability is increased slightly. Then, we have

$$\frac{dp_j}{d\alpha_{ij}} = \left[(1 - \eta_j) + \frac{(1 - p_j)\,\eta_j}{\sum_{i=1,\ i \neq j}^{J-1} \left(\frac{(1-p_i)\eta_i}{1-\eta_i} \right)} \right]^{-1} \frac{\partial E_j}{\partial \alpha_{mj}}$$

where $\frac{\partial E_j}{\partial \alpha_{mj}} = A\left(\max\left\{\sum_{k=1}^n r_i^k \mu^k(p_j), 0\right\}\right)^{\frac{1-\sigma}{\sigma+\beta-1}}$ for $A > 0$.

This technical lemma provides two important implications when the contents of the bracket is positive. First, an increase in α_{ij} increases party j's winning probability as long as she makes positive effort in equilibrium, which is dictated by the sign of candidate i's incentive term, $\sum_{k=1}^n r_i^k \mu^k(p_j)$. Second, the party can increase E_j by adjusting $q = (q^k)_{k=1}^n$ to shift weights $\sum_{k=1}^n r_i^k \mu^k(p_j)$ from low ability candidates to high ability candidates. The sign of the contents of the bracket term is assured to be positive (but is not necessary), if $1 > \eta_j$ is satisfied for all $j = 1, ..., n$. This condition can be considered as (individual) stability of party j's effort naturally. If η_j exceeds unity, it means that its best response is unstable: an increase in p_j increases p_j even more. *As a sufficient condition, we assume individual stability of each party's effort, and assume that each party chooses an assignment rule to maximize E_j given E_{-j} in the rest of the paper.*

Proposition 2. Suppose that $\eta_j < 1$ for all $j = 1, ..., J$. Then, for any $i = 1, ..., n$, $\frac{dp_j}{d\alpha_{ij}} > 0$ holds if and only if $\sum_{k=1}^n r_i^k \mu^k(p_j) > 0$.

6 Party-Optimal Rules

Party j maximizes its aggregate effort E_j by controlling $q : \mathcal{S} \to [0,1]$ with $\sum_{S\in\mathcal{S}(k)} q(S) = k$ for all $k = 1, ..., n$:

$$E_j = \left\{ V\left(\sum_{i=1}^n \alpha_{ij}\left[\max\left\{\sum_{k=1}^n r_i^k \mu^k(p_j), 0\right\}\right]^{\frac{1-\sigma}{\sigma+\beta-1}}\right)^{\frac{\sigma+\beta-1}{1-\sigma}}\right\}^{\frac{1}{\beta}}$$

where $r_i^k = \sum_{S\in\mathcal{S}_i(k)} q^k(S) \in [0,1]$ is the probability of candidate i goes to the parliament when party j wins k seats, and $\mu^k(p_j) = \frac{dP^k}{dp_j}(1-p_j)p_j$ which denotes the impact of increasing p_j (or aggregate effort E_j) on the probability of winning k seats. Since when k seats are won, k party j candidates need to go to the parliament, $\sum_{i=1}^n r_i^k = k$ must hold for all $k = 1, ..., n$.In order to solve this maximization problem, it would be easier to work on an $n \times n$ assignment matrix $R = (r_i^k)_{i=1,...,n;\ k=1,...,n}$ instead of mapping q. Then, the issue is whether or not we can choose an assignment matrix freely as long as $\sum_{i=1}^n r_i^k = k$ holds for all

$k = 1, ..., n$. The following lemma provides a positive answer.

Lemma 3. Any $n \times n$ assignment matrix R such that (i) $r_i^k \in [0,1]$ for all $i, k = 1, ..., n$, and (ii) $\sum_{i=1}^n r_i^k = k$ for all $k = 1, ..., n$, can be achieved by some allocation rule $q : \mathcal{S} \to [0,1]$ with $\sum_{S \in \mathcal{S}(k)} q^k(S) = k$ for all $k = 1, ..., n$.

Thus, party j's maximization problem becomes

$$\max_{(r_i^k)} \sum_{i=1}^n a_{ij}^{\frac{\beta}{\sigma+\beta-1}} \left[\max \left\{ \sum_{k=1}^n r_i^k \mu^k(p_j), 0 \right\} \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \quad \text{s.t.} \quad \sum_{i=1}^n r_i^k = k \text{ for all } k = 1, ..., n$$

where $\mu^k(p_j) = \frac{dP^k}{dp_j}(1-p_j)p_j$ with $\mu^k(p_j) \leq 0$ for all $k = 0, ..., k^* - 1$, and $\mu^k(p_j) > 0$ for all $k = k^*, ..., n$ by Single-Crossingness on Winning Probabilities.

Notice that the bracket in the above formula has power $\frac{1-\sigma}{\sigma+\beta-1}$. It turns out to be essential whether this power is more than unity ($\beta < 2(1-\sigma)$: a convex function), or less than unity ($\beta > 2(1-\sigma)$: a concave function). We will analyze these two cases separately.

6.1 Convex case: $\beta < 2(1-\sigma)$

In this case, we have the following result.

Lemma 4. Suppose that $\beta \leq 2(1-\sigma)$ holds. For $(r_i^k)_{k=1}^{n-1}$ and $(r_h^k)_{k=1}^{n-1}$ with $r_i^k, r_h^k > 0$ for at least some k and $a_{ij} \geq a_{hj}$, let $\tilde{r}_i^k = \min\left\{r_i^k + r_h^k, 1\right\}$ and $\tilde{r}_h^k = r_i^k + r_h^k - \tilde{r}_i^k$ for all $k = 1, ..., n-1$. Suppose $\sum_{k=1}^n r_i^k \mu^k(p_j) + \sum_{k=1}^n r_h^k \mu^k(p_j) > 0$. Then, we have

$$\begin{aligned} & a_{ij}^{\frac{\beta}{\sigma+\beta-1}} \left[\sum_{k=1}^n r_i^k \mu^k(p_j) \right]^{\frac{1-\sigma}{\sigma+\beta-1}} + a_{hj}^{\frac{\beta}{\sigma+\beta-1}} \left[\sum_{k=1}^n r_h^k \mu^k(p_j) \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \\ < \; & a_{ij}^{\frac{\beta}{\sigma+\beta-1}} \left[\sum_{k=1}^n \tilde{r}_i^k \mu^k(p_j) \right]^{\frac{1-\sigma}{\sigma+\beta-1}} + a_{hj}^{\frac{\beta}{\sigma+\beta-1}} \left[\sum_{k=1}^n \tilde{r}_h^k \mu^k(p_j) \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \end{aligned}$$

Thus, it suffices to consider a deterministic assignment matrix R in search for the optimal assignment function $q : \mathcal{S} \to [0,1]$ with $\sum_{S\in\mathcal{S}(k)} q(S) = k$ for all $k = 1, ..., n$.

We will consider deterministic assignment rules, so $r_i^k \in \{0,1\}$ for all $i, k = 1, ..., n$, since we assume $\beta \leq 2(1-\sigma)$. Rename candidates by their abilities in a decending order: $a_{1j} \geq a_{2j} \geq ... \geq a_{nj}$. Since $\beta \geq 2(1-\sigma)$, we need to find an weight function q that assigns the highest of the following sum of weights to $i = 1$, and the second highest to $i = 2$, and so on:

$$\sum_{k=1}^{n} r_k^i \mu^k(p_j)$$

in order to maximize E_j (thus, to maximize p_j given E_{-j}).

We can describe an assignment rule by the following $n \times n$ matrix R such that (i) each row represents candidate $i = 1, 2, ..., n$, and each column k represents the number of seats won in the election, (ii) each (i,k) argument r_{ik} is either 1 or 0, representing whether or not candidate i goes to the parliament when k seats are won, and (iii) for each column $k = 1, ..., n$, the elements in the kth column sum up to k. In order to describe the optimal (deterministic) mechanism, we will introduce some notations. Consider $k = 1, ..., n$ be the number of seats won by party j. For each $k = 1, ..., n$, party needs to send k candidates to the parliament. Let $\kappa(i) = (\kappa_1(i), ..., \kappa_k(i), ..., \kappa_n(i))$ be the number of seats available for each case k, and let $\nu(i)$ be the number of candidates left to be assigned when candidate i is going to be assigned: i.e., $\nu(i) = n - i + 1$. We will assign seats to candidates in order starting from the highest ability candidate $i = 1$ in a decending order. Let $\mathcal{M}(i) \equiv \{k \in \{1, ..., n\} : \kappa_k(i) > 0\}$ be the set of cases k in which candidate i can be sent to the parliament, and let $\mathcal{L}(i) \equiv \{k \in \{1, ..., n\} : \kappa_k(i) = \nu(i)\}$ be the set of cases k in which candidate i must be sent to the parliament (for feasibility: if not, k candidates cannot be sent to the parliament when k seats are won). Denote the effort-maximizing set of cases in which candidate i is sent to the parliament by $\zeta(i) \subseteq \{1, ..., k, ..., n\}$. Let $\kappa(i+1) = (\kappa_1(i+1), ..., \kappa_k(i+1), ..., \kappa_n(i+1))$ be such that $\kappa_k(i+1) = \kappa_k(i) - 1$ if $k \in \zeta(i)$, and $\kappa_k(i+1) = \kappa_k(i)$ otherwise. Initially, $\kappa(1) = (\kappa_1(1), ..., \kappa_k(1), ..., \kappa_n(1)) = (1, ..., k, ..., n)$, $\nu(1) = n$, $\mathcal{M}(1) \equiv \{1, ..., n\}$, and $\mathcal{L}(1) \equiv \{n\}$ hold. The optimal set of cases for candidate i to go to the parliament is defined by

$$\zeta(i) = \arg \max_{\mathcal{L}(i) \subseteq K \subseteq \mathcal{M}(i)} \sum_{k \in K} \mu_j(k)$$

for $i = 1, ..., n$. This $\zeta(i)$ gives candidate i the largest aggregate weights $\sum_{k \in \zeta(i)} \mu_j(k)$ available for her. The matrix is completed by setting $r_{ik} = 1$ if and only if $k \in \zeta(i)$ for all $i = 1, ..., n$ and all $k = 1, ..., n$.

From Single-Crossingness on Winning Probabilities, it is clear that $\zeta(1) = \{k^*, k^* + 1, ..., n\}$, since this set collects all positive $\mu^k(p_j)$s without having no negative $\mu^k(p_j)$s. How about $\zeta(2)$? It is still $\zeta(2) = \{k^*, k^* + 1, ..., n\}$ as long as $k^* \geq 2$ ($\kappa_{k^*}(2) \geq 1$), since $\mathcal{M}(2) \equiv \{1, ..., n\}$. We consider two cases: (Case 1) $k^* \leq \frac{n+1}{2}$, and (Case 2) $k^* > \frac{n+1}{2}$.

(Case 1: $k^* \leq \frac{n+1}{2}$) In this case, we can assign the top k^* candidates to $\{k^*, k^* + 1, ..., n\} = \zeta(1) = ... = \zeta(k^*)$. After that, as long as $i < n - k^* + 2$, we assign $\zeta(i) = \{i, i + 1, ..., n\}$. When $i = n - k^* + 2$ comes, we assign $\zeta(i) = \{k^* - 1\} \cup \{i, i + 1, ..., n\}$, and for $i = n - k^* + 3$, $\zeta(i) = \{k^* - 2, k^* - 1\} \cup \{i, i + 1, ..., n\}$, and so on. When $i = n$, $\zeta(n) = \{1, ..., k^* - 1\} \cup \{n\}$.

(Case 2: $k^* > \frac{n+1}{2}$) In this case, we can only assign the top $n - k^*$ candidates to $\{k^*, k^* + 1, ..., n\} = \zeta(1) = \zeta(n - k^*)$. Since $\kappa_{k^*-1}(n - k^* + 1) = \kappa_n(n - k^* + 1) = n - (n - k^* + 1) + 1 = \nu(n - k^* + 1)$, $\zeta(n - k^* + 1) = \{k^* - 1, k^*, ..., n\}$. Similarly, up to $i = n - k^* + 1$, $\zeta(i) = \{n - i + 1, ..., n\}$ is assigned. After that $\zeta(i) = \{n - i + 1, ..., k^* - 1\} \cup \{i, ..., n\}$.

Note that if $\sum_{k \in \zeta(i)} \mu_j(k) \leq 0$, then $e_{ij} = 0$ holds.

The **effort-maximizing rule** described by the optimal assignment matrix R has $q_{ik} = 1$ if and only if $k \in \zeta(i)$ for all $i = 1, ..., n$ and all $k = 1, ..., n$. This implies that the highest ability candidate 1 goes to the parliament if and only if party j wins k^* sets or more.

Proposition 3. Suppose $\beta \leq 2(1 - \sigma)$. Then, the optimal assignment rule is described by matrix R with $r_{ik} = 1$ if and only if $k \in \zeta(i)$ for all $i = 1, ..., n$ and all $k = 1, ..., n$.

In order to illustrate the optimal assignment rule, we provide an example below.

Example 1. Suppose $n = 7$. We consider three cases: $k^* = 3$, $k^* = 5$, and $k^* = 1$. The optimal assignment matrix is described by R_3^*, R_7^*, and R_1^* in the following:

$$R_3^* = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad R_7^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

When $k^* = 3$, the lowest ability candidate 7 goes to the parliament only when party j wins $k = 1, 2, 7$ seats, and candidate 6 goes to the parliament when party j wins $k = 2, 6, 7$ seats. When $k^* = 5$ (party j is a dominating party), then the highest ability candidates 1, 2, and 3 can go to the parliament only when party j wins 5 or more seats. This is because the party wants the highest ability candidates work very hard to be elected.

In contrast, when $k^* = 1$, the optimal assignment matrix exhibits monotonicity—it is in fact the list rule according to their abilities.

$$R_1^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Corollary 1. Suppose $\beta \leq 2(1 - \sigma)$. When $k^* = 1$, the optimal assignment rule is the list rule according to candidates' abilities.

Is there any justification to use a list rule from effort-maximization point of view when $k^* > 1$? One desirable property we may impose on the assignment matrix is monotonicity. A rule described by an assignment matrix R is **monotonic** if and only if $r_i^{k+1} \geq r_i^k$ for all $i = 1, ..., n$ and $k = 1, ..., n$. As is seen in Example 1, the optimal assignment matrix R does not necessarily satisfy monotonicity when $k^* > 1$. However, monotonicity is a very reasonable requirement. A particularly appealing property of a monotonic rule is that everybody exert a positive effort. This can be seen easily by rewriting $\sum_{k=1}^n r_i^k \mu^k(p_j)$:

$$\sum_{k=1}^n r_i^k \mu^k(p_j) = r_i^1 \sum_{k=1}^n \mu^k(p_j) + \left(r_i^2 - r_i^1\right) \sum_{k=2}^n \mu^k(p_j) + ... + \left(r_i^n - r_i^{n-1}\right) \mu^n(p_j)$$

By the first order stochastic dominance, $\sum_{k=m}^n \mu^k(p_j) > 0$ for all $m = 1, ..., n$. By monotonicity, $r_i^k - r_i^{k-1} \geq 0$ for all $k = 1, ..., n$ ($r_0^k = 0$). Thus, monotonicity implies:

$$\max\left\{\sum_{k=1}^n r_i^k \mu^k(p_j), 0\right\} = \sum_{k=1}^n r_i^k \mu^k(p_j) > 0$$

Proposition 4. Under any monotonic rule, every candidate exerts effort.

Under determinic rules, monotonicity requires that if candidate i is sent to parliament when k seats are won, she will be sent to the parliament if more than k seats are won. Let $Z(m) = \{m, ..., n\}$ for all $m = 1, ..., n$. Order $Z(m)$s by the value of $\sum_{k \in Z(m)} \mu^k(p_j)$ from the highest to the lowest: $Z_1^*, Z_2^*, ..., Z_n^*$. Each of which is assigned to candidates $1, 2, ..., n$, respectively. Denote the first element of Z_i^* by $m^*(i)$. Then, mapping $m^* : N_j \to \{1, ..., n\}$ is a one-to-one mapping.

Proposition 5. Suppose $\beta \leq 2(1 - \sigma)$. Then, the list rule $m^* : N_j \to \{1, ..., n\}$ is the optimal monotonic assignment rule.

Note that $m^*(1) = k^*$ and $Z_1^* = \zeta_1^*$. Thus, the top candidate's assignments are exactly the same in the optimal deterministic rule and the optimal list rule. Under the single-crossingness, it is easy to see that

$m^*(2)$ is either k^*+1 or k^*-1, and m^* orders candidates in such a way that it forms a single-peaked way at peak $m^*(1) = k^*$. If parties' winning seats realizations follow i.i.d. random variables and if a list rule $m : N_j \to \{1, ..., n\}$ is used, then we have the following results.

Lemma 5. Suppose that winning seat realization follows i.i.d. Then, we can write party j's aggregate effort as follows

$$E_j(p_j) = \left[V \left(p_j \left(1 - p_j\right) \sum_{m=1}^{n} \alpha_{mj} \omega_j(m) \right)^{\frac{\sigma+\beta-1}{1-\sigma}} \right]^{\frac{1}{\beta}}$$

where $\omega_j(m) = \left(mC(n, m) p_j^{m-1} \left(1 - p_j\right)^{n-m} \right)^{\frac{1-\sigma}{\sigma+\beta-1}}$.

Proposition 6 (Winning Probability and Efforts in i.i.d. Case). In each party j, each candidate's and the aggregated efforts are affected by the party's (expected) winning probability: (i) for each $m = 1, ..., n$, $\frac{\partial e_{mj}}{\partial p_j} \gtreqless 0 \iff m \gtreqless (n+1)\, p_j$, and (ii) $\frac{\partial E_j}{\partial p_j} \gtreqless 0 \iff \sum_{m=1}^{n} \alpha_{mj} \omega_j(m) \left(m - (n+1)\, p_j\right) \gtreqless 0$.

This proposition says that candidates who are ranked higher than $p_j\,(n+1)$ (i.e., $m < p_j(n+1)$) will reduce their efforts as party's winning probability goes up. In contrast, lower-ranked candidates (i.e., $m > p_j(n+1)$) will increase their efforts as p_j goes up. As a result, the effect of an increase in p_j on E_j is a rather complicated function of p_j. In order to maximize the winning probability of party j, the party leader needs to choose the ordering of candidates very carefully.

Remark. In this section we assumed convex case $\beta < 2(1-\sigma)$, and justified deterministic rules including optimal list rule as the optimal monotonic rule. If we confine our attentiion to deterministic rules, which may be well-justified in a party's political feasibility constraint, then the optimal list rule is the optimal deterministic monotonic rule even if $\beta \geq 2(1-\sigma)$ holds.

6.2 Concave Case $\beta > 2(1-\sigma)$

With strong complementarity of team members' efforts, the reward should not be concentrated to small set of members.

Thus, party j's maximization problem becomes

$$\arg\max_{(r_i^k)} \left\{ V\left(\sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}} \left[\sum_{k=1}^{n} r_i^k \mu^k(p_j) \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{\sigma+\beta-1}{1-\sigma}} \right\}^{\frac{1}{\beta}} \quad \text{s.t. } \sum_{i=1}^{n} r_i^k = k \text{ for all } k = 1, ..., n$$

or

$$\max_{(r_i^k)} \sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}} \left[\sum_{k=1}^{n} r_i^k \mu^k(p_j) \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \quad \text{s.t. } \sum_{i=1}^{n} r_i^k = k \text{ for all } k = 1, ..., n$$

The optimal mechanism is the solution of the above (rather complicated) problem when $\beta > 2(1-\sigma)$ holds. As k increases the set r_i^k will face more strict constraints (when $k = n$, $r_i^n = 1$ must hold: every candidate needs to be sent to the parliament). We know, however, that $\mu^k(p_j) = \frac{dP^k}{dp_j}(1-p_j)p_j < 0$ for all $k < k^*$ and $\frac{dP^k}{dp_j}(1-p_j)p_j > 0$ for all $k > k^*$, and that what matters is just the weighted sum of the shares in the bracket in achieving the optimal allocation. Intuitively, there will be a plenty of freedom using r^ks for low ks to achieve unequal allocations.

Supposing that the sum of reward $\bar{R} = \sum_{k=1}^{n} k \frac{dP^k}{dp_j}(1-p_j)p_j$ can be allocated freely to the candidates according to their abilities, the optimal allocation is described by solving the following problem.

$$\arg\max_{(R_i)_{i=1}^n} \sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}} R_i^{\frac{1-\sigma}{\sigma+\beta-1}} \qquad s.t. \quad \sum_{i=1}^{n} R_i = \bar{R} = \sum_{k=1}^{n} k \frac{dP^k}{dp_j}(1-p_j)p_j$$

The first order conditions generate the optimality conditions:

$$\frac{1-\sigma}{\sigma+\beta-1} a_{ij}^{\frac{\beta}{\sigma+\beta-1}} R_i^{\frac{1-\sigma}{\sigma+\beta-1}-1} = \frac{1-\sigma}{\sigma+\beta-1} a_{hj}^{\frac{\beta}{\sigma+\beta-1}} R_h^{\frac{1-\sigma}{\sigma+\beta-1}-1}$$

or

$$\frac{R_i}{R_h} = \left(\frac{a_{ij}}{a_{hj}}\right)^{\frac{\beta}{\beta-2(1-\sigma)}}$$

for all $i, h = 1, ..., n$.

Proposition 7. Suppose $\beta > 2(1-\sigma)$. Then, whenever feasible, the optimal assignment rule tries to allocate the chances of candidates to get a seat in the parliament proportionally to candidates' abilities (with power $\frac{\beta}{\beta-2(1-\sigma)}$).

Our Propositions 5 and 7 generate a generalized version of the result in Crutzen, Flamand, and Sahuguet (2017) as a corollary. When candidates are homogenous, $R_i = R_h$ holds for all $i, h = 1, ..., n$ when $\beta > 2(1-\sigma)$. Thus $q_{ik} = \frac{k}{n}$ for all $i, k = 1, ..., n$, which generates the egalitarian rule. We can allow for asymmetric parties (σ_j and β_j can be party-dependent: thus different parties can use different rules) and a non-i.i.d. probabilistic distribution.

Corollary 2. (Crutzen, Flamand, and Sahuguet, 2017) Suppose that all candidates have the same ability. Then, if $\beta \leq 2(1-\sigma)$ then the optimal monotonic mechanism is the list rule, while if $\beta > 2(1-\sigma)$ then the optimal mechanism is the egalitarian rule.

7 Conclusion

To be written.

Appendix

Here we provide an elementary proof of Lemma 1.

Proof of Proposition 1. From the formula of $E_j(p_j)$,

$$E_j(p_j)^\beta = V\left(\sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}}\left[\max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\}\right]^{\frac{1-\sigma}{\sigma+\beta-1}}\right)^{\frac{\sigma+\beta-1}{1-\sigma}}$$

$$= \left(\sum_{k=1}^{J} E_k(p_k)\right)^{\beta} p_j^\beta$$

The last equality holds by $p_j = \frac{E_j}{E_1+\ldots+E_J}$. This implies that $E_j(p_j) \gtreqless E_i(p_i)$ if and only if $p_j \gtreqless p_i$. We completed the proof.■

Proof of Proposition 2. Let the matrix in the left-hand side be D. Then, the determinant of D is

$$
\begin{aligned}
|D| &= \begin{vmatrix}
1-\frac{1}{E}\frac{\partial E_1}{\partial p_1} & 0 & 0 & 0 & -\frac{p_1}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
0 & 1-\frac{1}{E}\frac{\partial E_2}{\partial p_2} & 0 & 0 & -\frac{p_2}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\vdots & \ddots & & \vdots & \vdots \\
0 & 0 & 1-\frac{1}{E}\frac{\partial E_j}{\partial p_j} & 0 & -\frac{p_j}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\vdots & & \ddots & \vdots & \vdots \\
0 & 0 \cdots & 0 & \cdots\ 1-\frac{1}{E}\frac{\partial E_{J-2}}{\partial p_{J-2}} & -\frac{p_{J-2}}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\frac{p_{J-1}}{E}\frac{\partial E_1}{\partial p_1} & \frac{p_{J-1}}{E}\frac{\partial E_2}{\partial p_2} & \frac{p_{J-1}}{E}\frac{\partial E_j}{\partial p_j} & \frac{p_{J-1}}{E}\frac{\partial E_{J-2}}{\partial p_{J-2}} & 1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}+\frac{p_{J-1}}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}
\end{vmatrix} \\
&= \begin{vmatrix}
1-\frac{1}{E}\frac{\partial E_1}{\partial p_1} & 0 & 0 & 0 & -\frac{p_1}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
0 & 1-\frac{1}{E}\frac{\partial E_2}{\partial p_2} & 0 & 0 & -\frac{p_2}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\vdots & \ddots & & \vdots & \vdots \\
0 & 0 & 1-\frac{1}{E}\frac{\partial E_j}{\partial p_j} & 0 & -\frac{p_j}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\vdots & & \ddots & \vdots & \vdots \\
0 & 0 \cdots & 0 & \cdots\ 1-\frac{1}{E}\frac{\partial E_{J-2}}{\partial p_{J-2}} & -\frac{p_{J-2}}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\frac{p_{J-1}}{E}\frac{\partial E_1}{\partial p_1} & \frac{p_{J-1}}{E}\frac{\partial E_2}{\partial p_2} & \frac{p_{J-1}}{E}\frac{\partial E_j}{\partial p_j} & \frac{p_{J-1}}{E}\frac{\partial E_{J-2}}{\partial p_{J-2}} & 1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}
\end{vmatrix} \\
&\quad + \begin{vmatrix}
1-\frac{1}{E}\frac{\partial E_1}{\partial p_1} & 0 & 0 & 0 & -\frac{p_1}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
0 & 1-\frac{1}{E}\frac{\partial E_2}{\partial p_2} & 0 & 0 & -\frac{p_2}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\vdots & \ddots & & \vdots & \vdots \\
0 & 0 & 1-\frac{1}{E}\frac{\partial E_j}{\partial p_j} & 0 & -\frac{p_j}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\vdots & & \ddots & \vdots & \vdots \\
0 & 0 \cdots & 0 & \cdots\ 1-\frac{1}{E}\frac{\partial E_{J-2}}{\partial p_{J-2}} & -\frac{p_{J-2}}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\frac{p_{J-1}}{E}\frac{\partial E_1}{\partial p_1} & \frac{p_{J-1}}{E}\frac{\partial E_2}{\partial p_2} & \frac{p_{J-1}}{E}\frac{\partial E_j}{\partial p_j} & \frac{p_{J-1}}{E}\frac{\partial E_{J-2}}{\partial p_{J-2}} & \frac{p_{J-1}}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}
\end{vmatrix} \\
&= \prod_{i=1}^{J-1}\left(1-\frac{1}{E}\frac{\partial E_i}{\partial p_i}\right)+\prod_{i=1}^{J-1}\left(1-\frac{1}{E}\frac{\partial E_i}{\partial p_i}\right)\sum_{i=1}^{J-1}\frac{\frac{p_i}{E}\frac{\partial E_i}{\partial p_i}}{1-\frac{1}{E}\frac{\partial E_i}{\partial p_i}} \\
&= \prod_{i=1}^{J-1}(1-\eta_i)\sum_{j=1}^{J-1}\left(1+\frac{p_j\eta_j}{1-\eta_j}\right),
\end{aligned}
$$

where $\eta_i = \frac{1}{E}\frac{\partial E_j}{\partial p_j} = \frac{p_j}{E_j}\frac{\partial E_j}{\partial p_j}$ is ith party's aggregated effort elasticity of its probability of winning. If we impose stability on equilibrium, then it is natural to assume $\eta_i \in (0,1)$ for all $i = 1, ..., J$. Thus, under stability, $|D| > 0$ is assured. Now, we can conduct a comparative static analysis:

$$
\begin{aligned}
\frac{dp_j}{d\alpha_{mj}} &= \frac{1}{|D|}\begin{vmatrix}
1-\frac{1}{E}\frac{\partial E_1}{\partial p_1}+\frac{p_1}{E}\frac{\partial E_1}{\partial p_1} & \cdots & 0 & \cdots & \frac{p_1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}} \\
\vdots & \ddots & \vdots & & \vdots \\
\frac{p_j}{E}\frac{\partial E_1}{\partial p_1} & \cdots & \frac{\partial E_j}{\partial \alpha_{mj}} & \cdots & \frac{p_j}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}} \\
\vdots & & \vdots & \ddots & \vdots \\
\frac{p_{J-1}}{E}\frac{\partial E_1}{\partial p_1} & \cdots & 0 & \cdots & 1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}+\frac{p_{J-1}}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}
\end{vmatrix} \\
&= \frac{1}{|D|}\begin{vmatrix}
1-\frac{1}{E}\frac{\partial E_1}{\partial p_1} & 0 & 0 & 0 & -\frac{p_1}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
0 & 1-\frac{1}{E}\frac{\partial E_2}{\partial p_2} & 0 & 0 & -\frac{p_2}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{p_j}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}} & \frac{p_j}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}} & \frac{\partial E_j}{\partial \alpha_{mj}} & \frac{p_j}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}} & \frac{p_j}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}} \\
\vdots & & \vdots \quad \ddots & \vdots & \vdots \\
0 & 0 \quad \cdots & 0 \quad \cdots & 1-\frac{1}{E}\frac{\partial E_{J-2}}{\partial p_{J-2}} & -\frac{p_{J-2}}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\frac{p_{J-1}}{E}\frac{\partial E_1}{\partial p_1} & \frac{p_{J-1}}{E}\frac{\partial E_2}{\partial p_2} & 0 & \frac{p_{J-1}}{E}\frac{\partial E_{J-2}}{\partial p_{J-2}} & 1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}+\frac{p_{J-1}}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}
\end{vmatrix} \\
&= \frac{\frac{\partial E_j}{\partial \alpha_{mj}}}{|D|}\prod_{i=1, i\neq j}^{J-1}(1-\eta_i)\sum_{i=1, i\neq j}^{J-1}\left(1+\frac{p_i\eta_i}{1-\eta_i}\right) > 0
\end{aligned}
$$

Thus, by Cramer's rule, we have

$$\begin{aligned}
\frac{dp_j}{d\alpha_{mj}} &= \frac{\prod_{i=1,\ i\neq j}^{J-1}\left(1-\eta_i\right)\sum_{i=1,\ i\neq j}^{J-1}\left(1+\frac{p_i\eta_i}{1-\eta_i}\right)\frac{\partial E_j}{\partial \alpha_{mj}}}{\prod_{i=1}^{J-1}\left(1-\eta_i\right)\sum_{i=1}^{J-1}\left(1+\frac{p_i\eta_i}{1-\eta_i}\right)} \\
&= \frac{\sum_{i=1}^{J-1}\left(1+\frac{p_i\eta_i}{1-\eta_i}\right)-1-\frac{p_j\eta_j}{1-\eta_j}}{\left(1-\eta_j\right)\sum_{i=1}^{J-1}\left(1+\frac{p_i\eta_i}{1-\eta_i}\right)}\times\frac{\partial E_j}{\partial \alpha_{mj}} \\
&= \frac{\prod_{i=1,\ i\neq j}^{J-1}\left(1-\eta_i\right)\sum_{i=1,\ i\neq j}^{J-1}\left(1+\frac{p_i\eta_i}{1-\eta_i}\right)}{\prod_{i=1}^{J-1}\left(1-\eta_i\right)\left[\sum_{i=1,\ i\neq j}^{J-1}\left(1+\frac{p_i\eta_i}{1-\eta_i}\right)+1+\frac{p_j\eta_j}{1-\eta_j}\right]}\times\frac{\partial E_j}{\partial \alpha_{mj}} \\
&= \frac{1}{\left(1-\eta_j\right)\left[1+\frac{1+\frac{p_j\eta_j}{1-\eta_j}}{\sum_{i=1,\ i\neq j}^{J-1}\left(1+\frac{p_i\eta_i}{1-\eta_i}\right)}\right]}\times\frac{\partial E_j}{\partial \alpha_{mj}} \\
&= \left[\left(1-\eta_j\right)+\frac{\left(1-p_j\right)\eta_j}{\sum_{i=1,\ i\neq j}^{J-1}\left(\frac{(1-p_i)\eta_i}{1-\eta_i}\right)}\right]^{-1}\times\frac{\partial E_j}{\partial \alpha_{mj}}
\end{aligned}$$

Note that for all $m=1,...,n$ and all $j=1,...,J$, we have

$$\frac{\partial E_j}{\partial \alpha_{mj}}=\frac{\sigma+\beta-1}{\beta\left(1-\sigma\right)}E_j\times\frac{\omega_j(m)}{\sum_{m'=1}^{n}\alpha_{m'j}\omega_j(m')}>0.$$

Thus, we obtained natural comparative static results.$\square$

Proof of Lemma 3. We will prove the statement by induction. Our induction hypothesis is:

> When the number of candidates is m, for any $k=1,...,m-1$, and any element $r\in A(k,m)$, there is $q^k:\mathcal{S}(k,m)\to[0,1]$ with $\sum_{S\in\mathcal{S}(k,m)}q(S)=1$ such that $r_i=\sum_{S\in\mathcal{S}_i(k,m)}q^k(S)$ for all $i=1,...,m$. Then, when the number of candidates is $m+1$, for and $k=1,...,m$, and any element $r\in A(k,m+1)$, there is $q^k:\mathcal{S}(k,m+1)\to[0,1]$ with $\sum_{S\in\mathcal{S}(k,m+1)}q(S)=1$ such that $r_i=\sum_{S\in\mathcal{S}_i(k,m+1)}q^k(S)$ for all $i=1,...,m+1$.

First note that for any m, if $k=1$ or $k=m-1$, for any element $r\in A(k,m)$, there is $q^k:\mathcal{S}(k,m)\to[0,1]$ with $\sum_{S\in\mathcal{S}(k,m)}q(S)=1$ such that $r_i=\sum_{S\in\mathcal{S}_i(k,m)}q^k(S)$ for all $i=1,...,m$. This is because if $k=1$, each coalition is a singleton, and if $k = m-1$, then each coalition excludes only one candidate. Second note that when $m = 3$, the statement is correct by the first argument.

Now, we will start the induction argument. We suppose that when population is m, if for any $k = 1, ..., m-1$, and any element $r \in A(k, m)$, there is $q^k : \mathcal{S}(k, m) \to [0, 1]$ with $\sum_{S \in \mathcal{S}(k,m)} q(S) = 1$ such that $r_i = \sum_{S \in \mathcal{S}_i(k,m)} q^k(S)$ for all $i = 1, ..., m$. Consider population $m+1$ and $k = 2, ..., m-1$. Pick $\bar{r} = (\bar{r}_1, ...\bar{r}_m, \bar{r}_{m+1}) \in A(k, m+1)$. Let $(r_1^{k-1}, ..., r_m^{k-1}) = \frac{k-1}{\Sigma_{h=1}^m \bar{r}_h}(\bar{r}_1, ...\bar{r}_m)$ and $(r_1^k, ..., r_m^k) = \frac{k}{\Sigma_{h=1}^m \bar{r}_h}(\bar{r}_1, ...\bar{r}_m)$. By the induction hypothesis, there are $q^{k-1} : \mathcal{S}(k-1, m) \to [0, 1]$ with $\sum_{S \in \mathcal{S}(k-1,m)} q^{k-1}(S) = 1$ such that $(r_1^{k-1}, ..., r_m^{k-1}) = \sum_{S \in \mathcal{S}_i(k-1,m)} q^{k-1}(S)$ for all $i = 1, ..., m$, and $q^k : \mathcal{S}(k, m) \to [0, 1]$ with $\sum_{S \in \mathcal{S}(k,m)} q^k(S) = 1$ such that $(r_1^k, ..., r_m^k) = \sum_{S \in \mathcal{S}_i(k,m)} q^k(S)$ for all $i = 1, ..., m$. Take q^{k-1}, and add candidate $m+1$ to every size $k-1$ coalition S with $q^{k-1}(S) > 0$. This new assignment mapping $\bar{q}^{k-1} : \mathcal{S}(k, m+1) \to [0, 1]$ with $\sum_{S \in \mathcal{S}(k,m+1)} q^{k-1}(S) = 1$ satisfies $(r_1^{k-1}, ..., r_m^{k-1}, 1) = \sum_{S \in \mathcal{S}_i(k,m+1)} \bar{q}^{k-1}(S)$ for all $i = 1, ..., m+1$. Now, take q^k, and apply the same mapping on $\mathcal{S}(k, m+1)$: i.e., candidate $m+1$ is never selected. Then, this new assignment mapping $\bar{q}^k : \mathcal{S}(k, m+1) \to [0, 1]$ with $\sum_{S \in \mathcal{S}(k,m+1)} q^k(S) = 1$ satisfies $(r_1^k, ..., r_m^k, 0) = \sum_{S \in \mathcal{S}_i(k,m+1)} \bar{q}^k(S)$ for all $i = 1, ..., m+1$. Consider a convex combination of $\bar{q}^{k-1} : \mathcal{S}(k, m+1) \to [0, 1]$ and $\bar{q}^k : \mathcal{S}(k, m+1) \to [0, 1]$ with weights $\bar{r}_{m+1}$ and $1 - \bar{r}_{m+1}$, and name it $\bar{q} : \mathcal{S}(k, m+1) \to [0, 1]$. This mapping achieves $\bar{r} = (\bar{r}_1, ...\bar{r}_m, \bar{r}_{m+1}) \in A(k, m+1)$. Hence, the induction hypothesis is proven.□

Proof of Lemma 4. It is easy to see

$$\sum_{k=1}^{n} r_i^k \mu^k(p_j) + \sum_{k=1}^{n} r_h^k \mu^k(p_j) = \sum_{k=1}^{n} \tilde{r}_i^k \mu^k(p_j) + \sum_{k=1}^{n} \tilde{r}_h^k \mu^k(p_j)$$

Since $\frac{1-\sigma}{\sigma+\beta-1} > 1$ (convex function) and $a_{ij} \geq a_{hj}$, the desired inequality holds.■

Proof of Lemma 5. Under a list rule, the mth candidate's payoff is written as

$$B_{mj} - C_{mj} = V \sum_{k=m}^{n} C(n, k) p_j^k (1 - p_j)^{n-k} - \frac{1}{\beta} e_{mj}^{\beta}$$

Differentiating the above with respect to e_{mj}, we obtain,

$$\begin{aligned}
& \frac{\partial B_{mj}}{\partial e_{mj}} - \frac{dC_{mj}}{de_{mj}} \\
= \;& V \sum_{k=m}^{n} C(n,k) \left[k p_j^{k-1} \left(1-p_j\right)^{n-k} - (n-k)\, p_j^{k} \left(1-p_j\right)^{n-k-1} \right] \frac{E_{-j}}{\left(E_{-j}+E_j\right)^2} \frac{\partial E_j}{\partial e_{mj}} - e_{mj}^{\beta-1} \\
= \;& V \sum_{k=m}^{n} \left[\frac{n!}{(k-1)!\,(n-k)!} p_j^{k-1} \left(1-p_j\right)^{n-k} - \frac{n!}{k!\,(n-k-1)!} p_j^{k} \left(1-p_j\right)^{n-k-1} \right] \frac{E_{-j}}{\left(E_{-j}+E_j\right)^2} \frac{\partial E_j}{\partial e_{mj}} - e_{mj}^{\beta-1} \\
= \;& V m C(n,m) p_j^{m-1} \left(1-p_j\right)^{n-m} \frac{E_{-j}}{\left(E_{-j}+E_j\right)^2} \frac{\partial E_j}{\partial e_{mj}} - e_{mj}^{\beta-1} \\
= \;& V m C(n,m) p^{m} \left(1-p\right)^{n-m+1} \frac{1}{E_j} a_{mj} E_j^{\sigma} e_{mj}^{-\sigma} - e_{mj}^{\beta-1} \\
= \;& V E_j^{\sigma-1} m C(n,m) p^{m} \left(1-p\right)^{n-m+1} a_{mj} e_{mj}^{-\sigma} - e_{mj}^{\beta-1} \\
= \;& 0
\end{aligned}$$

Thus, we have

$$e_{mj} = V^{\frac{1}{\sigma+\beta-1}} E_j^{\frac{\sigma-1}{\sigma+\beta-1}} \left(m C(n,m) p^{m} \left(1-p\right)^{n-m+1} \right)^{\frac{1}{\sigma+\beta-1}} a_{mj}^{\frac{1}{\sigma+\beta-1}}$$

Since $E_j = \left(\sum_{i=1}^{n} a_{ij} e_{ij}^{1-\sigma} \right)^{\frac{1}{1-\sigma}}$, we have

$$\begin{aligned}
E_j &= V^{\frac{1}{\sigma+\beta-1}} E_j^{\frac{\sigma-1}{\sigma+\beta-1}} \left(\sum_{m=1}^{n} a_{mj} \left(m C(n,m) p^{m} \left(1-p\right)^{n-m+1} \right)^{\frac{1-\sigma}{\sigma+\beta-1}} a_{mj}^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{1}{1-\sigma}} \\
&= V^{\frac{1}{\sigma+\beta-1}} E_j^{\frac{\sigma-1}{\sigma+\beta-1}} \left(\sum_{m=1}^{n} a_{mj}^{\frac{\beta}{\sigma+\beta-1}} \left(m C(n,m) p^{m} \left(1-p\right)^{n-m+1} \right)^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{1}{1-\sigma}}
\end{aligned}$$

Thus, we have

$$E_j^{\frac{\beta}{\sigma+\beta-1}} = V^{\frac{1}{\sigma+\beta-1}} \left(\sum_{m=1}^{n} \alpha_{mj} \left(m C(n,m) p_j^{m} \left(1-p_j\right)^{n-m+1} \right)^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{1}{1-\sigma}}$$

or

$$\begin{aligned}
E_j &= \left[V \left(\sum_{m=1}^{n} \alpha_{mj} \left(mC(n,m) p_j^m (1-p_j)^{n-m+1} \right)^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{\sigma+\beta-1}{1-\sigma}} \right]^{\frac{1}{\beta}} \\
&= \left[p_j(1-p_j) V \left(\sum_{m=1}^{n} \alpha_{mj} \left(mC(n,m) p_j^{m-1} (1-p_j)^{n-m} \right)^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{\sigma+\beta-1}{1-\sigma}} \right]^{\frac{1}{\beta}}
\end{aligned}$$

Proof of Proposition 5. First, differentiating e_{mj} with respect to p_j, we obtain

$$\begin{aligned}
\frac{\partial e_{mj}}{\partial p_j} &= V^{\frac{1}{\sigma+\beta-1}} E_j^{\frac{\sigma-1}{\sigma+\beta-1}} \frac{1}{\sigma+\beta-1} \left(mC(n,m) p_j^m (1-p_j)^{n-m+1} \right)^{\frac{1}{\sigma+\beta-1}-1} a_{mj}^{\frac{1}{\sigma+\beta-1}} \\
&\quad \times mC(n,m) p_j^{m-1} (1-p_j)^{n-m} \left\{ m(1-p_j) - (n-m+1) p_j \right\}
\end{aligned}$$

The contents of the last brace can be written as $m-(n+1)p_j$. Thus, we have a desired result.

Second, differentiating E_j with respect to p_j, we obtain

$$\begin{aligned}
\frac{\partial E_j}{\partial p_j} &= \frac{1}{\beta} E_j^{1-\beta} V \left[(1-2p_j) \left(\sum_{m=1}^{n} \alpha_{mj} \left(mC(n,m) p_j^{m-1} (1-p_j)^{n-m} \right)^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{\sigma+\beta-1}{1-\sigma}} \right. \\
&\quad + p_j(1-p_j) \left(\sum_{m=1}^{n} \alpha_{mj} \left(mC(n,m) p_j^{m-1} (1-p_j)^{n-m} \right)^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{\sigma+\beta-1}{1-\sigma}-1} \\
&\quad \left. \times \sum_{m=1}^{n} \alpha_{mj} \left(mC(n,m) p_j^{m-1} (1-p_j)^{n-m} \right)^{\frac{1-\sigma}{\sigma+\beta-1}-1} mC(n,m) p_j^{m-1} (1-p_j)^{n-m} \left((m-1)\frac{1}{p_j} - (n-m)\frac{1}{1-p_j} \right) \right] \\
&= \frac{1}{\beta} E_j^{1-\beta} V \left[(1-2p_j) \left(\sum_{m=1}^{n} \alpha_{mj} \left(mC(n,m) p_j^{m-1} (1-p_j)^{n-m} \right)^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{\sigma+\beta-1}{1-\sigma}} \right. \\
&\quad + \left(\sum_{m=1}^{n} \alpha_{mj} \left(mC(n,m) p_j^{m-1} (1-p_j)^{n-m} \right)^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{\sigma+\beta-1}{1-\sigma}-1} \\
&\quad \left. \times \sum_{m=1}^{n} \alpha_{mj} \left(mC(n,m) p_j^{m-1} (1-p_j)^{n-m} \right)^{\frac{1-\sigma}{\sigma+\beta-1}} \left((m-1)(1-p_j) - (n-m) p_j \right) \right]
\end{aligned}$$

$$= \frac{1}{\beta}E_j^{1-\beta}V\left[(1-2p_j)\left(\sum_{m=1}^{n}\alpha_{mj}\left(mC(n,m)p_j^{m-1}(1-p_j)^{n-m}\right)^{\frac{1-\sigma}{\sigma+\beta-1}}\right)^{\frac{\sigma+\beta-1}{1-\sigma}}\right.$$
$$+\left(\sum_{m=1}^{n}\alpha_{mj}\left(mC(n,m)p_j^{m-1}(1-p_j)^{n-m}\right)^{\frac{1-\sigma}{\sigma+\beta-1}}\right)^{\frac{\sigma+\beta-1}{1-\sigma}-1}$$
$$\left.\times\sum_{m=1}^{n}\alpha_{mj}\left(mC(n,m)p_j^{m-1}(1-p_j)^{n-m}\right)^{\frac{1-\sigma}{\sigma+\beta-1}}(m-1-(n-1)p_j)\right]$$

$$= \frac{1}{\beta}E_j^{1-\beta}V\left[(1-2p_j)\left(\sum_{m=1}^{n}\alpha_{mj}\left(mC(n,m)p_j^{m-1}(1-p_j)^{n-m}\right)^{\frac{1-\sigma}{\sigma+\beta-1}}\right)^{\frac{\sigma+\beta-1}{1-\sigma}}\right.$$
$$+\left(\sum_{m=1}^{n}\alpha_{mj}\left(mC(n,m)p_j^{m-1}(1-p_j)^{n-m}\right)^{\frac{1-\sigma}{\sigma+\beta-1}}\right)^{\frac{\sigma+\beta-1}{1-\sigma}}$$
$$\left.\times\frac{\sum_{m=1}^{n}\alpha_{mj}\left(mC(n,m)p_j^{m-1}(1-p_j)^{n-m}\right)^{\frac{1-\sigma}{\sigma+\beta-1}}\times(m-1-(n-1)p_j)}{\sum_{m=1}^{n}\alpha_{mj}\left(mC(n,m)p_j^{m-1}(1-p_j)^{n-m}\right)^{\frac{1-\sigma}{\sigma+\beta-1}}}\right]$$
$$= \frac{1}{\beta}E_j^{1-\beta}\times\frac{E_j^{\beta}}{p_j(1-p_j)}\times\frac{\sum_{m=1}^{n}\alpha_{mj}\left(mC(n,m)p_j^{m-1}(1-p_j)^{n-m}\right)^{\frac{1-\sigma}{\sigma+\beta-1}}\times((1-2p_j)+m-1-(n-1)p_j)}{\sum_{m=1}^{n}\alpha_{mj}\left(mC(n,m)p_j^{m-1}(1-p_j)^{n-m}\right)^{\frac{1-\sigma}{\sigma+\beta-1}}}$$
$$= \frac{E_j}{\beta p_j(1-p_j)}\times\frac{\sum_{m=1}^{n}\alpha_{mj}\left(mC(n,m)p_j^{m-1}(1-p_j)^{n-m}\right)^{\frac{1-\sigma}{\sigma+\beta-1}}\times(m-(n+1)p_j)}{\sum_{m=1}^{n}\alpha_{mj}\left(mC(n,m)p_j^{m-1}(1-p_j)^{n-m}\right)^{\frac{1-\sigma}{\sigma+\beta-1}}}$$

The sign of the derivative is dictated by the sign of the last line. The proof is completed.■

References

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