Incentivized Kidney Exchange*

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Abstract

Within the last decade, kidney exchange has become a mainstream paradigm to increase the number of transplants. However, compatible pairs do not participate, and the full benefits from exchange can be realized only if they do. In this paper, we propose a new scheme, incentivizing participation of compatible pairs in exchange via insurance for a future renal failure in the patient. Efficiency and equity analyses of this scheme are conducted and compared with that of living-donor exchange in a new dynamic continuum model of kidney transplantation. We also calibrate the model with data from the US and quantify our predictions.

Keywords: Market design, organ allocation, kidney exchange, equity, efficiency, compatible pairs
JEL codes: D47, C78

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1 Introduction

The National Organ Transplant Act (NOTA) of 1984 called for an Organ Procurement and Transplantation Network (OPTN) to be created and run by a private, nonprofit organization under federal contract. The federal *Final Rule* provides a regulatory framework for the structure and operation of the OPTN: the primary goal of the OPTN is "to increase and ensure the effectiveness, efficiency, and equity of organ sharing in the national system of organ allocation," and "to increase the supply of donated organs available for transplantation" (Duda, 2005). As in most resource-allocation problems, tension often emerges between the dual objectives of efficiency and equity in the context of organ transplantation.

This paper's ultimate objective is the introduction and advocacy of a new organ-allocation policy that has strong potential not only to increase the supply of organs available for transplantation (thus increasing the efficiency of the organ-allocation system), but also to decrease its inequity. To our knowledge, the proposed policy is the first to enhance both the efficiency and equity of the system. To introduce our policy proposal, it will be helpful to explain two other contributions of the paper. We introduce a new and analytically tractable dynamic large-market model of organ transplantation that can be used to analyze the efficiency and equity implications of various technologies and policies.[1] Unlike former models that focus on a single organ-allocation technology (such as deceased-donor organ allocation or living-donor organ exchange), our model can be used to analyze the impact of various technologies and policies that are often used together and interact with each other. Another contribution is a formal analysis of the efficiency and equity implications of the following three primary organ-transplantation technologies: deceased-donor transplantation, living-donor transplantation, and living-donor organ exchange.

With the introduction of each of these technologies, the supply of donated organs available for transplantation potentially increases. Thus, each innovation potentially increases the efficiency of the organ-allocation system. However, for organs that require blood-type compatibility, the introduction of living-donor transplantation can potentially increase the inequity between various patient groups. That has been happening in the US for the case of kidneys. Similarly, living-donor organ exchange can further increase the inequity between certain patient groups. There is an intuitive explanation for this phenomenon: It is much harder for blood-type O patients to benefit from live donation or living-donor organ exchange than patients of other blood types. That is because, in the absence of other complications, while an O patient needs a O kidney for transplantation, a patient of blood-type A or B can receive a transplant from either a same-blood-type donor or an O donor, and a patient of blood-type AB can receive a transplant from any blood-type donor.

Using our model, we analyze the impact of each new technology on the number of patients of various groups who receive a transplant and characterize the average waiting time for those patients

[1] While traditional matching models mostly focus on discrete settings, the use of large-market and continuum models had become increasingly common over the last decade, especially in the context of market design applications. These models include Kojima and Pathak (2009), Che and Kojima (2010), Lee (2011), Azevedo and Budish (2012), Azevedo and Leshno (2016), Kojima, Pathak, and Roth (2013), Liu and Pycia (2013), Ashlagi and Roth (2014).

who are fortunate enough to receive one. Our results support the empirical observation that while living-donor transplantation and living-donor organ exchange both enhance the overall welfare of the patient population, they are potentially detrimental to equity across patients of different blood types.

To introduce our policy proposal, it will be helpful to give some background on the current status of living-donor organ exchange. This practice is in its infancy, with a handful of exchanges in the world. Moreover, it currently accounts for about three percent of US kidney transplants. Transplants from kidney exchanges only increased in the last decade, benefiting from a successful collaboration between economists and members of the transplantation community. Building on existing practices in kidney transplantation, Roth, Sönmez, and Ünver (2004, 2005b, 2007) formulated kidney exchange as a market-design problem and analyzed how an efficient and incentive-compatible system of exchanges might be organized and what its welfare implications might be.[2] Kidney-exchange research in the last decade revealed that the following four elements are especially important in the design and implementation of a successful kidney exchange program:

1. organization and optimization of the exchange,
2. utilization of gains from larger exchanges,
3. integration of good samaritan donors to exchange via kidney chains, and
4. inclusion of compatible pairs.

Of these four elements, the first three have been largely embraced by the transplantation community and successfully utilized by several kidney-exchange programs, but the success of the last element has, so far, been limited. For kidney exchange to realize its full promise, it is important to address the failure to include compatible pairs in exchange pools.

We introduce an incentive program that will encourage participation of compatible pairs. On the one hand, countless O patients with non-O donors are waiting for a potential exchange, and, on the other hand, many O donors donate directly to their non-O recipients. These non-O recipients thus use up kidneys that are more sought after. That is why inclusion of compatible pairs in exchange is so critical. How can compatible pairs be incentivized to participate in kidney exchange? A natural possibility is offering cash incentives, but cash incentives are currently taboo in much of the world. What we propose instead is the following incentive program:

New Policy Proposal: If an O donor with a compatible non-O patient (or if an AB patient with a compatible non-AB donor) participate in kidney exchange, even though they do not need to, then the patient is given priority in the deceased-donor queue in case he needs another kidney in the future.

Under our proposed incentive scheme, participation of compatible pairs is incentivized with "insurance" for a potential future renal failure. This insurance is of value to patients because transplanted kidneys last well below 20 years on average, and about 15 percent of kidney transplants are repeat transplants. Our policy proposal might receive wider acceptance in the medical community than cash incentives because such priority is already given to living donors: If a previous living

[2]See Segev et al. (2005) as a study in the medical literature, which advocated adoption of some of these techniques by the medical profession.

donor needs a kidney transplant in the future, she is prioritized in the deceased donor-queue. If adopted, our incentive scheme might confer a major advantage on the US national kidney exchange program run by the United Network for Organ Sharing (UNOS), since UNOS is also in charge of the deceased-donor queue. It would not be unrealistic to expect the national kidney exchange program to thrive under this new policy. Using our model, we analyze the impact of the introduction of our incentive scheme on the welfare of the patient population and analytically show that it increases the welfare of all patient groups. Moreover, for realistic parameters, it also decreases inequity across patients of different blood types. Besides general analysis of this policy, we quantify these welfare and equity gains numerically by plugging in kidney transplant data statistics as our model parameters.[3]

In organ transplantation, equity among different blood types emerges as an important policy objective. This is an indirect way of increasing equity among different ethnic groups, since different ethnic groups have different blood-type distributions. For example, B patients have to wait longer for kidney transplants. Blood-type B is more common in African-American and Asian minorities than in white populations. As a result, policies are being adopted that increase the access of B patients to organs. For example, some subtype A patients can feasibly donate to B and O patients. New policies make sure that B patients can receive such A subtype deceased-donor kidneys, while O patients are not offered such a policy.[4] Kidney exchange policies have also been tailored according to their impact on blood-type equity. An example is "indirect exchange," in which a blood-type incompatible patient-donor pair, such as O patient-A donor, donates to an A patient on the queue to receive priority on the blood-type O waiting list for the next deceased donor. Ethical criticism of this policy (for example, see Ross and Woodle, 2000; Ross, 2006) for its potential negative impact on O deceased-donor waiting times is one of the main reasons why it is not commonly practiced.

Dynamic kidney exchange has recently been an active area of research. Ünver (2010) considers a discrete dynamic market with a reduced state space and characterizes optimal policies under different exchange-size constraints using dynamic programming tools. Our two-way exchange policy is a continuum variant of the optimal two-way policy in that paper. Thus, our continuum model can be seen as a limit version of such a model. Some more recent papers inspect near-optimal dynamic policies in markets with more complicated state spaces. For example, Anderson et al. (2017) show that a greedy matching is near optimal asymptotically as the probability of compatibility decreases between a donor and a patient (i.e., a thin market assumption, unlike ours). They also show that the larger exchange cycles/chains are, the higher the asymptotic gains. On the other hand, Akbarpour, Li, and Oveis-Gharan (2017) study a dynamic market in which agents expire after a certain time. In such a market, they show that waiting and greedily matching the agents just before they expire is near-optimal under similar assumptions to the previous paper. These papers do not consider blood-type compatibility and inspect only near-optimal policies under the current exchange

[3]In an earlier draft of this paper, Sönmez and Ünver (2015), we also considered a model where patients can be listed in multiple exchange programs, and show that at equilibrium the national program that adopts our incentive program emerges as the only major program.

[4]See page 83 of OPTN kidney allocation guidelines retrieved from
`https://optn.transplant.hrsa.gov/media/1200/optn_policies.pdf#nameddest=Policy_08` on 06/05/2017.

paradigms. We propose a new policy paradigm for compatible pairs, do not seek out necessarily time-optimal allocation policies, and take first-in-first-out structure of the deceased donor queue as given. Another recent paper to study dynamic markets, albeit outside of the organ allocation context, is Baccara, Lee, and Yariv (2016). This paper finds and compares optimal centralized and decentralized matching policies as a function of thickness of the market when arriving agents have heterogeneous but common preferences, unlike our paper.[5]

The idea of including compatible pairs in exchange is not new and was initially proposed by Ross and Woodle (2000). This idea was further explored by Roth, Sönmez, and Ünver (2004, 2005a), Sönmez and Ünver (2014), and Nicolò and Rodríguez-Álvarez (2017) in market-design settings. Although the medical community initially concluded this idea should not be pursued because of ethical reasons, they later became proponents of it (for example, see Veatch, 2006, Kranenburg et al., 2006, Gentry et al., 2007, Ratner et al., 2010, Steinberg, 2011, and Ferrari et al., 2017). The proof of concept involving exchanges with compatible pairs is documented in Ratner et al. (2010). This study also reports the results of a survey conducted among compatible patient-donor pairs. The pairs' attitudes toward exchange were largely positive, especially if the patient benefits from the exchange in some form. From a medical ethics perspective, Veatch (2006) and Steinberg (2011) also advocated for incentives. The literature explored providing incentives through exchanging the donor of a compatible pair with a younger or genetically closer donor (see Roth, Sönmez, and Ünver, 2004, Ferrari et al., 2017, and Nicolò and Rodríguez-Álvarez, 2017). Such schemes can incentivize only a limited number of compatible pairs. Moreover, they induce uncertain and prolonged waiting times for compatible pairs, as their eventual participation will be determined by the characteristics of incompatible pairs currently available in a given pool. This can deter willing and suitable compatible pairs from participating in exchange pool in the first place. Our proposal is the first one that we are aware of that can globally and ex ante provide incentives to compatible pairs using tools that are already acceptable within the transplantation community.[6]

2 A Dynamic Model of Transplant Patients

We consider a comprehensive, dynamic kidney-transplantation model in which the deceased-donor queue, live donation, and living-donor exchange can be incorporated. To this end, we use a continuum-flow model where the cardinality of patients and donors who arrive at the same time are measured through a one-dimensional Lebesgue measure. We refer to this cardinality per unit time

[5]In the computer science literature, adaptive dynamic kidney exchange models that use non-parametric regression techniques on past data to determine optimal policy for the future have been also introduced. An important forerunner of this approach is Dickerson, Procaccia, and Sandholm (2012).

[6]Indeed, after the initial draft of our paper became available, Veale et al. (2017) reported three uses of a variant of our intertemporal insurance scheme, leading to 25 transplants through chain exchanges. This scheme is utilized as follows: The old live donor paired with a patient, who will likely need a kidney transplant in the future, initiates a chain of exchanges now by donating to an incompatible pair. In return, this patient receives a guaranteed priority in the deceased-donor queue if her kidney indeed fails in the future. The donor has a short donation window due to her old age, and the insurance scheme helps other pairs to receive transplants through chain exchanges now, in addition to insuring the potential patient originally paired with the donor.

as **measure**.

Consider patients who need a particular organ transplant. Each **patient** is represented by his blood type $X \in \mathcal{T} = \{A, B, AB, O\}$. Suppose p_X refers to the probability of having the X blood type in the population distribution. We refer to the arrival measures of patients or donors as **inflow rates**. We assume that π_X is the inflow rate of new X patients. Hence, $\pi_X dt$ is the two-dimensional Lebesgue measure of patients who enter in a small time interval dt. By a slight abuse of terminology, throughout the paper we refer to the two-dimensional Lebesgue measure of agent sets, such as $\pi_X dt$, as **mass**.

Suppose that in the population of new patients, the expected lifetime while living with the disease is distributed with a strictly increasing differentiable distribution function $F(\cdot)$ on the interval $[0, T]$. Therefore, the probability density function is well defined and positive in $(0, T)$. In addition, the measure of X patients who are alive after t years is given by $\pi_X[1 - F(t)]$.

In the long run, when a transplantation option is not present, the total mass of X patients is $\int_0^T \pi_X[1 - F(t)]dt$.

2.1 Organ Transplantation

The best remedy for organ failure is transplantation. A donor must be both blood- and tissue-type compatible with the patient before her organ(s) can be transplanted. O donors are blood-type compatible with all patients. A donors are blood-type compatible with type A and AB patients, and type B donors are blood-type compatible with B and AB patients. On the other hand, AB donors are blood-type compatible only with AB patients. **Blood-type compatibility** is formally defined through a partial order $\triangleright$ over blood types, such that $X \triangleright Y$ means that X donors are blood-type compatible with Y patients. Blood type distribution among US ethnic groups is reported in Table 1.[7] In general, O blood type is the most common, while AB is the rarest; A is observed more commonly than B, while their rates vary substantially across ethic groups: B has a strong presence among Asian- and African-American groups, while this is not the case for white Americans.

	Blood Types				Pop. %
	O	A	B	AB	— (1992)
African American	49%	27%	20%	4%	12.4%
Asian American	40%	28%	27%	5%	3.3%
Native American	79%	16%	4%	1%	0.8%
White American	45%	40%	11%	4%	83.4%
US all	45.6 %	37.8%	12.6%	4%	

Table 1: Blood-Type Distribution in the US.

Once a donor is deemed blood-type compatible with a patient, she also has to be tissue-type compatible with the patient. Tissue-type compatibility requires that the patient's body has no

[7]Retrieved from http://bloodbook.com on 03/18/2013. The US general population is constructed using the ethnicity distribution and could be slightly different from the general distributions reported in other sources.

pre-formed antibodies against the donor's DNA. We assume that each donor has a tissue type. There are k distinct tissue types. The probability that a donor is of tissue type i is $m_{i,k} > 0$, so $\sum_i m_{i,k} = 1$. Let $\theta_{i,k}$ be the **tissue rejection probability** between any patient and a donor of tissue type i. If a patient is tissue-type compatible with a type i donor, then the patient is tissue-type compatible with all donors of tissue type i. In our analysis, we take the limit as $k \to \infty$ and make some assumptions about how the market grows (see Appendix C). These assumptions are satisfied for a range of parameters. However, for the ease of exposition in the main text, we assume that $\theta_{i,k} = \theta$ for every i in the limit as $k \to \infty$ and that $m_{i,k}$ decays proportionally to $1/k$, that is $m_{i,k} = \Theta(1/k)$. More explicitly, for every donor type i, there exist constants $c_{i1} > 0$, $c_{i2} > 0$, and $k_0 \in \mathbb{N}$ such that, $c_{i1}\frac{1}{k} \leq m_{i,k} \leq c_{i2}\frac{1}{k}$ for every $k > k_0$. In the more general case, we can assume that **the expected tissue rejection probability** in the limit is θ.

A common source of donation across organs is deceased donors. The deceased-donor queue is governed by a central entity in most countries. For example, in the US, for all organ types, UNOS is the federal contractor that is in charge of the queue. We assume throughout the paper that any patient enrolled in the queue remains in the queue until he receives a transplant or he dies.

We assume that patients prefer earlier compatible transplants, and we assume that they are indifferent among compatible deceased donors. Thus, when a patient is offered a compatible transplant, the best option for her is to take it.

We denote the inflow rate of the X deceased donors as $\delta_X < \pi_X$ per unit time. Across blood types, the ratio δ_X/π_X need not be constant. For example, it is well known that among minority communities organ failure is more prominent than among the white American population, even though deceased-donation rates are not that significantly different. As the blood-type distribution of minorities is different from that of white Americans, the ratio δ_X/π_X is not constant across blood types in the US: while a very high percentage of the donors, live or deceased, are white, the patient rate of white Americans is much lower than their donation rate for kidneys and is higher only for lungs. On the other hand, for kidneys and hearts, the patient rate of African-Americans is higher than their donation rate; while for kidneys and livers, the patient rate of Asian-Americans is higher than their donation rate.[8] Although these rates are distorted by many other factors such as live-donation possibilities, we can conclude that the ratio δ_B/π_B is lower than that for other blood types.

When a transplanted kidney fails, the recipient reenters the deceased-donor queue as if he were a new patient. We assume that repeat patients' survival function on the deceased-donor queue is similar to that of new entrants. We also assume that ϕ^d is the steady-state fraction of the previous recipients whose kidneys fail and who reenter the deceased-donor queue per new deceased-donor

[8]From the US Department of Health and Human Services - The Office of Minority Health web page for organ donation https://minorityhealth.hhs.gov/templates/browse.aspx?lvl=3 & lvlid=12 retrieved on 02/25/2013.

transplant conducted.[9,10] Thus, if at steady state an ε measure of the X patients receive a deceased-donor kidney at each instance, then a $\phi^d\varepsilon$ measure of previous recipients reenter the queue at each instance.

2.2 Deceased-Donor Allocation Policies

The deceased-donor organs are allocated by UNOS through the points system of OPTN, which is a priority mechanism. When a deceased donor arrives, the point total for each compatible patient is determined. The organ is offered to the patient with the highest point total. If it is rejected by the patient or his doctor for any reason, then the organ is offered to the next patient, and so on. In general, different point schemes are used for different organs. Deceased-donor allocation policies usually differ across organs and across geographic transplant regions, although usually a centralized mechanism is used in allocation. For example, for kidneys, *ABO-identical* allocation policies are applied, while for organs with greater medical urgency, *ABO-compatible* allocation is more common. That is, in the **ABO-identical (ABO-i)** allocation policy, organs of blood-type X are offered only to X patients.[11] On the other hand, in the ABO-compatible (ABO-c) allocation policy, organs can be offered to any compatible patient. We study the welfare and distributional consequences of the ABO-i policy here. The welfare and distributional consequences of the ABO-c policy are investigated in a companion paper (Sönmez and Ünver, 2015).

Given ABO-i policy, the waiting time of a patient is often the most significant contributor to the patient's points in deceased-donor allocation. Therefore, we model deceased donor allocation using **first-in-first-out** (**FIFO** from now on) matching technology.[12]

[9] Fraction ϕ^d is formally calculated as follows: Suppose a measure ε of patients receive transplants at steady state at each instance. If the patient's life with a healthy graft ends, two things could be the reason: either the patient dies, or the patient stays alive but his graft fails. Of the patients who leave the status of "living with a healthy kidney," let $h_1(t)$ be the fraction that die t years after the transplant and $h_2(t)$ be the fraction whose kidneys fail t years after the transplant. Thus, we assume that a random patient's expected lifetime with a healthy kidney is distributed with a differentiable distribution function $H(\cdot)$ in some interval $[0, S]$ such that $\frac{dH(t)}{dt} \equiv h(t) \equiv h_1(t) + h_2(t)$ where t refers to the years that have passed since the transplant. We assume that this distribution is independent of how long the patient initially waited in the queue to receive his previous transplant. Then the inflow rate of patients reentering the deceased donor queue is given by $\int_0^S \varepsilon h_2(t)dt = \varepsilon \int_0^S h_2(t)dt$. We set $\phi^d = \int_0^S h_2(t)d(t)$. Observe that $\phi^d < \int_0^S h(t)d(t) = 1$.

[10] For simplicity, we assume that ϕ^d is constant, although it may possibly change as the age distribution of the patients receiving transplants in the deceased-donor queue changes, i.e., it may be a function of the waiting time.

[11] In the event that no X patient is available, then the organ is offered to a compatible patient. However, this is the application in the US. On the other hand, Eurotransplant uses full ABO - compatible scheme, and UK Transplant permits O organs to be transplanted to B patients, especially for kidneys (cf. Canadian Council of Transplantation documentation for "Deceased donor allocation in US, Europe, Australia, and New Zealand" released in October 2006).

[12] OPTN recently switched to a new deceased-donor kidney allocation scheme that uses a quality-based allocation scheme for 20 percent of all allocations, while 80 percent of all allocations continue to be done through its previous FIFO-type policy.

2.3 Living-Donor Transplantation

Live donation is common for kidneys. In 2011, 34 percent of all kidney transplants in the US were from living donors.

We refer to a living donor as a **paired donor**. We assume that each patient has at most one paired donor and that a $\lambda \in [0,1]$ fraction of incoming patients have a paired donor. We also assume that the blood types of the patient and the donor are independent and uncorrelated.We refer to a patient with a paired donor as a **paired patient** and a patient without a paired donor as a **single patient**. The patient and his paired donor are represented as a **pair**. The blood types of the pair, $X - Y \in \mathcal{T} \times \mathcal{T}$, X being the patient's and Y being the donor's blood type, determine the **type of the pair**.

If the paired donor of a patient is both blood- and tissue-type compatible, we refer to the pair as a **compatible** pair, and otherwise as an **incompatible** pair. Given a paired patient, the probability that his paired donor is blood type X is assumed to be the probability of having X in the population, which is p_X.

Consistent with donation rates throughout the world, in the rest of the paper we assume the following:

Assumption 1 *There is a shortage of deceased-donor organs, even if paired patients are removed from the patient pool, i.e.,* $(1-\lambda)\pi_X + \phi^d \delta_X \geq \delta_X$ *for all* $X \in \mathcal{T}$.

We assume that patients prefer earlier compatible transplants to later ones, regardless of the kind of donor, deceased or living. Also we assume that they are indifferent among compatible organs from living donors (as well as deceased donors). We do not need to model whether they prefer deceased-donor transplants or living-donor transplants. For kidneys, typically living donors are preferred.

As a result, deceased-donor kidneys are transplanted as soon as they become available using FIFO matching technology in which the measure of donated kidneys is equal to the measure of patients receiving transplants. Therefore, in any state, we still keep track of the measure of (X,t) pairs where X is the blood type and t is the number of years spent waiting after being diagnosed. However, when live donation or exchange is a possibility, we also keep track of the measure of $(X-Y,t)$ where X is the blood type of the patient, Y is the blood type of the donor, and t is the number of years spent waiting after being diagnosed.

A **state** is formally defined through the measure of $(X-Y,t)$ and (X,t) pairs. We say that the patient population under a given policy of transplantation is at a **steady state** when the measures of all $(X-Y,t)$ and (X,t) pairs are constant through time, i.e., the state does not change over time.

Transplanted organs from living donors can also fail, as in the case of transplants from deceased donors. Like before, we assume that reentering patients have the same survival function $1-F$ as new patients. However, it is well known that living-donor grafts survive longer than deceased donor grafts. We assume that $\phi^l \leq \phi^d$ is the fraction of live-donation recipients reentering the deceased-donor queue per each living-donor organ transplant at steady state. We further assume

that the reentrants (who received a graft previously from either a deceased donor or a living donor) are single (and no longer paired) upon reentry.

3 Living-Donor Exchange

In this section, we analyze the effect of having a living-donor exchange program on waiting times of different patient groups. In practice, a paired donor usually donates directly to her paired patient, and the patient leaves the pool before he ever enters the deceased-donor queue. For the incompatible pairs, we assume that a living-donor exchange program operates in parallel with the deceased-donor queue: Incompatible pairs are listed in the exchange program. While waiting for a deceased-donor organ in the queue, patients also wait for an exchange with another incompatible pair.

Formally, a two-way **exchange** matches two pairs, where the patient of the first pair is compatible with the donor of the second pair and the patient of the second pair is compatible with the donor of the first pair. We refer to such pairs as **mutually compatible** pairs.[13] Any such exchange is conducted between pairs that preserve the measure. For example, if we are conducting exchange between A-B pairs and B-A pairs then the measures of A-B and B-A pairs are the same. We also say that if the donor of the first pair is blood-type compatible with the patient of the second pair and vice versa, then these pairs are **mutually blood-type compatible**. We refer to the queue of pairs in the exchange program as the **exchange pool**. An **exchange matching** is a set of exchanges between mutually compatible pairs such that each pair is matched in at most one exchange. For a given pair type $X-Y$, we refer to $Y-X$ as its **reciprocal** type.

Note that far fewer incompatible A-O patient-donor pairs exist in an exchange pool than O-A pairs. The reason is that A-O pairs are incompatible only if there is tissue incompatibility between the A patient and O donor, while O-A pairs are always incompatible. Based on this observation, we make the following assumption.

Assumption 2 *For any incompatible pair type $X-Y$ such that $X \neq Y$ and $X \triangleright Y$, its inflow rate to the exchange pool is not less than the inflow rate of its reciprocal type $Y-X$, i.e., $\theta p_X \pi_Y \leq p_Y \pi_X$.*[14]

To give an idea of how easily this assumption is satisfied, recall that for kidneys, we have $\theta \approx 0.1$. For all organs with exchange programs, this inequality holds with a good deal of slack for all populations.

Another assumption concerns the prevalence of A-B and B-A types. This assumption is made for notational convenience; symmetric versions of our results hold if the inequality in the assumption is reversed.

[13] We can also think of exchanges that can match more than two pairs, such as three-way, four-way, etc. For simplicity, we focus on two-way exchanges in our analysis. However, our results can easily be extended to cover larger exchange sizes as in Roth, Sönmez, and Ünver (2007). Any size of exchange greater than four does not change the results as reported in that paper.

[14] A simple requirement that would make the assumption hold is that the ratio of live donation and patient inflow rates are similar across blood types; i.e., $p_X/\pi_X \approx p_Y/\pi_Y$ for all blood types X, Y. This would be ensured if live donation and illness rates are not too different for different blood types in a given population.

Assumption 3 *A-B pairs do not inflow to the exchange pool any slower than B-A pairs, i.e.,* $p_A\pi_B \leq p_B\pi_A$.

Through Assumptions 2 and 3, all incompatible $X-Y$ pairs with $Y \triangleright X$ and $X-Y = B-A$ pairs can be matched immediately with $Y-X$ pairs, as $Y-X$ pairs will always be more in mass than $X-Y$ pairs in the exchange pool. Observe that the probability of mutual compatibility between an $X-Y$ pair and a $Y-X$ pair is $(1-\theta)^2 > 0$. As a result, all $X-Y$ pairs can be matched with probability one. Lemmata 4 and 5 in Appendix C formalize this idea.

Using the terminology in Ünver (2010), we classify the pairs into several categories, based on their desirability in exchange: **Overdemanded pair types** are ones with a blood type donor who can donate to her patient's blood type but is not of the same blood type. These are $A-O, B-O, AB-A, AB-B$, and $AB-O$ types. **Underdemanded pair types** are those with a blood type donor who cannot feasibly donate to her patient's blood type, excluding types A-B and B-A. That is, underdemanded types are reciprocals of overdemanded types, i.e., $O-A, O-B, A-AB, B-AB$, and $O-AB$. **Reciprocally demanded pair types** are A-B and B-A, as they can be matched with each other in a donor exchange, if tissue incompatibility does not exist. Finally, **self-demanded pair types** are those with the same blood-type donor and patient: $O-O, A-A, B-B, AB-AB$.

Next, we study how the exchange pool and deceased-donor queue evolve at steady state. Recall that only incompatible pairs participate in exchange. It turns out that we can conduct optimal two-way exchanges in an ABO-identical manner as well. More precisely, we can match $X-Y$ pairs with $Y-X$ pairs as they become available. We show that this is the optimal exchange policy.

Theorem 1 (ABO-identical exchange is optimal) *Suppose Assumptions 2 and 3 hold. Then the exchange policy where an arriving pair is immediately matched with a compatible pair of its reciprocal type maximizes the measure of exchange transplants of pairs that arrive at that instance.*

Moreover, this policy maximizes the mass of pairs who arrive in an interval that can be matched within that interval. In particular, it matches a larger mass of pairs than the alternative policy of running the exchange only once at the end of the time interval.

Note that the optimal exchange can also accommodate the FIFO matching technology, where a pair is matched with one of the longest-waiting mutually compatible pairs. This technology is important in practice for reasons of fairness, and it can indeed be implemented in the optimal exchange.

A result that is similar but logically independent from ours was proven in Ünver (2010) for discrete problems with waiting costs. In our setting, there is no waiting cost per se, but patients can die while waiting for transplants.

With the availability of exchange, we separate patients into different groups based on their blood type and donor status as single, paired with a compatible donor, or paired with an incompatible donor. We can measure the efficiency and equity effects of the proposed policy on these groups. There are 29 patient groups based on these two criteria.

Since compatible and incompatible pairs that are blood-type compatible receive transplants at time 0 (under the optimal exchange), we do not distinguish them in our discussion. Therefore, we

denote each patient group by the pair type $X-Y$ if the patient is paired and by the blood type X if the patient is single.

Through Theorem 1, we compute the **measure of X patients matched through exchange** under the above-described optimal exchange policy in steady state at any instance, denoted as $\mathbf{e}_X$ for all $X \in \mathcal{T}$:

$$\begin{aligned}
\mathbf{e}_O &= \underbrace{\theta p_O \lambda \pi_O}_{O-O \text{ pairs}} + \underbrace{\theta p_O \lambda (\pi_A + \pi_B + \pi_{AB})}_{O-A,\, O-B,\, O-AB \text{ pairs}}, \\
\mathbf{e}_A &= \underbrace{\theta p_A \lambda \pi_A}_{A-A \text{ pairs}} + \underbrace{\theta p_O \lambda \pi_A}_{A-O \text{ pairs}} + \underbrace{p_A \lambda \pi_B}_{A-B \text{ pairs}} + \underbrace{\theta p_A \lambda \pi_{AB}}_{A-AB \text{ pairs}}, \\
\mathbf{e}_B &= \underbrace{\theta p_B \lambda \pi_B}_{B-B \text{ pairs}} + \underbrace{\theta p_O \lambda \pi_B}_{B-O \text{ pairs}} + \underbrace{p_A \lambda \pi_B}_{B-A \text{ pairs}} + \underbrace{\theta p_B \lambda \pi_{AB}}_{B-AB \text{ pairs}}, \\
\mathbf{e}_{AB} &= \underbrace{\theta p_{AB} \lambda \pi_{AB}}_{AB-AB \text{ pairs}} + \underbrace{\theta (p_O + p_A + p_B) \lambda \pi_{AB}}_{AB-O,\, AB-A,\, AB-B \text{ pairs}}.
\end{aligned} \tag{1}$$

Let us explain one of these calculations, say $\mathbf{e}_O$, more explicitly. The O-O pairs are matched as soon as they arrive, so their contribution to this term is their inflow rate $\theta p_O \lambda \pi_O$: only λ fraction of the O patients have a donor, which happens with the inflow rate π_O, p_O is the probability that the donor has O blood type, and these are multiplied by the probability of tissue rejection probability θ since compatible pairs do not enter the exchange. The measure of O-A pairs that are matched is equal to the measure of the A-O pairs, which is equal to $\theta p_O \lambda \pi_A$, as in the previous calculation. The contribution of O-B and O-AB pairs can be calculated similarly.

Likewise, let $\mathbf{l}_X = p_X^l \lambda \pi_X$ be the inflow rate of X patients with compatible donors where p_X^l is the probability that a random donor is compatible with X patient, which we calculate explicitly using the primitives of the model in the Appendix. Then $\mathbf{l}_X/\pi_X$ is the **live donation transplant ratio** for X patients, δ_X/π_X is the X **deceased donation transplant ratio**, and $\mathbf{e}_X/\pi_X$ is the blood type X **exchange-transplant ratio**.

We use these measures to analyze how the availability of exchange affects the waiting times in the deceased-donor queue. As more patients receive living-donor transplants under exchange technology, the waiting times of patients improve across all blood types. Some of these pairs are matched immediately as they enter the pool. These belong to overdemanded or self-demanded types, or the less abundant reciprocal type, B-A. And some pairs are matched only after waiting in the pool. As a result, not all of them receive transplants, since some of the paired patients die while waiting. These pairs belong to underdemanded types or the more abundant reciprocal type, A-B. They wait in the exchange pool and the deceased-donor queue simultaneously, and either

- are "pooled" with single patients of the same blood type in the deceased-donor queue, so that simultaneously some of them will receive deceased-donor organs and some will participate in exchange; or
- wait for less time than their cohort of single patients and participate exclusively in exchange.

We are ready to state some inequity consequences of exchange. Although all patients benefit from exchange, O and AB patients benefit the least, and B patients benefit the most under mild conditions. We use the exchange transplant ratios, $\{\frac{\mathbf{e}_X}{\pi_X}\}$, for this comparison. In general, transplant ratios reflect the ex-ante probability of a particular blood-type patient receiving living-donor transplantation under various policies. Thus, they are ex-ante measures of probability of access to living-donor transplantation for each patient. Making them as close to each other as possible without sacrificing the total number of transplants would be an inequity-improving endeavor. For example, an egalitarian welfare function on chances of receiving a transplant under ABO-i constraints would care about this objective.

When we consider living-donor and exchange-transplant ratios together, $\{\frac{\mathbf{l}_X+\mathbf{e}_X}{\pi_X}\}$, we see that O patients benefit the least, A and B patients benefit more than O, and AB patients benefit the most. These results hold in a benchmark model where no blood type is more likely to be a live donor than to get sick, i.e., where live-donation propensities are independent of blood type. Thus, although B is behind A in its living-donor transplant ratio (provided that $p_B < p_A$, as in the general population in the US and most of the world) the increase from B's exchange-transplant ratio makes its living-donor and exchange-transplant ratios level with those of A's.

Theorem 2 (Living donor exchange and inequity in transplant ratios) *Suppose Assumption 1 holds. Consider a benchmark model where the ratio of the living-donation rate to the patient-inflow rate is the same among blood types, i.e., $\frac{p_X}{\pi_X}$ is the same among all $X \in \mathcal{T}$. Then transplant ratios satisfy:*

- *For exchange only: $\frac{\mathbf{e}_O}{\pi_O} = \frac{\mathbf{e}_{AB}}{\pi_{AB}} < \frac{\mathbf{e}_A}{\pi_A}, \frac{\mathbf{e}_B}{\pi_B}$. If additionally $p_A > p_B$, then $\frac{\mathbf{e}_A}{\pi_A} < \frac{\mathbf{e}_B}{\pi_B}$.*
- *For living-donor transplantation and exchange together: $\frac{\mathbf{l}_O+\mathbf{e}_O}{\pi_O} < \frac{\mathbf{l}_A+\mathbf{e}_A}{\pi_A} = \frac{\mathbf{l}_B+\mathbf{e}_B}{\pi_B} < \frac{\mathbf{l}_{AB}+\mathbf{e}_{AB}}{\pi_{AB}}$.*

The intuition behind the first result comes from the fact that A and B have the additional advantage of exchange from two tissue-type-compatible pairs that are blood-type incompatible, i.e., exchanges between A-B and B-A pairs. In exchanges including type AB or type O patients, at least one pair should be tissue-type incompatible, and this pair becomes available for exchange with $\theta < 1$ probability. Additionally, if $p_A > p_B$, then $\pi_A > \pi_B$ holds as well in the benchmark model. Although A-B and B-A pair types participate in exchanges in equal measures, such exchanges are percentage-wise more beneficial for B patients, and, thus, B has the highest exchange-transplant ratio.

However, the exchange technology's contribution is not sufficient by itself to change the inequity caused by living-donor transplantation in transplant ratios, as indicated by the second part of the theorem. In addition, note that the transplant ratios of blood types A and B come very close to that of blood type AB as a result of the exchange technology. To see this, observe that the added benefit for AB over A or B of living-donor transplantation and exchange is that AB patients get direct live donation from AB donors, while A or B patients cannot. As the AB blood type is rare in the population, the aforementioned transplant ratios are very close.

4 A New Proposal: Incentivized Exchange

One shortcoming of current living-donor exchange practices is that they utilize almost exclusively *incompatible* pairs. As a result, many non-O patients receive transplants from an O donor without participating in exchange, effectively utilizing type O organs inefficiently. However, if *compatible* pairs can be incentivized to participate in exchange, then the lack of balance between reciprocal-type pairs will be mitigated. One sensible way of incentivizing compatible pairs to participate is to give their patients priority in the deceased-donor queue if their transplanted graft fails in the future. This is especially important because it may be harder to find an additional compatible donor when the patient needs another organ. As noted in the Introduction, living donors are already similarly incentivized: If a living donor's organ fails in the future, he will get priority in the deceased-donor queue. A similar practice of prioritizing not only the donor but also the patient of a compatible pair may face little resistance in the medical community.

In this section, we analyze the efficiency and equity effects of such an incentive scheme by using the tools we developed earlier. Thus, when a paired patient with a compatible donor receives a transplant through exchange and this graft later fails, the FIFO structure of deceased allocation is altered. In particular, such reentrants, who we refer to as **prioritized reentrants**, are placed at the front of the queue. (Not all compatible pairs need to be prioritized in the optimal policy, as we explain below.) In this section, we analyze the welfare effects of incentivized exchange with respect to its alternative, regular exchange.

Suppose that a proportion ρ of all compatible pairs takes up the incentivized exchange option. We will maintain the following assumption for the rest of the paper.

Assumption 4 *Compatible pairs may join the exchange pool only if an exchange is immediately available, and thus exchange does not involve a waiting cost; that is, the inflow rate of any underdemanded type $X-Y$ (i.e., $X \triangleright Y$ and $X \neq Y$) and its reciprocal overdemanded type $Y-X$, satisfy $[\rho(1-\theta)+\theta]p_X\pi_Y \leq p_Y\pi_X$.*

This assumption ensures that the inflow rate of any underdemanded type is greater than the inflow rate of its reciprocal type, who are either incompatible or compatible and willing to use the incentivized-exchange option. This is a simplification. If this is not the case, the excess inflow of paired patients with compatible donors will not wait for exchange, but will instead receive transplants from their donors immediately. As a result, compatible pairs never wait.[15]

We start with an analogue of Theorem 1.

Theorem 3 *Suppose Assumptions 1, 3, and 4 hold. Under incentivized-exchange technology, the following policy maximizes the measure of exchange transplants for pairs that arrive at that instance:*

[15]This assumption also endogenizes ρ to some degree. In a general equilibrium of this model, ρ would be endogenously maximized to match the maximum possible number of underdemanded pairs through exchange, so that if a non-participating compatible pair were to try to participate in incentivized exchange, it would not be able to participate in exchange immediately and have to wait, contradicting equilibrium conditions. Hence, a version of Assumption 4 would hold endogenously.

- *for any self-demanded type, immediately match incompatible pairs of this type with each other, and match each compatible self-demanded type with itself outside of the exchange (so the patients are not prioritized later if they reenter the deceased-donor queue), and*

- *for any underdemanded type or type B-A, match the longest waiting pairs of this type with their reciprocal* incompatible *or* willing compatible *pairs whenever feasible.*

Moreover, this policy maximizes the mass of pairs that can be matched within any closed time interval, and, in particular, matches a larger mass of pairs than waiting for the pairs to arrive and running the exchange once at the end of the time interval.

Before we compare incentivized exchange with regular exchange, let us note two simple observations of the optimal policy. First, no compatible self-demanded pairs participate in incentivized exchange (since almost all incompatible self-demanded pairs can be matched with each other). Therefore, each compatible self-demanded pair will not participate in the incentivized exchange but will be matched with itself (as noted above). Hence, only compatible overdemanded pairs participate. The second observation is that no O reentrants are ever prioritized. This follows from the first observation and the fact that the only type of compatible pair with O patients is O-O. On the other hand, compatible overdemanded pairs with A, B, and AB patients participate in exchange. Therefore, a positive measure of these patients reenters at steady state and gets prioritized.

The following theorem outlines the predictable differences in outcomes between exchange with incentivized compatible pairs and regular exchange.

Theorem 4 (Incentivized exchange and its efficiency and equity consequences) *Suppose Assumptions 1, 3, and 4 hold. At steady state, under the incentivized-exchange technology, with respect to regular exchange, the following hold:*

- *A weakly higher measure of patients is matched for each patient group. Furthermore, underdemanded type pairs are matched with a strictly higher measure.*

- *For underdemanded types, waiting times strictly decrease. For other pair types with waiting times of* 0 *under regular exchange, waiting times do not change. Moreover, if A-B pairs did not receive deceased-donor transplant under regular exchange, their waiting time does not change. For single O patients and nonprioritized single A and B patients, waiting times may decrease or increase. In particular, if some O patients with A, B, and AB live donors receive deceased-donor transplant under regular exchange, then for single O patients, waiting time decreases. If the measure of A-AB and B-AB pairs are sufficiently small, then for nonprioritized single A and B patients, waiting times slightly increase. For nonprioritized AB patients, waiting time slightly increases.*

The proof of this theorem, especially of the second statement, is also of independent interest. It quantifies the conflicting forces that affect waiting times when we switch from the regular exchange to the incentivized exchange.

With real-life parameters, we expect O-A, O-B, and O-AB pairs to receive deceased-donor transplants under regular exchange. Thus, an implication of Theorem 4 is that the waiting time for single O patients decreases under incentivized exchange. In theory, if such pairs do not receive deceased-donor transplants under regular exchange, then incentivized exchange will help only them, directly or indirectly. As a result, increasing O reentrants will cause the single O patient waiting time to increase, but only slightly, as it is a function of the steady-state fraction of the previous recipients from living donors who reenter the deceased-donor queue, ϕ^l.

Similarly, as the number of A-AB and B-AB pairs are expected to be sufficiently small in real life, we expect nonprioritized single A and B patients to wait slightly longer. Indeed, our model predictions in Section 5 are consistent with these theoretical results, although not all parameters used in that section are consistent with the theoretical model.

We also inspect the equity consequences of incentivized exchange in terms of transplant ratios. To this end, we state the marginal measures of transplants (in addition to regular-exchange technology) due to the compatible pair participation. Let $\mathbf{i}_X$ denote the additional measure of $X \in \mathcal{T}$ patients who receive transplants at each instant. For demonstration, consider the measure of additional O patients who receive grafts from donors of compatible pairs. All of these patients are from underdemanded pairs. In particular, the compatible overdemanded pairs with O donors now donate to O patients in underdemanded pairs. The measure of compatible A-O pairs who take up the incentivized exchange option is $\rho(1-\theta)p_O\lambda\pi_A$. Likewise, the measure of compatible B-O and AB-O pairs who participate are $\rho(1-\theta)p_O\lambda\pi_B$ and $\rho(1-\theta)p_O\lambda\pi_{AB}$, respectively. When we sum these terms we get $\mathbf{i}_O$. Similarly, we calculate $\mathbf{i}_X$ for all blood types $X \in \mathcal{T}$:

$$\begin{aligned} \mathbf{i}_O &= \rho(1-\theta)p_O\lambda(\pi_A+\pi_B+\pi_{AB}), \qquad & \mathbf{i}_A &= \rho(1-\theta)p_A\lambda\pi_{AB}, \\ \mathbf{i}_B &= \rho(1-\theta)p_B\lambda\pi_{AB}, \qquad & \mathbf{i}_{AB} &= 0. \end{aligned} \tag{2}$$

Thus, $\mathbf{i}_X/\pi_X$ is the **marginal incentivized-exchange transplant ratio** for blood type X.

We have the following result under a benchmark assumption:

Theorem 5 (Incentivized exchange and decrease of inequity in transplant ratios) *Suppose Assumption 1 holds and live donation rates are equal among blood types, i.e., p_X/π_X is the same among all $X \in \mathcal{T}$. Then, at steady state, incentivized exchange benefits O patients the most, followed by A and B equally, and does not benefit AB patients at all. That is, $0 = \frac{\mathbf{i}_{AB}}{\pi_{AB}} < \frac{\mathbf{i}_A}{\pi_A} = \frac{\mathbf{i}_B}{\pi_B} < \frac{\mathbf{i}_O}{\pi_O}$. Moreover, overall transplant ratios under incentivized exchange, except deceased-donor transplants, satisfy*

$$\frac{\mathbf{l}_O+\mathbf{e}_O+\mathbf{i}_O}{\pi_O} \leq \frac{\mathbf{l}_A+\mathbf{e}_A+\mathbf{i}_A}{\pi_A} = \frac{\mathbf{l}_B+\mathbf{e}_B+\mathbf{i}_B}{\pi_B} \leq \frac{\mathbf{l}_{AB}+\mathbf{e}_{AB}+\mathbf{i}_{AB}}{\pi_{AB}},$$

where weak inequalities all hold with equality if and only if $\rho = 1$.

Thus, incentivized exchange reverses – to some degree – the increasing inequity caused by the previous technologies in waiting times and transplant ratios for O patients. Next, we analyze the effects of incentivized exchange with some simulations.

5 Numerical Model Predictions

In this section, we inspect how our proposal, incentivized exchange, affects the efficiency and equity of transplant policies by calibrating our model with real-life US data statistics as our model parameters. In our view, these numerical model predictions are important as (1) our theorems (for example Theorem 4) do not predict an exact direction of change for some waiting times, (2) some model parameters we took as constant across blood types for simplicity are variable in real life, and (3) the magnitudes of changes could not be predicted in our theoretical results. These results are not statistical in nature; instead, they are numerical calibrations of theorems in the main text and appendices. First, we explain our calibration parameters.

In Table 2, survival rates, $1 - F(t)$, are listed using US Renal Data System (USRDS) data for dialysis patients.[16] We fit an exponential duration curve (for t measured in years) as

$$\hat{F}(t) = 1 - \hat{a}e^{\hat{b}t} \tag{3}$$

and obtain estimates of $\hat{a} = 0.9427$ with a 95 percent confidence interval of $(0.8945, 0.9909)$ and $\hat{b} = -0.1667$ with a 95 percent confidence interval of $(-0.1922, -0.1411)$ using non-linear least squares.

	Data				
Time:	3 mo.	1 yr.	2 yr.	3 yr.	5 yr.
On dialysis	0.9215	0.7824	0.6648	0.5694	0.4230

Table 2: Survival rates $(1 - F(t))$ for kidney failures in the US for 2009 entrants.

We use the US OPTN kidney data for the year 2009 (see Table 3).[17] We estimate other backbone parameters, such as inflow rates for blood types, $\{\pi_X\}$, from the data in Table 3 using our model (see Table 4). We also uncover the paired donor rate λ based on unobserved intended live donations that did not materialize. For example, an O-A pair is not detectable from the data, as the A donor could not donate to O, and hence, there is no recorded evidence for the existence of such pairs. We estimate these parameters under two different assumptions on the inflow rate of patients. In particular, we derive a lower-bound and an upper-bound on inflow rate for each blood type, which

[16] These 2009 estimates for dialysis patients are reported in the National Kidney Organization 2016 Annual Report Chapter 6 (retrieved from `https://www.usrds.org/2016/download/v2_c06_Mortality_16.pdf` on 05/12/2017) vol 2 Table 6.3, obtained from hemodialysis and peritoneal dialysis survival rates and weighted by average 2009-2014 percentages of patients on hemodialysis vs peritoneal dialysis reported in vol 2 Figure 1.2 of Chapter 1 (retrieved from `https://www.usrds.org/2016/download/v2_c01_IncPrev_16.pdf` on 05/12/2017).

[17] Year 2009 is used because this is the latest year for which five-year dialysis survival rates are available as of May, 2017. Data is obtained from OPTN using the "national data" option from `http://optn.transplant.hrsa.gov` on 05/15/2017. Median years is the norm for reporting time averages, as all patients should exit the queue for mean calculation, while 50 percent of patients should exit the queue for median. 2003-2004 entrants is the cohort used for median time for transplant data (using data available on 10/12/2012). Although we assume in our model that inflow and donor rates, as well as the survival function $1-F$ do not change over time, these are all growing, at different rates, due to population characteristics and medical advances in dialysis. To capture the changing underlying parameters, we repeated the same calculations for different years' cohorts. We report the results for the latest year for which we can calibrate. See also Footnote 20.

are abbreviated as "Est. Low"/"Est. High", and use them in our policy experiments. We assume that no/all pairs that participated in exchange arrived in 2009 for these bounds, respectively.

Data					
Blood Types:	**O**	**A**	**B**	**AB**	**Total**
Total Additions to the Queue	16,323	11,090	4,919	1325	33,657
Living Donation Recipients on Queue	2,446	2,016	717	209	6,388
Total Living Donation Recipients	2,878	2,422	844	244	5,388
Exchange Participants	128	96	58	8	290
Deceased Donation Recipients (σ_X)	4,726	3,818	1,347	554	10,442
Reentrants	2,062	1,513	580	198	4,353
Deceased Donors	3,458	2,722	850	218	7,248
Median Years To Transplant (2003-04 Entrants)	5.07	3.31	5.30	2.34	

Table 3: Arrivals to and transplants from the kidney deceased-donor queue for 2009 entrants. 7,248 deceased donors with 10,442 transplants leads to an average harvest rate of 1.441 kidneys per deceased donor. Living donor blood types are assumed to come from the distribution in Table 1.

Estimates					
Blood Types:	**O**	**A**	**B**	**AB**	**Total**
Deceased Donor Organ Inflow Rate (δ_X)	4,982	3,922	1225	314	10,442
Reentry Rate (ϕ)	27.12%	24.26%	26.47%	24.81%	25.86%
Entrant Inflow Rate (π_X) (Est. Low)	14,565	9,887	4,408	1,154	30,014
(Est. High)	14,693	9,983	4,466	1,162	30,304
Paired Donor Rate (λ) (Est. Low)	39.55%	26.87%	30.75%	20.35%	33.34%
(Est. High)	37.15%	25.34%	27.89%	19.44%	31.22%
Normalized Live Donor ($\frac{p_X}{\pi_X/(\sum_Y \pi_Y)}$) (Est. Low)	0.9398	1.1474	0.8572	1.0427	
to Patient Ratio (Est. High)	0.9406	1.1474	0.8543	1.0455	
Normalized Deceased Donor ($\frac{\delta_X/(\sum_Y \delta_Y)}{\pi_X/(\sum_Y \pi_Y)}$) (Est. Low)	0.9832	1.1401	0.7985	0.7823	
to Patient Ratio (Est. High)	0.9840	1.1400	0.7958	0.7844	

Table 4: Estimates regarding parameters, assuming that our model would have generated the data in Table 3. Deceased-donor numbers are reported for each blood type separately, but not the actual number of grafts transplanted. Using the empirical fact that on average 1.441 kidneys are harvested from each deceased donor, we found the number of deceased-donor grafts available for each blood type. "Est. Low"/"Est. High" reflect the estimates under the assumptions that no/all pairs that participated in exchange arrived in 2009.

One immediate observation from the estimates in last two rows of Table 4 is that B and AB blood types get end-stage renal disease more often. Minorities are known to be more prone to kidney disease. Moreover, B blood protein is more common among minorities such as African-Americans and Asian-Americans (See Table 1). Thus, this finding is not very surprising (as predicted in Section 2.1). Moreover, when the previous two rows in Table 4 are inspected, we observe that the assumed donation rates for live donors are more balanced (i.e., closer to 1 for B and AB) than those for deceased donors. Hence, the constant p_X/π_X assumption used in Theorems 2, 5, and 8 has some validity in the US, although it is not perfect.

We estimate the waiting time for each group of patients by using the formula $\hat{t} = \hat{F}^{-1}(1 - \min\{1, \Delta/\Pi\})$. Here, Δ is the measure of donors/pairs matched with the group of patients inspected, and Π is the measure of the entry cohort of the group of patients inspected. We implicitly use Lemmata 3-6 given in Appendix C to conclude that when $\Delta < \Pi$, almost all Δ measure of the donor/pair group examined in the numerator at time $\hat{t}$ can be matched with a Δ measure of the patient/pair group examined in the denominator (among total measure Π). When $\Pi \geq \Delta$, almost all patients are matched without waiting by the same results, implying $\hat{t} = 0$. The fitted $\hat{F}$ formula is given in Equation 3. We find each relevant Δ and Π through our analysis at steady state in Appendix B.

Deceased-donor allocation is done on a more regional than national basis. Moreover, a graft can easily go bad if a suitable patient is not found in time. Thus, in practice, it turns out that on many occasions AB patients benefit and receive transplants from other blood types (the same observation goes for A and B patients, who receive from O deceased donors more often than necessary). Hence, in the OPTN guidelines, the ultimate decision is left to the physician, although ABO-i is practiced first, especially for B and O.[18] We refer to this actual allocation policy as **de facto deceased allocation**. Since the AB blood type is seen in only $3 - 4$ percent of the population, even a few violations of FIFO cause dramatic decreases in AB patients' waiting time. Hence, in our policy discussion we will mostly ignore AB and focus on A, B, and O. We use σ_X in Table 3 instead of δ_X (found in Table 4) for calculating "de facto" predictions.

A second observation in Table 3 is noteworthy. As actual waiting times differ substantially across blood types, certain blood-type patients appear to be "looking for" paired donors more intensely than others, and possibly they find donors after they join the pool. In our model, the paired donor rate λ is constant for all blood types, and a pair is formed (or not) as soon as the patient joins the queue. However, in the data λ is different across blood types: $19.44 - 20.35$ percent for AB, $25.34 - 26.87$ percent for A, $27.89 - 30.75$ percent for B, and $37.15 - 39.55$ percent for O. As we know, the O blood type is at a disadvantage; it looks like they try hardest to find a compatible paired donor while waiting in the queue.[19] We use these rates for each blood type in what follows.

Next, we discuss our numerical predictions. Table 5 reflects the predicted number of patients receiving transplants for each blood type. In terms of both the efficiency and equity consequences of the step-wise changes across the 4 technologies, from deceased-donor transplantation to living-donor transplantation, from living-donor transplantation to regular exchange, and finally from regular exchange to incentivized exchange with $\rho = 100$ percent participation, we observe the following: O patients are predicted to experience 46.88 percent, 6.84 percent, and 21.90 percent increases in number of transplants, respectively. These numbers are 47.90 percent, 20.15 percent, and 1.06 percent for B; 49.23 percent, 9.84 percent and 1.21 percent for A (and 36.28 percent, 2.29 percent, and 0.00 percent for AB), respectively. Thus:

[18] See page 82 of the OPTN kidney allocation guidelines retrieved from `https://optn.transplant.hrsa.gov/media/1200/optn_policies.pdf#nameddest=Policy_08` on 06/05/2017.

[19] Also cultural issues, such as family composition among different ethnic groups, can play a role in paired-donor rates. This contributes to the observed disparity. For example, consider the B blood type. Although its waiting time is as long as O's or even longer, its patients' pairing rate is not as high.

Model Predictions – Est. High: Patients Receiving Transplants (in numbers)				
Blood Types:	**O**	**A**	**B**	**AB**
Total Living Donor Transplants				
Living Donor Transplantation	2,215.67 (15.08%)	1,878.04 (18.81%)	645.21 (14.45%)	201.00 (17.30%)
Regular Exchange ($\rho = 0\%$)	2,690.26 (18.31%)	2,438.08 (24.42%)	1,046.61 (23.44%)	225.84 (19.44%)
Incentivized Exchange with $\rho = 25\%$	3,096.31 (21.07%)	2,457.07 (24.61%)	1,052.94 (23.58%)	225.84 (19.44%)
Incentivized Exchange with $\rho = 50\%$	3,502.36 (23.84%)	2,476.06 (24.80%)	1,059.27 (23.72%)	225.84 (19.44%)
Incentivized Exchange with $\rho = 100\%$	4,314.45 (29.36%)	2,514.05 (25.18%)	1,071.92 (24.00%)	225.84 (19.44%)
Deceased-Donor Transplants				
De Facto Deceased-Donor Allocation	4,726.00 (32.16%)	3,815.00 (38.21%)	1,347.00 (30.16%)	554.00 (47.68%)
ABO-i	4,981.85 (33.91%)	3,921.51 (39.28%)	1,224.57 (27.42%)	314.07 (27.03%)

Table 5: Model predictions for the number of patients estimated to receive transplants under various policies. The percentages in parentheses are with respect to inflow rates π_X. Deceased-donor transplants are the same for a given allocation method, de facto or ABO-i, for all policies.

- A patients are predicted to experience the highest gain from deceased-donor transplantation to living-donor transplantation, and AB patients the lowest,
- B patients are predicted to experience the highest gain from living-donor transplantation to regular exchange, and AB and O patients the lowest, and
- O patients are predicted to experience the highest gain from regular exchange to incentivized exchange, and AB patients the lowest.

Observe that these estimates are consistent with our earlier predictions, even though our theory does not assume any heterogeneity among behavioral and medical characteristics of different blood-type patients and donors, as the data reflect (cf. Theorems 2, 4, 5, and 8).

Model Predictions – Est. High: Average Time to Transplant (in years)				
Blood Types:	**O**	**A**	**B**	**AB**
Deceased-Donor Transplantation				
De Facto Deceased-Donor Allocation	6.95	5.95	7.30	4.76
ABO-i	6.66	5.80	7.83	7.88
Living Donor Transplantation				
De Facto Deceased-Donor Allocation	4.19	3.29	4.43	2.85
ABO-i	4.07	3.22	4.63	4.25
Regular Exchange ($\rho = 0\%$)				
De Facto Deceased-Donor Allocation	3.99	2.95	3.66	2.78
ABO-i	3.87	2.86	3.81	4.07

Model Predictions – Est. High: Average Time to Transplant (in years)				
Blood Types:	**O**	**A**	**B**	**AB**
Incentivized Exchange with $\rho = 25\%$				
De Facto Deceased-Donor Allocation	3.80	2.95	3.65	2.81
ABO-i	3.68	2.88	3.81	4.05
Incentivized Exchange with $\rho = 50\%$				
De Facto Deceased-Donor Allocation	3.64	2.95	3.64	2.84
ABO-i	3.52	2.90	3.81	4.02
Incentivized Exchange with $\rho = 100\%$				
De Facto Deceased-Donor Allocation	3.15	2.93	3.63	2.89
ABO-i	3.05	2.87	3.74	3.94

Table 6: Model predictions reflecting *average waiting time* to a transplant using the Est. High model discussed. These are the average waiting-time estimates for all patients who receive transplants, including those who receive (1) transplants immediately through exchange, direct live donation, or prioritized deceased donation, (2) living-donor transplants through exchange after waiting some time, and (3) deceased-donor transplants after waiting in the deceased-donor queue under different technologies.

In terms of average waiting times for any transplant (for de facto), we observe that each new technology decreases the average waiting time for O patients from 6.95 years under deceased-donor

Model Predictions – Est. High: Time to Deceased-Donor Transplant or Median Time to Transplant (in years)				
Blood Types:	**O**	**A**	**B**	**AB**
Deceased-Donor Transplantation				
De Facto Deceased-Donor Allocation	6.95	5.95	7.30	4.76
ABO-i	6.66	5.80	7.83	7.88
Living Donor Transplantation				
De Facto Deceased-Donor Allocation	6.16	4.91	6.55	3.89
ABO-i	5.87	4.76	7.08	6.97
Regular Exchange ($\rho = 0\%$)				
De Facto Deceased-Donor Allocation	6.00	4.83	6.01	3.91
ABO-i	5.73	4.68	6.60	7.00

Model Predictions – Est. High: Time to Nonprioritized Deceased-Donor Transplant or Median Time to Transplant (in years)				
Blood Types:	**O**	**A**	**B**	**AB**
Incentivized Exchange with $\rho = 25\%$				
De Facto Deceased-Donor Allocation	5.57	4.90	6.14	4.04
ABO-i	5.32	4.75	6.73	7.23
Incentivized Exchange with $\rho = 50\%$				
De Facto Deceased-Donor Allocation	5.22	4.98	6.26	4.17
ABO-i	4.98	4.82	6.87	7.47
Incentivized Exchange with $\rho = 100\%$				
De Facto Deceased-Donor Allocation	5.07	5.19	6.65	4.46
ABO-i	4.79	5.03	7.24	7.99

Table 7: Model predictions reflecting *waiting times for deceased-donor transplants.* These are the waiting time estimates for patients who receive deceased-donor transplants under different technologies. The prioritized reentrants are *excluded* from the calculation for incentivized exchange. These are also the *median waiting times for all types of transplants.*

Model Predictions - Est High: Time to Transplant for Blood-Type-Incompatible Pairs When Compatible Pairs are Prioritized (in years)							
% of Comp.	**Pair Types**						
Pairs in	**O – A**	**O – B**	**O – AB**	**A – B**	**A – AB**	**B – A**	**B – AB**
$\rho = 0\%$	pooled w **O**	pooled w **O**	pooled w **O**	0	pooled w **A**	1.99	pooled w **B**
$\rho = 25\%$	pooled w **O**	pooled w **O**	pooled w **O**	0	pooled w **A**	1.99	pooled w **B**
$\rho = 50\%$	pooled w **O**	4.32	pooled w **O**	0	4.21	1.99	pooled w **B**
$\rho = 100\%$	3.13	0.79	4.17	0	0.68	1.99	3.02

Table 8: Model predictions reflecting *regular and incentivized-exchange waiting times for pairs* for transplant. Upon reentry, the patient of a compatible $X, -Y$ pair participating in exchange receives an X deceased-donor kidney. Here "pooled" means that some $X - Y$ pairs receive deceased-donor transplants along with single nonprioritized X patients while some other $X - Y$ pairs simultaneously participate in exchange. Their waiting times are reflected in Table 7 in the columns regarding the X blood type.

transplantation to 4.19 with living-donor transplantation, to 3.99 years with regular exchange.[20] It further falls to 3.80, 3.64, and 3.15 years with incentivized exchange with $\rho = 25$ percent, $\rho = 50$ percent, and $\rho = 100$ percent, respectively. Most importantly, we observe that with increasing ρ, average waiting time for all blood types decreases, except AB, for which waiting time slightly increases. In terms of equity, we have the following consequences: Once regular exchange policy is available, the B patient waiting time is 0.71 years longer than that of A patients, which is our

[20] As shown in Table 3, because of the de facto deceased-allocation policy, the actual *median* waiting time for a live-donor or deceased-donor transplant for AB is the shortest (2.34 years), while B is the longest, but very close to O (5.30 versus 5.07 years). Furthermore, A's waiting time is less than these two blood types (at 3.31 years). Moreover, our model predictions using the de facto deceased-allocation regime are for *averages* of 2.85 years for AB, 3.29 years for A, 4.19 years for O, and 4.43 years for B (Table 6, left pane, middle row for de facto, i.e., under the living donor transplantation modality) while the *medians* are 3.89 years for AB, 4.91 for A, 6.16 for O, and 6.55 for B for 2009 entrants. The median transplant is always a deceased-donor transplant in our model, simply because there are more deceased-donor transplants than live-donor transplants, and living donation never occurs later than deceased donation. These times are given in Table 7 (left middle row for de facto). We also did the same calculations for the 2003 cohort using both 2003/2009 survival function estimates and obtained the median waiting times as 2.52/3.26 years for AB, 3.25/4.16 for A, 4.13/5.26 for O and 4.87/6.18 for B, respectively. Actual median time for O is closer to the higher calculation while actual median times for other blood types are closer to the lower calculations.

reference point. The gap between O and A is 1.03 years on the other hand. These numbers are 0.91 years for B and 1.14 years for O under living-donor transplantation policy. Under incentivized-exchange policy, O starts to close the gap with A. The difference falls to 0.67 years with $\rho = 50$ percent and then to 0.22 years with $\rho = 100$ percent (cf. Theorems 4 and 5).

Both A and B have prioritized reentrants who receive a deceased-donor kidney as soon as they enter. As a result, the waiting times for nonprioritized deceased donation increase for both A and B under incentivized exchange (see Table 7's rows for "de facto allocation"). These are consistent with our predictions of Theorem 4.

As expected, increasing compatible-pair participation under incentivized exchange decreases waiting times of underdemanded pair types substantially (see Table 8): as $\rho = 100$ percent they no longer receive any deceased-donor transplants but they participate only in exchange with their reciprocal types.

6 Conclusion

Over the last decade, living-donor organ exchange has emerged as an important transplantation technology. While the analysis of the efficiency and equity implications of individual technologies has been an important focus for researchers, doctors, and health care policymakers, there has been no study to date that assesses the interaction between them and their collective implications. As the share of transplants from living-donor transplantation and organ exchange increased, a need for a model that studies the interaction of these technologies as well as their collective implications has arisen. Our model is a first attempt to fill this gap in the literature. Using this model, we analyze the welfare and equity implications of existing organ transplantation technologies, shedding light on the empirical patterns that are observed in practice.

Our final, and perhaps major, contribution is the introduction of a new policy that has the potential not only to increase overall patient welfare, but also to decrease waiting time inequity across patients of different blood types. Currently compatible pairs very rarely participate in organ exchange, and their reluctance to do so results in considerable welfare loss. To incentivize the participation of compatible pairs in exchange, our policy prioritizes patients of such pairs at the deceased-donor queue for possible future organ failures of the transplanted organ. This policy has the potential to decrease inequity across various patient populations while at the same time increasing the overall welfare.

References

Akbarpour, Mohammad, Shengwu Li, and Shayan Oveis-Gharan (2017). "Thickness and information in dynamic matching markets." Working paper.

Anderson, Ross, Itai Ashlagi, David Gamarnik, and Yash Kanoria (2017). "Efficient dynamic barter exchange." *Operations Research*, forthcoming.

Ashlagi, Itai and Alvin E. Roth (2014). "Free riding and participation in large scale, multi-hospital kidney exchange." *Theoretical Economics*, 9 (3), 817–863.

Azevedo, Eduardo M. and Eric Budish (2012). "Strategyproofness in the large." Working paper.

Azevedo, Eduardo M and Jacob D Leshno (2016). "A supply and demand framework for two-sided matching markets." *Journal of Political Economy*, 124 (5), 1235–1268.

Baccara, Mariagiovanna, SangMok Lee, and Leeat Yariv (2016). "Optimal dynamic matching." Working paper.

Che, Yeon-Koo and Fuhito Kojima (2010). "Asymptotic equivalence of probabilistic serial and random priority mechanisms." *Econometrica*, 78 (5), 1625–1672.

Dickerson, John P, Ariel D Procaccia, and Tuomas Sandholm (2012). "Dynamic matching via weighted myopia with application to kidney exchange." *AAAI*.

Duda, Lara (2005). "National organ allocation policy: the final rule." *Virtual Mentor*, 7 (9).

Ferrari, Paolo, Linda Cantwell, Joseph Ta, et al. (2017). "Providing better-matched donors for HLA mismatched compatible pairs through kidney paired donation." *Transplantation*, 101 (3).

Gale, David (1957). "A theorem on flows in networks." *Pacific Journal of Mathematics*, 7 (2), 1073–1082.

Gentry, S. E., D. L. Segev, M. Simmerling, and R. A. Montgomery (2007). "Expanding kidney paired donation through participation by compatible pairs." *American Journal of Transplantation*, 7 (10), 2361–2370.

Kojima, Fuhito and Parag A. Pathak (2009). "Incentives and stability in large two-sided matching markets." *American Economic Review*, 99 (3), 608–27.

Kojima, Fuhito, Parag A. Pathak, and Alvin E. Roth (2013). "Matching with couples: stability and incentives in large markets*." *The Quarterly Journal of Economics*, 128 (4), 1585–1632.

Kranenburg, Leonieke W., Willij Zuidema, Willem Weimar, et al. (2006). "One donor, two transplants: willingness to participate in altruistically unbalanced exchange donation." *Transplant International*, 19 (12), 995–999.

Lee, SangMok (2011). "Incentive compatibility of large centralized matching markets." Working paper.

Liu, Qingmin and Marek Pycia (2013). "Ordinal efficiency, fairness, and incentives in large markets." Working paper.

Nicolò, Antonio and Carmelo Rodríguez-Álvarez (2017). "Age-based preferences in paired kidney exchange." *Games and Economic Behavior*, 102, 508–524.

Ratner, Lloyd E., Abbas Rana, Emily R. Ratner, et al. (2010). "The altruistic unbalanced paired kidney exchange: proof of concept and survey of potential donor and recipient attitudes." *Transplantation*, 89, 15–22.

Ross, Laine Friedman and E. Steve Woodle (2000). "Ethical issues in increasing living kidney donations by expanding kidney paired exchange programs." *Transplantation*, 69, 1539–1543.

Ross, Lainie Friedman (2006). "The ethical limits in expanding living donor transplantation." *Kennedy Institute of Ethics Journal*, 16 (2), 151–172.

Roth, Alvin E., Tayfun Sönmez, and M. Utku Ünver (2004). "Kidney exchange." *The Quarterly Journal of Economics*, 119 (2), 457–488.

Roth, Alvin E., Tayfun Sönmez, and M. Utku Ünver (2005a). "A kidney exchange clearinghouse in New England." English. *The American Economic Review*, 95 (2), pp. 376–380.

Roth, Alvin E., Tayfun Sönmez, and M. Utku Ünver (2005b). "Pairwise kidney exchange." *Journal of Economic Theory*, 125 (2), 151 –188.

Roth, Alvin E., Tayfun Sönmez, and M. Utku Ünver (2007). "Efficient kidney exchange: coincidence of wants in markets with compatibility-based preferences." English. *The American Economic Review*, 97 (3), pp. 828–851.

Segev, Dorry, Sommer Gentry, Daniel S. Warren, Brigette Reeb, and Robert A. Montgomery (2005). "Kidney paired donation: optimizing the use of live donor organs." *Journal of the American Medical Association*, 293, 1883–1890.

Sönmez, Tayfun and M. Utku Ünver (2014). "Altruistically unbalanced kidney exchange." *Journal of Economic Theory*, 152. Working paper, 105–129.

Sönmez, Tayfun and M. Utku Ünver (2015). "Enhancing the efficiency of and equity in transplant organ allocation via incentivized exchange." Available as Boston College Working Paper 868.

Steinberg, David (2011). "Compatible-incompatible live donor kidney exchanges." *Transplantation*, 91 (3).

Ünver, M. Utku (2010). "Dynamic kidney exchange." *The Review of Economic Studies*, 77 (1), 372–414.

Veale, Jeffrey L., Alexander M. Capron, Nima Nassiri, et al. (2017). "Vouchers for future kidney transplants to overcome 'chronological incompatibility' between living donors and recipients." *Transplantation*, Online First.

Veatch, R. M. (2006). "Organ exchanges: fairness to the O-blood group." *American Journal of Transplantation*, 6 (1), 1–2.

Symbol	Meaning
p_X	probability of having blood-type X in the population
π_X	inflow rate of new blood-type X patients
θ	the expected tissue rejection probability
δ_X	inflow rate of blood-type X deceased donors
ϕ^d	steady-state fraction of the previous recipients from deceased donors who reenter the deceased-donor queue
λ	fraction of incoming patients with donors
ϕ^l	steady-state fraction of the previous recipients from living donors who reenter the deceased-donor queue
ρ	fraction of compatible pairs that take up the incentivized-exchange option
$\mathbf{l}_X$	rate of X patients receiving direct donation when $\rho = 0$ (i.e., inflow rate of X patients with compatible donors)
$\mathbf{e}_X$	rate of X patients matched through exchange when $\rho = 0$
$\mathbf{i}_X$	marginal rate of transplants for X patients under incentivized exchange beyond regular exchange (as a function of ρ)

Table 9: Notation for Main Text

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Appendix A Remaining Proofs

The following lemma is useful in the proof of Theorem 1. Similar results also appear in Roth, Sönmez, and Ünver (2007) and Ünver (2010), so we skip its proof.

Lemma 1 (Exchange blood-type feasibility) *An underdemanded type pair can be matched only with an overdemanded type pair in an exchange. An overdemanded type pair can be matched with an overdemanded, underdemanded, reciprocally demanded, or self-demanded type pair. A reciprocally demanded type pair can be matched with a (reciprocal of its type) reciprocally demanded or overdemanded type pair. A self-demanded type pair can be matched with a same type or overdemanded type pair. In particular, the following holds:*

- *An underdemanded O-A (or O-B) pair can be matched only with a pair from overdemanded types A-O (or B-O) or AB-O. An underdemanded A-AB (or B-AB) pair can be matched only with a pair from overdemanded types AB-A (or AB-B) or AB-O. An underdemanded O-AB pair can be matched only with an overdemanded AB-O pair.*

- *A reciprocally demanded A-B (or B-A) pair can be matched only with a pair from the other reciprocally demanded type B-A (or A-B); or overdemanded types B-O (or A-O), AB-A (or AB-B), or AB-O.*

- *A self-demanded $X - X$ pair can be matched with a same type pair. Additionally, an O-O pair can be matched only with a pair from overdemanded types $A - O, B - O$, or AB-O; an*

A-A (or B-B) pair can be matched only with a pair from overdemanded types AB-A (or AB-B) or AB-O; and an AB-AB pair can be matched only with a pair from overdemanded types $AB-A, AB-B$, or AB-O.

Proof of Theorem 1. Under the proposed policy, by Lemma 6 in Appendix C, all self-demanded pairs can be matched with their own type pairs as soon as they arrive, and all pairs of type B-A that have a lower inflow rate than A-B pairs (by Assumption 3) can be matched as soon as they arrive with their reciprocal-type pairs (Lemma 4 in Appendix C). Hence, under this policy only A-B pairs will remain in the exchange pool at any point in time. These pairs can only be matched with overdemanded pairs by Lemma 1, as B-A pairs are already committed to other A-B pairs.

Next consider underdemanded type pairs. These are $Y-X$ type pairs such that $Y \neq X$ and $Y \triangleright X$. By Assumption 2, we have $\theta p_Y \pi_X \leq p_X \pi_Y$. By Lemma 1, they can only be matched with overdemanded types. Recall that the inflow rate of each $Y-X$ type pair to the exchange pool is $p_X \lambda \pi_Y$. Their reciprocal type $X-Y$, which is overdemanded, has the inflow rate $\theta p_Y \lambda \pi_X \leq p_X \lambda \pi_Y$. Hence, we can match all such overdemanded pairs $X-Y$ as soon as they enter the pool with their reciprocal type pairs (by Lemma 5 in Appendix C). As all overdemanded, self-demanded, and type B-A reciprocally demanded pairs are matched as soon as they arrive, by Lemma 1, the proposed policy achieves the maximum measure of pairs matched. At steady state, as no incompatible overdemanded, self-demanded, and B-A type pair waits in the pool, gets immediately matched, and saves one additional pair, the maximum mass of possible exchanges is also conducted in this manner in any closed time interval.

On the other hand, if we do not conduct the exchanges immediately whenever they become available, but only after a closed time interval, then some of the patients of overdemanded, self-demanded, and B-A type pairs who have arrived earlier will not survive. Hence, when we conduct the exchanges at the end of the time interval, we will match a strictly smaller mass of possible pairs than we would have matched under the proposed policy. ■

Proof of Theorem 2. Consider $\{\mathbf{e}_X\}_{X \in \mathcal{T}}$, the overall measures of pairs with X blood type participating in exchange for each $X \in \mathcal{T}$ reported in Equation 1. Observe that

$$\begin{aligned}
\frac{\mathbf{e}_O}{\pi_O} =& \theta p_O \lambda + \theta p_O \lambda \frac{\pi_A + \pi_B + \pi_{AB}}{\pi_O} = \theta(p_O + p_A + p_B + p_{AB})\lambda \\
\frac{\mathbf{e}_A}{\pi_A} =& \theta p_A \lambda + \theta p_O \lambda + p_A \lambda \frac{\pi_B}{\pi_A} + \theta p_A \lambda \frac{\pi_{AB}}{\pi_A} = (\theta p_O + \theta p_A + p_B + \theta p_{AB})\lambda \\
\frac{\mathbf{e}_B}{\pi_B} =& \theta p_B \lambda + \theta p_O \lambda + p_A \lambda + \theta p_B \lambda \frac{\pi_{AB}}{\pi_B} = (\theta p_O + p_A + \theta p_B + \theta p_{AB})\lambda \\
\frac{\mathbf{e}_{AB}}{\pi_{AB}} =& \theta(p_{AB} + p_O + p_A + p_B)\lambda
\end{aligned}$$

where the second equality in each line (except the last) follows from the assumption that p_X/π_X is a constant among all $X \in \mathcal{T}$. Since $\theta < 1$, we have $\mathbf{e}_O/\pi_O = \mathbf{e}_{AB}/\pi_{AB} < \mathbf{e}_A/\pi_A, \mathbf{e}_B/\pi_B$. With the additional assumption $p_A > p_B$, we obtain $\mathbf{e}_A/\pi_A < \mathbf{e}_B/\pi_B$.

Now, recall that $\mathbf{l}_X$ is the flow rate of blood-type X patients with compatible donors. We can write them out as follows: $\mathbf{l}_O = (1-\theta)p_O \lambda \pi_O$, $\mathbf{l}_A = (1-\theta)(p_O + p_A)\lambda \pi_A$, $\mathbf{l}_B = (1-\theta)(p_O + p_B)\lambda \pi_B$,

and $\mathbf{l}_{AB} = (1-\theta)(p_O + p_B + p_A + p_{AB})\lambda\pi_{AB}$. Next consider $\{\mathbf{l}_X + \mathbf{e}_X\}_{X\in\mathcal{T}}$, direct living-donor and exchange transplants. We have

$$\begin{aligned}
\frac{\mathbf{l}_O + \mathbf{e}_O}{\pi_O} &= (1-\theta)p_O\lambda + \theta(p_O + p_A + p_B + p_{AB})\lambda = (p_O + \theta p_A + \theta p_B + \theta p_{AB})\lambda \\
\frac{\mathbf{l}_A + \mathbf{e}_A}{\pi_A} &= (1-\theta)(p_O + p_A)\lambda + (\theta p_O + \theta p_A + p_B + \theta p_{AB}) = (p_O + p_A + p_B + \theta p_{AB})\lambda \\
\frac{\mathbf{l}_B + \mathbf{e}_B}{\pi_B} &= (1-\theta)(p_O + p_B)\lambda + (\theta p_O + p_A + \theta p_B + \theta p_{AB})\lambda = (p_O + p_A + p_B + \theta p_{AB})\lambda \\
\frac{\mathbf{l}_{AB} + \mathbf{e}_{AB}}{\pi_{AB}} &= (1-\theta)(p_{AB} + p_O + p_A + p_B)\lambda + \theta(p_{AB} + p_O + p_A + p_B)\lambda = (p_{AB} + p_O + p_A + p_B)\lambda
\end{aligned}$$

Since $\theta < 1$, we have, $\frac{\mathbf{l}_O+\mathbf{e}_O}{\pi_O} < \frac{\mathbf{l}_A+\mathbf{e}_A}{\pi_A} = \frac{\mathbf{l}_B+\mathbf{e}_B}{\pi_B} < \frac{\mathbf{l}_{AB}+\mathbf{e}_{AB}}{\pi_{AB}}$. ■

Proof of Theorem 3. Using Assumption 4 instead of Assumption 2, the proof follows verbatim the proof of Theorem 1, after noting that no self-demanded type can be used to save additional underdemanded or A-B type pairs. ■

Proof of Theorem 4. Let $\psi^{\mathbf{i}}$ be the optimal policy explained in Theorem 3 under incentivized exchange, and $\varphi^{\mathbf{e}}$ be the optimal policy explained in Theorem 1 under regular exchange. Recall that any reentrant is classified as a single patient. Under $\psi^{\mathbf{i}}$, no unwilling compatible pairs and compatible self-demanded pairs participate in exchange. And under $\varphi^{\mathbf{e}}$, no compatible pairs participate in exchange. Such compatible pairs' patients immediately receive transplants from their paired donors. All overdemanded type pairs are matched through exchange with their reciprocal types under both $\psi^{\mathbf{i}}$ and $\varphi^{\mathbf{e}}$ upon entry immediately (by Assumption 4). We prove the first statement and then the second.

Proof of the First Part: First consider underdemanded pairs. Suppose that an underdemanded $X-Y$ pair type is not pooled with blood-type X single patients for deceased donation under $\varphi^{\mathbf{e}}$. Under $\psi^{\mathbf{i}}$, that type of pair is matched at the rate

$$\pi^{\mathbf{i}}_{Y-X} = [\rho(1-\theta) + \theta]p_X\lambda\pi_Y, \tag{4}$$

i.e., the inflow rate of Y-X pairs for incentivized exchange. At each point in time while under $\varphi^{\mathbf{e}}$, they are matched at the rate

$$\pi^{\mathbf{e}}_{Y-X} = \theta p_X\lambda\pi_Y, \tag{5}$$

which is strictly smaller.

Next, suppose that pair types $X_1-Y_1, \ldots, X_\ell-Y_\ell$ are pooled together for deceased donation, and suppose that among these pair types, $X_{\ell^*}-Y_{\ell^*}$ is underdemanded. Note that all of these pair types are either underdemanded or A-B. Each X_k-Y_k is matched at the rate $\pi^{\mathbf{e}}_{X_k-Y_k} + \varepsilon^{\mathbf{e}}_{X_k-Y_k}$ under $\varphi^{\mathbf{e}}$, where the rate $\varepsilon^{\mathbf{e}}_{X_k-Y_k} > 0$ is the measure of X_k-Y_k pairs whose patients receive deceased donation and $\pi^{\mathbf{e}}_{X_k-Y_k}$ is defined as in Equation 5. Under $\psi^{\mathbf{i}}$, $\pi^{\mathbf{i}}_{Y_k-X_k}$ is the measure of the reciprocal X_k-Y_k pairs (who are on high demand) willing to participate in exchange, which is strictly larger than $\pi^{\mathbf{e}}_{Y_k-X_k}$, while the rate of deceased donation does not change. Hence, while $\pi^{\mathbf{i}}_{Y_k-X_k} - \pi^{\mathbf{e}}_{Y_k-X_k}$ more

$X_k - Y_k$ pairs participate in exchange under $\psi^{\mathbf{i}}$, fewer such pairs *may* receive deceased donation. Suppose that $\varepsilon^{\mathbf{i}}_{X_k-Y_k}$ is the rate of $X_k - Y_k$ pairs receiving deceased donation under $\psi^{\mathbf{i}}$. We will show that $\mathbf{i}_{X_k-Y_k} = [\pi^{\mathbf{i}}_{Y_k-X_k} + \varepsilon^{\mathbf{i}}_{X_k-Y_k}] - [\pi^{\mathbf{e}}_{Y_k-X_k} + \varepsilon^{\mathbf{e}}_{X_k-Y_k}] > 0$ for all k. Suppose not for some k. In particular, if there are multiple such k, let k be chosen with the smallest $\mathbf{i}_{X_k-Y_k} \leq 0$. Hence, as waiting time of all pairs $X_1 - Y_1, \ldots, X_\ell - Y_\ell$ is the same, under $\varphi^{\mathbf{e}}$, $X_k - Y_k$'s waiting time increases the most among all pairs or stays the same and no other pair's waiting time increases under $\psi^{\mathbf{i}}$. Hence, $X_k - Y_k$ continues to be pooled with X_k single patients under $\psi^{\mathbf{i}}$. As $\pi^{\mathbf{i}}_{Y_{\ell^*}-X_{\ell^*}} - \pi^{\mathbf{e}}_{Y_{\ell^*}-X_{\ell^*}} > 0$, and for all $k^* \neq \ell^*$, we have, $\pi^{\mathbf{i}}_{Y_{k^*}-X_{k^*}} - \pi^{\mathbf{e}}_{Y_{k^*}-X_{k^*}} \geq 0$, then a higher share of deceased donors should go to $X_k - Y_k$ pairs under $\psi^{\mathbf{i}}$ with respect to $\varphi^{\mathbf{e}}$. Hence, $\varepsilon^{\mathbf{i}}_{X_k-Y_k} - \varepsilon^{\mathbf{e}}_{X_k-Y_k} > 0$, implying that $\mathbf{i}_{X_k-Y_k} > 0$, a contradiction.

Hence, unless A-B is pooled by itself with blood-type A single patients under $\varphi^{\mathbf{e}}$, any pooled paired group with blood-type X single patients has a strictly higher measure of being matched at each point in time under $\psi^{\mathbf{i}}$.

We continue with other patient groups. All overdemanded pairs and self-demanded pairs receive live donation under both $\psi^{\mathbf{i}}$ and $\varphi^{\mathbf{e}}$ immediately after their arrival. We already showed that underdemanded pairs strictly benefit from $\psi^{\mathbf{i}}$. Moreover, by Assumption 3, Lemma 4, and Theorems 1 and 3, all B-A pairs are matched with A-B pairs through exchange as soon as they enter the exchange pool. This and the proof for underdemanded pairs imply that A-B pairs either benefit under $\psi^{\mathbf{i}}$ (if they are pooled with an underdemanded type for deceased donation under $\varphi^{\mathbf{e}}$) or they remain indifferent between the two technologies (otherwise). Thus, all reciprocally demanded pairs have weakly less waiting times under incentivized exchange.

Finally, consider any $X \in \mathcal{T}$ blood-type single patients. As more underdemanded-type pairs are matched through exchange and the same measure of A-B pairs participate in exchange under $\psi^{\mathbf{i}}$, overall fewer underdemanded-type and A-B type pairs will be left from the same cohort for deceased donation. Furthermore, the measure of incentivized patients from willing compatible pairs returning for deceased-donor kidneys is only a fraction of this number. Hence, weakly more blood-type X single patients receive deceased donation under $\psi^{\mathbf{i}}$.

Proof of the Second Part: First observe that the waiting times of underdemanded types strictly decreases by the first part. The waiting times of reciprocally demanded B-A type pairs and A-B type pairs do not increase by the first part. Moreover, self-demanded and overdemanded type pairs do not wait and get immediately matched under both technologies. Finally, we consider single patients. To see how their waiting times are affected, we consider the change in the exchange rates for compatible and incompatible pairs first. We do this analysis for all blood types separately.

1. Blood-type O patients:

   Compatible pairs: O-O is the only compatible type with blood-type O patients. However, incompatible O-O pairs are already immediately matched with each other in exchange. Hence, a $\mathbf{c}_O = 0$ measure of compatible pairs with blood-type O patients participates in exchange.

   Incompatible pairs: A measure of $[\rho(1-\theta)+\theta]p_O\lambda[\pi_A + \pi_B + \pi_{AB}]$ incompatible pairs with blood-type O patients is matched through exchange with their reciprocal type pairs at each

point in time. This is a net increase of $\mathbf{i}_O = \rho(1-\theta)p_O\lambda[\pi_A+\pi_B+\pi_{AB}]$ with respect to regular exchange. If some of these pair types are pooled for deceased donation under exchange with incentivized compatible pairs, then they are also pooled for deceased donation under regular exchange.

Single patients:

* *Prioritized reentrants*: As no blood-type O reentrants are prioritized, all blood-type O deceased donors are still given to blood-type O single patients, and there is a $\phi^l\mathbf{c}_O = 0$ measure of prioritized blood-type O reentrants per unit time.

* *Nonprioritized single patients*: On the other hand, because some additional blood-type O patients are saved through exchange, an additional measure of $\phi^l\mathbf{i}_O = \phi^l[\rho(1-\theta)]p_O\lambda[\pi_A + \pi_B+\pi_{AB}]$ of blood-type O patients reenters with respect to regular exchange. These reentrants join the nonprioritized deceased-donor queue. However, if some underdemanded pairs with blood-type O patients receive deceased-donation under exchange, then some of these fall from competition for deceased donation under incentivized exchange. Depending on the size of this fallout, the net effect on the net inflow rate of blood-type O single patients can be negative or positive, but this additional inflow rate to the nonprioritized deceased donation queue will be no more than $\phi^l\mathbf{i}_O$. Depending on which of the above effects dominates, the waiting time for nonprioritized blood-type O single patients can slightly increase or decrease under incentivized exchange. However if O-A, O-B, and O-AB pairs received deceased-donor transplants under regular exchange, then the first effect dominates, and their waiting time decreases. More formally, in Theorem 9 in Appendix B see the waiting time in Equation 14, $t = F^{-1}\left(1-\frac{\Delta}{\Pi}\right)$ for appropriately defined $\Delta < \Pi$, there. Suppose ρ increases above 0 sufficiently but not too much so that O-A, O-B, and O-AB still receive deceased-donor transplants. Then the increase in Δ is $\mathbf{i}_O$ and the increase in Π is $\phi^l\mathbf{i}_O$. Thus, t decreases as Δ/Π increases. As ρ gets higher, so that such pairs are no longer pooled, the numerator Δ does not change while the denominator Π gets larger, and hence, t continues to increase.

2. Blood-type A patients:

   Compatible pairs: A measure $\mathbf{c}_A = \rho(1-\theta)p_O\lambda\pi_A$ of A-O type compatible pairs participates in exchange to save O-A type pairs. Self-demanded A-A type compatible pairs do not participate in exchange.

   Incompatible pairs: A measure $[\rho(1-\theta)+\theta]p_A\lambda\pi_{AB}$ of underdemanded type pairs A-AB is matched through exchange in every point in time. This is a net increase of $\mathbf{i}_A = \rho(1-\theta)p_A\lambda\pi_{AB}$ with respect to regular exchange. If some of these pair types are pooled for deceased donation under incentivized exchange, then they are also pooled for deceased donation under regular exchange.

   The reciprocally demanded pair type A-B continues to run a deficit as B-A inflow is – by Assumption 3 – lower than A-B inflow. If A-B type pairs wait both for B-A type pairs and deceased donors under incentivized exchange, see the case for single patients to understand

the effect of incentivized exchange on their waiting times below. On the other hand, if they are waiting exclusively for B-As under incentivized-exchange policies, then A-B types wait for the same time under both regular and incentivized exchange, and exactly the same measure of them gets matched.

Single patients:

* *Prioritized reentrants*: Patients of some of the A-O type compatible pairs that previously participated in exchange reenter as their grafts fail. Their inflow is $\phi^l \mathbf{c}_A = \phi^l \rho(1-\theta) p_O \lambda \pi_A$. These A reentrants, who no longer have living donors, go directly to the top of the blood-type A deceased-donor queue instead of going to the bottom as under regular exchange. We will refer to this as the *incentivized exchange burden*. This is also the rate of the deceased donors reserved for these patients.

* *Nonprioritized single patients*: An additional $\mathbf{i}_A$ measure of A-AB pairs are saved by AB-A types through exchange. A measure of $\phi^l \mathbf{i}_A = \phi^l \rho(1-\theta) p_A \lambda \pi_{AB}$ blood-type A patients reenters and joins in the nonprioritized queue with the single blood-type A patients. However, if some A-AB pairs receive deceased donation under regular exchange, then some of these fall from competition for deceased donation under incentivized exchange. Depending on the size of this fallout, the net effect on the net inflow of blood-type A single patients for the nonprioritized queue can be negative or positive, but this additional inflow will be no more than $\phi^l \mathbf{i}_A - \phi^l \mathbf{c}_A$. Moreover a $\phi^l \mathbf{c}_A$ measure of blood-type A deceased-donor kidneys will be reserved for the prioritized blood-type A reentrants. If A-AB is sufficiently small in measure, then $\mathbf{i}_A$ will be negligible, and as a result the waiting time will be $t' \approx F^{-1}\big(1 - \frac{\Delta - \phi^l \mathbf{c}_A}{\Pi - \phi^l \mathbf{c}_A}\big)$, while the waiting time with regular exchange was $t = F^{-1}\big(1 - \frac{\Delta}{\Pi}\big)$ for appropriately defined Δ and Π, as in Theorem 9 in Appendix B in Equation 14 with $\Delta < \Pi$. Therefore, $t' > t$.

As a result, the waiting time for new blood-type A single patients can slightly increase or decrease under exchange with incentivized compatible pairs with respect to parameters.

3. Blood-type B patients:

   Symmetric version of blood-type A patients, except that B-A's are immediately matched with A-B's when they enter the pool by the assumption that B-A's are on the short side.

4. Blood-type AB patients:

   Compatible pairs: A total measure of $\mathbf{c}_{AB} = \rho(1-\theta)[p_O + p_A + p_B]\lambda \pi_{AB}$ compatible AB-O, AB-A, and AB-B type pairs participate in exchange to save their reciprocals at each point in time. Self-demanded compatible AB-AB type pairs do not participate in exchange.

   Incompatible pairs: All incompatible pairs with blood-type AB patients are either self-demanded or overdemanded. Hence, they are matched immediately when they arrive through exchange with their reciprocal types under both regular exchange and exchange with incentivized compatible pairs. Hence, additionally an $\mathbf{i}_{AB} = 0$ measure of incompatible pairs with blood-type AB patients is matched under the new regime.

Single patients:

* *Prioritized reentrants*: The reentry burden of blood-type AB patients from previous compatible pairs that participated in exchange is $\phi^l \mathbf{c}_{AB} = \phi^l \rho(1-\theta)[p_O + p_A + p_B]\lambda\pi_{AB}$, which is the rate of prioritization for blood-type AB reentrants to the deceased-donor queue. This is also the rate of the deceased donors reserved for these patients.

* *Nonprioritized single patients*: On the other hand, the same measure of blood-type AB patients reenters at each point in time under both regular exchange and exchange with incentivized compatible pairs. No pairs with blood-type AB patients are pooled for deceased donation under either regular exchange or exchange with incentivized compatible pairs. Hence, a $\phi^l \mathbf{i}_{AB} = 0$ measure of additional blood-type AB reentrants from previous incompatible pairs reenters the deceased-donor queue. The net increase of rate of entry to the nonprioritized blood-type AB deceased-donor queue is negative and equal to $-\phi^l \mathbf{c}_{AB}$. As a result, the waiting time for nonprioritized blood-type AB single patients unambiguously slightly increases under exchange with incentivized compatible pairs. This holds as all of the prioritized blood-type AB patients receive deceased donation under exchange with incentivized compatible pairs, while some patients from the same population would have died and not received deceased donation under the alternative regime, regular exchange.

■

Proof of Theorem 5. From the proof of Theorem 2, we have the following:

$$\begin{aligned}
\frac{\mathbf{l}_O + \mathbf{e}_O}{\pi_O} =& (p_O + \theta p_A + \theta p_B + \theta p_{AB})\lambda, \\
\frac{\mathbf{l}_A + \mathbf{e}_A}{\pi_A} =& (p_O + p_A + p_B + \theta p_{AB})\lambda, \\
\frac{\mathbf{l}_B + \mathbf{e}_B}{\pi_B} =& (p_O + p_A + p_B + \theta p_{AB})\lambda, \text{ and} \\
\frac{\mathbf{l}_{AB} + \mathbf{e}_{AB}}{\pi_{AB}} =& (p_{AB} + p_O + p_A + p_B)\lambda.
\end{aligned}$$

Plugging in the values of $\mathbf{i}_X/\pi_X$ for each $X \in \mathcal{T}$ and using the assumption that p_X/π_X is a constant among all $X \in \mathcal{T}$ give us the following:

$$\begin{aligned}
\frac{\mathbf{l}_O + \mathbf{e}_O + \mathbf{i}_O}{\pi_O} =& p_O\lambda + (p_A + p_B + p_{AB})\lambda[\theta + \rho(1-\theta)], \\
\frac{\mathbf{l}_A + \mathbf{e}_A + \mathbf{i}_A}{\pi_A} =& (p_O + p_A + p_B)\lambda + p_{AB}\lambda[\theta + \rho(1-\theta)], \\
\frac{\mathbf{l}_B + \mathbf{e}_B + \mathbf{i}_B}{\pi_B} =& (p_O + p_A + p_B)\lambda + p_{AB}\lambda[\theta + \rho(1-\theta)], \text{ and} \\
\frac{\mathbf{l}_{AB} + \mathbf{e}_{AB} + \mathbf{i}_{AB}}{\pi_{AB}} =& (p_{AB} + p_O + p_A + p_B)\lambda.
\end{aligned}$$

Therefore, we get $\frac{\mathbf{l}_O+\mathbf{e}_O+\mathbf{i}_O}{\pi_O} \leq \frac{\mathbf{l}_A+\mathbf{e}_A+\mathbf{i}_A}{\pi_A} = \frac{\mathbf{l}_B+\mathbf{e}_B+\mathbf{i}_B}{\pi_B} \leq \frac{\mathbf{l}_{AB}+\mathbf{e}_{AB}+\mathbf{i}_{AB}}{\pi_{AB}}$. Furthermore, since $\lambda > 0$, weak inequalities hold with equality if and only if $\rho = 1$. ■

Appendix B Steady States of Transplantation Policies

In this appendix, we characterize the steady state waiting times for transplantation using the primitives of our model for all the transplantation policies. Using these results, we calculate the steady state waiting times in our policy experiments with calibrated real-data statistics in Section 5.

B.1 Deceased-Donor Transplantation

In this subsection, we characterize the steady state of the deceased-donor transplantation policy under ABO-i FIFO allocation policy.

We start with the following result, which directly follows from Lemma 3 in Appendix C:

Lemma 2 *Under the ABO-i deceased-donor allocation policy, a δ_X measure of blood-type X patients receive deceased-donor transplantation per unit time at steady state. Moreover, a $\phi^d\delta_X$ measure of previous recipients reenter the deceased-donor queue per unit time due to graft failure at steady state.*

Let the receiving cohort have arrived $t_X^{\mathbf{d},dec}$ years before the current point in time. As there is a $[\pi_X + \phi^d\delta_X][1 - F(t_X^{\mathbf{d},dec})]$ measure of patients in this cohort, including reentries and new arrivals, we should have

$$[\pi_X + \phi^d\delta_X][1 - F(t_X)] = \delta_X.$$

Hence, at steady state, the time spent on the X queue by the receiving cohort can be found through $t_X^{\mathbf{d},dec} = F^{-1}(1 - \frac{\delta_X}{\pi_X+\phi^d\delta_X}) < T = F^{-1}(1)$. This is also the transplant waiting time for blood-type X patients. Based on this analysis, we state the following characterization of the deceased-donor queue at steady state.

Theorem 6 (ABO-i deceased-donor transplantation) *Under the ABO-i deceased-donor transplantation technology, at steady state, the waiting time for blood-type X patients for transplant is*

$$t_X^{\mathbf{d},dec} = F^{-1}\big(1 - \tfrac{\delta_X}{\pi_X+\phi^d\delta_X}\big), \tag{6}$$

which is also the average waiting time for transplant. Moreover, $\frac{\delta_X}{\pi_X+\phi^d\delta_X}$ is the probability of a patient ever receiving a transplant.

Proof. Immediately follows from the analysis preceding the theorem.[21] ■

B.2 Living Donor Transplantation

We continue with the characterization of waiting times when direct live donation is also feasible. We calculate the inflow rates of compatible and incompatible pair types as follows:

[21] Average and deceased-donor waiting times are identical, as the only means of transplantation is deceased donors under this technology.

For blood-type O patients: $(1-\theta)p_O\lambda\pi_O$ is the inflow rate of blood-type O patients with a compatible paired donor. Given $Y \in \{A, B, AB\}$, $p_Y\lambda\pi_O$ is the measure of $O-Y$ pairs, which are always incompatible.

For blood-type A and B patients, given $X \in \{A, B\}$: Given $Y \in \{X, O\}$, $(1-\theta)p_Y\lambda\pi_X$ is the inflow rate of blood-type X patients with a compatible blood-type Y live donor. Given $Y \in \{X, O\}$, $\theta p_Y\lambda\pi_X$ is the measure of incompatible $X-Y$ pairs. Given $Y \in \{A, B, AB\} \setminus \{X\}$ $p_Y, \lambda\pi_X$ as the inflow rate of $X-Y$ pairs, which are always incompatible.

For blood-type AB patients: For any given Y: $(1-\theta)p_Y\lambda\pi_{AB}$ is the inflow rate of compatible $AB-Y$ pairs, and $\theta p_Y\lambda\pi_{AB}$ is the inflow rate of incompatible $AB-Y$ pairs.

For a paired patient of blood type X, let p_X^l denote the probability that his paired donor is compatible with the patient. Thus, $p_X^l\lambda\pi_X$ is the inflow rate of blood-type X patients with compatible living donors. These patients receive organs from their paired donors upon entry, and they do not wait in the deceased-donor queue.

The total inflow rate of patients entering or reentering the blood-type X deceased-donor queue under the ABO-i allocation policy is given as

$$\pi_X^{\mathbf{l},dec} = \underbrace{\pi_X}_{\text{new patients}} + \underbrace{\phi^d\delta_X}_{\text{reentry / deceased}} + \underbrace{\phi^l p_X^l\lambda\pi_X}_{\text{reentry / live}} - \underbrace{p_X^l\lambda\pi_X}_{=\imath_X\text{: compatible pairs}} \tag{7}$$

Above, "reentry / deceased" and "reentry / live" refer to the reentering previous deceased- and living-donor organ recipients, respectively. Equation 7 and Lemma 3 imply that the ABO-i allocation waiting time in the blood-type X deceased-donor queue is given by

$$t_X^{\mathbf{l},dec} = F^{-1}\left(1 - \frac{\delta_X}{\pi_X^{\mathbf{l},dec}}\right) \tag{8}$$

The average waiting time for transplant for patients under living-donor transplantation technology is substantially less for all blood types than those under the deceased-donor transplantation. Many patients have compatible living donors, and they immediately receive a transplant without waiting. Hence, the average waiting time for transplant is

$$t_X^{\mathbf{l},ave} = \frac{\delta_X t_X^{\mathbf{l},dec}}{\delta_X + p_X^l\lambda\pi_X} \tag{9}$$

under the ABO-i deceased-donor allocation policy.

We are ready to make a more detailed analysis of how different blood types are affected by the availability of live donation. Due to the partial-order structure of blood-type compatibility across blood types, not all blood types will be affected equally when live donation is possible. For example, O blood-type paired patients are at a disadvantage in finding a compatible paired donor. In general, A blood type is more prominent in the population than B. Therefore, at random, blood-type A paired patients will have a higher chance of finding a compatible donor than B types, given that they can all receive from O blood-type donors as well as their own types. Finally, AB

blood-type paired patients have the highest chance of a compatible paired donor.

However, depending on the exact shape of the survival function, $1-F$, and the deceased-donor-to-new-patient inflow rate ratios across blood types, δ_X/π_X, O blood type does not necessarily experience the lowest decrease in waiting time, and AB blood type does not necessarily experience the greatest improvement.

On the other hand, for the benchmark case, where the deceased donor to new patient inflow rate ratio δ_X/π_X, is the same for each blood type, we can make unambiguous predictions.[22]

Theorem 7 (Living-donor transplantation) *Suppose Assumption 1 holds. For all blood types with respect to deceased-donor transplantation, living-donor transplantation will unambiguously decrease the steady state deceased-donor waiting time, which is now only relevant for single patients and patients with incompatible donors, will make overall waiting time zero for patients with compatible donors, and will decrease the overall average waiting times for transplant.*

Consider the benchmark case that the ratio δ_X/π_X is constant across all $X \in \mathcal{T}$. The following hold:

- *blood-type O patients have the lowest waiting-time decrease,*
- *blood-type AB patients have the highest waiting-time decrease, and*
- *provided that $p_A > p_B$, blood-type A patients have a higher waiting-time decrease than blood-type B patients.*

In particular, if $p_A > p_B$, then the waiting times for deceased-donor transplant satisfy $t_O^{\mathbf{l},dec} > t_B^{\mathbf{l},dec} > t_A^{\mathbf{l},dec} > t_{AB}^{\mathbf{l},dec}$, and the average waiting times for transplant satisfy $t_O^{\mathbf{l},ave} > t_B^{\mathbf{l},ave} > t_A^{\mathbf{l},ave} > t_{AB}^{\mathbf{l},ave}$.

Theorem 8 (Living donor transplantation and inequity in transplant ratios) *Suppose Assumption 1 holds. Living-donor transplantation unambiguously increases transplant ratios for all blood types at steady state. Moreover, live-donation transplant ratios satisfy $\mathbf{l}_O/\pi_O < \mathbf{l}_A/\pi_A, \mathbf{l}_B/\pi_B < \mathbf{l}_{AB}/\pi_{AB}$, i.e., blood-type O patients benefit marginally the least and blood-type AB patients marginally the most from living-donor transplantation technology. Additionally, if $p_A > p_B$, then $\mathbf{l}_B/\pi_B < \mathbf{l}_A/\pi_A$, i.e., blood-type A patients marginally benefit more than blood-type B patients.*

Proof of Theorems 7 and 8. Observe that we have $p_O^l = p_O(1-\theta)$, $p_A^l = (p_O+p_A)(1-\theta)$, $p_B^l = (p_O+p_B)(1-\theta)$, and $p_{AB}^l = 1-\theta$. Hence, $p_O^l < p_A^l, p_B^l < p_{AB}^l$. Since $\mathbf{l}_X/\pi_X = p_X^l\lambda$ (recall that $\mathbf{l}_X = p_X^l\lambda\pi_X \in (0,\pi_X)$ is the inflow rate of compatible pairs with blood-type X patients), we obtain Theorem 8.

For Theorem 7, first consider the ABO-i deceased-donor allocation policy. By Equation 8, for any X,

$$t_X^{\mathbf{l},dec} = F^{-1}\big(1 - \tfrac{\delta_X}{(\pi_X-(1-\phi^l)\mathbf{l}_X)+\phi^d\delta_X}\big). \tag{10}$$

[22] Although these conclusions seem to have been reached with the help of our assumption that blood types of patients are uncorrelated with their paired donors, a version of this result will also hold true even if there is positive correlation in a pair's blood types; however, the magnitude of the difference in eventual waiting times will not be as extreme.

As $t_X^{\mathbf{l},dec}$ is increasing in net patient inflow rate, comparing Equation 6 with Equation 10, we conclude for all X, $t_X^{\mathbf{l},dec} < t_X^{\mathbf{d},dec}$.

In the rest of the proof, we analyze the benchmark case where δ_X/π_X is constant across all blood types X. Then $\mathbf{l}_O \leq \mathbf{l}_X$ for all X, and $\mathbf{l}_{AB} \geq \mathbf{l}_X$ for all X. These in turn imply that $t_O^{\mathbf{l},dec} \geq t_X^{\mathbf{l},dec}$ for all X, and $t_{AB}^{\mathbf{l},dec} \leq t_X^{\mathbf{l},dec}$ for all X, respectively, since $t_X^{\mathbf{l},dec}$ is decreasing in $\mathbf{l}_X$. We also have

$$\frac{\delta_O}{\pi_O - (1-\phi^l)\mathbf{l}_O} \leq \frac{\delta_A}{\pi_A - (1-\phi^l)\mathbf{l}_A}\,,\ \frac{\delta_B}{\pi_B - (1-\phi^l)\mathbf{l}_B} \leq \frac{\delta_{AB}}{\pi_{AB} - (1-\phi^l)\mathbf{l}_{AB}}.$$

Further assume that $p_A > p_B$. Then $p_A^l > p_B^l$. Therefore, $\mathbf{l}_A > \mathbf{l}_B$, which in turn implies $\frac{\delta_B}{\pi_B - (1-\phi^l)\mathbf{l}_B} < \frac{\delta_A}{\pi_A - (1-\phi^l)\mathbf{l}_A}$, and hence, $t_A^{\mathbf{l},dec} < t_B^{\mathbf{l},dec}$.

Given this result, comparing Equation 9 across blood types together with the fact that $p_O^l < p_A^l, p_B^l < p_{AB}^l$ leads to the analogous result for the overall average waiting times for deceased and living donors. If $p_B^l < p_A^l$, then we get the required result in Theorem 7. ■

B.3 Living-Donor Exchange: Regular and Incentivized

Suppose $\rho \in [0,1]$ ratio of compatible pairs participate in exchange. We use terms "regular exchange" and "incentivized exchange with $\rho = 0$," interchangeably. To determine steady state, for each blood type X and $Y \neq X$, let

$$\pi_{X-Y}^{\mathbf{i}} = \begin{cases} [\theta + \rho(1-\theta)]p_Y \lambda \pi_X & \text{if } Y \triangleright X \\ p_Y \lambda \pi_X & \text{otherwise} \end{cases} \tag{11}$$

refer to the **exchange pool $X - Y$ inflow rate**,[23] and let

$$\pi_X^{\mathbf{i}} = \underbrace{(1-\lambda)\pi_X}_{\text{new single}} + \underbrace{\phi^d \delta_X}_{\text{reentry / deceased}} + \underbrace{\phi^l[\mathbf{l}_X + \mathbf{e}_X + \mathbf{i}_X - \mathbf{c}_X]}_{\text{reentry / all live minus incentivized}} \tag{12}$$

be the **nonprioritized single X patient inflow rate** where the **incentivized compatible pair rate** is given by[24]

$$\mathbf{c}_X = \rho(1-\theta)\Big(\sum_{Y \triangleright X \& Y \neq X} p_Y\Big)\lambda \pi_X.$$

We calculate the following ratios for each blood type X:

(1) The deceased-donor inflow rate reserved for nonprioritized single patients is $\delta_X - \phi^l \mathbf{c}_X$. The ratio of this rate to nonprioritized single patient inflow rate is defined as

$$r_X = \frac{\delta_X - \phi^l \mathbf{c}_X}{\pi_X^{\mathbf{i}}}$$

[23]These were defined only for overdemanded types in Equation 4 before.

[24]These were previously defined throughout the Proof of Theorem 4 before.

(2) For each underdemanded type $X-Y$ (i.e., $X \neq Y$ and $X \triangleright Y$), the ratio of the incompatible $Y-X$ inflow rate to the $X-Y$ inflow rate :

$$r_{X-Y} = \frac{\pi^{\mathbf{i}}_{Y-X}}{\pi^{\mathbf{i}}_{X-Y}} = \frac{[\theta + \rho(1-\theta)]p_X \lambda \pi_Y}{p_Y \lambda \pi_X}.$$

(3) For reciprocal type A-B,

$$r_{A-B} = \frac{\pi^{\mathbf{i}}_{B-A}}{\pi^{\mathbf{i}}_{A-B}} = \frac{p_A \lambda \pi_B}{p_B \lambda \pi_A}.$$

Ratio r_X would be relevant if we wanted to allocate all X deceased donors to only blood-type X single patients. For an underdemanded type $X-Y$ or $X-Y=A-B$, ratio $r_{X-Y} = \frac{\pi^{\mathbf{i}}_{Y-X}}{\pi^{\mathbf{i}}_{X-Y}}$ would be relevant if we did not want $X-Y$ pairs to receive deceased donation, but only to participate in ABO-i optimal exchange. In these cases, the waiting time for deceased-donor transplant for nonprioritized single blood-type X patients would be $t_X = F^{-1}\big(1 - \frac{\delta_X - \phi^l \mathbf{c}_X}{\pi^{\mathbf{i}}_X}\big)$, and the waiting time of $X-Y$ pairs would be $t_{X-Y} = F^{-1}\big(1 - \frac{\pi^{\mathbf{i}}_{Y-X}}{\pi^{\mathbf{i}}_{X-Y}}\big)$.[25]

However, underdemanded or reciprocally demanded $X-Y$ pairs have another option besides waiting for their reciprocal type pairs. If available deceased donors arrive earlier, they can receive deceased-donor transplants. We assume that patients accept the first donor who is offered to them through deceased-donor allocation or exchange. Hence, the patient of an $X-Y$ type pair will never wait for a $Y-X$ pair for exchange if a deceased organ comes first, i.e., if $t_{X-Y} < t_X$. As time is decreasing in r ratios, all we need to do is to compare these ratios in an iterative manner to decide whether any underdemanded type or A-B type will receive a deceased-donor transplant:

Exchange technology pooling procedure for single and paired patients:
(1) Let $X-Y_1, \ldots, X-Y_k$ be the ordered list of underdemanded or reciprocally demanded types ascending in r_{X-Y} ratio. Define for each $\ell = 0, \ldots, k$:

$$r_{X,X-Y_1,\ldots,X-Y_\ell} \;=\; \frac{\delta_X - \phi^l \mathbf{c}_X + \pi^{\mathbf{i}}_{Y_1-X} + \ldots + \pi^{\mathbf{i}}_{Y_\ell - X}}{\pi^{\mathbf{i}}_X + \pi^{\mathbf{i}}_{X-Y_1} + \ldots + \pi^{\mathbf{i}}_{X-Y_\ell}}. \tag{13}$$

(2) For $\ell \in \{0, \ldots, k-1\}$, suppose types $X-Y_1, \ldots, X-Y_\ell$ have already been deemed to be receiving both deceased-donor and exchange transplants.
(*) If $r_{X-Y_{\ell+1}} < r_{X,X-Y_1,\ldots,X-Y_\ell}$ then $X-Y_{\ell+1}$ pairs receive both exchange transplants and deceased-donor transplants with the rest of the blood-type X single patients and $X-Y_1, \ldots, X-Y_\ell$ pairs. We continue with Step 2 with $\ell := \ell + 1$.
(*) If $r_{X-Y_{\ell+1}} \geq r_{X,X-Y_1,\ldots,X-Y_\ell}$ then all types $X-Y_{\ell+1}, \ldots, X-Y_k$ only receive exchange transplants, but no transplants from deceased donors. We terminate the procedure.

Based on this procedure, we state the following theorem:

[25] The waiting time of B-A is 0, as this type is on the shorter side of the market when compared to A-B, by Assumption 3.

Theorem 9 (Regular and incentivized exchange) *Suppose Assumptions 1, 2, and 4 hold. Consider the ABO-i deceased-donor allocation and incentivized-exchange policies with a given $\rho \in [0,1]$ compatible pair participation rate. Consider a blood type X. Then the following statements hold.*
(1) *Blood-type X paired patients with non-exchange-participating compatible donors immediately receive their donor's organ upon entry.*
(2) *Blood-type X reentrants who previously participated in incentivized exchange immediately receive deceased donor donation upon entry.*
(3) *Blood-type X paired patients who are part of incompatible or exchange-participating overdemanded type pairs or self-demanded type pairs and, if $X = B$, then of B-A type pairs, immediately participate in an exchange upon entry.*
(4) *Suppose patients of underdemanded and reciprocally demanded types $X - Y_1, \ldots, X - Y_\ell$ receive both deceased-donor and exchange transplants while patients of underdemanded and reciprocally demanded types $X - Y_{\ell+1}, \ldots, X - Y_k$ receive only exchange transplants. Then*
(*) *Other blood-type X single patients and patients of $X - Y_1, \ldots, X - Y_\ell$ pairs wait for a deceased-donor or exchange transplant for*

$$t_X^{\mathbf{i},dec} \;=\; F^{-1}\left(1 - \frac{\delta_X - \phi^l \mathbf{c}_X + \pi^{\mathbf{i}}_{Y_1-X} + \ldots + \pi^{\mathbf{i}}_{Y_\ell - X}}{\pi^{\mathbf{i}}_X + \pi^{\mathbf{i}}_{X-Y_1} + \ldots + \pi^{\mathbf{i}}_{X-Y_\ell}}\right). \tag{14}$$

(*) *For all $m \in \{\ell+1, \ldots, k\}$, patients of $X - Y_m$ type pairs wait for an exchange transplant for*

$$t^{\mathbf{i}}_{X-Y_m} \;=\; F^{-1}\left(1 - \frac{\pi^{\mathbf{i}}_{Y_m - X}}{\pi^{\mathbf{i}}_{X-Y_m}}\right). \tag{15}$$

The average waiting time to a transplant for all blood-type X patients is

$$t_X^{\mathbf{i},ave} \;=\; \frac{\left[\delta_X - \phi^l \mathbf{c}_X + \sum_{m=1}^{\ell} \pi^{\mathbf{i}}_{Y_m - X}\right] t_X^{\mathbf{i},dec} + \sum_{m=\ell+1}^{k} \left[\pi^{\mathbf{i}}_{Y_m - X} t^{\mathbf{i}}_{X-Y_m}\right]}{\delta_X + \mathbf{l}_X + \mathbf{e}_X + \mathbf{i}_X} \tag{16}$$

Proof. It follows from the procedure discussed before the statement of the theorem. For Equation 16, the blood-type X patient inflow with compatible living donors, $p_X^l \lambda \pi_X$ and the blood-type X patient inflow with incompatible but blood-type compatible donors have 0 waiting time. ■

Appendix C Discrete Limit

In this appendix we weaken our limit assumptions on the donor tissue types. Each donor has a tissue type. There are k distinct tissue types. The probability that a donor is of tissue type i is $m_{i,k} > 0$, so $\sum_i m_{i,k} = 1$. Let $\theta_{i,k}$ be the tissue rejection probability between any patient and donor of tissue type i. If a patient is tissue-type compatible with a type i donor, then the patient is tissue-type compatible with all donors of tissue type i.

C.1 Matching Deceased-Donor Kidneys

We first consider the case when deceased-donor kidneys are matched with patients. We make the following regularity assumption on the frequency and tissue rejection probability of donor types.

Assumption 5 *For every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$, such that for every $k > k_0$ and $l \leq k$ and every permutation σ of donor types,*

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq 1 - \prod_{i=1}^{l} \theta_{\sigma(i),k}.$$

When $\epsilon \to 0$, the regularity assumption can be rewritten as $\sum_{i=l+1}^{k} m_{\sigma(i),k} \geq \prod_{i=1}^{l} \theta_{\sigma(i),k}$. It implies that if you take a set of patients and a set of donors with the same measure, then for any set of donor types the measure of donors with those types is greater than or equal to the measure of the set of patients who are tissue-type incompatible with all the other donors.

Now let us look at the implications of this assumption in detail for the special case when $\theta_{i,k} = \theta$ for every donor type i. The assumption is equivalent to $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq 1 - \theta^l$, which is satisfied, for example, when $m_{i,k} = \Theta(1/k)$. More explicitly, for every donor type i, if there exist constants $c_{i1} > 0$, $c_{i2} > 0$, and $k_0 \in \mathbb{N}$ such that, $c_{i1}\frac{1}{k} \leq m_{i,k} \leq c_{i2}\frac{1}{k}$ for every $k > k_0$, then the regularity assumption is satisfied.

Under this assumption, we get the following result, which is used explicitly in establishing Theorems 6, 7, 8, and 9 in Appendix B and implicitly in all other results regarding steady state of various policies:

Lemma 3 *Consider a measurable set of patients and deceased-donor kidneys that are blood-type compatible with all the patients such that the measure of the set of kidneys is weakly greater than the measure of the set of patients. Suppose these sets are formed randomly using the governing population distributions. Then, as the number of donor types k goes to infinity, almost every patient can be matched with a compatible kidney under Assumption 5.*

Proof. Without loss of generality, consider the case when the measures of the two sets are the same and equal to one. Fix a small $\epsilon > 0$. By Assumption 5, there exists k_0 such that, for every $k > k_0$, $l \leq k$, and permutation σ, $\frac{1-\prod_{i=1}^{l}\theta_{\sigma(i),k}}{\sum_{i=1}^{l} m_{\sigma(i),k}} \geq 1-\epsilon$. Consider any such k.

We use Gale's Supply-Demand Theorem (Gale, 1957) to show that $1-\epsilon$ measure of the patients can be matched with compatible kidneys. Then the result follows by taking the limit as $\epsilon \to 0$ by taking $k \to \infty$. To apply Gale's Supply-Demand Theorem, consider a random measurable subset of

kidneys with measure $1-\epsilon$. Since the subset is chosen randomly, the compatibility of kidneys with the patients can still be formed randomly using the governing population. We need to show that for any subset of kidneys, the measure of patients who are compatible with at least one kidney is weakly greater than the measure of kidneys. Without loss of generality, instead of considering any set of kidneys we can consider the set of all kidneys that have tissue types from any given set. For any such set, the desired inequality is:

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq 1-\prod_{i=1}^{l}\theta_{\sigma(i),k}.$$

Here, the set of tissue types that we consider is $\{\sigma(1),\ldots,\sigma(l)\}$. The measure of kidneys that have a tissue type in this set is $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}$. The measure of patients who are incompatible with all such types is $\prod_{i=1}^{l}\theta_{\sigma(i),k}$ because the measure of patients is one. Therefore, the measure of patients who are compatible with at least one kidney in the set is $1-\prod_{i=1}^{n}\theta_{\sigma(i),k}$.

The desired inequality holds by construction. The claim of the lemma follows by taking the limit as $\epsilon \to 0$ and $k \to \infty$. ■

C.2 Matching Reciprocal-Type Pairs

We next consider the case when we match reciprocally demanded pairs (A-B pairs with B-A pairs). For any such pair, tissue-type compatibility is not known because the pair is blood-type incompatible. Therefore, for any such pair, the tissue-type incompatibility is determined randomly as in the overall population.

We make the following assumption on how the market grows, which guarantees that we can match almost every patient in two measurable sets of A-B pairs and B-A pairs that have the same measure.

Assumption 6 *For every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$, $l \leq k$, and every permutation σ of donor types,*

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq \sum_{i=1}^{k} m_{\sigma(i),k}[1-\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))].$$

Consider two measurable sets of A-B and B-A pairs with the same measure. As $\epsilon \to 0$, the assumption guarantees that for any measurable set of reciprocal-type pairs, say B-A, the measure of this set is smaller than the measure of A-B pairs that are compatible with at least one B-A pair in this set.

When $\theta_{i,k} = \theta$ for every donor type i, the assumption is equivalent to $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq 1-(1-(1-\theta)^2)^l$ and it is satisfied when $m_{i,k} = \Theta(1/k)$.

Lemma 4 *Consider two measurable sets of A-B and B-A pairs that have the same measure. Suppose these sets are formed randomly using the governing population distributions. Then, as the number of donor types k goes to infinity, almost every pair can be matched with a compatible pair under Assumption 6.*

Proof. Without loss of generality, consider the case when the measures of the two sets are the same and equal to one. Fix a small $\epsilon > 0$. By Assumption 6, there exists k_0 such that, for every $k > k_0$, $l \leq k$, and permutation σ, $\frac{\sum_{i=1}^{k} m_{\sigma(i),k}[1-\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]}{\sum_{i=1}^{l} m_{\sigma(i),k}} \geq 1-\epsilon$. Consider any such k.

Like before, we use Gale's Supply-Demand Theorem (Gale, 1957) to show that $1-\epsilon$ measure of the B-A pairs can be matched with compatible A-B pairs. Then the result follows by taking the limit as $\epsilon \to 0$ and $k \to \infty$. To apply Gale's Supply-Demand Theorem, consider a random measurable subset of B-A pairs with measure $1-\epsilon$. Since the subset is chosen randomly, the compatibility of kidneys with patients can still be formed randomly using the governing population. We need to show that for any subset of B-A pairs, the measure of A-B pairs who are compatible with at least one B-A pair in the chosen set is weakly greater than the measure of the chosen set of B-A pairs. Without loss of generality, instead of considering any set of B-A pairs, we can consider the set of all B-A pairs with donors that have tissue types from any given set. For any such set, the desired inequality is:

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq \sum_{i=1}^{k} m_{\sigma(i),k}[1-\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))].$$

Here, the set of tissue types that we consider is $\{\sigma(1),\ldots,\sigma(l)\}$. The measure of B-A pairs with donor kidneys that have a tissue type in this set is $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}$. The measure of A-B pairs with donor tissue type i who are incompatible with all such pairs is $m_{\sigma(i),k}\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))$. Therefore, the measure of A-B pairs with donor tissue type i who are compatible with at least one B-A pair from the chosen set is $m_{\sigma(i),k}[1-\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]$. Hence, the measure of A-B pairs that are compatible with at least one B-A pair in the chosen set is $\sum_{i=1}^{k} m_{\sigma(i),k}[1-\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]$. This sum is greater than the measure of chosen B-A pairs $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}$ by construction.

The proof that $1-\epsilon$ measure of B-A pairs can be matched follows. The lemma follows by taking $k \to \infty$ and $\epsilon \to 0$. ■

C.3 Matching Overdemanded Pairs with Underdemanded Pairs

We next consider the case when we match overdemanded pairs with underdemanded pairs. Underdemanded pairs are blood-type incompatible, so their tissue-type compatibility is determined as in the general population. However, overdemanded types are blood-type compatible. Therefore, in the regular exchange they are tissue-type incompatible, whereas in the incentivized exchange some of them are tissue-type compatible. To guarantee that almost every pair can be matched, we make the following assumption.

Assumption 7 *For every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$, $l \leq k$, $a \in [0,1]$, and every permutation σ of donor types,*

$$(1-\epsilon)\frac{\sum_{i=1}^{l} m_{\sigma(i),k}[a\theta_{\sigma(i),k}+(1-a)(1-\theta_{\sigma(i),k})]}{\sum_{i=1}^{k} m_{i,k}[a\theta_{i,k}+(1-a)(1-\theta_{i,k})]} \leq \sum_{i=1}^{k} m_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))].$$

For overdemanded type pairs, tissue-type incompatible ones participate in the regular exchange. However, in the incentivized exchange, compatible pairs also participate. As a result, a fraction of the overdemanded pairs are compatible, while the rest are incompatible. Hence, the assumption above has a parameter $a \in [0,1]$ that changes as the fraction of incompatible pairs to compatible pairs changes. It guarantees that, for any set of overdemanded pairs, the set of underdemanded pairs that are compatible with at least one pair in the set has a greater measure as $\epsilon \to 0$.

In the special case when $\theta_{i,k} = \theta$ for every donor type i, the assumption is equivalent to $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq 1-(1-(1-\theta)^2)^{l-1}$. Like before, it is satisfied when $m_{i,k} = \Theta(1/k)$.

Lemma 5 *Consider two measurable sets of overdemanded $X-Y$ pairs and underdemanded $Y-X$ pairs with the same measure. Suppose that a fraction of overdemanded $X-Y$ pairs are known to be tissue-type incompatible and the rest are known to be tissue-type compatible but otherwise these sets are formed randomly using the governing population distributions. Then, as the donor types goes to infinity, almost every pair can be matched with a compatible pair from the other side under Assumption 7.*

Proof. Without loss of generality consider the case when the measures of the two sets are the same and equal to one. Then, for underdemanded $Y-X$ pairs, $m_{i,k}$ measure of the donors have type i for every i. For overdemanded $X-Y$ pairs, some are known to be tissue-type compatible while others are tissue-type incompatible. The incompatible pairs with type i donors have measure proportional to $m_{i,k}\theta_{i,k}$, whereas among the compatible pairs those with type i donors have measure proportional to $m_{i,k}(1-\theta_{i,k})$. Let $M \equiv \sum_{i=1}^{k} m_{i,k}[a\theta_{i,k} + (1-a)(1-\theta_{i,k})]$, where $a \in [0,1]$ is determined by the ratio of compatible $X-Y$ pairs to incompatible $X-Y$ pairs. Then the measure of overdemanded $X-Y$ pairs with type i donors is $m_{i,k}[a\theta_{i,k} + (1-a)(1-\theta_{i,k})]/M$.

Fix a small $\epsilon > 0$. By Assumption 7, there exists k_0 such that, for every $k > k_0$, $l \leq k$, $a \in [0,1]$ and permutation σ,

$$\frac{\sum_{i=1}^{k} m_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]}{\sum_{i=1}^{l} m_{\sigma(i),k}[(a\theta_{\sigma(i),k}+(1-a)(1-\theta_{\sigma(i),k}))]/M} \geq 1-\epsilon.$$

Consider any such k.

Like before, we use Gale's Supply-Demand Theorem (Gale, 1957) to show that $1-\epsilon$ measure of the overdemanded $X-Y$ pairs can be matched with compatible underdemanded $Y-X$ pairs. Then the result follows by taking the limit as $\epsilon \to 0$ and $k \to \infty$. To apply Gale's Supply-Demand Theorem, consider a random measurable subset of overdemanded $X-Y$ pairs with measure $1-\epsilon$. Since the subset is chosen randomly, the compatibility of kidneys with the patients can still be formed randomly using the governing population. We need to show that, for any subset of overdemanded $X-Y$ pairs, the measure of underdemanded $Y-X$ pairs who are compatible with at least one overdemanded $X-Y$ pair is weakly greater than the measure of overdemanded $X-Y$ pairs. Without loss of generality, instead of considering any set of overdemanded $X-Y$ pairs, we can consider the set of all overdemanded $X-Y$ pairs with donors that have tissue types from any given set. For any such set, the desired inequality is:

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}[(a\theta_{\sigma(i),k}+(1-a)(1-\theta_{\sigma(i),k}))]/M \leq$$
$$\sum_{i=1}^{k} m_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))].$$

Here, the set of tissue types that we consider is $\{\sigma(1),\ldots,\sigma(l)\}$. The measure of overdemanded $X-Y$ pairs with donor kidneys that have tissue types in this set is $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}[(a\theta_{\sigma(i),k}+(1-a)(1-\theta_{\sigma(i),k}))]/M$. The measure of A-B pairs with donor tissue-type i who are incompatible with all such pairs is $m_{\sigma(i),k}\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))$. Therefore, the measure of underdemanded $Y-X$ pairs with donor tissue type i who are compatible with at least one overdemanded $X-Y$ pair from the chosen set is at least $m_{\sigma(i),k}[1-\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]$. Note here that we do not consider possible matchings with overdemanded $X-Y$ pairs with donor tissue-type i. Hence, the measure of underdemanded $Y-X$ pairs that are compatible with at least one overdemanded $X-Y$ pair in the chosen set is $\sum_{i=1}^{k} m_{\sigma(i),k}[1-\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]$. This sum is greater than the measure of chosen overdemanded $X-Y$ pairs $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}[(a\theta_{\sigma(i),k}+(1-a)(1-\theta_{\sigma(i),k}))]/M$ by construction.

The proof that $1-\epsilon$ measure of B-A pairs can be mathced follows. The lemma follows by taking $k\to\infty$ and $\epsilon\to 0$. ■

C.4 Matching Self-Demanded Type Pairs

In this section, we consider the case when we match self-demanded type pairs. Fix any self-demanded type pair $X-X$ where $X\in\mathcal{T}$. Any such pair in the exchange pool is tissue-type incompatible. We match these pairs with each other. Therefore, in contrast with the previous sections, this is a one-sided matching problem.

We make the following assumption to show that almost every pair can be matched in the limit.

Assumption 8 *For every $\epsilon>0$, there exists $k_0\in\mathbb{N}$ such that for every $k>k_0$, $l\leq k$, and every permutation σ of donor types,*

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}\theta_{\sigma(i),k} \leq \sum_{i=1}^{k} m_{\sigma(i),k}\theta_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))].$$

When $\theta_{i,k}=\theta$ for every donor type i, the assumption reduces to $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}\leq 1-(1-(1-\theta)^2)^{l-1}$. As before, this is satisfied when $m_{i,k}=\Theta(1/k)$.

Lemma 6 *Consider a set of self-demanded type pairs $X-X$ that are tissue-type incompatible with positive measure. Assume that this set is formed randomly using the governing population distributions. Then, as the donor types goes to infinity, almost every pair can be matched with a compatible pair, Assumption 8.*

Proof. Since the pairs are tissue-type incompatible, but otherwise formed randomly using the governing population distributions, for each donor tissue type i, the measure of pairs with donor type i is proportional to $m_i\theta_i$.

Fix a small $\epsilon>0$. By Assumption 7, there exists k_0 such that, for every $k>k_0$, $l\leq k$, and permutation σ,

$$\frac{\sum_{i=1}^{k} m_{\sigma(i),k}\theta_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]}{\sum_{i=1}^{l} m_{\sigma(i),k}\theta_{\sigma(i),k}} \geq 1-\epsilon.$$

Consider any such k.

We use Gale's Supply-Demand Theorem (Gale, 1957) to show that $1-\epsilon$ fraction of the self-demanded $X-X$ pairs can be matched with compatible self-demanded $X-X$ pairs. To show this, we first construct a two-sided matching problem with these pairs. For any donor tissue-type i, we split the set of pairs with donor tissue-type i into two sets with equal measure. These sets are then added to one side of the market. As a result, we get a two-sided matching problem where each side has $X-X$ pairs where those with donor type i have a measure proportional to $m_i\theta_i$. For ease of exposition, suppose that the measure is exactly $m_i\theta_i$.

Consider one side of the market. To apply Gale's Supply-Demand Theorem, consider a random measurable subset of pairs on one side of the market with measure $1-\epsilon$. Since the subset is chosen randomly, the compatibility of kidneys can still be formed randomly using the governing population. We need to show that for any subset of pairs, the measure of pairs on the other side of the market that are compatible with at least one pair is weakly greater than the measure of chosen pairs. Without loss of generality, instead of considering any set of donor types, we can consider the set of all donor types that have tissue types from any given set. For any such set, the desired inequality is:

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}\theta_{\sigma(i),k} \leq \sum_{i=1}^{k} m_{\sigma(i),k}\theta_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))].$$

But this inequality holds by assumption, so $1-\epsilon$ fraction of pairs on both sides of the market can be matched. As we take $\epsilon \to 0$ and $k \to \infty$, we establish the desired result that almost every pair is matched with a compatible pair. ■