On the Role of Stability and Strategy-proofness under Multi-unit Demand*

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Abstract

We examine the indivisible resource allocation problem with multi-unit demand. In the literature on matching, stability and strategy-proofness have been important concepts. We pay our attention to the role of these concepts and show the following results in matching markets with multi-unit demand. First, the stable and strategy-proof rule, if exists, is unique, which coincides with the agent-optimal stable rule. Next, the stable and strategy-proof rule is robustly stable. Moreover, from the point of view of decentralized procedure, the stable and strategy-proof rule is Nash implementable.

1 Introduction

The indivisible resource allocation problem treats a problem where each object is assigned to agents through a centralized clearinghouse, based on the preferences agents

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submit. Typical examples include school choice problem. In such problem, schools are merely regarded as objects, so incentives and efficiency are involved in only agents. Same assumption is put on our paper.

In the literature, two central concepts have played an essential role in various matching contexts. A matching rule is *stable* if any agent and any agent-object pair cannot block the matching induced by the rule. A matching rule is *strategy-proof* if submitting truthfully makes no agent worse, compared with the matching under the untruthful behavior. Gale and Shapley [5] do the great work of discovering the algorithm, called the Deferred Acceptance (DA) algorithm.[1] This prominent algorithm always gives the stable matching, and more marvelously, the stable matching becomes an optimal in the sense that all agents weakly prefer the matching to any other stable matching. Fortunately, the DA algorithm is also endowed with strategy-proofness in different settings (Dubins and Freedman [4] and Roth [12]).

Observe that the results associated with strategy-proofness hold true in the setup where each agent obtains at most one object. Meanwhile, Kojima [10] shows that there is no stable and strategy-proof rule in the setting which allows each agent to get multiple objects, and he also finds the necessary and sufficient condition for the existence of the rule.

Our paper builds on the result of Kojima [10] and considers the following questions: if there exists a stable and strategy-proof rule, then how many such rules are present and whether do they also meet other properties ? To answer these questions, we will first show that the stable and strategy-proof rule is at most one. In addition, from the result of Kojima [10], the rule can be specified to the agent-optimal stable one. Then, we seek to consider the property like merging stability with stratgey-

[1] Note that they consider the two-sided matching market, where an agent in each side has a preference ranking over agents in the other side. Our paper, by contrast, examines the one-sided version.

proofness, called *robust stability*. What we will next show is that the rule indeed satisfies robust stability. In turn, the last part of this paper hits light on the strategic interaction among agents since submitting preferences forms the preference revelation game. Finally, we will show that the set of matchings induced by the desirable rule corresponds to the set of Nash equilibrium outcomes in some mechanism, that is, Nash implementability. Recently, several experiments reveal that people in the real world may take untruthful actions, even though it is best to act honestly. The last result says that by setting alternative mechanism, we can reach the outcome produced by the desirable rule, under the situation that agents play the weaker solution concept.

1.1 Related literature

The most related paper is Kojima [10]. He shows the equivalence in the same setup as ours that a stable matching rule is strategy-proof if and only if essential homogeneity holds. *Essential homogeneity* means the similar rankings objects have over agents. Also, Hatfield, Kojima, and Narita [7] analyze the many-to-many matching market and investigate the (in)compatibility between stability and strategy-proofness. However, their approach focuses on the two-sided matching, assuming the "max-min" preferences. As to the uniqueness theorem, Alcalde and Barberà [2], Sakai [13], and Hirata and Kasuya [8] tackle this issue. Concerning robust stability, Kojima [9] first introduces it and Afacan [1] extends to the group robust stability. Regarding implementability, Danilov [3] and Yamato [15] propose the necessary and sufficient condition for Nash implementability. Particularly, in the context of matching, Haeringer and Klijn [6], and Kumano [11] show the Nash implementability of agent-optimal stable rule in different environments.

2 Model

Let N and X be a finite set of agents and a finite set of objects, respectively. Each agent $i \in N$ has a preference relation R_i which is complete, transitive, and antisymmetric over the set of subsets of X. P_i is the strict part associated with R_i. Let $R = (R_i)_{i \in N}$ and $R_{-i} = (R_j)_{j \in N \setminus \{i\}}$. Also, let $\mathcal{R}_i$ be the set of all preference relations for $i \in N$ and define $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$. Besides the preference relation, each agent $i \in N$ has a quota $q_i \in \mathbb{N}$. We assume that the preference relation is **responsive**, that is, for all $i \in N$,[2]

- for all $X' \subseteq X$ with $|X'| \leq q_i$, $x \in X \setminus X'$ and $y \in X'$, $X' \cup \{x\} \setminus \{y\} R_i X'$ if and only if $x R_i y$,
- for all $X' \subseteq X$ with $|X'| \leq q_i$ and $x \in X'$, $X' R_i X' \setminus \{x\}$ if and only if $x R_i \emptyset$, and
- $\emptyset P_i X'$ for all $X' \subseteq X$ with $|X'| > q_i$.

For convenience of notation, we sometimes write the preference relation as, say, $R_i : a, b, \cdots, \emptyset$. The responsiveness of the preference relations allows us to use this formulation. A subset of objects $X' \subseteq X$ is **acceptable** to agent $i \in N$ if $X' R_i X''$ for all $X'' \subseteq X'$. Each object $x \in X$ has a strict priority (linear order) $\succ_x$ over N and has a quota $q_x \in \mathbb{N}$. Let $\succ = (\succ_x)_{x \in X}$ and $q_X = (q_x)_{x \in X}$. We call the pair $(\succ, q_X)$ a **priority structure**. Throughout the paper, fix the priority structure.

A **matching** $\mu : N \to X$ is a correspondence that satisfies (1) $\mu(i) \subseteq X$ for all $i \in N$, (2) $|\mu(i)| \leq q_i$ for all $i \in N$, and (3) each object $x \in X$ is matched to at most q_x agents. We write $\mu_x = \{i \in N | x \in \mu(i)\}$. Let $\mathcal{M}$ be the set of all matchings. A matching μ is **individually rational** under R if for all $i \in N$, $\mu(i)$ is acceptable

[2] For convenience, we frequently denote a singleton set $\{x\}$ as x.

to agent i under R_i. A matching μ is **blocked** by a pair $(i, x) \in N \times X$ under R if $i \notin \mu_x$ and either (1) $xP_i\emptyset$ and $|\mu(i)| < q_i$ or (2) xP_iy for some $y \in \mu(i)$, and either (1) $|\mu_x| < q_x$ or (2) $i \succ_x j$ for some $j \in \mu_x$. A matching μ is **stable** under R if μ is individually rational and no pair $(i, x) \in N \times X$ blocks μ under R. A matching μ **dominates** another matching ν under R if $\mu(i)R_i\nu(i)$ for all $i \in N$ and $\mu(i)P_i\nu(i)$ for some $i \in N$.

A **rule** is a function from each preference profile to a matching. We denote the assignment of i at the matching $f(R)$ by $f_i(R)$. A rule f is **stable** if for all $R \in \mathcal{R}$, $f(R)$ is stable under R. A rule f is **agent-optimal stable** if for all $R \in \mathcal{R}$, $f(R)$ is stable under R and $f(R)$ is not dominated by any other stable matching under R. A rule f is **strategy-proof** if for all $i \in N, R \in \mathcal{R}$ and $R'_i \in \mathcal{R}_i$, $f_i(R)R_if_i(R'_i, R_{-i})$. Both stability and strategy-proofness are regarded as desirable. However, no such rule exists in many-to-many matching markets (Kojima [10]). The DA rule is a rule which produces a matching via the following algorithm.

> **step 1**: Each agent i applies to q_i most acceptable objects if any. Each object rejects the lowest-ranking agents in excess of its supply among those who applied to it, keeping the remaining agents tentatively.
>
> **step t**: Each agent i who was not tentatively matched to q_i objects in step $t-1$ applies to the next highest acceptable objects up to demand if any. Each object considers these agents and agents who are tentatively held from the previous step together, and rejects the lowest-ranking agents in excess of its supply, keeping the remaining agents tentatively.

The algorithm terminates when no agent applies to an object. Each agent tentatively accepted by an object at the last step is allocated a seat in that object, resulting in a matching. $\diamondsuit$

Indeed, the DA rule sustains the stability in many-to-many setting, which implies that the DA rule is not strategy-proof by the impossibility result. The next section displays some results concerning the stable and strategy-proof rule.

3 Some results

3.1 Uniqueness of the stable and strategy-proof rule

First, we pay our attention to the number of the stable and strategy-proof rules. From an application point of view, it becomes unnecessary to wonder which rule to use as the number of the desirable rule is smaller. In fact, as Theorem 1 indicates, the stable and strategy-proof rule is unique.

Theorem 1. *There exists at most one stable and strategy-proof rule.*

Proof. In the Appendix. $\square$

What kind of rule can be a candidate for the stable and strategy-proof rule ? Theorem 1 in Kojima [10] together with Theorem 1 in our paper implies the following theorem.

Theorem 2. *The agent-optimal stable rule is the unique candidate for the stable and strategy-proof rule.*

Proof. By Kojima [10], if the stable and strategy-proof rule exists, then the DA rule is one candidate for it. By Theorem 1, the DA rule can be the unique candidate for the stable and strategy-proof rule. Also, the DA rule coincides with the agent-optimal stable rule in our setup. $\square$

3.2 Robust stability

In this subsection, further analysis is put on the property of mixing stability and strategy-proofness, which takes an element of after-market. Roughly speaking, the property eliminates the situation where any agent makes more desirable object match another agent with lower priority than him, and then he blocks the previous matching, cooperating with the object. More specifically,

Definition 1. A rule f is **robustly stable** if it satisfies the following three properties:

- f is stable,
- f is strategy-proof, and
- there exist no $i \in N, R \in \mathcal{R}$ and $R'_i \in \mathcal{R}_i$ such that (1) $X' P_i f_i(R)$ for some $X' \subseteq X$ and (2) for all $x \in X'$, either $i \succ_x j$ for some $j \in f_x(R'_i, R_{-i})$ or $|f_x(R'_i, R_{-i})| < q_x$.

Theorem 3. *The stable and strategy-proof rule is robustly stable.*

Proof. In the Appendix. □

By Theorem 3, we can exclude another kind of manipulable deviation. In the next section, we seek to consider the practical application of the stable and strategy-proof rule while considering strategic interaction among agents.

4 Preference revelation game

So far we have discussed the desirability of the stable and strategy-proof rule. From now on, we consider how to produce the matching via the stable and strategy-proof rule, by regarding the rule as a goal we would like to achieve. Let M_i be the set of strategies for $i \in N$. Denote the strategy profile by $m \in M = \times_{i \in N} M_i$. An outcome function is a function g mapping each strategy profile to a matching. We call the pair (M, g) a **mechanism**. The assignment of i at g under $m \in M$ is denoted by $g_i(m)$. A strategy profile $m \in M$ is **Nash equilibrium** at (M, g) under $R \in \mathcal{R}$ if for all $i \in N$ and all $m'_i \in M_i$, $g_i(m) R_i g_i(m'_i, m_{-i})$. Let $N(M, g, R)$ be the set of Nash equilibria at (M, g) under $R \in \mathcal{R}$, and define $g(N(M, g, R)) = \cup_{m \in N(M,g,R)} \{g(m)\}$. A matching rule f is **Nash implementable** if there exists a mechanism (M, g) such that $\{f(R)\} = g(N(M, g, R))$ for all $R \in \mathcal{R}$.

Let $L(\mu, R_i) = \{\nu \in \mathcal{M} | \mu(i) R_i \nu(i)\}$, that is, the lower contour set of a matching μ for $i \in N$ under R_i. For all $i \in N$ and all $M' \subseteq \mathcal{M}$, $\mu \in M'$ is **essential** for i in M' if there exists some preference profile $R \in \mathcal{R}$ such that $L(\mu, R_i) \subseteq M'$ and $\mu = f(R)$. Denote the set of essential matchings for $i \in N, f$, and $M' \subseteq \mathcal{M}$ by $ESS(f, i, M')$. The following two notions of monotonicity play a significant role for Nash implementability.

Definition 2. A matching rule f is **essentially monotonic** if for all $R, R' \in \mathcal{R}$,

$$\forall i \in N \quad ESS(f, i, L(f(R), R'_i)) \subseteq L(f(R), R'_i) \Rightarrow f(R') = f(R).$$

Definition 3. A matching rule f is **Maskin monotonic** if for all R,

$$\forall i \in N \quad L(f(R), R_i) \subseteq L(f(R), R'_i) \Rightarrow f(R') = f(R).$$

Remark 1. Essential monotonicity implies Maskin monotonicity by definition.

Remark 2. Yamato [15] shows that a social choice correspondence (i.e., a rule) is Nash implementable if and only if it is essentially monotonic. His environment contains ours.

Now we are ready to state the implementability result. From the next theorem, we can accomplish the matching via the stable and strategy-proof rule (which is also robustly stable) by playing a Nash equilibrium in some mechanism.

Theorem 4. *The stable and strategy-proof rule is Nash implementable.*

Proof. In the Appendix. □

This result gives a certain solution to the problem that real world humans often do not take optimal action. That is, by setting up an alternative mechanism, we can achieve a desirable matching, while we weaken the equilibrium concept.

5 Environment in which the desirable rule is guaranteed

In the final section, we note the existence of the desirable rule. In many-to-many matching, there does not necessarily exist a stable rule which is compatible with strategy-proofness.[3] The necessary and sufficient condition for the existence of such desirable rule is characterized by a class of the priority structure.

Definition 4. The priority structure $(\succ, q_X)$ is **essentially homogeneous** if there do not exist $i, j \in N$ and $x, y \in X$ such that

- $i \succ_x j, \quad j \succ_y i \quad$ and
- there exist distinct sets $N_x, N_y \subseteq N \setminus \{i, j\}$ such that $|N_x| = q_x - 1,\ |N_y| = q_y - 1,\ k \succ_x j$ for all $k \in N_x$, and $k \succ_y i$ for all $k \in N_y$.

[3]See Kojima [10] for the example which shows the impossibility.

Roughly speaking, essential homogeneity requires that objects should have the similar rankings. Due to the next theorem shown by Kojima [10], our analysis will cover the class of the priority structure satisfying essential homogeneity.

Theorem . *(Kojima [10]) There exists a stable and strategy-proof rule if and only if the priority structure satisfies essential homogeneity.*

Appendix: Omitted proof

The following Lemma plays an essential role in showing Theorem 1.

Lemma 1. *Suppose that there exist two distinct stable matchings μ and ν under R. Then, there exists an agent $i \in N$ such that $\mu(i) \nsubseteq \nu(i)$ and $\nu(i) \nsubseteq \mu(i)$; there is no case where $\mu(i) \subseteq \nu(i)$ or $\nu(i) \subseteq \mu(i)$ for all $i \in N$.*

Proof. Take any $R \in \mathcal{R}$. Let μ and ν be two distinct stable matchings under R. Suppose, by way of contradiction, that $\mu(i) \subseteq \nu(i)$ or $\nu(i) \subseteq \mu(i)$ for all $i \in N$.

Case 1: For all $i \in N$, $\mu(i) \subseteq \nu(i)$.

Let $i \in N$ be an agent such that $\mu(i) \subsetneq \nu(i)$. By the individual rationality of ν, and by strict preferences, we have $\nu(i) P_i \mu(i)$. By responsiveness, for all $x \in \nu(i) \backslash \mu(i)$, (1) $x P_i \emptyset$. Note that (2) $i \notin \mu_x$ and (3) $|\mu(i)| < q_i$ since $|\nu(i)| \leq q_i$ and $\mu(i) \subsetneq \nu(i)$. Also, since ν is a matching, $|\nu_x| \leq q_x$. Since $\mu(i) \subseteq \nu(i)$ for all $i \in N$, there exists no agent j such that $x \in \mu(j)$ and $x \notin \nu(j)$. By noting that $i \in \nu_x$ but $i \notin \mu_x$, we have (4) $|\mu_x| < q_x$.

Therefore, by (1)–(4), we can get to the contradiction that the μ is not stable under R.

Case 2: For all $i \in N$, $\nu(i) \subseteq \mu(i)$.

We can apply the same argument as that of Case 1 to this case.

Case 3: For some $i, j \in N$ with $i \neq j$, $\mu(i) \subsetneq \nu(i)$ and $\nu(j) \subsetneq \mu(j)$.

By the individual rationality of ν, and by strict preferences, we have $\nu(i)P_i\mu(i)$. By responsiveness, for all $x \in \nu(i) \setminus \mu(i)$, (1) $xP_i\emptyset$. Note that (2) $i \notin \mu_x$ and (3) $|\mu(i)| < q_i$ since $|\nu(i)| \leq q_i$ and $\mu(i) \subsetneq \nu(i)$.

Case 3–1: $|\mu_x| < q_x$.

The matching μ is blocked by the pair (i, x) under R, a contradiction.

Case 3–2: $|\mu_x| = q_x$.

For all $k \in \mu_x$, $k \succ_x i$, otherwise the pair (i, x) blocks μ due to (1)–(3). Since $i \in \nu_x$, $i \notin \mu_x$, and $|\mu_x| = q_x$, there exist $k \in \mu_x$ such that (4) $k \notin \nu_x$. By the assumption that $\mu(i) \subseteq \nu(i)$ or $\nu(i) \subseteq \mu(i)$ for all $i \in N$, we have $\nu(k) \subsetneq \mu(k)$. By responsiveness, (5) $xP_k\emptyset$. Also, (6) $|\nu(k)| < q_k$. Moreover, (7) $i \in \nu_x$ and $k \succ_x i$.

Therefore, by (4)–(7), we can get to the contradiction that ν is not stable under R.

In either case, we can get to the contradiction, and the proof is complete. □

Proof of Theorem 1

Proof. Let f and g be the stable and strategy-proof rules. Suppose, by way of contradiction, that for some R, $f(R) \neq g(R)$. Let $Ac(R_i)$ be the set of objects contained in *some* acceptable set X', that is,

$$Ac(R_i) = \{x \in X |\ x \in X' \text{ for some } X' \subseteq X \text{ which is acceptable to agent } i\ \}.$$

Let $\tilde{R}$ be a minimal one in terms of total number of acceptable objects: that is,

$f(\tilde{R}) \neq g(\tilde{R})$ and

$$f(R) \neq g(R) \Rightarrow \sum_{i \in N} |Ac(R_i)| \geq \sum_{i \in N} |Ac(\tilde{R}_i)|.$$

Since $f(\tilde{R})$ and $g(\tilde{R})$ is stable under $\tilde{R}$, by Lemma 1, there exists an agent $i \in N$ such that

$$f_i(\tilde{R}) \nsubseteq g_i(\tilde{R}) \text{ and } g_i(\tilde{R}) \nsubseteq f_i(\tilde{R}). \tag{1}$$

Assume, without loss of generality, $f_i(\tilde{R})\tilde{P}_i g_i(\tilde{R})$ by strict preferences. Let $\hat{R}_i$ be a preference for $i \in N$ such that $f_i(\tilde{R})$ is only acceptable objects, that is, $Ac(\hat{R}_i) = \{f_i(\tilde{R})\}$.[4] Define $\hat{R} = (\hat{R}_i, \tilde{R}_{-i})$. Observe that $f_i(\hat{R}), g_i(\hat{R}) \in \{f_i(\tilde{R}), \emptyset\}$, otherwise contradicts to the individual rationality of f and g.

By strategy-proofness of f, $f_i(\hat{R})\hat{R}_i f_i(\tilde{R})$. By the construction of $\hat{R}_i$,

$$f_i(\hat{R}) = f_i(\tilde{R}) \neq \emptyset. \tag{2}$$

Also, by assumption, $f_i(\tilde{R})\tilde{P}_i g_i(\tilde{R})$. By strategy-proofness of g, $g_i(\tilde{R})\tilde{R}_i g_i(\hat{R})$. Thus, these two imply that $f_i(\tilde{R})\tilde{P}_i g_i(\hat{R})$, meaning that $f_i(\tilde{R}) \neq g_i(\hat{R})$ and $g_i(\hat{R}) = \emptyset$ since $g_i(\hat{R}) \in \{f_i(\tilde{R}), \emptyset\}$. Hence, $f_i(\hat{R}) \neq g_i(\hat{R})$, and so $f(\hat{R}) \neq g(\hat{R})$.

However, by (1), there exists an object $x \in X$ such that $x \in g_i(\tilde{R}) \setminus f_i(\tilde{R})$ and $x\tilde{P}_i\emptyset$. This means that $|Ac(\tilde{R}_i)| > |Ac(\hat{R}_i)|$. By noting that $\tilde{R}_j = \hat{R}_j$ for all $j \neq i$,

$$\sum_{i \in N} |Ac(\tilde{R}_i)| > \sum_{i \in N} |Ac(\hat{R}_i)|,$$

which contradicts to the minimality of the number of acceptable objects in $\tilde{R}$.

Therefore, $f(R) = g(R)$ for all R, and the uniqueness of the stable and strategy-

[4] Note that $f_i(\tilde{R}) \neq \emptyset$, otherwise contradicts to the stability of g under $\tilde{R}$ because of $f_i(\tilde{R})\tilde{P}_i g_i(\tilde{R})$.

proof rule has been proved. $\square$

A rule f is **efficient** if for all R, there exists no matching that dominates $f(R)$ under R. A rule f is **group strategy-proof** if there exist no $N' \subseteq N, R \in \mathcal{R}$, and $R'_{N'} \in \times_{i \in N'} \mathcal{R}_i$ such that $f_i(R'_{N'}, R_{-N'}) R_i f_i(R)$ for all $i \in N'$ and $f_j(R'_{N'}, R_{-N'}) P_j f_j(R)$ for some $j \in N'$. A rule f is **nonbossy** if for all $i \in N, R \in \mathcal{R}$, and $R'_i \in \mathcal{R}_i$, $f_i(R'_i, R_{-i}) = f_i(R)$ implies $f(R'_i, R_{-i}) = f(R)$. Observe that if a rule is group strategy-proof, then it is nonbossy. The following proposition is useful for the proof of Theorem 3. Proposition 1 is very similar to Theorem 1 in Kojima [10], but we can strengthen the second statement, from strategy-proofness to group strategy-proofness.

Proposition 1. *The following statements are equivalent.*

(1) The agent-optimal stable rule is efficient.

(2) The agent-optimal stable rule is group strategy-proof.

(3) The priority structure is essentially homogeneous.

Proof. We have already known that (1) is equivalent to (3). Since group strategy-proofness implies strategy-proofness, we obtain that (2) implies (3). Regarding that (3) implies (2), let f be the agent-optimal stable rule and suppose that f is not group strategy-proof. Then, there exist $N' \subseteq N, R \in \mathcal{R}$, and $R'_{N'} \in \times_{i \in N'} \mathcal{R}_i$ such that $f_i(R'_{N'}, R_{-N'}) R_i f_i(R)$ for all $i \in N'$ and $f_j(R'_{N'}, R_{-N'}) P_j f_j(R)$ for some $j \in N'$. This implies that there exists $x \in X$ such that $j \in f_x(R'_{N'}, R_{-N'}) \setminus f_x(R)$. From now, proof proceeds exactly the same as Kojima [10]. $\square$

Proof of Theorem 3

Proof. Let f be the stable and strategy-proof rule. Suppose that f is not robustly stable. Then, there exist $i \in N$, $x \in X, R$, and R'_i such that (1) $X' P_i f_i(R)$ for some $X' \subseteq X$ and (2) for all $x \in X'$, either $i \succ_x j$ for some $j \in f_x(R'_i, R_{-i})$ or $|f_x(R'_i, R_{-i})| < q_x$.

Case 1: $f_i(R'_i, R_{-i}) = \emptyset$.

Let $R''_i : X', \emptyset$ be another preference for $i \in N$. We divide two cases.

Case 1-1: $f_i(R''_i, R_{-i}) = X'$.

Then, we have $f_i(R''_i, R_{-i}) = X' P_i f_i(R)$, implying a contradiction to the strategy-proofness of f.

Case 1-2: $f_i(R''_i, R_{-i}) = \emptyset$.

Then, by definition of R''_i, we have

$$X' P''_i f_i(R''_i, R_{-i}) \tag{3}$$

In addition, since $(\succ, q)$ is essentially homogeneous whenever the stable and strategy-poof mechanism exists, f also satisfies group strategy-proofness by Proposition 1. Thus, f is nonbossy. Hence, $f(R'_i, R_{-i}) = f(R''_i, R_{-i})$. This property implies by supposition that for all $x \in X'$, either $i \succ_x j$ for some $j \in f_x(R''_i, R_{-i})$ or $|f_x(R''_i, R_{-i})| < q_x$. This fact together with (3) contradicts to the stability of f under (R''_i, R_{-i}).

Case 2: $f_i(R'_i, R_{-i}) \neq \emptyset$.

Since f is strategy-proof and $X' P_i f_i(R)$, it follows that $f_i(R'_i, R_{-i}) \neq X'$. Let $R''_i : X', f_i(R'_i, R_{-i}), \emptyset$. Strategy-proofness of f implies that $f_i(R''_i, R_{-i}) \in \{f_i(R'_i, R_{-i}), \emptyset\}$, otherwise i can manipulate by misreporting R''_i at R.

Case 2-1: $f_i(R''_i, R_{-i}) = \emptyset$.

Then, $f_i(R'_i, R_{-i}) P''_i \emptyset = f_i(R''_i, R_{-i})$, which contradicts to strategy-proofness of f.

Case 2-2: $f_i(R''_i, R_{-i}) = f_i(R'_i, R_{-i})$.

Then, nonbossiness implies $f(R_i'', R_{-i}) = f(R_i', R_{-i})$. This implies by supposition that for all $x \in X'$, either $i \succ_x j$ for some $j \in f_x(R_i'', R_{-i})$ or $|f_x(R_i'', R_{-i})| < q_x$. This fact together with $X' P_i'' f_i(R_i'', R_{-i})$ contradicts to the stability of f under (R_i'', R_{-i}). $\square$

Proof of Theorem 4

Proof. Let f be the stable and strategy-proof rule. Suppose that f is not essentially monotonic. Then, there exist R and R' such that

$$\forall i \in N \quad ESS(f, i, L(f(R), R_i')) \subseteq L(f(R), R_i') \ \text{ and } \ f(R') \neq f(R).$$

Let $\mu = f(R)$. We will divide three cases and reach a contradiction in either case.

Case 1: μ is not individually rational under R'.

Then, there exists $i \in N$ such that $\emptyset P_i' \mu(i)$. Consider another matching ν and another preference profile $\tilde{R}$:

$$\nu = \begin{cases} \nu(i) = \emptyset & \\ \nu(j) = \mu(j) \quad \text{if} & \mu(j) \cap X' \neq \emptyset \text{ for some } X' P_i \mu(i) \\ \nu(k) = \emptyset \quad \text{otherwise} & \end{cases}$$

and

$$\tilde{R}_i : X'', \emptyset$$

$$\tilde{R}_j : \nu(j), \emptyset$$

$$\tilde{R}_k : \emptyset$$

where X'' is the same as the part above $\mu(i)$ of R_i. Then, $L(\nu, \tilde{R}_i) = L(\mu, R_i)$.

Also, ν is stable under $\tilde{R}$ because for all $l \in N$, ν is individually rational, and l cannot make a blocking pair (if so, then such blocking pair also blocks μ under R). Moreover, ν is the agent-optimal stable matching under $\tilde{R}$ since otherwise i can match more preferred objects at ν, but this is impossible because the quotas of such objects are already occupied by other agents. Hence, $\nu = f(\tilde{R})$, meaning that $\nu \in ESS(f, i, L(\mu, R_i))$ by $L(\nu, \tilde{R}_i) \subseteq L(\nu, \tilde{R}_i)$ and $L(\nu, \tilde{R}_i) = L(\mu, R_i)$. By assumption, $\nu \in L(\mu, R'_i)$. Therefore, $\mu(i) R'_i \nu(i) = \emptyset P'_i \mu(i)$, a contradiction.

Case 2: There exists a blocking pair $(i, x) \in N \times X$ under R'.

Then, $i \notin \mu_x$ and either (1) $x P'_i \emptyset$ and $|\mu(i)| < q_i$ or (2) $x P'_i y$ for some $y \in \mu(i)$, and either (1) $|\mu_x| < q_x$ or (2) $i \succ_x j$ for some $j \in \mu_x$. We divide two cases depending on i's conditions.

Case 2–1: $x P'_i \emptyset$ and $|\mu(i)| < q_i$.

By responsiveness, $\mu(i) \cup \{x\} P'_i \mu(i)$. We observe that $\mu(i) R_i \mu(i) \cup \{x\}$ since otherwise μ is not stable under R. Consider another matching ν and another preference profile $\tilde{R}$:

$$\nu = \begin{cases} \nu(i) = \mu(i) \cup \{x\} & \\ \nu(j) = \mu(j) & \text{if} \quad \mu(j) \cap X' \neq \emptyset \text{ for some } X' P_i \mu(i) \\ \nu(k) = \emptyset & \text{otherwise} \end{cases}$$

and

$$\tilde{R}_i : X'', \mu(i) \cup \{x\}, \emptyset$$
$$\tilde{R}_j : \nu(j), \emptyset$$
$$\tilde{R}_k : \emptyset$$

where X'' is the same as the part above $\mu(i)$ of R_i. Then, $L(\nu, \tilde{R}_i) = L(\mu, R_i)$. Similar to Case 1, $\nu = f(\tilde{R})$, meaning that $\nu \in ESS(f, i, L(\mu, R_i)) \subseteq L(\mu, R'_i)$. Therefore, $\mu(i) R'_i \nu(i) = \mu(i) \cup \{x\} P'_i \mu(i)$, a contradiction.

Case 2–2: $x P'_i y$ for some $y \in \mu(i)$.

By responsiveness, $(\mu(i) \setminus \{y\}) \cup \{x\} P'_i \mu(i)$. We observe that $\mu(i) R_i (\mu(i) \setminus \{y\}) \cup \{x\}$ since otherwise μ is not stable under R. Consider another matching ν and another preference profile $\tilde{R}$:

$$\nu = \begin{cases} \nu(i) = (\mu(i) \setminus \{y\}) \cup \{x\} & \\ \nu(j) = \mu(j) & \text{if} \qquad \mu(j) \cap X' \neq \emptyset \text{ for some } X' P_i \mu(i) \\ \nu(k) = \emptyset & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} &\tilde{R}_i : X'', (\mu(i) \setminus \{y\}) \cup \{x\}, \emptyset \\ &\tilde{R}_j : \nu(j), \emptyset \\ &\tilde{R}_k : \emptyset \end{aligned}$$

where X'' is the same as the part above $\mu(i)$ of R_i. Then, $L(\nu, \tilde{R}_i) = L(\mu, R_i)$. Similar to Case 1, $\nu = f(\tilde{R})$, meaning that $\nu \in ESS(f, i, L(\mu, R_i)) \subseteq L(\mu, R'_i)$. Therefore, $\mu(i) R'_i \nu(i) = (\mu(i) \setminus \{y\}) \cup \{x\} P'_i \mu(i)$, a contradiction.

Case 3: μ is stable, but not agent-optimal stable under R'.

Then, there is the agent-optimal stable matching ν under R'. That is, $\nu(i) R'_i \mu(i)$ for all $i \in N$ and $\nu(j) P'_j \mu(j)$ for some $j \in N$. Let $N' = \{j \in N | \nu(j) P'_j \mu(j)\}$.

Case 3–1: For some $j \in N'$, $\mu(j) P_j \nu(j)$.

Take any $j \in N'$ such that $\mu(j) P_j \nu(j)$. Consider another matching η and another preference profile $\tilde{R}$:

$$\eta = \begin{cases} \eta(j) = \nu(j) & \\ \eta(k) = \mu(k) & \text{if} \qquad \mu(k) \cap X' \neq \emptyset \text{ for some } X' P_j \mu(j) \\ \eta(l) = \emptyset & \text{otherwise} \end{cases}$$

and

$$\tilde{R}_i : X'', \nu(j), \emptyset$$
$$\tilde{R}_j : \mu(k), \emptyset$$
$$\tilde{R}_k : \emptyset$$

Then, $L(\eta, \tilde{R}_j) = L(\mu, R_j)$ and η is agent-optimal stable under $\tilde{R}$, that is, $\eta = f(\tilde{R})$. Thus, $\eta \in ESS(f, j, L(\mu, R_j)) \subseteq L(\mu, R'_j)$. Therefore, $\mu(j) R'_j \eta(j) = \nu(j) P'_j \mu(j)$, a contradiction.

Case 3–2: For all $j \in N'$, $\nu(j) P_j \mu(j)$.

Consider another preference profile $\tilde{R}$:

$$\forall j \in N', \quad \tilde{R}_j : \nu(j), \mu(j), \emptyset$$
$$\forall k \in N \setminus N', \quad \tilde{R}_k : \mu(k), \emptyset$$

By antisymmetry of preferences, each agent who is not in N' gets the same objects under μ and ν. Then, on one hand, for all $i \in N$, $L(\nu, R'_i) \subseteq L(\nu, \tilde{R}_i)$, so $\tilde{R} \in MT(\nu, R')$. Note here that group strategy-proofness implies Maskin monotonicity by Takamiya [14]. So, f satisfies Maskin monotonicity, which implies $\nu = f(\tilde{R})$. On the other hand, for all $i \in N$, $L(\mu, R_i) \subseteq L(\mu, \tilde{R}_i)$, so $\tilde{R} \in MT(\mu, R)$. By Maskin monotonicity of f, $\mu = f(\tilde{R})$. Therefore, $\mu = \nu$, a contradiction.

□

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