

Equilibrium Selection in the War of Attrition under Complete Information*

Youngsoo Kim, George Georgiadis, and H. Dharma Kwon[†]

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Abstract

We consider a two-player game of war of attrition under complete information. It is well-known that this class of games admits equilibria in pure, as well as mixed strategies, and much of the literature has focused on the latter. We show that if the players' payoffs whilst in "war" vary stochastically and their exit payoffs are heterogeneous, then the game admits Markov Perfect equilibria in pure strategies *only*. This is true irrespective of the degree of randomness and heterogeneity, thus highlighting the fragility of mixed-strategy equilibria to a perturbation of the canonical model. In contrast, when the players' flow payoffs are deterministic *or* their exit payoffs are homogeneous, we show that the game admits equilibria in pure *and* mixed strategies.

1 Introduction

In the classic war of attrition, the first player to quit concedes a prize to his opponent. Each player trades off the cost associated with fighting against the value of the prize. These features are common in many managerial and economic problems. Oligopolists in a declining industry may bear losses in anticipation of profitability following a competitor's exit (Ghemawat and Nalebuff, 1985). For example, the rise of Amazon in the mid-1990s made the business model of Barnes & Noble and Borders obsolete, turning traditional bookselling into a declining market. As the demand shrank sharply, these two major players at the time had to cut down slack in their capacities, but each would prefer its competitors to carry the painful burden of closing stores or exit the market altogether (Newman, 2011). Similarly, the presently low price of crude oil is often attributed to

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[†]Y. Kim: Gies College of Business, University of Illinois at Urbana-Champaign; ykim180@illinois.edu.;
G. Georgiadis: Kellogg School of Management, Northwestern University; g-georgiadis@kellogg.northwestern.edu.;
H.D. Kwon: Gies College of Business, University of Illinois at Urbana-Champaign; dhkwon@illinois.edu.

a war of attrition among Saudi Arabia, its Persian Gulf OPEC allies, and non-OPEC rivals such as Russia and the many shale-oil producers in the United States (Reed, 2016). Other examples of wars of attrition include the provision of public goods (Bliss and Nalebuff, 1984), lobbying (Becker, 1983), labor disputes (Greenhouse, 1999), court of law battles (McAfee, 2009), races to dominate a market (Ghemawat, 1997), technology standard races (Bulow and Klemperer, 1999), price cycles in oligopolistic collusion (Maskin and Tirole, 1988), all-pay auctions (Krishna and Morgan, 1997), and bargaining games (Abreu and Gul, 2000).

It is well-known that the canonical model of war of attrition admits equilibria in both pure and mixed strategies (Tirole (1988), Fudenberg and Tirole (1996), Levin (2004), and others). Moreover, much of the (applied) literature has focused on the mixed-strategy equilibria of the game, owing to the fact that *only* they feature certain attractive properties such as attrition (*i.e.*, costly waste of resources), and symmetry in certain cases. In this paper, we study a simple model of war of attrition under complete information, and we show that if the players' flow payoffs whilst fighting for the prize are stochastic and their exit payoffs are heterogeneous, then the game admits only pure-strategy Markov Perfect equilibria (hereafter MPE). Moreover, this is true irrespective of the degree of randomness and heterogeneity. As it is rare for players to be precisely identical or payoffs to be deterministic (in applications), this result highlights the instability of the mixed-strategy equilibria in wars of attrition under complete information. This result also has implications for a growing literature that aims to empirically characterize strategies in real-world games of war of attrition; see for example Wang (2009) and Takahashi (2015).

In our model, two competing oligopolists contemplate exiting a market. While both firms remain in the market, each receives a flow payoff that depends on the stochastically fluctuating market conditions (*e.g.*, the price of a relevant commodity); hereafter the *state*. At every moment, each firm can exit the market and collect its outside option. Its rival then obtains a (higher) *winner's* payoff, which depends on the state at the time of exit; *i.e.*, the net present discounted monopoly profit in that market. We assume that all payoff-relevant parameters are common knowledge, so this is a game with complete information. Noting that the state follows a Markov process and the mixed-strategy equilibria characterized in the literature are typically stationary (see for example, Tirole (1988)), we restrict attention to Markov strategies, wherein at every moment, each firm conditions its probability of exit on the current state.

We begin by characterizing the best response of a firm that anticipates its rival will never exit the market. We show that a firm will optimally exit at the first moment the state drifts below a threshold. Moreover, this *single decision-maker's optimal* threshold increases in the firm's outside option. Intuitively, the better is a firm's outside option, the less it is willing to endure poor market conditions before exiting the market. Proposition 1 shows that there exists a pure-strategy MPE in which the firm with the larger outside option exits the market at its single decision-maker's optimal

threshold, and its rival never exits. If the heterogeneity in outside options is not too large, then there exists another pure-strategy MPE in which the firm with the lower outside option exits the market at its own single decision-maker's optimal threshold and its rival never exits.

Towards our main result, we establish two lemmas. The first shows that in any mixed-strategy MPE, even if the state evolves deterministically, (a) both firms must be randomizing between remaining in the market and exiting on a common set of states (*i.e.*, their strategies must have common support), and (b) strategies must be continuous in the interior of their support (*i.e.*, during any interval $(t, t + dt)$, the probability that a firm exits the market must be of order dt). The second lemma shows that if the state evolves stochastically, then each firm's strategy must be continuous everywhere, including at the boundary of its support, and its support must be equal the set of states below its single decision-maker's optimal exit threshold.

The main result follows immediately: If the market conditions are stochastic and the firms have heterogeneous outside options, in which case their single decision-maker's optimal exit thresholds differ, then the game admits no mixed-strategy MPE. Both of these ingredients are necessary for the game to admit only pure-strategy MPE: if the firms have identical outside options *or* the market conditions are deterministic, then there exists a mixed-strategy MPE which we characterize.

Related Literature

First and foremost, this paper contributes to the literature on wars of attrition, which has received widespread attention since the seminal work of Maynard Smith (1974). Our model is closest to Hendricks et al. (1988) and Murto (2004). The former provides a complete characterization of the equilibria (in both pure and mixed strategies) in a war of attrition under complete information, in which the players' payoffs vary deterministically over time. The latter considers stochastic payoffs, but restricts attention to pure-strategy Markov Perfect equilibria. In contrast, we allow payoffs to vary stochastically, and we show that if players are heterogeneous, then the game admits MPE in pure strategies *only*.

Our paper also contributes to a strand of literature that contemplates equilibrium selection in the war of attrition. Fudenberg and Tirole (1986) consider a game of exit in a duopoly, in which players are uncertain about their rivals' cost of remaining in the market. In the unique equilibrium, each firm exits at a deterministic time that decreases in its cost. In Kornhauser et al. (1989), with a small probability, each player is irrational and never quits. They show that this approach of introducing a small probability of irrational behavior selects a unique equilibrium in which the weaker player quits immediately. Kambe (1999), and Abreu and Gul (2000) analyze a bargaining game in which two players seek to divide some surplus, and each player behaves irrationally with some probability. They show that this possibility of irrational behavior gives rise to a unique equilibrium that entails delay. See also Myatt (2005) and the references therein. In contrast, players are rational and possess

complete information about the parameters of the game in our model.

Touzi and Vieille (2002) introduces the concept of mixed strategies in continuous-time Dynkin games (a class of stopping games), and proves that the game admits minimax solutions in mixed strategies. With this notion of mixed strategies, Seel and Strack (2016) investigates a war of attrition (all-pay auction) with privately observed Brownian motions, and Steg (2015) characterizes equilibria in both pure and mixed strategies in a family of continuous-time stochastic timing games. Whereas these papers consider games with identical players, we focus on games with non-homogeneous players and show that the set of equilibria differ drastically from the case with homogenous players. Riedel and Steg (2017) examines mixed-strategy equilibria in continuous-time stopping games with heterogeneous players, but they restrict attention to pre-emption games, whereas our model is one of a war of attrition.

Finally, our paper is also related to the literature in real option games in the context of timing decisions with externalities under uncertainty. Dixit and Pindyck (1994) establishes the fundamental framework for analyzing real options and real option games. Grenadier (2002), Lambrecht and Perraudin (2003), and Mason and Weeds (2010) examine the interplay between the option value of waiting and externalities due to competition, learning, and network effects. However, these papers focus on the role of a preemptive threat in real option games while our work is focused on a free-riding incentive.

2 Model

We consider a war of attrition with complete information between two oligopolistic firms. Time is continuous, and both firms discount time at rate $r > 0$. At every moment, each firm decides whether to remain or exit the market.

While both firms remain in the market, each firm earns a flow profit $\pi(X_t)$, where $\pi : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and strictly increasing, while X_t is a scalar that captures the *market conditions* that the firms operate in (*e.g.*, the size of the market, or the price of raw materials).¹ The market conditions fluctuate over time according to

$$dX_t^x = \mu(X_t^x)dt + \sigma(X_t^x)dB_t, \quad (1)$$

where X_t^x is defined on $\mathcal{I} := (a, b) \subseteq \mathbb{R}$, $X_0^x = x$, the functions $\mu : \mathcal{I} \rightarrow \mathbb{R}$ and $\sigma : \mathcal{I} \rightarrow \mathbb{R}_+$ are

¹For simplicity, we assume that the firms earn identical flow profits. Our results are generalizable to allow heterogeneous flow profits.

continuous, and B_t is a Wiener process.^{2,3,4} Let $(\Omega, \{\mathcal{F}_t\}_{t \geq 0})$ denote the sample space Ω and a natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$ with respect to the process $\{B_t\}_{t \geq 0}$ (or equivalently $\{X_t\}_{t \geq 0}$). Throughout the paper, we let $\mathbb{E}[\cdot]$ denote the expected values. In particular, we often use the notation $\mathbb{E}^x[\cdot] := \mathbb{E}[\cdot | X_0 = x]$ for expected values conditional on the initial value $X_0 = x$.

If firm i chooses to exit at time t , then it receives its outside option l_i , and firm $-i$ receives $w(X_t^x)$, where $w : \mathcal{S} \rightarrow \mathbb{R}$ is the expected (finite) payoff associated with being the sole remaining firm; *i.e.*, the expected net present value of monopoly profits. We say that firm j is the *winner*, and firm i is the *loser*. We set the convention that $l_1 \leq l_2$; *i.e.*, firm 2 has a larger outside option than firm 1. We assume that $w(x) > l_2$ for all $x \in \mathcal{S}$ so that the winner's reward is always larger than that of the loser. The game ends as soon as a firm exits the market. If both firms exit at the same moment, then each firm obtains the outside option l_i or $w(X_t^x)$ with probability $1/2$ each.

Finally, we make the following technical assumptions (see also Alvarez, 2001):

1. $\pi(\cdot)$ satisfies the absolute integrability condition $\mathbb{E}^x \left[\int_0^\infty |e^{-rt} \pi(X_t^x)| dt \right] < \infty$.
2. For each i , there exists some $x_{ci} \in \mathcal{S}$ such that $\pi(x_{ci}) = rl_i$.

The first assumption ensures that each firm's payoff is well-defined, whereas the second guarantees the existence of an optimal exit threshold in the interior of \mathcal{S} .

2.1 Markov Strategies

At every moment t , each firm chooses the probability with which to remain in the market to maximize its expected discounted payoff. We assume that both firms employ Markov strategies, so for any x , their decision at time t depends only on the current state X_t^x . We make the definition of a Markov strategy mathematically precise below. Each firm i 's strategy can be defined as a family of cumulative distribution functions (hereafter CDF) $G_i := (G_i^x)_{x \in \mathcal{S}}$ of stopping times with respect to $\{\mathcal{F}_t\}_{t \geq 0}$ for each $x \in \mathcal{S}$.⁵ We say that a pair (G_1, G_2) is a *strategy profile*. For each i , G_i must be time-consistent, or equivalently, conform to Bayes' rule; *i.e.*, for any $t \geq s \geq 0$, $G_i^x(t) = G_i^x(s^-) + [1 - G_i^x(s^-)]G_i^{X_s^x}(t - s)$.

For example, suppose that neither firm has yet exited by time t and $X_t = x$. Then firm i employs the strategy G_i^x , and if neither firm exits during $(t, t + dt)$, then the state evolves to $x + dX_t$ at time

²We use the superscript in X_t^x to denote its dependence on the initial value x at time 0.

³Special cases in which $\sigma(\cdot) = 0$ have been analyzed extensively (Ghemawat and Nalebuff, 1985, Hendricks et al., 1988, and others). Therefore, we restrict attention to $\sigma(\cdot) > 0$ in the main body of this paper, and for completeness, we revisit the case in which $\sigma(\cdot) = 0$ in Appendix A.

⁴The boundary points a and b are assumed to be *natural* (Borodin and Salminen, 1996, p.18-20); *i.e.*, neither a , nor b can be reached by X_t^x in finite time. For example, if X_t is a standard diffusion process, then $\mathcal{S} = \mathbb{R}$. If X_t is a geometric diffusion process, then $\mathcal{S} = (0, \infty)$.

⁵By a CDF of stopping times, we refer to an \mathcal{F}^X -adapted, right-continuous, and non-decreasing process that ranges in the interval $[0, 1]$, where $\mathcal{F}^X := \{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration generated by X .

$t + dt$, at which moment firm i employs the strategy $G_i^{x+dX_t}$. This definition extends the concept of randomized stopping times (Touzi and Vieille, 2002) and subgames (Steg, 2015) to a continuous-time game with a stochastic state variable.

Note that firm i exits at time t with positive probability when G_i^x either has an (upward) jump at t , or it is continuously (strictly) increasing at t . One characteristic of a Markov strategy is that any jump in G_i^x occurs at a hitting time $\tau_E = \inf\{t \geq 0 : X_t \in E\}$ for some set $E \subset \mathcal{S}$, and that the probability of exit at τ_E depends only on $X_{\tau_E}^x$. The other defining characteristic of a Markov strategy is that the hazard rate of exit is a function of X_t^x alone whenever G_i^x continuously increases in t .

A special case of a Markov strategy G_i is one in which there exists a stopping time (a hitting time of a set E) τ_i at which G_i^x jumps from 0 to 1. We call a strategy of this form a *pure* Markov strategy and denote it by $H(\tau_i)$, where $G_i^x(t) = H^x(t; \tau_i) := \mathbf{1}_{\{t \geq \tau_i\}}(t)$. In contrast, if a Markov strategy G_i cannot be represented by $H(\tau_i)$ for any stopping time τ_i , then we refer to it as a *mixed* Markov strategy.

Lastly, we define the *support* of a mixed-strategy as a subset of the state space in which firm i randomizes between remaining in the market and exiting; *i.e.*,

$$\text{supp}(G_i) := \left\{ x \in \mathcal{S} : \left. \frac{dG_i^y(t)}{dt} \right|_{t=\tau} > 0 \text{ or } \Delta G_i^y(\tau) \in (0, 1) \text{ for any } y \in \mathcal{S} \text{ whenever } X_\tau^y = x \right\},$$

where $\Delta G_i^y(\tau) = G_i^y(\tau) - G_i^y(\tau^-)$ denotes a jump at time τ .

3 Preliminaries

In this section, we introduce notation and establish a lemma that will be helpful for the subsequent analysis. In particular, in Section 3.1, we characterize each firm's expected discounted payoff given an arbitrary strategy profile. Then in Section 3.2, we characterize the best response of a firm which anticipates that its rival will never exit the market.

3.1 Payoffs

We begin by defining the conditional expected payoff of firm i , given the history \mathcal{F}_t of X starting at $X_0^x = x$ and its rival exiting the market at t , at which time it becomes the *winner*:

$$W_i^x(t) = \int_0^t \pi(X_s^x) e^{-rs} ds + w(X_t^x) e^{-rt}. \quad (2)$$

Firm i receives the flow payoff $\pi(X_s^x)$ during $[0, t)$, whereas at time t , its rival, firm $-i$ exits and firm i receives the winner's payoff $w(X_t^x)$. Similarly, we define the conditional expected payoff of firm i ,

given the history \mathcal{F}_t of X starting at $X_0^x = x$ and it exiting the market at t , at which time it becomes the *loser*:

$$L_i^x(t) = \int_0^t \pi(X_s^x) e^{-rs} ds + l_i e^{-rt}, \quad (3)$$

If both firms exit at t , then we assume that either firm becomes the winner with equal probability, so each firm obtains conditional expected payoff $M_i^x(t) = (L_i^x(t) + W_i^x(t))/2$. We define

$$S_i^x(t; G_{-i}) = \int_0^{t^-} W_i^x(s) dG_{-i}^x(s) + M_i^x(t) \Delta G_{-i}^x(t) + L_i^x(t) [1 - G_{-i}^x(t)]. \quad (4)$$

Note that $S_i^x(t; G_{-i})$ denotes the conditional expected payoff of firm i , given the history \mathcal{F}_t of X starting at $X_0^x = x$, it exiting at t , and its rival employing strategy G_{-i} . The first term captures the payoff associated with becoming the winner at any time before t . The second term captures the payoff associated with both firms exiting simultaneously at t , and the last term captures the payoff associated with becoming the loser at t .

Finally, define firm i 's expected payoff under an arbitrary strategy profile (G_i, G_{-i}) starting at $X_0^x = x$ by

$$V_i(x; G_i, G_{-i}) = \mathbb{E}^x \left[\int_0^\infty S_i(t; G_{-i}) dG_i^x(t) \right]. \quad (5)$$

We say that a strategy profile (G_1^*, G_2^*) is an MPE if for each i , $V_i(x; G_i^*, G_{-i}^*) \geq V_i(x; G_i, G_{-i}^*)$ for all $x \in \mathcal{I}$ and any G_i .

3.2 Best Response to $H(\infty)$

As a building block towards characterizing the MPE of the game, we begin by characterizing firm i 's best response to $H(\infty)$; *i.e.*, the best response of firm i when its opponent's strategy is to never exit the market. In this case, $G_{-i}^x(t) = 0$ for any $x \in \mathcal{I}$ and $t < \infty$, so firm i 's best response can be determined by solving the following optimal stopping problem of a single decision maker:

$$\sup_{\tau_i} V_i(x; H(\tau_i), H(\infty)) = \sup_{\tau_i} \mathbb{E}^x [L_i(\tau_i)] = \sup_{\tau_i} \mathbb{E}^x \left[\int_0^{\tau_i} \pi(X_t) e^{-rt} dt + l_i e^{-r\tau_i} \right]. \quad (6)$$

We use Proposition 2 in Alvarez (2001, p.334) to establish the following lemma.

Lemma 1 *For each $i \in \{1, 2\}$, there exists a unique threshold θ_i^* such that firm i optimally exits at*

$$\tau_i^* = \inf \{t \geq 0 : X_t^x \leq \theta_i^*\}; \quad (7)$$

i.e., at the first time such that $X_t^x \leq \theta_i^*$. If $l_1 < l_2$, then $\theta_1^* < \theta_2^*$.

The firm's value of remaining in the market decreases as the market conditions deteriorate, and once they become sufficiently poor, the firm is better off exiting and collecting its outside option. As the firms earn identical flow payoffs while they remain in the market, the firm with the higher outside option optimally exits at a higher threshold.

4 Markov Perfect Equilibria

In this section, we characterize the MPE of this game. We begin by characterizing the pure-strategy MPE in Section 4.1. In Section 4.2, we establish our main result: if the market conditions fluctuate stochastically (*i.e.*, $\sigma(\cdot) > 0$) and the firms are heterogeneous (*i.e.*, $l_1 < l_2$), then this game has no mixed-strategy MPE.

4.1 Pure-strategy MPE

The following result shows that the strategy profile $(H(\infty), H(\tau_2^*))$ constitutes an MPE, and under certain conditions, $(H(\tau_1^*), H(\infty))$ also constitutes an MPE, where τ_1^*, τ_2^* are defined in Lemma 1.

Proposition 1 *The strategy profile $(G_1, G_2) = (H(\infty), H(\tau_2^*))$ is a pure-strategy MPE. Moreover, there exists a threshold $\kappa > 0$ that is independent of l_1 such that $(G_1, G_2) = (H(\tau_1^*), H(\infty))$ is also a pure-strategy MPE if $l_2 < l_1 + \kappa$.*

If firm i expects its rival to never exit the market, then by Lemma 1, it will optimally exit at the first time such that $X_t^x \leq \theta_i^*$. Therefore, it suffices to show that if firm i employs the strategy $H(\tau_i^*)$, then its opponent's best response is to never exit.

Suppose that firm 1 expects its rival to exit at the first moment that $X_t^x \leq \theta_2^*$. Recall that firm 2 has a better outside option than firm 1 (*i.e.*, $l_2 \geq l_1$), so by Lemma 1, $\theta_1^* \leq \theta_2^*$, which implies that firm 1 has no incentive to exit until at least $X_t^x \leq \theta_1^*$. Therefore, firm 1 expects that the game will end before the state drifts below θ_1^* , and hence the strategy $G_1 = H(\infty)$ is incentive compatible. If instead firm 2 anticipates that its rival employs the strategy $H(\tau_1^*)$, then it can sustain the strategy $H(\infty)$ as long as it does not need to wait too long in the time interval (τ_2^*, τ_1^*) until firm 1 exits, and so $H(\infty)$ is a best response for firm 2 as long as $l_2 - l_1$ is not too large.

Note that we restrict attention to single-threshold strategies, so $(H(\tau_1^*), H(\infty))$ and $(H(\infty), H(\tau_2^*))$ are the sole candidates for pure-strategy MPE. As shown in Murto (2004), there may also exist pure-strategy equilibria with multiple exit thresholds.⁶ However, these pure-strategy MPE with multiple

⁶In particular, Murto (2004) shows that there may exist an equilibrium in which each firm i exits at the first moment such that $X_t^x \in (-\infty, a_i] \cup [b_i, \theta_i^*]$ for some $a_i < b_i$; *i.e.*, firm i does not exit within some interval (a_i, b_i) below the threshold

thresholds do not affect our characterization of mixed-strategy MPE, and so we do not consider them in this paper.

4.2 Mixed-strategy MPE

We begin by establishing two Lemmas, which outline a set of necessary conditions that any mixed-strategy MPE must satisfy. Below we let Γ° denote the interior of a set Γ , and $\overline{\Gamma^\circ}$ denote the closure of Γ° .⁷

Lemma 2 *Suppose that (G_1, G_2) constitutes a mixed-strategy MPE. Then:*

- (a) *The supports of G_1 and G_2 have common interior Γ° .*
- (b) *If $x \in \overline{\Gamma^\circ}$, then both $G_1^x(t)$ and $G_2^x(t)$ are continuous at any $t = \tau$ such that $\Pr(X_\tau^x \in \overline{\Gamma^\circ}) > 0$.*

It is helpful to convey the intuition with a heuristic derivation. (The formal arguments are relegated to Appendix C.) In the interior of the support of G_i , firm i must be indifferent between exiting immediately and remaining in the market; *i.e.*,

$$l_i = \frac{dG_{-i}^x(t)}{1 - G_{-i}^x(t)} w(X_t^x) + \left(1 - \frac{dG_{-i}^x(t)}{1 - G_{-i}^x(t)} \right) [\pi(X_t^x)dt + (1 - rdt)l_i]. \quad (8)$$

where $dG_{-i}^x(t)/[1 - G_{-i}^x(t)]$ represents the probability that firm $-i$ will exit during $(t, t + dt)$, conditional on not having exited until t . The left-hand-side of (8) represents firm i 's payoff in case it exits at t . If it remains in the market, then with probability $dG_{-i}^x(t)/[1 - G_{-i}^x(t)]$ it receives the winner's payoff $w(X_t^x)$, whereas with the complementary probability, it earns the flow payoff $\pi(X_t^x)$ during $(t, t + dt)$, and its (discounted) continuation profit l_i at $t + dt$.⁸ It follows from (8) that firm $-i$'s probability of exit during $(t, t + dt)$, where $dt \simeq 0$, must equal

$$\frac{dG_{-i}^x(t)}{1 - G_{-i}^x(t)} = \frac{rl_i - \pi(X_t^x)}{w(X_t^x) - l_i} dt. \quad (9)$$

Notice that if $\pi(X_t^x) > rl_i$, then the right-hand-side of (8) is strictly larger than l_i , and so firm i strictly prefers to remain in the market regardless of its rival's strategy; *i.e.*, any x with $\pi(x) > rl_i$ does not belong to the support of G_i .

θ_i^* . Intuitively, for $x \in (a_1, b_1)$, if $x - a_1$ or $b_1 - x$ is sufficiently small, then firm 2 can be better off waiting until X_t^x hits a_1 or b_1 and becoming the winner rather than exiting immediately. Finally, note that if the initial state $x \geq \max\{\theta_1^*, \theta_2^*\}$, then the outcome of this equilibrium coincides with the outcome of the equilibrium characterized in Proposition 1.

⁷Clearly, $\overline{\Gamma^\circ}$ is always a subset of $\overline{\Gamma}$. Note also that it can be a *proper* subset; *i.e.*, $\overline{\Gamma^\circ} \subsetneq \overline{\Gamma}$, if and only if there are some *point* components of Γ that are not in Γ° . A point component of a set A is defined as a singleton point set $\{a\}$ such that $a \in A$ but disconnected from $A \setminus \{a\}$.

⁸We ignore the event that both firms exit the market simultaneously. As the proof shows, this is an innocuous simplification.

Towards a contradiction, suppose that there exists a non-empty interval that is in the interior of the support of G_i but not of G_{-i} . Then for at least some x in that interval, we must have $\pi(x) < rl_i$ and $dG_{-i}^x(0) = 0$. This implies that the right-hand-side of (8) is strictly smaller than l_i , so firm i strictly prefers to exit. However, this contradicts that x is in the interior of the support of G_i , so we conclude that the supports of G_1 and G_2 share the same interior.

Second, observe from (9) that in the interior of the common support of G_1 and G_2 , $dG_i^x(t)/dt$ is finite for each i , which implies that strategies are continuous. If a firm's strategy were discontinuous at some state in the interior of its support, then its rival would strictly prefer to remain in the market when that state is reached in order to increase the probability of obtaining the winner's payoff.

Lemma 2 holds irrespective of whether the market conditions fluctuate stochastically (*i.e.*, $\sigma(\cdot) > 0$), or deterministically. The following lemma establishes two additional necessary conditions that any mixed-strategy MPE must satisfy when $\sigma(\cdot) > 0$.

Lemma 3 *Suppose that $\sigma(\cdot) > 0$, and (G_1, G_2) constitutes a mixed-strategy MPE. Then:*

- (a) $G_1^x(t)$ and $G_2^x(t)$ are continuous in t for all $x \in \mathcal{I}$, *i.e.*, they have no mass points (discontinuities of the CDFs).
- (b) The support $\Gamma = (a, \theta_1^*) = (a, \theta_2^*)$, where θ_i^* is given in Lemma 1.

Lemma 3(a) establishes that if $\sigma(\cdot) > 0$, then CDFs of an MPE must be continuous even if the initial value x is not in $\bar{\Gamma}^\circ$. To see why, we first recall that a discontinuity of a CDF must be a hitting time $\tau_E = \inf\{t \geq 0 : X_t^x \in E\}$ for some set $E \subset \mathbb{R}$ and for all x such that $\Pr(\tau_E < \infty) > 0$. Then because X is irreducible if $\sigma(\cdot) > 0$, Lemma 2(b) implies that if $\sigma(\cdot) > 0$, then a discontinuity of a CDF cannot take place while X is within $\bar{\Gamma}^\circ$ for $\forall x \in \mathcal{I}$, *i.e.*, irrespective of the initial value x . Hence, a mass point can exist only outside $\bar{\Gamma}^\circ$, in which case Γ must have a point component. However, we can further show that Γ cannot have a point component. Suppose to the contrary that $\{y\}$ is a point component of Γ . Then both firms assign a non-zero probability of exit when X hits y . However, in that case, one firm may decide never to exit when $X = y$ thereby increasing the probability of being the winner. Thus, a point component of Γ cannot be part of an equilibrium, and so $\bar{\Gamma} = \bar{\Gamma}^\circ$. Thus, we obtain the result that the CDFs of an MPE are continuous in time.

Next, recall that even if firm i anticipates that its rival will never exit the market, it is not willing to exit before X goes below θ_i^* . Hence, if firm i expects its rival to exit in finite time with positive probability, then this would, *ceteris paribus*, only decrease firm i 's incentive to exit. Consequently, firm i *always* strictly prefers to remain in the market whenever $X > \theta_i^*$, which implies $\Gamma \subseteq (a, \theta_i^*)$. To see that this inclusion is indeed an equality, suppose that, for some $\theta < \theta_i^*$, firm $-i$ (and hence firm i) exits with positive probability on (a, θ) . Then because firm $-i$ does not exit at any $X > \theta$ and its strategy has no mass points, starting at θ_i^* , firm i 's expected payoff from exiting at the first time

that $X_t^{\theta_i^*} \leq \theta$ is strictly less than l_i by Lemma 1. Therefore, firm i strictly prefers to exit at θ_i^* , which contradicts the premise that $\Gamma = (a, \theta)$ where $\theta < \theta_i^*$.

Recall from Lemma 1 that if $l_1 < l_2$, then $\theta_1^* < \theta_2^*$. Therefore, we have the following immediate implication.

Theorem 1 *Suppose that $\sigma(\cdot) > 0$ and $l_1 < l_2$. Then this game admits no mixed-strategy MPE.*

While the assumptions that payoffs are deterministic and firms are symmetric may be a good approximation of a particular setting, in reality, payoffs are not set in stone and no firms are exactly alike. This theorem, together with Proposition 1, shows that in this case, the game admits either one, or two pure-strategy MPE only.

Because the result holds irrespective of the degree of uncertainty (and heterogeneity), it shows that mixed-strategy MPE are unstable to a natural perturbation of the canonical model, and provides an equilibrium selection argument for wars of attrition under complete information.

Finally, we point out that both ingredients are necessary to eliminate mixed-strategy MPE. To highlight this point, in the following section and in Appendix A, we characterize a mixed-strategy MPE for the case in which firms are homogeneous and payoffs are deterministic, respectively.

Special Case: Homogeneous Firms ($l_1 = l_2$)

In this section, we consider the case in which the firms are homogeneous (*i.e.*, $l_1 = l_2$), and we characterize the unique mixed-strategy MPE. It follows from Lemmas 2 and 3 that if (G_1, G_2) constitutes a mixed-strategy MPE, then each G_i must satisfy (8) on $\Gamma = (a, \theta_1^*)$, where $\theta_1^* = \theta_2^*$ is given in Lemma 1. Solving (8) subject to the boundary condition $G_i^x(0) = 0$ for every i and $x \in \Gamma$ yields

$$G_i^x(t) = 1 - \exp \left[- \int_0^t \frac{\mathbf{1}_{\{X_s^x \in \Gamma\}}(s) [rl_{-i} - \pi(X_s^x)]}{w(X_s^x) - l_{-i}} ds \right]. \quad (10)$$

Observe that $G_i^x(t)$ is a CDF of stopping times because it is right-continuous with left limits and non-decreasing in t . Moreover, its hazard rate depends only on the state X_t^x , confirming that it is a Markov strategy. The following Proposition shows that the strategy profile $(G_1, G_2) = (G_1^x, G_2^x)_{x \in \mathcal{J}}$ indeed constitutes the unique mixed-strategy MPE, which is analogous to the symmetric mixed strategy equilibrium obtained by Steg (2015).

Proposition 2 *Suppose that $\sigma(\cdot) > 0$ and $l_1 = l_2$. Then $(G_1, G_2) = (G_1^x, G_2^x)_{x \in \mathcal{J}}$, where $\Gamma = (a, \theta^*)$ and $\theta^* = \theta_1^* = \theta_2^*$ constitutes a mixed-strategy MPE.*

5 Concluding Remarks

It is well known that canonical war of attrition games under complete information admit equilibria in pure, as well as in mixed strategies; *e.g.*, Tirole (1988), Levin (2004), and others. We study such a two-player model and show that if the players' in-war payoffs are stochastic, *and* their exit payoffs are heterogeneous, then the game admits only pure-strategy MPE. That is, any degree of Brownian uncertainty in the players' in-war payoffs and any amount of heterogeneity in their exit payoffs is sufficient to destabilize the mixed-strategy MPE. The implication of this result is that it may be more prudent to focus on pure-strategy MPE in this class of games. (This is in contrast to much of the extant literature that has focused on mixed-strategy equilibria.)

The issue of equilibrium selection is interesting to explore further. First, whether non-Markov strategies make it possible to construct mixed-strategy equilibria in the war of attrition remains an open question. Second, analyzing the set of equilibria in the presence of asymmetric information constitutes a plausible avenue for future research. For instance, firms in a declining industry may have private information about their exit barriers (Fudenberg and Tirole, 1986; Takahashi, 2015). Because a game with private types involves multiple types of players, the equilibrium selection requires understanding games between asymmetric types. In particular, games with discrete types are known to have mixed strategy equilibria (Levin, 2004). It will be fruitful to examine the impact of stochasticity on the equilibrium selection in this case.

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A Mixed-strategy MPE when $\sigma(\cdot) = 0$

In this section, we consider the case in which $\sigma(\cdot) = 0$. This case was previously analyzed by Hendricks et al. (1988) using a similar model. We present it here for completeness, and to highlight that the non-existence of mixed-strategy MPE requires two ingredients: first, that the firms' flow payoffs while they remain in the market are stochastic, and second, that the firms have heterogeneous outside options. To facilitate the analysis and following Hendricks et al. (1988), we will assume in this section that $\mu(\cdot) \leq 0$; *i.e.*, the market conditions deteriorate over time.

We shall construct a mixed-strategy MPE in which both firms remain in the market whenever $X_t^x > \theta_1^*$, and they randomize on the set $\Gamma = (\alpha, \theta_1^*)$. Lemma 2 holds, and because $\mu(\cdot) \leq 0$, the set Γ is absorbing; *i.e.*, if $X_t^x \in \Gamma$, then $X_s^x \in \Gamma$ for all $s > t$ with probability 1. Therefore, for every $x \in \Gamma$, each firm i 's strategy $G_i^x(t)$ must satisfy (10). If $x \notin \Gamma$, then each firm i 's strategy may have a discontinuity of size $1 - p_i$ at $\tau_\Gamma = \inf\{t \geq 0 : X_t^x \in \Gamma\}$, and solving (8) subject to the boundary condition $G_i^x(\tau_\Gamma) = 1 - p_i$ where $p_i \in [0, 1]$ yields that each firm i 's strategy must satisfy

$$G_i^x(t) = \mathbf{1}_{\{X_t^x \in \Gamma\}}(t) \left\{ 1 - p_i \exp \left[- \int_0^t \frac{\mathbf{1}_{\{X_s^x \in \Gamma\}}(s) [rl_{-i} - \pi(X_s^x)]}{w(X_s^x) - l_{-i}} ds \right] \right\}. \quad (11)$$

The strategy $G_i^x(\cdot)$ stipulates that firm i exits with probability p_i when X_t^x first hits Γ , after which $G_i^x(t)$ is continuous in time. Observe that the strategy profile $(G_1^x, G_2^x)_{x \in \mathcal{X}} \big|_{p_1, p_2}$ is Markov, and the following proposition shows that for an appropriate choice of p_1 and p_2 , it constitutes a mixed-strategy MPE.

Proposition A.1 *Suppose that $\sigma(\cdot) = 0$. Then there exists $\kappa(l_2) > 0$ such that $(G_1^x, G_2^x)_{x \in \mathcal{X}} \big|_{p_1, p_2}$ is a mixed-strategy MPE with $0 < p_1 < \bar{p}(l_1, l_2)$ for some function $\bar{p}(l_1, l_2) \in (0, 1)$ and $p_2 = 1$ as long as $l_2 - l_1 < \kappa(l_2)$.*

This continuum of equilibria parameterized by p_1 are analogous to the continuum of mixed strategy equilibria obtained by Hendricks et al. (1988). The proof of the proposition appears in Appendix C. Recall from Proposition 1 that whenever $X_t^x \in (\theta_1^*, \theta_2^*]$, firm 1 strictly prefers to remain in the market, whereas firm 2 strictly prefers to exit immediately. In order for firm 2 to wait until X_t^x enters the randomization set Γ , in equilibrium, firm 1 must exit at the moment such that $X_t^x = \theta_1^*$ with sufficiently high probability. When $X_t^x \in \Gamma^o$, it follows from Lemma (2) that both firms must exit at the rate given by (9).

B Mathematical Preliminaries

This section lays out the preliminary facts necessary to construct mathematical proofs in Appendix C. We first define the following functions that will be used in Appendices B and C.

$$R(x) := \mathbb{E}^x \left[\int_0^\infty \pi(X_t) e^{-rt} dt \right], \quad (12)$$

$$\beta_i(x) := \frac{l_i - R(x)}{\phi(x)}, \quad (13)$$

where $\phi : \mathcal{I} \rightarrow \mathbb{R}$ satisfies the differential equation⁹ $\frac{1}{2}\sigma^2(x)\phi''(x) + \mu(x)\phi'(x) - r\phi(x) = 0$ with the properties of $\phi(\cdot) > 0$ and $\phi'(\cdot) < 0$. The function $R(\cdot)$ is well-defined because we assume that $\pi(\cdot)$ satisfies the absolute integrability condition in Section 2.

Lemma B.1 *The function $\beta_i(x)$ has a unique interior maximum at $\theta_i^* \leq x_{ci}$ where $\pi(x_{ci}) = rl_i$. Furthermore, $\beta_i'(x) > 0$ for $x < \theta_i^*$ and $\beta_i'(x) < 0$ for $x > \theta_i^*$.*

Proof of Lemma B.1: To prove this lemma, it is enough to examine the behavior of the first derivative of $\beta_i(x) = [l_i - R(x)]/\phi(x)$.

According to the theory of diffusive processes (Alvarez, 2001, p.319), the function $R(\cdot)$, given in (12), can be expressed as

$$R(x) = \frac{\phi(x)}{B} \int_a^x \psi(y)\pi(y)m'(y)dy + \frac{\psi(x)}{B} \int_x^b \phi(y)\pi(y)m'(y)dy. \quad (14)$$

Here, a and b are the two boundaries of the state space \mathcal{I} , $\psi(\cdot)$ and $\phi(\cdot)$ are the increasing and decreasing fundamental solutions to the differential equation $\frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x) - rf(x) = 0$, $B = [\psi'(x)\phi(x) - \psi(x)\phi'(x)]/S'(x)$ is the *constant* Wronskian determinant of $\psi(\cdot)$ and $\phi(\cdot)$, $S'(x) = \exp(-\int 2\mu(x)/\sigma^2(x)dx)$ is the density of the scale function of X , and $m'(y) = 2/[\sigma^2(y)S'(y)]$ is the density of the speed measure of X .

By virtue of (14), differentiation of $R(x)$ with respect to x leads to

$$R'(x)\phi(x) - R(x)\phi'(x) = S'(x) \int_x^b \phi(y)\pi(y)m'(y)dy. \quad (15)$$

⁹This second-order linear ordinary differential equation (ODE) always has two linearly independent fundamental solutions, one of which is monotonically decreasing (see Alvarez, 2001, p.319). Note that if $f(\cdot)$ solves this equation, then so does $cf(\cdot)$ for any constant $c \in \mathbb{R}$ because it is a homogeneous equation. Hence, we can always find the one which is always positive.

Moreover, because $l_i = \mathbb{E}^x[\int_0^\infty rl_i e^{-rt} dt]$, we can write

$$R(x) - l_i = \mathbb{E}^x \left[\int_0^\infty [\pi(X_t) - rl_i] e^{-rt} dt \right], \quad (16)$$

which implies that we can treat the functional $R(x) - l_i$ as the expected cumulative present value of a flow payoff $\pi(\cdot) - rl_i$. Combining (15) and (16), therefore, we obtain

$$\beta'_i(x) = -\frac{R'(x)\phi(x) - [R(x) - l_i]\phi'(x)}{\phi^2(x)} = -\frac{S'(x)}{\phi^2(x)} \int_x^b \phi(y)[\pi(y) - rl_i]m'(y)dy. \quad (17)$$

Now, because $\pi(\cdot)$ is strictly increasing and $\pi(x_{ci}) = rl_i$, it must be the case that $\pi(x) < rl_i$ for $x < x_{ci}$ and $\pi(x) > rl_i$ for $x > x_{ci}$. Thus, $\beta'_i(x) < 0$ for all $x > x_{ci}$. Note also that if $x < K < x_{ci}$, then

$$\begin{aligned} \int_x^b \phi(y)[\pi(y) - rl_i]m'(y)dy &= \int_x^K \phi(y)[\pi(y) - rl_i]m'(y)dy + \int_K^b \phi(y)[\pi(y) - rl_i]m'(y)dy \\ &\leq \frac{[\pi(K) - rl_i]}{r} \left(\frac{\phi'(K)}{S'(K)} - \frac{\phi'(x)}{S'(x)} \right) + \int_K^b \phi(y)[\pi(y) - rl_i]m'(y)dy \rightarrow -\infty, \end{aligned}$$

as $x \downarrow a$ because a is a natural boundary, which implies that $\lim_{x \downarrow a} \beta'_i(x) = \infty$. Here we use $\phi'(x) < 0$ and $\pi(x) < \pi(K) < rl_i$ for $x < K$. It thus follows that $\beta'_i(\theta_i^*) = 0$ for some $\theta_i^* \leq x_{ci}$, which implies that $\int_{\theta_i^*}^b \phi(y)[\pi(y) - rl_i]m'(y)dy = 0$ because $S'(x) > 0$ and $\phi(x) > 0$ in (17). Moreover, note that $\int_x^b \phi(y)[\pi(y) - rl_i]m'(y)dy$ is increasing in $x < x_{ci}$ because $\pi(y) < rl_i$ for $\forall y < x_{ci}$, thus yielding $\int_x^b \phi(y)[\pi(y) - rl_i]m'(y)dy < 0$ if $x < \theta_i^* \leq x_{ci}$ and $\int_x^b \phi(y)[\pi(y) - rl_i]m'(y)dy > 0$ if $\theta_i^* < x \leq x_{ci}$. Combining this with (17), we obtain the unique existence of θ_i^* such that $\beta'_i(x) > 0$ for $\forall x < \theta_i^*$ and $\beta'_i(x) < 0$ for $\forall x > \theta_i^*$, which completes the proof. \blacksquare

Lemma B.2 *A mixed-strategy G_i is a best response to a mixed-strategy G_{-i} if and only if, for each $x \in \mathcal{I}$,*

$$\mathbb{E}^x[S_i(\hat{\tau}; G_{-i})] = \sup_{\tau} \mathbb{E}^x[S_i(\tau; G_{-i})], \quad (18)$$

whenever $X_{\hat{\tau}}^x \in \overline{\text{supp}(G_i)}$ almost surely.

Lemma B.2 implies that each pure-strategy, which is involved in a mixed-strategy best response, must itself be a best response.

Proof of Lemma B.2: This lemma follows from Lemma 3.1. in Steg (2015). Define the right-continuous inverse of G_i^x as

$$\tau_i^{G_i, x}(z) := \inf\{s \geq 0 : G_i^x(s) > z\}, \forall z \in [0, 1], \quad (19)$$

which satisfies $\tau_i^{G,x}(z) \leq t$ if and only if $G_i^x(t) \geq z$. Then we can obtain the change-of-variable formula between G_i^x and $\tau_i^{G,x}(z)$ as the following:

$$\mathbb{E}^x \left[\int_0^\infty S_i(t; G_{-i}) dG_i^x(t) \right] = \mathbb{E}^x \left[\int_0^1 S_i(\tau_i^{G,x}(z); G_{-i}) dz \right].$$

Using this change-of-variable, we have

$$\begin{aligned} V_i(x; G_i, G_{-i}) &= \mathbb{E}^x \left[\int_0^\infty S_i(t; G_{-i}) dG_i^x(t) \right] = \mathbb{E}^x \left[\int_0^1 S_i(\tau_i^{G,x}(z); G_{-i}) dz \right] \\ &= \int_0^1 \mathbb{E}^x [S_i(\tau_i^{G,x}(z); G_{-i})] dz \\ &\leq \int_0^1 \sup_\tau \mathbb{E}^x [S_i(\tau; G_{-i})] dz = \sup_\tau \mathbb{E}^x [S_i(\tau; G_{-i})], \end{aligned} \quad (20)$$

where the first equality follows from (5) and the first inequality follows because $\tau_i^{G,x}(z)$ is a stopping time with respect to the state X for each z . Note that the relation holds for any pair of mixed strategies (G_i, G_{-i}) .

Now, suppose that $\mathbb{E}^x [S_i^x(\hat{\tau}; G_{-i})] = \sup_\tau \mathbb{E}^x [S_i^x(\tau; G_{-i})]$ whenever $X_{\hat{\tau}}^x \in \overline{\text{supp}(G_i)}$ almost surely. Observe that $X_{\tau_i^{G,x}(z)}^x$ is in $\overline{\text{supp}(G_i)}$ for any z ; this is because $t > \tau_i^{G,x}(z)$ implies that $G_i^x(t) > z = G_i^x(\tau_i^{G,x}(z))$ if $G_i^x(\cdot)$ has no jump at $\tau_i^{G,x}(z)$, or $\Delta G_i^x(\tau_i^{G,x}(z)) > 0$ if $G_i^x(\cdot)$ has a jump at $\tau_i^{G,x}(z)$. Hence, $\mathbb{E}^x [S_i^x(\tau_i^{G,x}(z); G_{-i})] = \sup_\tau \mathbb{E}^x [S_i^x(\tau; G_{-i})]$ for any $z \in [0, 1]$ by our assumption, which implies that, for any mixed-strategy \tilde{G}_i ,

$$\begin{aligned} V_i(x; G_i, G_{-i}) &= \mathbb{E}^x \left[\int_0^\infty S_i(t; G_{-i}) dG_i^x(t) \right] = \mathbb{E}^x \left[\int_0^1 S_i(\tau_i^{G,x}(z); G_{-i}) dz \right] \\ &= \int_0^1 \mathbb{E}^x [S_i(\tau_i^{G,x}(z); G_{-i})] dz = \int_0^1 \sup_\tau \mathbb{E}^x [S_i(\tau; G_{-i})] dz \\ &= \sup_\tau \mathbb{E}^x [S_i(\tau; G_{-i})] \geq V_i(x; \tilde{G}_i, G_{-i}), \end{aligned}$$

where the last inequality follows from (20). Thus, G_i is a best response to G_{-i} . Conversely, if G_i is a best response to G_{-i} and $X_{\hat{\tau}}^x \in \overline{\text{supp}(G_i)}$, then we must have $\mathbb{E}^x [S_i(\hat{\tau}; G_{-i})] = \sup_\tau \mathbb{E}^x [S_i(\tau; G_{-i})]$; otherwise firm i could earn higher payoff by exiting at τ such that $X_\tau^x \notin \overline{\text{supp}(G_i)}$, which is a contradiction. This completes the proof. \blacksquare

C Proofs

Proof of Lemma 1: The proof of this lemma is available in Alvarez (2001), but here, we provide a sketch of the proof based on the verification theorem (Oksendal, 2003, Theorem 10.4.1). To that end, we will use the optimality conditions, which are known as “value matching” and “smooth pasting” conditions (Samuelson, 1965; McKean, 1965; Merton, 1973).

First, the state space \mathcal{S} must be the union of $C := \{x \in \mathcal{S} : V_i^*(x) > l_i\}$ and $\Gamma := \{x \in \mathcal{S} : V_i^*(x) = l_i\}$, which are mutually exclusive: This is because (1) X is a stationary process and the time horizon is infinite, and (2) the value function $V_i^*(\cdot)$ from an optimal stopping policy must be always no less than the reward l_i from stopping immediately. Hence, the problem to find an optimal stopping policy can be reduced to identify C or Γ .

Next, we find the differential equation that $V_i^*(x)$ must satisfy if $x \in C$. Note that the optimal value function $V_i^*(x)$ is the maximum of the reward from waiting an instant and the reward from stopping immediately. For any $x \in C$, therefore, the optimal stopping policy is to wait an instant dt , and hence, the optimal value function must satisfy the following equation:

$$V_i^*(x) = \pi(x)dt + (1 - rdt)\mathbb{E}^x[V_i^*(x) + dV_i^*(X_t)]. \quad (21)$$

Then applying Ito formula to $V_i^*(X_t)$ and using $\mathbb{E}^x[dB_t] = 0$ yields

$$\mathbb{E}^x[dV_i^*(X_t)] = [\mu(x)V_i^{*'}(x) + \frac{1}{2}\sigma^2(x)V_i^{*''}(x)]dt. \quad (22)$$

By plugging (22) into (21) and ignoring the term smaller than dt , we have

$$V_i^*(x) = \pi(x)dt + V_i^*(x) + [-rV_i^*(x) + \mu(x)V_i^{*'}(x) + \frac{1}{2}\sigma^2(x)V_i^{*''}(x)]dt,$$

from which we obtain the following second-order linear differential equation:

$$\frac{1}{2}\sigma^2(x)V_i^{*''}(x) + \mu(x)V_i^{*'}(x) - rV_i^*(x) = -\pi(x). \quad (23)$$

Thus, $V_i^*(\cdot)$ can be obtained by solving the differential equation (23). In fact, it can be seen from a series of algebra with the relation (14) that the function $R(\cdot) + A\phi(\cdot)$ with some constant $A \in \mathbb{R}$ is a solution to (23), and hence, we can guess $V_i^*(x) = R(x) + A\phi(x)$ with some constant A .

Intuitively, firm i must find it optimal to exit and receive his outside option l_i as soon as the state X hits some lower threshold θ_i . Hence, assume at the moment that the optimal stopping policy is given as $\tau^* := \inf\{t \geq 0 : X_t^x \leq \theta_i\}$, which implies that θ_i is the boundary point of the region C . Now, we state the value matching condition and the smooth pasting condition, which results in two

boundary conditions to the boundary value problem (23) with the free boundary θ_i :

$$V_i^*(\theta_i) = R(\theta_i) + A\phi(\theta_i) = l_i \quad (24)$$

$$V_i^{*\prime}(\theta_i) = R'(\theta_i) + A\phi'(\theta_i) = 0. \quad (25)$$

The value matching condition (24) and the smooth pasting condition (25) are the conditions that $V_i^*(\cdot)$ must satisfy at the boundary θ_i of C . We can first obtain $A = [l_i - R(\theta_i)]/\phi(\theta_i) = \beta_i(\theta_i)$ from (24). Then the condition (25) is equivalent to

$$\begin{aligned} 0 &= R'(\theta_i) + \frac{l_i - R(\theta_i)}{\phi(\theta_i)}\phi'(\theta_i) \\ &= \frac{R'(\theta_i)\phi(\theta_i) + [l_i - R(\theta_i)]\phi'(\theta_i)}{\phi(\theta_i)} = -\phi(\theta_i)\beta_i'(\theta_i). \end{aligned}$$

Because $\phi(\cdot) > 0$, it can be seen from Lemma B.1 that this condition is satisfied if and only if $\theta_i = \theta_i^*$, which implies that $A = \beta_i(\theta_i^*)$.

Lastly, it can be easily verified that $R(x) + \beta_i(\theta_i^*)\phi(x) \geq l_i$ for $\forall x \geq \theta_i^*$ and $\pi(x) < rl_i$ for $\forall x \leq \theta_i^* < x_{ci}$. By the verification theorem (Oksendal, 2003, Theorem 10.4.1), therefore, the proposed value function $R(\cdot) + \beta_i(\theta_i^*)\phi(\cdot)$ is, in fact, the optimal value function $V_i^*(\cdot)$, as desired. ■

Proof of Proposition 1: (i) We first prove that $(H(\infty), H(\tau_2^*))$ is an MPE. Because it is shown in Lemma 1 that $G_2 = H(\tau_2^*)$, where $\tau_2^* = \inf\{t \geq 0 : X_t^x \leq \theta_2^*\}$ is given in (7), is firm 2's best response to $G_1 = H(\infty)$, it only remains to prove that $G_1 = H(\infty)$ is also firm 1's best response to $G_2 = H(\tau_2^*)$.

Let $H(\tau_1)$ be firm 1's best response to $H(\tau_2^*)$ and $V_{W_1}^*(x) := \sup_{\tau} V_1(x; H(\tau), H(\tau_2^*)) = V_1(x; H(\tau_1), H(\tau_2^*))$ be the corresponding payoff to firm 1. We denote the continuation region associated with τ_1 by C_1 , i.e., $\tau_1 = \inf\{t \geq 0 : X_t^x \notin C_1\}$, and its complement by $\Gamma_1 = \mathcal{S} \setminus C_1$.

First, we show that $\Gamma_1 \cap (\theta_2^*, \infty) = \emptyset$. Toward a contradiction, suppose this is not the case. Then pick some $x \in \Gamma_1 \cap (\theta_2^*, \infty)$ and observe that $V_{W_1}^*(x) = l_1$ due to $x \in \Gamma_1$. However,

$$\begin{aligned} V_{W_1}^*(x) &\geq V_1(x; H(\infty), H(\tau_2^*)) = \mathbb{E}^x \left[\int_0^{\tau_2^*} \pi(X_t) e^{-rt} dt + w(X_{\tau_2^*}^x) e^{-r\tau_2^*} \right] \\ &= R(x) + \left[\frac{w(\theta_2^*) - R(\theta_2^*)}{\phi(\theta_2^*)} \right] \phi(x) \\ &> R(x) + \left[\frac{l_1 - R(\theta_2^*)}{\phi(\theta_2^*)} \right] \phi(x) \\ &= R(x) + \beta_1(\theta_2^*)\phi(x) \\ &> R(x) + \beta_1(x)\phi(x) = l_1, \end{aligned}$$

where the first inequality follows because $w(X_{\tau_2^*}^x) = w(\theta_2^*) > l_1$ and $\mathbb{E}^x[e^{-r\tau_2^*}] = \phi(x)/\phi(\theta_2^*)$ for $x >$

θ_2^* , and the second inequality holds because $x > \theta_2^* > \theta_1^*$ and $\beta_1'(x) < 0$ for $x > \theta_1^*$ by Lemma B.1. This establishes the contradiction.

Second, we also prove that $\Gamma_1 \cap (-\infty, \theta_2^*] = \emptyset$. Towards a contradiction, suppose this is not the case. Then we can pick some $x \in \Gamma_1 \cap (-\infty, \theta_2^*]$ such that $V_{W_1}^*(x) = m_1(x)$ because $\tau_2^* = \inf\{t \geq 0 : X_t^x \leq \theta_2^*\}$. However,

$$V_{W_1}^*(x) \geq V_1(x; H(\infty), H(\tau_2^*)) = \mathbb{E}^x \left[\int_0^{\tau_2^*} \pi(X_t) e^{-rt} dt + w(X_{\tau_2^*}^x) e^{-r\tau_2^*} \right] = w(x) > m_1(x),$$

where the second equality uses that $\tau_2^* = 0$ when $X_0 = x \leq \theta_2^*$. This establishes the contradiction. Hence, we can conclude that $\Gamma_1 = \emptyset$ and $C_1 = \mathcal{S}$, which implies that $\tau_1 = \infty$.

(ii) Next, we prove the conditions under which $(H(\tau_1^*), H(\infty))$ is an MPE. Consider the following condition:

$$V_2(x; H(\tau_1^*), H(\infty)) = \mathbb{E}^x \left[\int_0^{\tau_1^*} \pi(X_t) e^{-rt} dt + w(X_{\tau_1^*}^x) e^{-r\tau_1^*} \right] > l_2 \quad \text{for all } x \in (\theta_1^*, \theta_2^*]. \quad (26)$$

First, we prove that (26) is a sufficient condition for $(G_1, G_2) = (H(\tau_1^*), H(\infty))$ to be an MPE. Let $H(\tau_2)$ be firm 2's best response to $H(\tau_1^*)$, i.e., $V_{W_2}^*(x) := \sup_{\tau} V_2(x; H(\tau_1^*), H(\tau)) = V_2(x; H(\tau_1^*), H(\tau_2))$ be the corresponding payoff. We denote the continuation region associated with τ_2 by C_2 , i.e., $\tau_2 = \inf\{t \geq 0 : X_t^x \notin C_2\}$, and its complement by $\Gamma_2 = \mathcal{S} \setminus C_2$.

We now claim that $\Gamma_2 \cap (\theta_2^*, \infty) = \emptyset$. Towards a contradiction, suppose not. Then we can pick some $x \in \Gamma_2 \cap (\theta_2^*, \infty)$, which implies that $V_{W_2}^*(x) = l_2$. However, because $\tau_1^* > \tau_2^*$ when $X_0 = x$, Lemma 1 implies that firm 2 could obtain a *strictly* higher payoff by exiting at $\tau_2^* > 0$ instead, i.e.,

$$V_{W_2}^*(x) \geq V_2(x; H(\tau_1^*), H(\tau_2^*)) = \mathbb{E}^x \left[\int_0^{\tau_2^*} \pi(X_t) e^{-rt} dt + l_2 e^{-r\tau_2^*} \right] > l_2,$$

which is a contradiction. We next claim that $\Gamma_2 \cap (\theta_1^*, \theta_2^*] = \emptyset$. Towards a contradiction, suppose not. Then we can pick $x \in \Gamma_2 \cap (\theta_1^*, \theta_2^*]$, which implies that $V_{W_2}^*(x) = l_2$. However, we have

$$V_{W_2}^*(x) \geq V_2(x; H(\tau_1^*), H(\infty)) = \mathbb{E}^x \left[\int_0^{\tau_1^*} \pi(X_t) e^{-rt} dt + w(X_{\tau_1^*}^x) e^{-r\tau_1^*} \right] > l_2,$$

where the last inequality follows from (26). This establishes the contradiction. We further claim that $\Gamma_2 \cap (-\infty, \theta_1^*] = \emptyset$. If not, then there exists $x \in \Gamma_2 \cap (-\infty, \theta_1^*]$, which implies that both firms exit simultaneously when $X_t^y = x$, and hence, $V_{W_2}^*(x) = m_2(x)$. Because $\tau_1^* = 0$ when $X_0 = x \leq \theta_1^*$, we

have

$$V_{W_2}^*(x) \geq V_2(x; H(\tau_1^*), H(\infty)) = \mathbb{E}^x \left[\int_0^{\tau_1^*} \pi(X_t) e^{-rt} dt + w(X_{\tau_1^*}^x) e^{-r\tau_1^*} \right] = w(x) > m_2(x),$$

which is a contradiction. Combining the three claims above, therefore, we conclude that $\Gamma_2 = \emptyset$, which implies that $C_2 = \mathcal{I}$, and hence, $\tau_2 = \infty$.

Second, define $\underline{w} := \inf\{w(x) : x \in \mathcal{I}\}$ and $\beta_W(\theta) := [\underline{w} - R(\theta)]/\phi(\theta)$. Note that $\beta_W(\theta) > \beta_2(\theta)$ for $\forall \theta \in \mathcal{I}$ because $\underline{w} > l_2$. Also, observe that for $\forall \theta < \theta_2^*$, we have

$$\begin{aligned} \beta'_W(\theta) &= \left\{ -R'(\theta)\phi(\theta) - \phi'(\theta)[\underline{w} - R(\theta)] \right\} / \phi^2(\theta) \\ &> \left\{ -R'(\theta)\phi(\theta) - \phi'(\theta)[l_2 - R(\theta)] \right\} / \phi^2(\theta) = \beta'_2(\theta) > 0 \end{aligned}$$

where the first inequality follows because $\phi'(\theta) < 0$, and the last inequality holds because $\beta'_2(\theta) > 0$ for $\theta < \theta_2^*$ from Lemma B.1. Next, pick $\kappa_\theta > 0$ such that

$$\beta_W(\theta_2^* - \kappa_\theta) = \beta_2(\theta_2^*), \quad (27)$$

where $\beta_2(\cdot)$ is defined in (13). If such κ_θ exists, it must be unique because $\beta'_W(\theta) > 0$ for $\theta < \theta_2^*$. If there does not exist κ_θ which satisfies 27, then we let $\kappa_\theta = \infty$.

Finally, we show that (26) is satisfied if $\theta_2^* - \theta_1^* < \kappa_\theta$, which will complete the proof; this is because we can always find the unique $\kappa_l > 0$ for any given $\kappa_\theta > 0$ such that $\theta_2^* - \theta_1^* < \kappa_\theta$ if and only if $l_2 - l_1 < \kappa_l$ from the fact that θ_i^* given in (7) strictly increases in l_i . Suppose now that $\theta_2^* - \theta_1^* < \kappa_\theta$, i.e., $\theta_1^* > \theta_2^* - \kappa_\theta$. Note that $\beta'_W(\theta) > 0$ for $\forall \theta < \theta_2^*$, and recall that $\theta_1^* < \theta_2^*$. Therefore, $\beta_W(\theta_1^*) > \beta_W(\theta_2^* - \kappa_\theta) = \beta_2(\theta_2^*)$ by (27). Thus, for any $x \in (\theta_1^*, \theta_2^*]$,

$$\begin{aligned} \mathbb{E}^x \left[\int_0^{\tau_1^*} \pi(X_t^x) e^{-rt} dt + w(\theta_1^*) e^{-r\tau_1^*} \right] &\geq \mathbb{E}^x \left[\int_0^{\tau_1^*} \pi(X_t^x) e^{-rt} dt + \underline{w} e^{-r\tau_1^*} \right] \\ &= R(x) + \phi(x)\beta_W(\theta_1^*) \\ &> R(x) + \phi(x)\beta_2(\theta_2^*) \\ &\geq R(x) + \phi(x)\beta_2(x) = l_2, \end{aligned}$$

where the first inequality holds from the definition of \underline{w} , the first equality holds because $\mathbb{E}^x[e^{-r\tau_1^*}] = \phi(x)/\phi(\theta_1^*)$ for $x > \theta_1^*$, the second inequality follows because $\beta_W(\theta_1^*) > \beta_2(\theta_2^*)$, the last inequality holds because $\beta_2(\cdot)$ achieves its maximum at θ_2^* by Lemma B.1, and the last equality follows by the definition of $\beta_2(\cdot)$. Hence, (26) is satisfied, which establishes the desired result for $\kappa_\theta > 0$. \blacksquare

Proof of Lemma 2(a): Suppose that (G_1, G_2) is a mixed-strategy MPE. First, let us define $D_i :=$

$\{x \in \mathcal{S} : \pi(x) > rl_i\}$. We will show that D_i is a subset of the continuation region for firm i , i.e., $\overline{\text{supp}(G_i)} \cap D_i = \emptyset$. Towards a contradiction, suppose there exists some $x \in \overline{\text{supp}(G_i)} \cap D_i$. Because $\pi(\cdot)$ is continuous, for sufficiently small $\varepsilon > 0$, $\pi(y) > rl_i$ is satisfied for all $y \in (x - \varepsilon, x + \varepsilon)$. Then using $\int_0^t -rl_i e^{-rs} ds = l_i(e^{-rt} - 1)$, we have

$$L_i(t) = l_i + \int_0^t [\pi(X_s^x) - rl_i] e^{-rs} ds > l_i, \forall t \in (0, \tau_\varepsilon], \quad (28)$$

where $\tau_\varepsilon := \inf\{t \geq 0 : X_t^x \notin (x - \varepsilon, x + \varepsilon)\}$, and the inequality follows because $\pi_i(X_s^x) > rl_i$ for $\forall s < \tau_\varepsilon$. Note that

$$\begin{aligned} \mathbb{E}^x[S_i(\tau_\varepsilon; G_j)] &= \mathbb{E}^x \left[\int_0^{\tau_\varepsilon} W_i(t) dG_j(t) + M_i(\tau_\varepsilon) \Delta G_j(\tau_\varepsilon) + L_i(\tau_\varepsilon) [1 - G_j(\tau_\varepsilon)] \right] \\ &> \mathbb{E}^x \left[\int_0^{\tau_\varepsilon} L_i(t) dG_j(t) + L_i(\tau_\varepsilon) \Delta G_j(\tau_\varepsilon) + L_i(\tau_\varepsilon) [1 - G_j(\tau_\varepsilon)] \right] \\ &> \mathbb{E}^x \left[\int_0^{\tau_\varepsilon} l_i dG_j(t) + l_i \Delta G_j(\tau_\varepsilon) + l_i [1 - G_j(\tau_\varepsilon)] \right] = l_i = \mathbb{E}^x[S_i(0; G_j)], \end{aligned}$$

where the first inequality follows because $W_i(t) > M_i(t) > L_i(t)$, the second inequality follows from (28). This contradicts the supposition that $x \in \overline{\text{supp}(G_i)}$ because firm i can obtain a strictly greater expected payoff by adopting the strategy τ_ε . Therefore, it must be the case that $\overline{\text{supp}(G_i)} \cap D_i = \emptyset$.

Next, we prove that the *interiors* of $\text{supp}(G_1)$ and $\text{supp}(G_2)$ must coincide, which establishes the statement of this lemma. Towards a contradiction, suppose that there exists an open interval $E \subseteq \text{supp}(G_i)$ but $E \not\subseteq \text{supp}(G_j)$. Consider an exit strategy $\tau_E := \inf\{t > 0 : X_t^x \notin E\}$ for firm i , where $x \in E$. Then $\tau_E > 0$ a.s. because E is an open set. Fix some $\tau \in (0, \tau_E)$, and note that Lemma B.2 implies that $\mathbb{E}^x[S_i(\tau; G_j)] = \mathbb{E}^x[S_i(0; G_j)]$. Moreover, because $x \in E \not\subseteq \text{supp}(G_j)$, it must be the case that $G_j^x(\tau) = 0$.

Recall that $\overline{\text{supp}(G_i)} \cap D_i = \emptyset$, so $E \cap D_i = \emptyset$, which implies that $\pi(X_s^x) < rl_i$ for all $s \in [0, \tau)$ because E is an open set and $\pi(\cdot)$ strictly increases. Hence, we have

$$\mathbb{E}^x[S_i(\tau; G_j)] = \mathbb{E}^x[L_i(\tau)] = l_i + \mathbb{E}^x \left[\int_0^\tau [\pi(X_t) - rl_i] e^{-rt} dt \right] < l_i = \mathbb{E}^x[S_i(0; G_j)],$$

where the first equality follows because $G_j^x(\tau) = 0$, and hence, $G_j^x(t) = \Delta G_j^x(t) = 0$ for all $t \leq \tau$, the second equality follows from the definition of $L_i(\cdot)$, and the inequality follows from $\pi(X_s^x) < rl_i$ for all $s \in [0, \tau)$. Therefore, firm i can obtain a strictly greater payoff by exiting immediately, which contradicts the supposition that E is in the support of G_i . This completes the proof. \blacksquare

Proof of Lemma 2(b): Suppose that (G_1, G_2) is a mixed-strategy MPE, and let $\overline{\Gamma^o}$ be the closure of the common interior of $\text{supp}(G_1)$ and $\text{supp}(G_2)$. Note that Γ^o comprises of all the (open) component

intervals of $\text{supp}(G_1)$ and $\text{supp}(G_2)$; it thus excludes all the point components, if any, of either $\text{supp}(G_1)$ or $\text{supp}(G_2)$. (A point component of $\text{supp}(G_i)$ is a singleton subset $\{c\}$ of $\text{supp}(G_i)$ that is disconnected from the rest of $\text{supp}(G_i)$.) $\overline{\Gamma^o}$ is simply augmented by the boundary points of all the component intervals of Γ^o .

Towards a contradiction, pick some $x \in \overline{\Gamma^o}$, some $i \in \{1, 2\}$, $j \in \{1, 2\} \setminus \{i\}$, and suppose that $G_j^x(\cdot)$ has a jump of size $q_\tau > 0$ at some τ such that $\mathbb{P}(X_\tau^x \in \overline{\Gamma^o}) > 0$. Defining $\tau' := \min\{\tau, \tau_E\}$ where $\tau_E := \inf\{t \geq 0 : X_t \notin \overline{\Gamma^o}\}$ is the exit time from $\overline{\Gamma^o}$, this supposition implies that $G_j^x(\cdot)$ has a jump of size $q_{\tau'} > 0$ at time τ' such that $\mathbb{P}(X_{\tau'}^x \in \overline{\Gamma^o}) = 1$. (Here we allow for the possibility that $G_j^x(\cdot)$ has an additional jump at time τ_E as well.) It must then follow from Lemma B.2 that $\mathbb{E}^x[S_i(\tau'; G_j)] \geq \mathbb{E}^x[S_i(\tau^+; G_j)]$.

Now, we compare $\mathbb{E}^x[S_i(\tau'; G_j)]$ and $\mathbb{E}^x[S_i(\tau^+; G_j)]$. First, observe that

$$\mathbb{E}^x[S_i(\tau'; G_j)] = \mathbb{E}^x \left[\int_0^{\tau'^-} W_i(t) dG_j^x(t) + M_i(\tau') \Delta G_j^x(\tau') + L_i(\tau') [1 - G_j^x(\tau')] \right],$$

where $\Delta G_j^x(\tau') = [1 - G_j^x(\tau'^-)]q_{\tau'}$ because we assume $G_j^x(\cdot)$ has a jump at τ' . On the other hand, we can similarly express $\mathbb{E}^x[S_i(\tau^+; G_j)]$ as

$$\mathbb{E}^x \left[\int_0^{\tau'} W_i(t) dG_j^x(t) + L_i(\tau') [1 - G_j^x(\tau')] \right] = \mathbb{E}^x \left[\int_0^{\tau'^-} W_i(t) dG_j^x(t) + W_i(\tau') \Delta G_j^x(\tau') + L_i(\tau') [1 - G_j^x(\tau')] \right],$$

where the equality follows by breaking down the integral over $[0, \tau'^-]$ and $(\tau'^-, \tau']$. Then because $W_i(\cdot) > M_i(\cdot)$ and $\mathbb{E}^x[\Delta G_j^x(\tau')] = \mathbb{E}^x[[1 - G_j^x(\tau'^-)]q_{\tau'}] > 0$, we conclude $\mathbb{E}^x[S_i(\tau'; G_j)] < \mathbb{E}^x[S_i(\tau^+; G_j)]$, which is a contradiction. Hence, if $x \in \overline{\Gamma^o}$, then $G_j^x(t)$ must be continuous as long as $X_t \in \overline{\Gamma^o}$. \blacksquare

Proof of Lemma 3(a): Suppose that $\mathcal{G} := (G_i, G_j)$ is a mixed-strategy MPE. First, we claim that there is no point component $c \in \mathcal{S}$ of $\text{supp}(G_i)$ such that $c \notin \text{supp}(G_j)$. Note that this claim combined with Lemma 2(a) establishes that $\text{supp}(G_i) = \text{supp}(G_j)$, thus yielding $\overline{\text{supp}(G_i)} = \overline{\text{supp}(G_j)}$.

To prove this claim, observe first that there is no singleton set $\{c'\}$ that is a point component of both $\text{supp}(G_i)$ and $\text{supp}(G_j)$. This is because exit simultaneously with the opponent yields lower expected payoff compared to exit in an infinitesimal time. Now, consider a component interval (d, θ) of the common interior Γ^o for some $d \in [\alpha, \theta)$ and $\theta < \infty$, and suppose that there is a point component $c \in \mathcal{S}$ of $\text{supp}(G_i)$ such that $c \notin \text{supp}(G_j)$. We further suppose that the interval (θ, c) does not contain any other point component of $\text{supp}(G_i)$ or $\text{supp}(G_j)$.

It follows from the proof of Lemma 2(a) that $c \notin D_i$ where $D_i := \{x : \pi(x) > rl_i\}$, which implies that (θ, c) does not intersect with D_i because $\pi(\cdot)$ strictly increases. Also, because our assumption $c \notin \text{supp}(G_j)$ and Lemma 2(b) imply that $G_j^c(0) = G_j^\theta(0) = 0$, we obtain $V_i(c; \mathcal{G}) = V_i(\theta; \mathcal{G}) = l_i$; otherwise c and θ do not belong to $\overline{\text{supp}(G_i)}$ by definition of a mixed-strategy equilibrium. Then

because $\pi(x) < rl_i$ for $\forall x \in (\theta, c)$, it is straightforward to verify that $V_i(x; \mathcal{G}) < l_i$ for $\forall x \in (\theta, c)$ (Oksendal, 2003). However, firm i can always achieve a higher payoff l_i from an immediate exit at any point $x \in (\theta, c)$, which contradicts the assumption that \mathcal{G} is an equilibrium. Hence, such a point component c of $\text{supp}(G_i)$ cannot exist *above* a component interval (d, θ) of Γ° . Lastly, because the exactly same procedure can be used to prove that there is no such point component c of $\text{supp}(G_i)$ *below* a component interval (θ, d) of Γ° , we can conclude that there is no point component $c \in \mathcal{S}$ of $\text{supp}(G_i)$ such that $c \notin \text{supp}(G_j)$. This has proved that $\text{supp}(G_i) = \text{supp}(G_j) = \Gamma$ and $\bar{\Gamma} = \bar{\Gamma}^\circ$.

Finally, we prove the statement of this lemma. Suppose that G_i^x is discontinuous in time. Because we have proved above that no point component of Γ can exist outside (disconnected from) $\bar{\Gamma}$ in equilibrium, discontinuities in $G_i^x(\cdot)$ cannot take place when $X_t \notin \bar{\Gamma}$, thus implying a discontinuity can only happen within $\bar{\Gamma}$. Also, by definition of the Markov strategy, a discontinuity can only happen at $\tau_E = \inf\{t \geq 0 : X_t \in E\}$ for some $E \subset \bar{\Gamma}$ irrespective of the initial point x . If $E \neq \emptyset$, because X is an irreducible Markov chain if $\sigma(\cdot) > 0$, we have $\tau_E < \infty$ with positive probability irrespective of the initial point x . However, Lemma 2(b) stipulates that such a set E cannot intersect $\bar{\Gamma}^\circ$ if the initial point x is within $\bar{\Gamma}^\circ$. Because $\bar{\Gamma} = \bar{\Gamma}^\circ$, and because a Markov strategy does not depend on the initial value of the state variable, these two statements contradict each other unless $E = \emptyset$. Therefore, G_i^x must be continuous in time irrespective of x . ■

Proof of Lemma 3(b): Suppose that (G_1, G_2) is a mixed-strategy MPE. We have shown in Lemma 3(a) that if $\sigma(\cdot) > 0$, then $G_1^x(\cdot)$ and $G_2^x(\cdot)$ are continuous for all $x \in \mathcal{S}$. Let $\Gamma := \text{supp}(G_1) = \text{supp}(G_2)$ and define $C := \mathcal{S} \setminus \Gamma = (a, b) \setminus \Gamma$. Recall from the proof of Lemma 2(a) that $\{x : \pi(x) > rl_i\} = (x_{ci}, b) \subset C$ for each $i \in \{1, 2\}$.

First, we show that Γ is of the form (a, θ) . Towards a contradiction, suppose that there exists an interval (c, d) such that $(c, d) \subseteq C$ and $c, d \in \bar{\Gamma}$. This implies that (c, d) is disconnected from $\{x : \pi(x) > rl_i\} = (x_{ci}, b)$ for each i . Pick $x \in (c, d)$ and define $\tau_{(c,d)} := \inf\{t \geq 0 : X_t^x \notin (c, d)\}$.

It follows from Lemma B.2 that $\mathbb{E}^x[S_i(\tau_{(c,d)}; G_j)] = \sup_\tau \mathbb{E}^x[S_i(\tau; G_j)]$. Observe that

$$\mathbb{E}^x[S_i(\tau_{(c,d)}; G_j)] = \mathbb{E}^x[L_i(\tau_{(c,d)})] = l_i + \mathbb{E}^x \left[\int_0^{\tau_{(c,d)}} [\pi(X_t) - rl_i] e^{-rt} dt \right] < l_i = \mathbb{E}^x[S_i(0; G_j)],$$

where the first equality follows from $G_j^x(\tau_{(c,d)}) = 0$, the second equality follows from the definition of $L_i(\cdot)$, and the inequality follows because $\pi(X_s^x) < rl_i$ for all $s \leq \tau_{(c,d)}$. This is a contradiction, which implies that $\Gamma = (a, \theta)$ for some $\theta \leq x_{ci} \wedge x_{cj}$.

Second, pick some $x > \theta$, and define the strategy $\tau := \inf\{t \geq 0 : X_t^x \leq \theta\}$. We show that it must be the case $\theta = \theta_i^*$ for each i where θ_i^* is given in Lemma 1.

Towards a contradiction, suppose that $\theta < \theta_2^*$ and recall that $\theta_1^* \leq \theta_2^*$ because $l_1 \geq l_2$ by convention. Let $\tau_2^* = \inf\{t \geq 0 : X_t^x \leq \theta_2^*\}$ and note that $G_1(\tau_2^*) \leq G_1(\tau) = 0$ because $\theta < \theta_2^*$. Therefore,

we have $\mathbb{E}^x[S_2(\tau; G_1)] = \mathbb{E}^x[L_2(\tau)] < \mathbb{E}^x[L_2(\tau_2^*)] = \mathbb{E}^x[S_2(\tau_2^*; G_1)]$ where the equalities follow from $G_1(\tau_2^*) = G_1(\tau) = 0$, and the inequality follows from Lemma 1. However, this contradicts that $\mathbb{E}^x[S_2(\tau; G_1)] = \sup_{\tilde{\tau}} \mathbb{E}^x[S_2(\tilde{\tau}; G_1)]$.

On the other hand, suppose that $\theta > \theta_2^*$, which implies that $\tau \leq \tau_2^*$. Recall that $w(x)$ is defined as the expected net present value of monopoly profits so that $w(x) = \mathbb{E}^x[\int_0^\infty \pi^M(X_t)e^{-rt} dt]$ for some $\pi^M(\cdot) > \pi(\cdot)$. It is implicitly assumed that $\pi^M(\cdot)$ satisfies the absolute integrability assumption. We first note that $W_2(\cdot)$ defined in (2) is a supermartingale:

$$\begin{aligned} \mathbb{E}^x \left[W_2(t) | \mathcal{F}_s \right] &= \mathbb{E}^x \left[\int_0^t \pi(X_v) e^{-rv} dv + e^{-rt} w(X_t) \mid \mathcal{F}_s \right] \\ &= \int_0^s \pi(X_v) e^{-rv} dv + \mathbb{E}^x \left[\int_s^t \pi(X_v) e^{-rv} dv + \mathbb{E}^x \left[\int_t^\infty \pi^M(X_v) e^{-rv} dv \mid \mathcal{F}_t \right] \mid \mathcal{F}_s \right] \\ &< \int_0^s \pi(X_v) e^{-rv} dv + \mathbb{E}^x \left[\int_s^\infty \pi^M(X_v) e^{-rv} dv \mid \mathcal{F}_s \right] = W_2(s), \end{aligned}$$

for any $s < t$. Here, we use the inequality $\pi^M(\cdot) > \pi(\cdot)$ and the tower rule of the conditional expectation for the inequality and we employ Markov property of X for the last equality. We next establish that

$$\begin{aligned} \mathbb{E}^x \left[\int_\tau^{\tau_2^*} W_2(t) dG_1(t) \mid \mathcal{F}_\tau \right] &= \int_0^1 \mathbb{E}^x \left[W_2(\tau_1^{G,x}(z)) \mathbf{1}_{[\tau, \tau_2^*]}(\tau_1^{G,x}(z)) \mid \mathcal{F}_\tau \right] dz \\ &\geq \int_0^1 \mathbb{E}^x \left[\mathbb{E}^x \left[W_2(\tau_2^*) \mid \mathcal{F}_{\tau_1^{G,x}(z)} \right] \mathbf{1}_{[\tau, \tau_2^*]}(\tau_1^{G,x}(z)) \mid \mathcal{F}_\tau \right] dz \\ &= \int_0^1 \mathbb{E}^x \left[\mathbb{E}^x \left[W_2(\tau_2^*) \mathbf{1}_{[\tau, \tau_2^*]}(\tau_1^{G,x}(z)) \mid \mathcal{F}_{\tau_1^{G,x}(z)} \right] \mid \mathcal{F}_\tau \right] dz \\ &= \mathbb{E}^x \left[W_2(\tau_2^*) \int_0^1 \mathbf{1}_{[\tau, \tau_2^*]}(\tau_1^{G,x}(z)) dz \mid \mathcal{F}_\tau \right] \\ &= \mathbb{E}^x \left[W_2(\tau_2^*) [G_1(\tau_2^*) - G_1(\tau)] \mid \mathcal{F}_\tau \right] > \mathbb{E}^x \left[L_2(\tau_2^*) G_1(\tau_2^*) \mid \mathcal{F}_\tau \right], \quad (29) \end{aligned}$$

where the first equality holds from the change-of-variable from $G_1(t)$ to $\tau_1^{G,x}(z)$ given in (19), the inequality holds because $W_2(\cdot)$ is a *supermartingale*, the second equality holds because $\mathbf{1}_{[\tau, \tau_2^*]}(\tau_1^{G,x}(z))$ is measurable with respect to the filtration $\mathcal{F}_{\tau_1^{G,x}(z)}$, the third equality follows from the tower rule of the conditional expectation because $\mathcal{F}_\tau \subseteq \mathcal{F}_{\tau_1^{G,x}(z)}$, the fourth equality follows from $\int_0^1 \mathbf{1}_{[\tau, \tau_2^*]}(\tau_1^{G,x}(z)) dz = G_1(\tau_2^*) - G_1(\tau)$ by the definition of $\tau_1^{G,x}(z)$, and the last inequality follows because $W_2(\cdot) > L_2(\cdot)$ and $G_1(\tau) = 0$ (recall that $\text{supp}(G_1) = (a, \theta)$).

Using (29), therefore, we obtain

$$\begin{aligned}
\mathbb{E}^x[S_2(\tau_2^*; G_1)] - \mathbb{E}^x[S_2(\tau; G_1)] &= \mathbb{E}^x \left[\mathbb{E}^x \left[\int_{\tau}^{\tau_2^*} W_2(t) dG_1(t) + L_2(\tau_2^*)[1 - G_1(\tau_2^*)] - L_2(\tau) \mid \mathcal{F}_{\tau} \right] \right] \\
&> \mathbb{E}^x \left[\mathbb{E}^x \left[L_2(\tau_2^*)G_1(\tau_2^*) + L_2(\tau_2^*)[1 - G_1(\tau_2^*)] - L_2(\tau) \mid \mathcal{F}_{\tau} \right] \right] \\
&= \mathbb{E}^x \left[\mathbb{E}^x \left[L_2(\tau_2^*) - L_2(\tau) \mid \mathcal{F}_{\tau} \right] \right] = \mathbb{E}^x \left[L_2(\tau_2^*) - L_2(\tau) \right] > 0,
\end{aligned}$$

where the last equality follows from the tower rule of the conditional expectation and the last inequality follows from Lemma 1.

Therefore, it must be the case that $\theta = \theta_2^*$. By a symmetric argument, one can show that it must be the case that $\theta = \theta_1^*$. \blacksquare

Proof of Theorem 1: By noting that $\theta_1^* = \theta_2^*$ if and only if $l_1 = l_2$, it is straightforward to see from Lemma 3(b) that if $l_1 \neq l_2$, then the game does not admit any mixed-strategy MPE, which completes the proof. \blacksquare

Proof of Proposition 2: To prove this proposition, we need to establish the following lemma:

Lemma C.1 *Define the process*

$$J_i(t) := \sup_{\tau \geq t} \mathbb{E}^x[L_i(\tau) \mid \mathcal{F}_t] = \int_0^t \pi(X_s^x) e^{-rs} ds + V_i^*(X_t^x) e^{-rt}, \quad (30)$$

where $V_i^*(\cdot)$ is defined in (6). Then we have the following results:

(a) $J_i(t) \geq L_i(t)$ for all $t \geq 0$, and the equality holds if $t \geq \tau_i^*$ where τ_i^* is given in (7). Moreover, $J_i(\cdot)$ can be expressed as $J_i(t) = N_i(t) - D_i^{\theta_i^*}(t)$ where $N_i(\cdot)$ is a uniformly integrable martingale, and $D_i^y(\cdot)$ is a non-decreasing and predictable process given by

$$dD_i^y(t) := \mathbf{1}_{\{X_t^x \leq y\}} [rl_i - \pi(X_t^x)] e^{-rt} dt \text{ with } D_i^y(0) = 0. \quad (31)$$

(b) For any stopping times τ_A, τ_B with $\tau_A \leq \tau_B$ and a mixed-strategy G_{-i} ,

$$\mathbb{E}^x \left[\int_{\tau_A}^{\tau_B} N_i(t) dG_{-i}^x(t) \mid \mathcal{F}_{\tau_A} \right] = -\mathbb{E}^x \left[N_i(\tau_B) [1 - G_{-i}^x(\tau_B)] \mid \mathcal{F}_{\tau_A} \right] + N_i(\tau_A) [1 - G_{-i}^x(\tau_A)], \quad (32)$$

where \mathcal{F}_t is the natural filtration generated by the state X (Oksendal, 2003).

Proof of Lemma C.1(a): Comparing (3) and (30), it can be clearly seen that $J_i(t) \geq L_i(t)$ for all $t \geq 0$. Also, $J_i(t) = L_i(t)$ if $t \geq \tau_i^*$ by the definition of $V_i^*(\cdot)$. According to the theory of optimal stopping, it is well-known that the process $J_i(\cdot)$ is the Snell envelope of the process $L_i(\cdot)$ and it is of Class D (Steg, 2015). Hence, we can apply Doob-Meyer decomposition theorem to $J_i(\cdot)$,

which implies that $J_i(\cdot)$ can be decomposed into a uniformly integrable martingale and a unique, non decreasing, predictable process. Because $J_i(t)$ (more precisely, $V_i^*(X_t^x)$) is a twice differentiable function of X , the exact form of $N_i(\cdot)$ and $D_i^{\theta^*}(\cdot)$ can be obtained by applying Ito formula to $J_i(\cdot)$. ■

Proof of Lemma C.1(b): For a mixed-strategy G_{-i} , consider the right-continuous inverse $\tau_{-i}^{G,x}(z)$ given in (19). Observe that

$$\mathbb{E}^x \left[\int_{\tau_A}^{\tau_B} N_i(t) dG_{-i}^x(t) | \mathcal{F}_{\tau_A} \right] = \mathbb{E}^x \left[\int_0^1 N_i(\tau_{-i}^{G,x}(z)) \mathbf{1}_{[\tau_A, \tau_B]}(\tau_{-i}^{G,x}(z)) dz | \mathcal{F}_{\tau_A} \right] \quad (33)$$

$$= \int_0^1 \mathbb{E}^x \left[N_i(\tau_{-i}^{G,x}(z)) \mathbf{1}_{[\tau_A, \tau_B]}(\tau_{-i}^{G,x}(z)) | \mathcal{F}_{\tau_A} \right] dz$$

$$= \int_0^1 \mathbb{E}^x \left[\mathbb{E}^x \left[N_i(\tau_B) | \mathcal{F}_{\tau_{-i}^{G,x}(z)} \right] \mathbf{1}_{[\tau_A, \tau_B]}(\tau_{-i}^{G,x}(z)) | \mathcal{F}_{\tau_A} \right] dz \quad (34)$$

$$= \int_0^1 \mathbb{E}^x \left[\mathbb{E}^x \left[N_i(\tau_B) \mathbf{1}_{[\tau_A, \tau_B]}(\tau_{-i}^{G,x}(z)) | \mathcal{F}_{\tau_{-i}^{G,x}(z)} \right] | \mathcal{F}_{\tau_A} \right] dz \quad (35)$$

$$= \int_0^1 \mathbb{E}^x \left[N_i(\tau_B) \mathbf{1}_{[\tau_A, \tau_B]}(\tau_{-i}^{G,x}(z)) | \mathcal{F}_{\tau_A} \right] dz \quad (36)$$

$$= \mathbb{E}^x \left[N_i(\tau_B) \int_0^1 \mathbf{1}_{[\tau_A, \tau_B]}(\tau_{-i}^{G,x}(z)) dz | \mathcal{F}_{\tau_A} \right]$$

$$= \mathbb{E}^x \left[N_i(\tau_B) [G_{-i}(\tau_B) - G_{-i}(\tau_A)] | \mathcal{F}_{\tau_A} \right] \quad (37)$$

$$= \mathbb{E}^x \left[-N_i(\tau_B) [1 - G_{-i}(\tau_B)] + N_i(\tau_B) [1 - G_{-i}(\tau_A)] | \mathcal{F}_{\tau_A} \right]$$

$$= -\mathbb{E}^x \left[N_i(\tau_B) [1 - G_{-i}(\tau_B)] | \mathcal{F}_{\tau_A} \right] + N_i(\tau_A) [1 - G_{-i}(\tau_A)], \quad (38)$$

where (33) holds from the change-of-variable from $G_{-i}(t)$ to $\tau_{-i}^{G,x}(z)$, (34) follows from the fact $\mathbb{E}^x \left[N_i(\tau_B) | \mathcal{F}_{\tau_{-i}^{G,x}(z)} \right] = N_i(\tau_{-i}^{G,x}(z))$ because $N_i(\cdot)$ is a martingale, (35) holds because $\mathbf{1}_{[\tau_A, \tau_B]}(\tau_{-i}^{G,x}(z))$ is measurable with respect to the filtration $\mathcal{F}_{\tau_{-i}^{G,x}(z)}$, (36) follows from the smoothing law of the conditional expectation because $\mathcal{F}_{\tau_A} \subseteq \mathcal{F}_{\tau_{-i}^{G,x}(z)}$, (37) follows from $\int_0^1 \mathbf{1}_{[\tau_A, \tau_B]}(\tau_{-i}^{G,x}(z)) dz = G_{-i}(\tau_B) - G_{-i}(\tau_A)$ by the definition of $\tau_{-i}^{G,x}(z)$, and (38) follows because $\mathbb{E}^x \left[N_i(\tau_B) | \mathcal{F}_{\tau_A} \right] = N_i(\tau_A)$ ($N_i(\cdot)$ is a martingale) and $G_{-i}(\tau_A)$ is measurable with respect to \mathcal{F}_{τ_A} . Thus, the desired relation is established. ■

Because $l_1 = l_2$, we must have $\theta^* := \theta_1^* = \theta_2^*$. Define $\tau^* := \inf\{t \geq 0 : X_t^x \leq \theta^*\}$ and it is enough to show that G_i is a best response to G_j by symmetry.

To that end, we will use Lemma B.2. More precisely, since it can be seen from (10) that the

closure of the support of G_i^x is $(-\infty, \theta^*]$, we only need to prove the following two relations:

$$\mathbb{E}^x[S_i(u_1; G_j)] = \mathbb{E}^x[S_i(u_2; G_j)] \text{ for any } u, v \geq \tau^*, \quad (39)$$

$$\mathbb{E}^x[S_i(\tau; G_j)] < \mathbb{E}^x[S_i(\tau^*; G_j)] \text{ for any } \tau < \tau^*. \quad (40)$$

To show (39), choose any stopping times $u, v > \tau^*$ with $v > u$, and observe that

$$\mathbb{E}^x[S_i(v; G_j)] - \mathbb{E}^x[S_i(u; G_j)] = \mathbb{E}^x \left[\int_u^v F_i(s) dG_j^x(s) + L_i(v)[1 - G_j^x(v)] - L_i(u)[1 - G_j^x(u)] \right]. \quad (41)$$

Then it is enough to prove that the right side of (41) is equal to 0. By differentiating (10) with respect to time, we obtain

$$dG_j^x(s) = [1 - G_j^x(s)] \frac{dD_i^{\theta^*}(s)}{F_i(s) - L_i(s)}, \quad (42)$$

where $D_i^{\theta^*}(\cdot)$ is defined in (31). By applying integration by parts to (42), we have

$$\begin{aligned} \int_u^v [F_i(s) - L_i(s)] dG_j^x(s) &= - \int_u^v [1 - G_j^x(s)] dD_i^{\theta^*}(s) \\ &= - \int_u^v D_i^{\theta^*}(s) dG_j^x(s) - D_i^{\theta^*}(v)[1 - G_j^x(v)] + D_i^{\theta^*}(u)[1 - G_j^x(u)]. \end{aligned}$$

Then it follows from (41) that

$$\begin{aligned} S_i(v; G_j) - S_i(u; G_j) &= \int_u^v F_i(s) dG_j^x(s) + L_i(v)[1 - G_j^x(v)] - L_i(u)[1 - G_j^x(u)] \\ &= \int_u^v [D_i^{\theta^*}(s) + L_i(s)] dG_j^x(s) + [1 - G_j^x(v)][D_i^{\theta^*}(v) + L_i(v)] \\ &\quad - [1 - G_j^x(u)][D_i^{\theta^*}(u) + L_i(u)]. \end{aligned}$$

By Lemma C.1(b), we now obtain

$$\begin{aligned} \mathbb{E}^x[S_i(v; G_j) - S_i(u; G_j)] &= \mathbb{E}^x \left[\int_u^v [D_i^{\theta^*}(s) + L_i(s) - N_i(s)] dG_j^x(s) + [1 - G_j^x(v)][D_i^{\theta^*}(v) + L_i(v) - N_i(v)] \right. \\ &\quad \left. - [1 - G_j^x(u)][D_i^{\theta^*}(u) + L_i(u) - N_i(u)] \right]. \end{aligned}$$

Lemma C.1(a) implies that $L_i(s) = J_i(s) = N_i(s) - D_i^{\theta^*}(s)$ for any $s \geq \tau^*$ and $u, v > \tau^*$, from which (39) follows.

To show (40), because $G_j^x(s) = 0$ for all $s \leq \tau^*$, we obtain, for any $\tau < \tau^*$,

$$\mathbb{E}^x[S_i(\tau; G_j)] = \mathbb{E}^x[L_i(\tau)] < \mathbb{E}^x[L_i(\tau^*)] = \mathbb{E}^x[S_i(\tau^*; G_j)],$$

where the inequality follows from Lemma 1. This establishes (40), which completes the proof. ■

Proof of Proposition A.1: Note that we do not need the expectation notation throughout the proof of this proposition because there is no uncertainty when $\sigma(\cdot) = 0$ in (1). However, Lemma 1 and Lemma B.2 are still valid when $\sigma(\cdot) = 0$ so that we use those lemmas without expectation notation for notational simplicity.

First, we show that G_2 is also a best response to G_1 if $\theta_2^* - \theta_1^* < \kappa_\theta$ where κ_θ is given in (27). We will use Lemma B.2 and will prove the following relations:

$$S_2(u; G_1) = S_2(v; G_1) \text{ for any } u, v > \tau_1^*, \quad (43)$$

$$S_2(t; G_1) < S_2(u; G_1) \text{ for any } t \leq \tau_1^* < u. \quad (44)$$

To prove (43), choose any $u, v > \tau^*$ with $v > u$, and observe that

$$S_2(v; G_1) - S_2(u; G_1) = \int_u^v F_2(s) dG_1^x(s) + L_2(v)[1 - G_1^x(v)] - L_2(u)[1 - G_1^x(v)]. \quad (45)$$

Then it is enough to prove that the right side of (45) is equal to 0. For any $s > \tau_1^*$, observe from (10) and (11) that

$$G_1^x(s) = \begin{cases} (1 - p_1) + p_1 G_1^{\theta_1^*}(s - \tau_1^*) & \text{for } s > \theta_1^*, \\ G_1^x(s) & \text{for } s \leq \theta_1^*. \end{cases} \quad (46)$$

In addition, we can obtain from differentiating (10) that for any $s > \tau_1^*$,

$$dG_1^x(s) = [1 - G_1^x(s)] \frac{[rl_2 - \pi(X_s^x)]}{w(X_s^x) - l_2} ds = [1 - G_1^x(s)] \frac{[rl_2 - \pi(X_s^x)]e^{-rs} ds}{W(s) - L_2(s)} = [1 - G_1^x(s)] \frac{-dL_2(s)}{W(s) - L_2(s)}, \quad (47)$$

where $\mathbf{1}_{\{s \geq \tau_1^*\}}$ disappears because $\sigma(\cdot) = 0$. Hence, we can use integration by parts and obtain

$$\int_u^v [F_2(s) - L_2(s)] dG_1^x(s) = - \int_u^v [1 - G_1^x(s)] dL_2(s) \quad (48)$$

$$= - \int_u^v L_2(s) dG_1^x(s) - L_2(v)[1 - G_1^x(v)] + L_2(u)[1 - G_1^x(u)]. \quad (49)$$

After $\int_u^v L_2(s) dG_1^x(s)$ is cancelled out on the left side of (48) and the right side of (49), the resulting equation implies that the equation (45) is equal to 0, which establishes (43).

To show (44), we will use $\lim_{t \uparrow \tau_1^*} G_1^x(t) = 0$ and $G_1^x(\tau_1^*) = (1 - p_1)$. These imply that $S_2(t; G_1) = L_2(t)$ for all $t < \tau_1^*$, and that, for any $u > \tau_1^*$

$$S_2(\tau_1^*; G_1) = (1 - p_1)M_2(\tau_1^*) + p_1L_2(\tau_1^*) < (1 - p_1)W_2(\tau_1^*) + p_1L_2(\tau_1^*) = S_2(u; G_1),$$

because $p_1 < 1$ and (43). Thus, (44) holds if

$$\sup_{\tau} L_2(\tau) = L_2(\tau_2^*) < (1 - p_1)W_2(\tau_1^*) + p_1L_2(\tau_1^*) = \int_0^{\tau_1^*} \pi(X_s) e^{-rs} ds + [(1 - p_1)w(X_{\tau_1^*}^x) + p_1l_2] e^{-r\tau_1^*}. \quad (50)$$

It can be then seen that (50) holds if, for any $x \in (\theta_1^*, \theta_2^*]$, we have

$$l_2 < \mathbb{E}^x \left[\int_0^{\tau_1^*} \pi(X_s^x) e^{-rs} ds + [(1 - p_1)w + p_1l_2] e^{-r\tau_1^*} \right] = R(x) + \beta_2^p(\theta_1^*)\phi(x), \quad (51)$$

where $\beta_2^p(\theta) := \{[(1 - p_1)w + p_1l_2] - R(\theta)\} / \phi(\theta)$ and $w = \inf\{W(x) : x \in \mathcal{S}\}$. However, because $\beta_2^p(\theta) < \beta_w(\theta)$ for all $p_1 < 1$ and $\theta \in \mathcal{S}$ where $\beta_w(\theta) = [w - R(\theta)] / \phi(\theta)$ was used in the proof of Proposition 1(b), (51) holds if (26) does. Because we already proved in the proof of Proposition 1(b) that (26) is implied by the condition $\theta_2^* - \theta_1^* < \kappa_\theta$, we can conclude that the desired result follows.

Conversely, we show that G_1 is a best response to G_2 . Since the closure of the support of G_1^x is $(\alpha, \theta_1^*]$, by the virtue of Lemma B.2, it is enough to establish the following relations:

$$S_1(u; G_2) = S_1(v; G_2) \text{ for any } u, v \geq \tau_1^*, \quad (52)$$

$$S_1(t; G_2) < S_1(u; G_2) \text{ for any } t < \tau_1^* \leq u. \quad (53)$$

To show (52), can be shown by using the arguments for (52) above with $p_2 = 0$.

To prove (53), because $G_2^x(\tau_1^*) = 0$, (52) implies that $S_1(u; G_2) = S_1(\tau_1^*; G_2) = L_1(\tau_1^*)$ for any $u > \tau_1^*$, and that $S_1(t; G_2) = L_1(t)$ for any $t < \tau_1^*$. Hence, for any $t < \tau_1^* \leq u$, we have $S_1(t; G_2) =$

$L_1(t) < L_1(\tau_1^*) = S_1(u; G_2)$ where the inequality is due to Proposition 1. ■