ROBUST VOTER PERSUASION

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Abstract. This paper studies persuasion of large electorates in a general environment with heterogeneous, private preferences. Persuasion is possible in a simple equilibrium under a weak condition on voter preferences. Persuasion is even possible just by releasing additional information when voters already have private signals and a version of the Condorcet Jury Theorem would otherwise hold in a large election. Persuasion does not require detailed knowledge the distribution of voters' preferences and one signal structure can be used uniformly across environments.

Elections are ubiquitous instruments of collective choice. This paper studies the manipulability of elections through persuasion: An interested party has information that is valuable for voters and tries to affect voters' choices by the strategic release of this information. Examples of interested parties holding and strategically releasing relevant information for voters are numerous. Consider the vote on a reform. The advantages of the reform are unknown to the public, and an informed politician can decide how to release information. Or consider the election of a CEO at an annual shareholder meeting. The Board of directors provides information on the candidates with

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the shareholder meeting brochure, through conversations, and presentations.

This paper revisits the general voting setting by Feddersen and Pesendorfer [1997]. There are two possible policies (outcomes), A and B. Voters' preferences over policies are heterogenous and depend on an unknown state, α or β, in a fairly general way: Some voters may prefer A in state α, some prefer A in state β, and some may prefer A independently of the state (while others always prefer B). Preferences are drawn independently and identically across voters. Their preferences are each voters' private information. The election determines the outcome by a simple majority rule.

Voters have a common prior over the states. We explore the possibility and limits of persuasion, (Kamenica and Gentzkow [2011]): Prior to the election, a manipulator commits to an anonymous signal, which is a joint distribution over states and signal realizations that are privately observed by the voters. We ask: Can the manipulator ensure that a majority supports his favorite policy in a large election by choosing an appropriate signal?

In this setting, Feddersen and Pesendorfer [1997] have shown that, within a broad class of monotone preferences and conditionally i.i.d. signals, equilibrium outcomes of large elections are equivalent to the outcome with publicly known states ('information aggregation'). This may suggest that elections are robust. Our main result shows that, nevertheless, under weak conditions on preferences for any possible state-contingent policy there is a signal structure and a natural equilibrium that ensures that the policy is supported by a majority with probability close to one. In particular, the supported policy can be the opposite of the outcome with publicly known states, for every state.

For clarity, at first, we assume that all information of voters comes from a manipulator. Specifically, the main result for this baseline model roughly shows that persuasion is possible if there is one belief about the likelihood that the state is α such that a voter with randomly drawn preferences prefers A with probability larger than $1/2$ given this belief and another belief such that the probability of preferring A given this belief is closer to $1/2$ (Theorem 1). Denote these beliefs by $\bar{q}$ and $\bar{r}$, respectively. Clearly, some such condition is necessary for persuasion to be effective: If, for all beliefs, each voter prefers A with probability less than $1/2$, then, whatever the induced beliefs, in a large election the expected share of voters supporting A will be less than $1/2$.

We show that the condition is sufficient by constructing a signal structure as follows. Roughly speaking, with high probability $1-\varepsilon$ the voters receive conditionally independent draws of a binary signal, a or b, with a being relatively more likely in state α and b relatively more likely in state β. With monotone preferences and $\varepsilon = 0$, this would generally allow for information aggregation in equilibrium as in Feddersen and Pesendorfer [1997]. However, with probability $\varepsilon > 0$, the manipulator induces an additional *state-of-confusion*: In this additional state, almost all voters will receive a common signal z while only few voters receive signals a or b. Conditional on observing z, a voter knows that most other voters have also observed z. The consequence is that, in contrast to the usual calculus of strategic voting, there is no further information about others' signals contained in the event of being pivotal. This is the critical observation and it implies that voters behave essentially sincerely conditional on z. By choosing the relative probability of z in the two states appropriately, the posterior conditional on z will be $\bar{r}$, meaning, each voter prefers A with probability $1/2$ and, hence, the election is close to being tied. Thus, even from the viewpoint of the few voters observing signals a or b, conditional on the election being tied, it is likely that the other voters received the common signal z. By appropriately choosing the probabilities of a and b in the state-of-confusion, the posterior conditional on the state-of-confusion and conditional on a or b is $\bar{q}$. Hence, in the standard state, when there are only signals a and b, a large majority supports A. The main idea of the construction is that one can first characterize equilibrium for voters receiving a z signal and then use that to extend the construction to the other voters. For ϵ converging to 0, the construction in the baseline model converges to conditionally i.i.d. signals as in Feddersen and Pesendorfer [1997]. For persuasion it will be sufficient to have the ability to 'block' information aggregation in a state with vanishing probability, as the electorate grows large.

We argue that persuasion is robust in various dimensions. First, the played equilibrium is simple and insures voters against errors. Specifically, the equilibrium profile is almost identical to voting sincerely given one's signal, conditional on the state-of-confusion. One may argue that this behavior is simple. In particular, voters just need interpret their own signal conditional on that state; they do not need to make any further inference about other voters' signals using the equilibrium strategy profile. Furthermore, as will be explained in detail later, sincere behavior is "safe" in the sense of being an ε best response conditional on being pivotal for a neighborhood around the actual environment. Thus, even if a voter's belief about the environment and the equilibrium is slightly wrong, the cost of this error is small (conditional on being pivotal).

Second, the played equilibrium is 'attracting'. Specifically, for almost any strategy, the best response to the best response lies in a neighborhood of the equilibrium. In particular, the best response dynamics converges to the equilibrium starting from almost any strategy profile (Proposition 1).

Third, the manipulator needs little information about the preference distribution of voters: The same signal structure works uniformly across environments; it is not finely tuned to a particular distribution of preferences and priors. In fact, all the manipulator needs to know is a neighborhood of the actual prior and neighborhoods of the beliefs $\bar{q}$ and $\bar{r}$ that induce a strict majority for A and a closer election, respectively. By way of contrast, as discussed momentarily, existing work assumes that the manipulator knows the exact preference of each individual voter and this knowledge is indeed used.

In the second part of the paper, we consider an setting in which voters already have access to information of the form studied in Feddersen and Pesendorfer [1997]. Thus, if the manipulator adds no further information, the outcome would be as with publicly known states. We show that, by adding additional information, the manipulator can still persuade the voters effectively (Theorem 2). In this setting, the manipulator does not have the ability to 'block' information in a small added state. However, the main idea of the construction of the baseline model works here, too. We can first characterise equilibrium for voters receiving z. In the added state (where almost all voters receive z), the game converges to a game with only the exogeneous binary signal. It is known that the equilibrium limit of such a game is uniquely determined. In particular, this pins down the behavior in the added state. We extend the construction to the other voters that received a or b.

We generalise, and show that implementation of any state-contingent policy - not only implementation of A in both states - is possible by constructing signal structures in a similar way (Proposition 2). Consequently, if an outside observer only knows that voters have the preferences as in Feddersen and Pesendorfer [1997] and access to information that is at least as fine as theirs, then it is not certain that information is aggregated in equilibrium. No robust prediction (Bergemann and Morris [2017]) is possible if the observer only knows that voters have a certain minimal information (Corollary 1).

In an extension, we discuss the possibility of persuasion with public signals. Suppose that preferences are monotone, voters have no private signals about the state and hold a prior at which a majority votes B. Revealing that the state is α clearly increases the probability of the outcome A. However, it is not possible to induce a consistenly higher prior at which a majority supports A. For public signals, Bayes-consistency implies that the expected prior is equal to the initial prior. So persuasion is only partial (Proposition 3).[1] When voters receive exogeneous private signals and preferences are monotone, persuasion is not possible with public signals: When adding a public signal to a setting as in Feddersen and Pesendorfer [1997], this is equivalent to a shift in the common prior. However, we know that information is aggregated for all possible non-degenerate priors. A degenerate prior can only be induced by revealing a state, but this only helps information aggregation.

The paper is related to work on persuasion in general and especially to work on persuasion with multiple receivers (e.g., Mathevet et al. [2016]) and

[1]However, when preferences are non-monotone, complete persuasion might be possible, for $n \to \infty$. These results are reminiscient of Alonso and Câmara [2015].

to persuasion of a receiver with private information about its preferences (Kolotilin et al. [2015], Guo and Shmaya [2017]). Persuasion in the context of elections has been studied in a number of papers under various restrictions. Alonso and Câmara [2015] study persuasion through a public signal that is observed by all voters simultaneously. Consequently, voters do not condition on being pivotal. We allow for private signals. In many settings, it is natural that signals are not commonly observed.[2] Bardhi and Guo [2016a] study persuasion with unanimity rule. With unanimity, every voter needs to be persuaded and hence the problem is more similar to a single-receiver persuasion problem. Wang [2013] studies private persuasion by conditionally independent signals. This rules out the type of persuasion through a state-of-confusion that we consider. We believe that correlation of signals is feasible in many natural applications. Chan et al. [2016] study persuasion with known and monotone preferences through private signals. Our work shares with theirs the observation that the voters' conditioning on being pivotal allows relaxing the Bayesian consistency requirement. However, in our work, persuasion is achieved differently, namely, through a state-of-confusion. Moreover, we allow for general preference heterogeneity and, in particular, voters' preferences are their private information. The latter means that the type of "targeted persuasion" that is studied in the related work is not feasible here. When the preferences of individual voters are known, signals can be individualized so that they make a specific individual just indifferent. Methodologically, with known preferences, a revelation principle argument implies that individual signals are binary without loss of generality. [3]

A more detailed discussion of the related literature is in Section 7. There, we also discuss in depth the existing work on failures of information aggregation, especially Mandler [2012], Feddersen and Pesendorfer [1997] (their extension to aggregate uncertainty about preferences), and Bhattacharya [2013].

The rest of the paper is organized as follows: In Section 1 we present the model. In Section 2 we discuss a binary-state version of Feddersen and Pesendorfer [1997] as in Bhattacharya [2013]. In Section 3 we present the persuasion possibility result for the baseline model (Theorem 1), and illustrate its robustness. In Section 4 we prove the possibility result (Theorem 2) for the setting where a manipulator can add information to exogeneous

[2]Note that z is an almost common signal.

[3]Furthermore, given that there is a deterministic relation between signals and induced votes, the signal structure can be chosen such that the signal 'vote A' is pivotal in different profiles from the signal 'vote B'. For example, with 11 voters, if the induced signal profile is 6 "vote A" and 5 'vote B' signals, then only voters with a 'vote A' signal are pivotal. By judiciously choosing distributions over such profiles across states, being pivotal with a 'vote A' signal implies that the state is α with certainty and being pivotal with a 'vote B' signal implies that the state is β with certainty. In our setting, the interpretation of being pivotal is independent of one's signal.

private signals. In Section 4.1 we show that any state-contingent policy can be implemented with probability close to 1 (Proposition 2). In Section 5 we discuss the possibility of persuasion with public signals (Proposition 3) and when voters do not have heterogeneous types (Proposition 4). Section 6 discusses feasibility, other equilibria (Proposition 5) and evidence for the strategic voter paradigm. In Section 7 we discuss the paper's contribution to the existing literature and compare our results especially to other results on voter persuasion and other reported failures of information aggregation.

1 The Model

There are $2n+1$ voters, two possible election outcomes A and B, and two states of the world $\omega \in \{\alpha, \beta\} = \Omega$. Voters hold a common prior. The prior probability of α is $p_0 \in (0,1)$, and the probability of β is $1 - p_0$.

Voters have heterogeneous preferences. The preferences are private information. A preference type is a pair $t = (t_\alpha, t_\beta) \in [-1,1]^2$, with t_ω the utility of A in ω. We normalise the utility of B to zero.[4] Preference types are independently and identically distributed according to a commonly known distribution G that has a strictly positive density.

An information structure π is a set of signals S and a joint distribution of signal profiles and states. We assume that, for all $\omega \in \Omega$, $\pi_{|\omega}$ is symmetric with respect to the voters. [5]

A symmetric strategy of the voters is a function of the signal s and the type t, and denoted by $\sigma : S \times [-1,1]^2 \to [0,1]$ where $\sigma(s,t)$ is the probability of type t to vote A after s.

Aggregate Preferences. For a given strategy σ, denote by *piv* the event in which, from the viewpoint of a given voter, n of the other $2n$ voters vote for A and n for B. This is the event in which the voter's vote is decisive. Given σ, a voter of type t who received s weakly prefers to vote A if and only if

$$\Pr(\alpha|s, piv; \sigma, \pi) \cdot t_\alpha + (1 - \Pr(\alpha|s, piv; \sigma, \pi)) \cdot t_\beta \quad \geq \quad 0. \tag{1.1}$$

A central object of our analysis is the *aggregate preference function*

$$\phi(p) := \Pr_G(p \cdot t_\alpha + (1-p) \cdot t_\beta > 0)$$

[4]Otherwise, we can view t_ω as the difference of the utilities of A and of B in ω.

[5]The joint distribution F of random variables $Y_1, \ldots, Y_{2n+1}$ is called symmetric if $\Pr_F(y_1 \leq z_1, \ldots, y_{2n+1} \leq z_{2n+1}) = \Pr_F(y_{h(1)} \leq z_{h(1)}, \ldots, y_{h(2n+1)} \leq z_{h(2n+1)})$ whenever h is a permutation of $\{1, \ldots, 2n+1\}$.

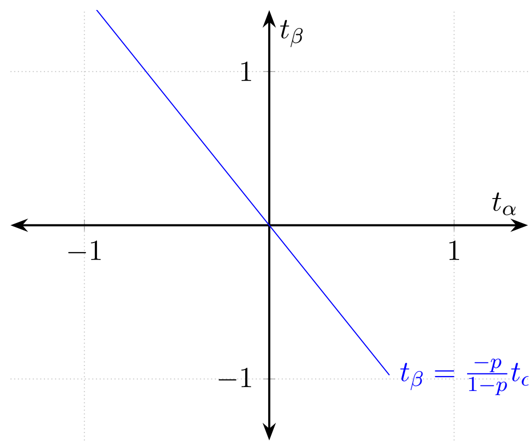


FIGURE 1. For any given belief $p = \Pr(\alpha) \in (0,1)$ the curve of indifferent types is $t_\beta = \frac{-p}{1-p} t_\alpha$.

which maps a common belief p to the probability that a random type t prefers A under p. Note that ϕ is continuous, since G is continuous.

Equilibrium. We analyse symmetric Bayes-Nash-equilibria in weakly undominated, pure strategies and call them (voting) equilibria. A strategy σ is a *cut-off-strategy* if for all $s \in S$ there exists $p_s \in [0,1]$ such that $\sigma(s,t) = 1 \Leftrightarrow t_\alpha \cdot p_s + t_\beta \cdot (1 - p_s)) \geq 0$. Any best reponse is a cut-off strategy with cut-offs $p_s = \Pr(\alpha|s, piv; \sigma, \pi)$ by 1.1. Voter types $t \gg 0$ (*A-partisans*) have the weakly dominant strategy to vote for A. Voter types $t \ll 0$ (*B-partisans*) have the weakly dominant strategy to vote for B. The restriction to undominated equilibria rules out trivial equilibria, because because G puts strictly positive probability on voter types $t >> 0$ and $t << 0$ by the assumption that it has a strictly positive density. The restriction to equilibria in pure strategies is without loss, because, by 1.1 and the continuity of G, a voter has a unique strict best response with probability 1.

Convergence. Convergence of strategies means pointwise convergence (up to measure 0). A sequence of cut-off-strategies σ_n with cut-offs $(p_{s,n})_{s\in S}$ converges to a cut-off strategy σ with cut-offs $(p_s)_{s\in S}$ if and only if $p_{s,n} \overset{n\to\infty}{\to} p_s$ for all $s \in S$. When we speak of distances $\|\sigma - \sigma'\|$ between cut-off strategies, we mean the Euclidean distance.

Remark 1. The collection of posteriors conditional on *piv* and s, namely $(\Pr(\alpha|s, piv; \sigma, \pi))_{s\in S}$, is a sufficient statistic for the unique undominated best response. The possibility of writing equilibria in terms of posteriors is what makes our model easily amenable to the Bayesian Persuasion literature.

Remark 2. Imagine that a sender, who we do not model, commits to π. By analysing the equilibria of $\Gamma(\pi)$, we implicitly analyse the scope of voter persuasion.

Remark 3. Given the general preference distribution G, the model nests almost common values. Besides, it does not only include the case in which a majority prefers A when it is known that α holds, and B when it is known that β holds, but also all cases in which the majority preference does not match the state.

The *sincere strategy* $\hat{\sigma}_n(q,r)$ is the strategy that acts upon the posteriors conditional on the signal s only.

Definition 1. The sincere strategy $\hat{\sigma}_n(q,r)$ is the pure strategy given by

$$\hat{\sigma}(q,r)(s,t) = 1 \qquad \Leftrightarrow t_\alpha \cdot \Pr(\alpha|s;q,r) + t_\beta \cdot (1 - \Pr(\alpha|s;q,r)) \geq 0.$$

Definition 2. For any $(q,r,\delta) \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0} \times [0,1]$, the δ-perturbed game $\Gamma_n(q,r,\delta)$ is the game of n voters induced by $\pi_n(q,r)$ in which each voter, with probability δ, plays $\hat{\sigma}$ and votes sincerely.[6]

2 Benchmark: Condorcet Jury theorem

Consider the case when voters receive a binary signal $\{u,d\}$ that is independently, and identically distributed conditional on the state ω. The informativeness of signals is bounded, namely the signal probabilities satisfy $1 > \Pr(u|\alpha) > \Pr(u|\beta) > 0$.The following are **benchmark assumptions** from the literature:

1. The aggregate preference function $\phi(p)$ is strictly increasing, that is, the probability that a random voter prefers A for a given belief p on α strictly increases with p. We say that preferences are monotone.
2. It holds that $\phi(1) > \frac{1}{2}$ and $\phi(0) < \frac{1}{2}$. So, the full information outcome, namely the outcome which is prefered by a majority of voters under perfect information on the state, is A in α and B in β.

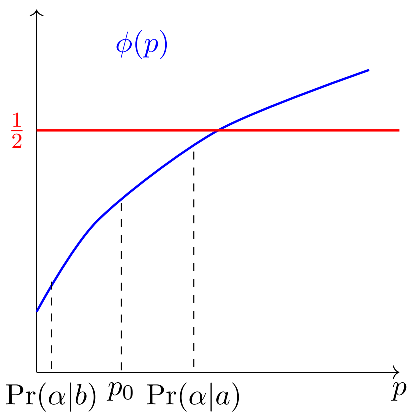
The information aggregation literature concerns with the question if strategies imply the full information outcome when the electorate n grows large.

Sincere Voting. When voters vote sincerely and the prior is sufficiently extreme, sincere voting does not aggregate information: For example, if p_0 is sufficiently low such that $\phi(\Pr(\alpha|u)) < \frac{1}{2}$, a random voter votes A with probability smaller $\frac{1}{2}$ after any signal. The law of large numbers implies that B is elected with probability converging to 1. However, when voters vote sincerely and priors are sufficiently unbiased, the full information outcome

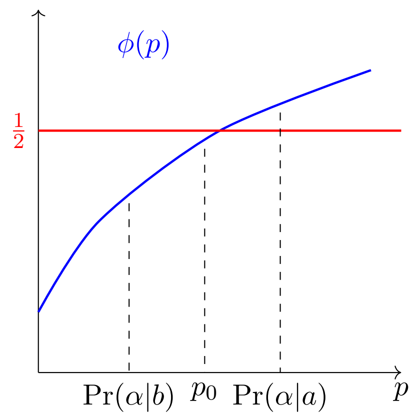
[6]Formally, when a voter chooses to play σ in the δ-perturbed games, he effectively plays the perturbed strategy $(1-\delta) \cdot \sigma(s,t) + \delta \cdot \hat{\sigma}(s,t)$. Sometimes we highlight with a superscript δ that for example the effective probabilities to vote A are meant, $\Pr^\delta(\sigma(s,t) = 1) := (1-\delta)\Pr(\sigma(s,t)=1) + \delta\Pr(\hat{\sigma}(s,t)=1)$.

is elected with probability converging to 1. For example, if for given signal precision $\frac{\Pr(u|\alpha)}{\Pr(u|\beta)} > 1$, the prior is sufficiently close to $\frac{1}{2}$ we have $\phi(\Pr(\alpha|u)) > \frac{1}{2}$.

This instance of the Condorcet Jury Theorem (Condorcet [1793]) is illustrated in the following figure on the right hand.



Under $\hat{\sigma}$, information aggregation fails with sufficiently extreme priors.



Under $\hat{\sigma}$, information is aggregated with sufficiently unbiased priors.

Strategic Voting. When the benchmark assumptions hold, and signals are identically distributed conditional on the state ω, the model in this paper describes a binary-state version of Feddersen and Pesendorfer [1997]. In this exact setting, Bhattacharya [2013] has replicated a result by Feddersen and Pesendorfer [1997], namely that the Condorcet Jury Theorem extends to strategic voting. Moreover, under strategic voting information is aggregated even with extreme priors.

Theorem 0. *(Bhattacharya [2013])*[7]
Under the benchmark assumptions: When voters receive binary signals that are independently, and identically distributed conditional on the state ω*, with* $1 > \Pr(a|\alpha) > \Pr(a|\beta) > 0$*, and when preferences are monotone, then the probability that the full information outcome is elected, converges to* 1 *in any sequence of equilibria.*

Proof. In the Appendix.

[7] This theorem is implied by Theorem 1 in Bhattacharya [2013] The notation for the function $\phi(\cdot)$ is $h(\cdot)$ in Bhattacharya [2013].

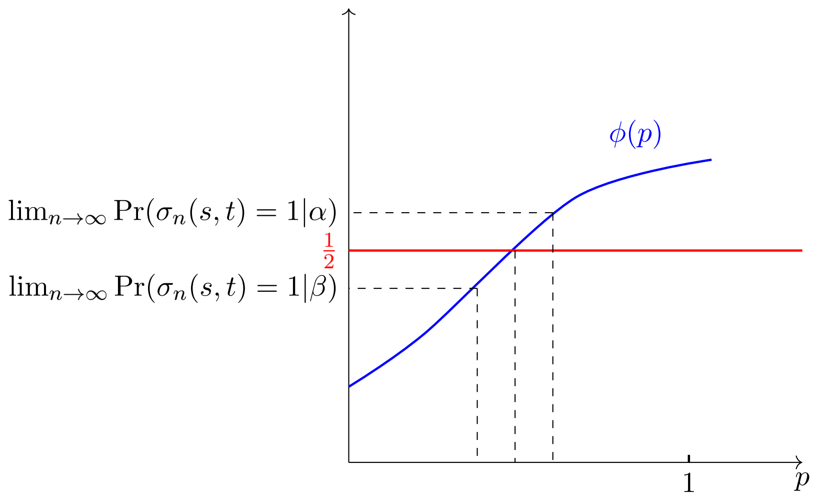
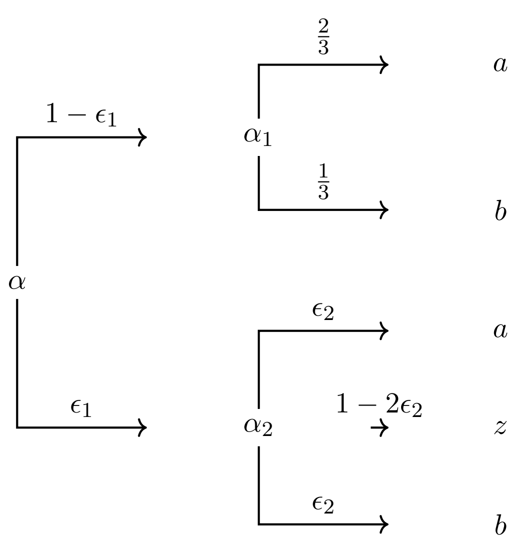


FIGURE 2. Condorcet Jury Theorem in Bhattacharya [2013]: In any equilibrium sequence , $\lim_{n\to\infty} \Pr(\sigma_n(s,t) = 1|\alpha) - \frac{1}{2} = \lim_{n\to\infty} \frac{1}{2} - \Pr(\sigma_n(s,t) = 1|\beta)$ holds.

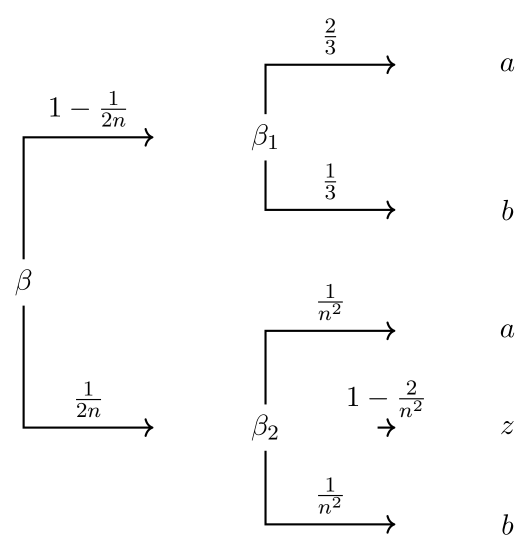
3 Possibility of Persuasion

3.1 Information

We consider a parametric class of information structures $\pi_n(q,r)$ with $S = \{a, b, z\}$. We illustrate $\pi_n(q,r)$ by two diagrams:



Information structure $\pi_n(q,r)$ in α.



Information structure $\pi_n(q,r)$ in β.

We set $\epsilon_1 = \frac{1-p_0}{p_0}\frac{r}{1-r}\frac{1}{2n}$ and $\epsilon_2 = \frac{1-r}{r}\frac{q}{1-q}\frac{1}{n^2}$. First, nature draws the state $\omega \in \{\alpha, \beta\}$ according to p_0. Then, a sub-state ω_j with $j \in \{1, 2\}$ is drawn. Conditional on ω_j voters receive independently and identically distributed signals $s \in \{a, b, z\}$. The probabilities by which the sub-states ω_j are drawn and the probabilities by which the signals are sent to voters conditional on ω_j are indicated along the arrows.

The parameters q, r have an easy interpretation: By definition, the posteriors conditional on the signal and $\Omega_2 := \{\alpha_2, \beta_2\}$[8] satisfy

$$\lim_{n\to\infty} \Pr(\alpha|z, \Omega_2; q, r) = r, \quad \text{and} \tag{3.1}$$

$$\Pr(\alpha|s, \Omega_2; q, r) = q \quad \text{for } s \in \{a, b\} \text{ and all } n \in \mathbb{N}. \tag{3.2}$$

Remark 4. When the electorate grows large, $\pi_n(q, r)$ converge to signals as in the benchmark 2. Only in states Ω_2, the information structures $\pi_n(q, r)$ differ from an information structure in the benchmark. In Ω_2, voters receive an almost public signal z. Intuitively, in Ω_2, in contrast to the usual calculus of strategic voting, there is no further information about others' signals contained in the event of being pivotal: Information aggregation is 'blocked'. By concentrating the analysis on $\pi_n(q, r)$ we restrict a potential sender to 'blocking' information aggregation in states of vanishing probability.

Remark 5. The following story shall illustrate $\pi_n(q, r)$: Imagine a politician who places ads (signals) on a webpage. Most of the time in α, ad a appears with probability $\frac{1}{3}$ to a visitor, and ad b appears with probability $\frac{2}{3}$, and vice versa in β. However, in state α, with small probability ϵ_1, ad a and ad b appear each with probability ϵ_2, and ad z appears with probability $1 - 2\epsilon_2$. In state β, with small probability $\frac{1}{2n}$, ad a and ad b appear each with probability $\frac{1}{n^2}$, and ad z appears with probability $1 - \frac{2}{n^2}$. This ad placement strategy implements $\pi_n(q, r)$ when the arrival of voters to the webpage is independently, and identically distributed.

3.2 Result

From now on, consider preferences G such that there exists a common belief on α under which less than a majority of voters prefers A. Formally, this requires that $\phi(p)$ takes values strictly below $\frac{1}{2}$.[9]

Definition 3. The Ω_2-sincere strategy $\hat{\sigma}_{\Omega_2}(q, r)$ is the pure strategy that votes A if and only if [10]

$$t_\alpha \cdot \Pr(\alpha|s, \Omega_2; q, r) + t_\beta \cdot (1 - \Pr(\alpha|s, \Omega_2; q, r)) \geq 0.$$

[8]Similarly, we define $\Omega_1 := \{\alpha_1, \beta_1\}$ and we denote the generic element of Ω_i by ω_i.

[9]If $\phi(p) > \frac{1}{2}$ for all $p \in [0, 1]$, A is the election outcome for $n \to \infty$ under the 'null' information structure that always sends the same signal.

[10]Recall the interpretation of the parameters q and r in terms of the posteriors $\Pr(\alpha|s, \Omega_2; q, r)$ in 3.1 and 3.2.

The following Theorem 1 gives a weak condition on G under which the parameters q and r of the information structure $\pi_n(q,r)$ can be chosen such that, for large n, A is elected with probability close to 1 in an equilibrium of the unperturbed game. The equilibrium in which A is elected is simple as it converges to Ω_2-sincere voting.

Theorem 1. *If there exists a common belief $\bar{q} \in [0,1]$ such that a strict majority of voters prefers A under $\bar{q}$, $\phi(\bar{q}) > \frac{1}{2}$, then there exists a belief $\bar{r}$ such that $\hat{\sigma}_{\Omega_2}(\bar{q},\bar{r})$ is the limit of some equilibrium sequence σ_n in the games $\Gamma_n(\bar{q},\bar{r},0)$, and it holds that $\lim_{n\to\infty} \Pr(A \textit{ is elected} | \sigma_n; \bar{q}, \bar{r}, 0) = 1$.*

Weakened Bayes-Consistency Constraints: Note that under $\hat{\sigma}_{\Omega_2}(\bar{q},\bar{r})$, for $n \to \infty$, agents act upon the posterior $\bar{q}$ with probability converging to 1. The Bayes consistency constraints for persuasion of multiple voters vanish completely for $n \to \infty$. This is in stark contrast to persuasion of a single receiver where posteriors have the martingale property (Kamenica and Gentzkow [2011]).

The sufficient condition of Theorem 1 requires that there must exist a common belief p under which a majority of voters prefers A. This is clearly a necessary condition for A to be elected. Intuitively, the condition describes two aspects of G:

No majority of B-partisans. The voters that prefer to vote B regardless of their belief do not represent a majority, for $n \to \infty$. These are the types with $t << 0$.

Asymmetry of Information-Sensitive Types. There must be an asymmetry between the voter types who prefer A only in state α, that is, those for which $t_\alpha > 0$ and $t_\beta < 0$, and the voter types who prefer A only in state β, that is, those for which $t_\alpha < 0$ and $t_\beta > 0$. If both groups of voter types are equally likely, and the density of G is symmetric, meaning that it takes the same values at (t_α, t_β) and at $(-t_\alpha, -t_\beta)$ for all (t_α, t_β) with $t_\alpha > 0$ and $t_\beta < 0$, then the function $\phi(p) = \Pr_G(p \cdot t_\alpha + (1-p) \cdot t_\beta > 0)$ is constant in p. Then the condition can only be fulfilled if $\phi(p) > \frac{1}{2}$ for all p, hence persuasion is trivially possible. So, in the fully symmetric situation, the condition of Theorem 1 either fails or persuasion is trivially possible.

3.3 Sketch of Proof

In the states Ω_2, almost all voters receive the common signal z, for any q, r and δ. Conditional on observing z, a voter knows that most other voters have also observed z. Intuitively, in contrast to the usual calculus of strategic voting, there is no further information about others' signals contained in the event of being pivotal. Formally, we we establish Lemma

1, (i) in the Appendix and apply it to $x_n = \Pr^\delta(\sigma_n(s,t) = 1|\alpha_2; \sigma_n, q, r, \delta)$, and $y_n := \Pr^\delta(\sigma_n(s,t) = 1|\beta_2; \sigma_n, q, r, \delta)$. We record

Lemma 2. *For any sequence* $(\sigma_n)_{n\in\mathbb{N}}$ *of weakly undominated strategies, it holds that* $\lim_{n\to\infty} \frac{\Pr(piv|\alpha_2;\sigma_n,q,r,\delta)}{\Pr(piv|\beta_2;\sigma_n,q,r,\delta)} = 1$.

Proof. In the Appendix.

Consequently, $\lim_{n\to\infty} \frac{\Pr(\alpha|z,piv;\sigma_n,q,r,\delta)}{\Pr(\beta|z,piv;\sigma_n,q,r,\delta)} = \lim_{n\to\infty} \frac{\Pr(\alpha|z;q,r,\delta)}{\Pr(\beta|z;q,r,\delta)}$. Therefore, after signal z agents behave sincerely. This means that we can control the behavior of agents getting z perfectly - and in particular make the election arbitrarily close to being tied in ω_2 by choosing $r \overset{3.1}{=} \lim_{n\to\infty} \Pr(\alpha|z, \Omega_2; q, r)$ appropriately. The assumption of Theorem 1 requires that there exists $\bar{q}$ with $\phi(\bar{q}) > \frac{1}{2}$. Since ϕ is continuous, we can apply the intermediate value theorem and obtain $\bar{r}$ with $\phi(\bar{r}) = \frac{1}{2}$, that is under the belief $\bar{r}$ a random voter prefers A with probability $\frac{1}{2}$. So, given $\pi_n(q, \bar{r})$ and any equilibrium sequence σ_n, it holds

$$\begin{aligned} &\lim_{n\to\infty} \Pr(\sigma_n(z,t) = 1|\sigma_n; q, \bar{r}, \delta) \\ \overset{1.1}{=}\ &\Pr_G(t_\alpha \cdot \bar{r} + t_\beta \cdot (1-\bar{r}) > 0) = \phi(\bar{r}) = \frac{1}{2}. \end{aligned} \tag{3.3}$$

Remark 6. We call the states Ω_2 the *'states-of-confusion'*, because information aggregation is not possible in Ω_2, since voters receive an almost public signal z. Also, by 3.3, the election outcome is purposefully highly uncertain in Ω_2.

Lemma 3. *If*

$$\begin{aligned} &\min_\omega \| \lim_{n\to\infty} \Pr^\delta(\sigma_n(s,t) = 1|\omega_2; q, r, \delta) - \frac{1}{2} \| \\ < \ &\min_\omega \| \lim_{n\to\infty} \Pr^\delta(\sigma_n(s,t) = 1|\omega_1; q, r, \delta) - \frac{1}{2} \|, \end{aligned} \tag{3.4}$$

holds, then $\lim_{n\to\infty} \frac{\Pr(\alpha|s,piv;\sigma_n,q,r,\delta)}{\Pr(\beta|s,piv;\sigma_n,q,r,\delta)} = \frac{\Pr(\alpha|s,\Omega_2;q,r,\delta)}{\Pr(\beta|s,\Omega_2;,q,r,\delta)}$ *for* $s \in \{a, b\}$*, and the unique best response to* σ_n *in the games* $\Gamma_n(q, r, \delta)$ *converges to* $\hat{\sigma}_{\Omega_2}(q, r)$ *for* $n \to \infty$*.*

Proof. In the Appendix.

Lemma 3 shows that if the limit of the expected margin of victory in the states ω_1 is strictly larger than the limit of the expected margin of victory in the states ω_2 (call this the *'margin-of-victory condition'*), the unique best response converges to Ω_2-sincere voting $\hat{\sigma}_{\Omega_2}(q, r)$ for $n \to \infty$. Intuitively, when the margin-of-victory-condition holds, conditional on being tied, the states ω_2 are much more likely than the states ω_1. Hence, being pivotal contains the information that the states Ω_1 do not hold, but no information beyond that, by Lemma 2. This is precisely the information that Ω_2-sincere

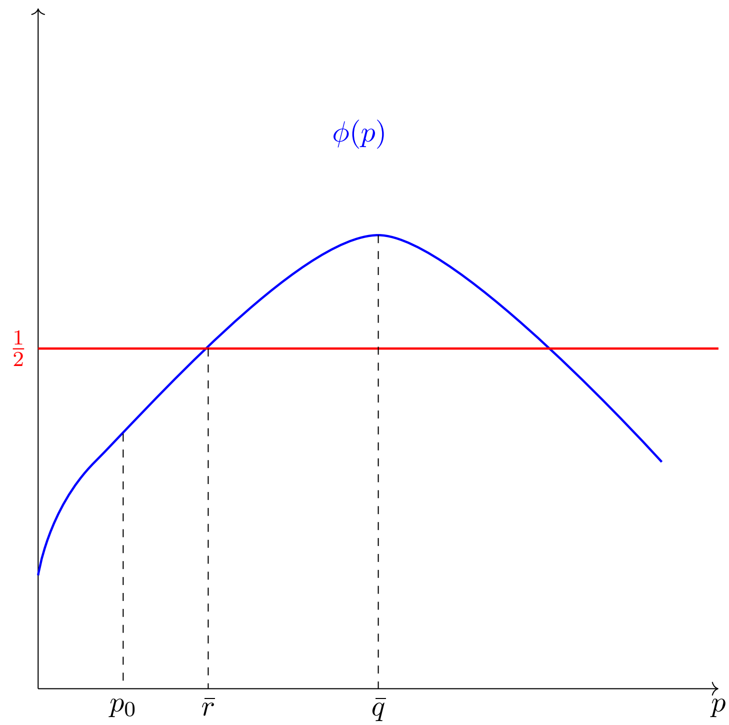
voters condition on. Hence, the best reply converges to $\hat{\sigma}_{\Omega_2}$.
More precisely, the margin-of-victory condition implies $\lim_{n\to\infty} \frac{\Pr(\omega_1|s,piv;\sigma_n,q,r,\delta)}{\Pr(\omega_2'|s,piv;\sigma_n,q,r,\delta)} = \lim_{n\to\infty} \frac{\Pr(\omega_1|s;q,r)}{\Pr(\omega_2'|s;q,r,\delta)} \frac{\Pr(piv|\omega_1;\sigma_n,q,r)}{\Pr(piv|\omega_2;\sigma_n,q,r,\delta)} = 0$ for $s \in \{a,b\}$ and any $\omega, \omega' \in \Omega$. This can be seen as follows: The probability of the election being tied is decreasing exponentially faster in states ω_1 than in states ω_2. Conditional on the signal $s \in \{a,b\}$, the states ω_2 are less likely than the states ω_1. However, note that the ratios $\frac{\Pr(\omega_1|s;q,r,\delta)}{\Pr(\omega_2|s;q,r,\delta)}$ are only increasing at a rate proportional to n^3 for $s \in \{a,b\}$. So, the exponentially decreasing terms $\frac{\Pr(piv|\omega_1;\sigma_n,q,r,\delta)}{\Pr(piv|\omega_2;\sigma_n,q,r,\delta)}$ dominate. So, the posteriors conditional on being pivotal and conditional on $s \in \{a,b\}$ vanish on ω_1. Being pivotal contains the information that the states Ω_1 do not hold for $n \to \infty$.

Equilibrium Construction. We can control the limit behaviour of agents getting $s \in \{a,b\}$ by choosing $q \overset{3.1}{=} \Pr(\alpha|s,\Omega_2;q,r,\delta)$ appropriately by Lemma 3. By assumption, there exists a commmon belief $\bar{q} \in (0,1)$ such that under $\bar{q}$ a majority of voters prefers to vote for A,

$$\phi(\bar{q}) > \frac{1}{2}. \tag{3.5}$$

In the games $\Gamma_n(\bar{q},\bar{r},0)$, under the Ω_2-sincere strategy $\hat{\sigma}_{\Omega_2}(\bar{q},\bar{r})$ and for large n, a strict majority of agents votes A after getting $s \in \{a,b\}$: $\lim_{n\to\infty} \Pr(\hat{\sigma}_{\Omega_2}(s,t) = 1|s;\bar{q},\bar{r},0) \overset{\text{Definition 3 + 3.2}}{=} \Pr_G(t_\alpha \cdot \bar{q} + t_\beta(1-\bar{q}) > 0) = \phi(\bar{q}) \overset{3.5}{>} \frac{1}{2}$ for $s \in \{a,b\}$. By 3.3, under $\hat{\sigma}_{\Omega_2}(\bar{q},\bar{r})$, the limit of the expected margin of victory in the states ω_2 is zero. So, $\hat{\sigma}_{\Omega_2}(\bar{q},\bar{r})$ satisfies the margin-of-victory condition 3.4 of Lemma 3. So when agents vote Ω_2-sincerely or use a cut-off strategy close-by[11] to $\hat{\sigma}_{\Omega_2}(\bar{q},\bar{r})$, the best reply converges to $\hat{\sigma}_{\Omega_2}(\bar{q},\bar{r})$. A fixed-point argument, Lemma 4, closes the loop of best replies. Hence, there exists a sequence of equilibria that converge to $\hat{\sigma}_{\Omega_2}(\bar{q},\bar{r})$. This shows the claim of Theorem 1, because under any strategy close-by to $\hat{\sigma}_{\Omega_2}(\bar{q},\bar{r})$, A gets elected with certainty, for $n \to \infty$.
We illustrate the equilibrium construction:

[11]By close-by we mean that the Euclidean distance of the cut-offs is small.



Under $\hat{\sigma}_{\Omega_2}(\bar{q},\bar{r})$ citizens vote A with probability $\phi(\bar{r}) = \frac{1}{2}$ after z; citizens vote A with probability $\phi(\bar{q}) > \frac{1}{2}$ after a and b. This supports the belief that conditional on being pivotal the states Ω_1 do not hold, and justifies that voting according to $\Pr(\alpha|\Omega_2, z; \bar{q}, \bar{r}) = \bar{r}$ after z and $\Pr(\alpha|\Omega_2, s; \bar{q}, \bar{r}) = \bar{q}$ after $s \in \{a, b\}$ is optimal, for $n \to \infty$ (see Lemma 3).

Remark 7. (*Stability of* $\hat{\sigma}_{\Omega_2}(\bar{q},\bar{r})$) There exists $\epsilon > 0$ such that the margin-of-victory condition 3.4 holds for all $\sigma \in B_\epsilon(\hat{\sigma}_{\Omega_2}(q,r))$. Consequently, $\lim_{n\to\infty} BR(\sigma) = \hat{\sigma}_{\Omega_2}(\bar{q},\bar{r})$ for all $\sigma \in B_\epsilon(\hat{\sigma}_{\Omega_2}(q,r))$ by Lemma 3. Hence, there exists $\bar{n} \in \mathbb{N}$ such that any cut-off strategy $\sigma \in B_\epsilon(\hat{\sigma}_{\Omega_2}(q,r))$ is ϵ-close to its best reponse $BR(\sigma)$ for all $n \geq \bar{n}$. Formally, this means that the cut-offs of σ and $BR(\sigma)$ are ϵ-close. So, after any signal s, any type that makes different choices under σ and $BR(\sigma)$ must be ϵ-close to the indifferent type (the cut-off of $BR(\sigma)$); consequently the type's loss is smaller than ϵ conditional on being pivotal. We say that σ is a *conditional* ϵ*-equilibrium*.[12] The strategy $\hat{\sigma}_{\Omega_2}(\bar{q},\bar{r})$ is *stable* or 'safe' in the sense that all strategies in a neighbourhood of $\hat{\sigma}_{\Omega_2}(\bar{q},\bar{r})$ are conditional ϵ-equilibria.

[12]The classical notion of ϵ-equilibrium (see e.g. Radner [1980]) is void for the voting games analysed, since the probability of being pivotal converges to 0 for $n \to \infty$. Therefore any strategy is an ϵ-equilibrium for n large enough.

Computational Example. We specify the preferences, the prior and the information structure by the assumptions that $\Pr(t_\alpha > 0, t_\beta < 0) = 1$[13], that $\Pr(\frac{-t_\beta}{t_\alpha + t_\beta} \leq p) = p$ for all $p \in [0,1]$[14] which implicitly defines G, and that $p_0 = \frac{1}{4}$. Further we set $r = \frac{1}{2}$, and $q = \frac{3}{4}$. In the Appendix we show that under these primitives an equilibrium σ_n close to conditional sincere voting exists for $n \geq 200$. Additionally, in this equilibrium A is elected with a probability of more than 99 percent.
To do so, we show that under the specified primitives the best reponse is a self-map on the set of strategies σ satisfying $\Pr(A|s) \geq 0.7$ for $s \in \{a, b\}$, and $\Pr(A|z) \in [0.45, 0.54]$ for $n \geq 200$. This yields an equilibrium in which voters with an a-or b-signal vote A with at least 70%.

3.4 Robustness

Conditional Sincere Voting is Simple. The voting strategy is simple to operationalise: If we want to tell a voter to behave conditionally sincere, then this will only require the voter to calculate his personal beliefs. It would not require knowledge of G or the strategies of others. It is simple to rationalise: Conditional sincere voting is an equilibrium (limit) by the simple logic that it is optimal to condition on the states Ω_2 if the expected margin of victory is smallest in Ω_2 (cf. Lemma 3); if all voters actually condition on Ω_2, the underlying assumption on the order of the margin of victories is indeed true, intuitively, because voters receive an almost public signal in Ω_2 which (by construction) induces a close election outcome.

Best Response Robustness. Recall that we use a stability argument in the equilibrium construction of the proof of Theorem 1 (we use Lemma 4, recall also Remark 7). This implies that, in the games $\Gamma_n(\bar{q}, \bar{r}, 0)$, conditional sincere voting has a non-trivial basin of attraction with respect to the best response dynamics for n sufficiently large. We prove a more general result in the following. Denote by $BR(\sigma)$ the best response to a strategy σ, and by BR^2 the the twice iterated best response, $BR^2(\sigma) = BR(BR(\sigma))$. Further for any $\epsilon > 0$, and $n \in \mathbb{N}$ define

$$\Sigma^2(\epsilon, n) := \{\sigma : \sigma \text{ cut-off strategy for which } \|BR^2(\sigma) - \hat{\sigma}_{\Omega_2}(\bar{q}, \bar{r})\| < \epsilon\}$$

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[13]Note that this is slightly inconsistent with the assumption that G has a strictly positive density, but made for simplicity of presentation.

[14]Note that this assumption is equivalent to saying that $\phi(p) = p$ for all $p \in (0,1)$. One distribution G on $[0,1] \times [-1,0]$ that induces such a uniform distribution of 'thresholds of doubt' is given by the density

$$g(t_\alpha, t_\beta) = \begin{cases} \sqrt{1 + (\frac{t_\beta}{t_\alpha})^2} \cdot (2 \cdot \int_{\|t_\alpha\| > \|t_\beta\|} \sqrt{1 + (\frac{t_\beta}{t_\alpha})^2} dt)^{-1} & \text{if} \quad \frac{-t_\beta}{t_\alpha - t_\beta} \leq \frac{1}{2}, \\ \sqrt{1 + (\frac{t_\alpha}{t_\beta})^2} \cdot (2 \cdot \int_{\|t_\alpha\| > \|t_\beta\|} \sqrt{1 + (\frac{t_\beta}{t_\alpha})^2} dt)^{-1} & \text{if} \quad \frac{-t_\beta}{t_\alpha - t_\beta} \geq \frac{1}{2}. \end{cases}$$

for $\bar{r}$ and $\bar{q}$ satisfying 3.3 and 3.5 respectively. We can identify cut-off-strategies with their cut-offs $(p(s))_{s\in S}$, and therefore understand $\Sigma^2(\epsilon, n)$ as a subset of $[0,1]^3$.

Proposition 1. *(Global Basin of Attraction)*[15]
Under the benchmark assumptions: For any $\epsilon > 0$, the measure of $\Sigma^2(\epsilon, n)$ in the space of cut-off-strategies $[0,1]^3$ converges to 1, for $n \to \infty$. In particular, there exists $n(\epsilon) \in \mathbb{N}$ such that all cut-off strategies σ for which

$$(3.6)\quad \|\|\Pr(\sigma(s,t)=1|\alpha_1) - \frac{1}{2}\| - \|\Pr(\sigma(s,t)=1|\beta_1) - \frac{1}{2}\|\| > n^{-\frac{1}{4}} \quad \text{and}$$

$$(3.7)\quad \|\min_{\omega} \|\Pr(\sigma(s,t)=1|\omega_1) - \frac{1}{2}\| - \min_{\omega} \|\Pr(\sigma(s,t)=1|\omega_2) - \frac{1}{2}\|\| > n^{-\frac{1}{4}}$$

hold, are elements of $\Sigma^2(\epsilon, n)$ for $n \geq n(\epsilon)$.

Proof. In the Appendix.

Proposition 1 implies that the best response dynamics converges to $\hat{\sigma}_{\Omega_2}(q,r)$ for almost any starting point, when n is large.

Sketch of Proof. Whenever the expected vote shares are sufficiently larger in Ω_1 than in Ω_2, than - similar to the margin-of-victory condition 3.4 - the best response converges to conditional sincere voting $\hat{\sigma}_{\Omega_2}(q,r)$. Conversely, whenever the expected vote shares are sufficiently smaller in Ω_1 than in Ω_2, for $n \to \infty$, being pivotal contains the information that states Ω_2 do not hold. However, by the same reasoning if the difference in expected vote shares in α_1 and β_1 is sufficiently large, for $n \to \infty$,being pivotal contains the information that either α_1 does not hold or β_1 does not hold after signals a and b. In any case under the best response voter behaviour in Ω_1 is almost as if it is known that a specific state holds. Under the benchmark assumptions, a strict majority of voters prefers A in α and B in β. Since, in any equilibrium the limit of the expected margin-of-victory is zero in Ω_2, the best response satisfies the margin-of-victory condition 3.4. Consequently, the twice iterated best response converges to $\hat{\sigma}_{\Omega_2}(q,r)$ by Lemma 3.

Conditional ϵ-equilibrium $BR^2(\sigma)$. By Remark 7 any strategy σ in a neighbourhood of conditional sincere voting $\hat{\sigma}_{\Omega_2}(q,r)$ is ϵ-close to its best reponse $BR(\sigma)$; σ is a conditional ϵ-equilibrium. So, Proposition 1 implies that for almost any σ, the twice iterated best response $BR^2(\sigma)$ is a conditional ϵ-equilibrium.

Level k-Implementability: Note that Proposition 1 and the remark after loosely relate to the concept of level k-implementability (de Clippel

[15]The result holds more generally. If we consider any random (not necessarily cut-off) strategy as starting point, for any $\epsilon > 0$ the probability that the twice iterated best response lies in an ϵ-neighbourhood of conditional sincere voting converges to 1, for $n \to \infty$.

et al. [2016]). For approximately any strategy (a 'behavioral anchor'), the level-2-consistent strategies are conditional ϵ-equilibria and ϵ-close to conditional sincere voting for $n \geq n(\epsilon)$. In this sense, alternative A is level-2-implementable.

Perturbation Robustness. Consider an agent with a misspecified belief $G' \neq G$ (or alternatively a misspecified prior $p_0' \neq p_0$. Consequently, he has a wrong belief on the margin-of-victory in the states ω_1 and ω_2 under conditional sincere voting $\hat{\sigma}_{\Omega_2}(\bar{q}, \bar{r})$. If the misspecification is small, he believes that $\hat{\sigma}_{\Omega_2}(\bar{q}, \bar{r})$ satisfies the margin-of-victory condition 3.4, so his best response converges to equilibrium limit play $\hat{\sigma}_{\Omega_2}(\bar{q}, \bar{r})$ for $n \to \infty$.

Sender Robustness (Informational Requirements). Recall that $\pi_n(q, r)$ are functions of the prior. Consider a sender who has a misspecified prior p_0. Suppose, the sender commits to $\pi_n(q', r')$ such that under a believed prior p_0', the induced posteriors conditional on the signal and conditional on Ω_2, satisfy $\lim_{n\to\infty} \Pr(\alpha|z, \Omega_2; q', r', p_0') = r'$, and $\Pr(\alpha|s, \Omega_2; q', r', p_0') = q'$ for $s \in \{a, b\}$. If the true prior is p_0, the actual posteriors satisfy $\lim_{n\to\infty} \frac{\Pr(\alpha|z,\Omega_2;q',r',p_0)}{\Pr(\beta|z,\Omega_2;q',r',p_0)} = r' \cdot \frac{p_0}{p_0'} \cdot \frac{1-p_0}{1-p_0'}$. Suppose the sender chooses r' such that $\phi(r') = \frac{1}{2}$ (compare 4.2), and q' with $\phi(q') > \frac{1}{2}$. If his misspecification on the prior is small, also the actual posteriors q and r satisfy $\|\phi(q) - \frac{1}{2}\| - \|\phi(r) - \frac{1}{2}\| > 0$. Then, the argument of the proof of Theorem 1 goes through and there exists an equilibrium sequence that converges to conditional sincere voting for $n \to \infty$. A similar argument can be applied if the sender has a slightly misspecified belief on G.

4 Exogeneous Private Signals

The results by Feddersen and Pesendorfer [1997] and Bhattacharya [2013] (see Theorem 0) have shown that, within a broad class of monotone preferences and conditionally i.i.d. signals, equilibrium outcomes of large elections are equivalent to the outcome with publicly known states, a version of the Condorcet Jury Theorem holds. In the first part of the paper, we showed a sender can manipulate elections by creating states-of-confusion Ω_2 in which voters receive an almost public signal z. (Theorem 1). This 'blocks' information aggregation in Ω_2 and we explained how this allows to steer voter behaviour even in the states where no voter receives z.

The possibility of almost public signals relies on the assumption that the sender is a monopolistic information provider. This is restrictive. For example, the independent media are a major source of information for voters. This section investigates the following natural question: *Is manipulation possible just by releasing additional information when voters already have* *private signals and a version of the Condorcet Jury Theorem would otherwise hold in a large election?*

In this section we consider a variation of the voting games $\Gamma_n(q,r,0)$ from Section 3. The information structure of voters contains signals $S = \{a,b,z\} \times \{u,d\}$ from two sources. We look at the scenario in which voters receive both signals from $\pi_n(q,r)$, and from an exogeneous information structure $\bar{\pi}$, which sends binary signals $\{u,d\}$ that are independently, and identically distributed conditional on the state of the world $\omega \in \Omega$. Signals from $\bar{\pi}$ and $\pi_n(q,r)$, are send independently of each other conditional on the state of the world $\omega \in \Omega$. We make the usual assumption on informativeness of signals, $1 > \Pr(u|\alpha) > \Pr(u|\beta) > 0$. So,

$$\frac{\Pr(\alpha|u,\bar{\pi})}{\Pr(\beta|u,\bar{\pi})} > \frac{\Pr(\alpha|d,\bar{\pi})}{\Pr(\beta|d,\bar{\pi})}. \tag{4.1}$$

For $s \in \{a,b\}$, and $v \in \{u,d\}$, in this setting, it holds that

$$\lim_{n\to\infty} \frac{\Pr(\alpha|v,z;\Omega_2,\pi_n,\bar{\pi})}{\Pr(\beta|v,z,\Omega_2;\pi_n,\bar{\pi})} = \frac{r}{1-r}\cdot\frac{\Pr(v|\alpha)}{\Pr(v|\beta)}, \quad \text{and} \tag{4.2}$$

$$\frac{\Pr(\alpha|v,s,\Omega_2;\pi_n,\bar{\pi})}{\Pr(\beta|v,s,\Omega_2;\pi_n,\bar{\pi})} = \frac{q}{1-q}\cdot\frac{\Pr(v|\alpha)}{\Pr(v|\beta)} \tag{4.3}$$

where we used 3.1, and 3.2, and the independence of $\bar{\pi}$ and $\pi_n(q,r)$. In comparison with 3.1, and 3.2 the additional terms $\frac{\Pr(v|\alpha)}{\Pr(v|\beta)}$ come from the learning through $\bar{\pi}$. The definitions of Section 3 are adapted to the setup with exogeneous private signals $\bar{\pi}$; in particular Definition 1 of sincere voting $\hat{\sigma}$, Definition 3 of Ω_2-sincere voting $\hat{\sigma}_{\Omega_2}$, and the definition of the voting games $\Gamma_n(q,r,0)$. The possibility result still holds:

Theorem 2. *Under the benchmark assumptions: For any $\bar{\pi}$ there exist $\bar{q} > \bar{r}$, such that $\hat{\sigma}_{\Omega_2}(\bar{q},\bar{r})$ is the limit of an equilibrium sequence σ_n in $\Gamma_n(\bar{q},\bar{r},0)$ and it holds that* $\lim_{n\to\infty}\Pr(A \textit{ is elected}|\sigma_n;\pi_n(\bar{q},\bar{r}),\bar{\pi}) = 1$.

Contrary to the case without exogeneous private signals $\bar{\pi}$, there is not an almost public signal in Ω_2. So, the expected vote share for A in α_2 is not necessarily close to the expected vote share for A in β_2, and conditional on Ω_2 being pivotal can obtain information, for $n \to \infty$ (compare with Lemma 2). So voters do not neccesarily vote sincerely after z. However, similarly, voter behaviour after z is independent of the behaviour after a and b in equilibrium and for $n \to \infty$, as the Lemma 6 below shows. For this, note that conditional on Ω_2, the voting game approximately describes a game with binary signals u,d that are conditionally independent given the state α_2 or β_2. From Bhattacharya [2013] we know that such a game has a unique equilibrium limit for $n \to \infty$. The proof is made in three steps:
The first step establishes uniqueness of the limit of equilibrium play after z.
The second step shows that we can rationalise the equilibrium play after z

as sincere voting. The third step constructs an equilibrium sequence along the lines of the proof of Theorem 1. However, here the fixed point argument is applied to a modified best response function.

Proof.

Definition 4. *(Partial games)*
Any information structure $\pi_n(q,r)$ induces a collection of (unperturbed) Bayesian games of n voters, in which the voter behaviour after the signal combinations s and v for $s \in \{a,b\}$ and $v \in \{u,d\}$ is fixed by a given strategy. We call these games the partial games (of $\pi_n(q,r)$).

Step 1: Equilibrium Play After z. After receiving z, and $v \in \{u,d\}$, a voter weakly prefers to vote A if and only if

$$t_\alpha \cdot \Pr(\alpha|piv,z,v;\sigma_n,\pi_n(q,r),\bar{\pi}) + t_\beta \cdot (1 - \Pr(\alpha|piv,z,v;\sigma_n,\pi_n(q,r),\bar{\pi}) \geq 0.$$

with $\frac{\Pr(\alpha|piv,z,v;\sigma_n,\pi_n(q,r),\bar{\pi})}{1-\Pr(\alpha|piv,z,v;\sigma_n,\pi_n(q,r),\bar{\pi})} = \frac{\Pr(\alpha|piv,z;\sigma_n,\pi_n(q,r),\bar{\pi})}{1-\Pr(\alpha|piv,z;\sigma_n,\pi_n(q,r),\bar{\pi})} \cdot \frac{\Pr(v|\alpha,\bar{\pi})}{\Pr(v|\beta,\bar{\pi})}$ by independence of $\pi_n(q,r)$ and $\bar{\pi}$. So $r_z := \Pr(\alpha|z,piv;\sigma_n)$ is a sufficient statistic for equilibrium behaviour after z. Identify $r_z \in (0,1)$ with the pure strategy in a partial game that votes optimally with respect to the belief $p(v)$ after z and $v \in \{u,d\}$, for $\frac{p(v)}{1-p(v)} = \frac{r_z}{1-r_z} \cdot \frac{\Pr(v|\alpha)}{\Pr(v|\beta)}$. So under r_z a citizen votes A if and only if $t_\alpha \cdot p(v) + t_\beta \cdot (1-p(v)) \geq 0$.

Lemma 5. *(Equal Margins of Victory in α_2, β_2.)*
For any given partial game, $\lim_{n\to\infty} \|\Pr(\sigma(s,t)=1|\alpha_2) - \frac{1}{2}\| - \lim_{n\to\infty} \|\Pr(\sigma(s,t)=1|\beta_2) - \frac{1}{2}\|$ *is strictly increasing in r_z. Moreover, there exists a unique strategy $\bar{r}_z \in (0,1)$ under which the margin of victory in α_2 is equal to the margin of victory in β_2 for $n \to \infty$, that is*

$$\lim_{n\to\infty} \|\Pr(\sigma(s,t)=1|\alpha_2) - \frac{1}{2}\| \quad = \quad \lim_{n\to\infty} \|\Pr(\sigma(s,t)=1|\beta_2) - \frac{1}{2}\|.$$

Proof. In the Appendix.

We understand the best response in a partial game as a function of $r_z \in (0,1)$.

Lemma 6. *There exists $n_0 \in \mathbb{N}$ such that: Let any partial game be given. There exists a unique equilibrium r_z^* in this partial game if $n \geq n_0$, and it holds that* $\lim_{n\to\infty} r_z^* = \bar{r_z}$.

Proof. In the Appendix.

Step 2: Sincere Voting After z. Recall that r is the limit of the posterior conditional on Ω_2 and conditional on z in the games $\Gamma_n(q,r,0)$ by 3.1. So, for $r = \bar{r_z}$, conditional on Ω_2 being pivotal contains no information in equilibrium, for $n \to \infty$, since Lemma 6 implies that $\lim_{n\to\infty} \Pr(\alpha|z,piv) = \bar{r}_z = r = \lim_{n\to\infty} \Pr(\alpha|z,\Omega_2)$. Hence, voters vote sincerely after z, for $n \to \infty$. This is analogous to Lemma 2 in the situation without exogeneous private signals $\bar{\pi}$.

Step 3: Equilibrium Construction. As Lemma 3 followed from Lemma 2, its analogue (with the necessary notational changes) holds with exogeneous private signals $\bar{\pi}$ when $r = \bar{r_z}$. We write $\bar{r} = \bar{r_z}$ in the following.

Lemma 7. *If for all $\omega, \omega' \in \{\alpha, \beta\}$,*

$$\begin{aligned} (4.4) \qquad & \| \lim_{n\to\infty} \Pr(\sigma(s,t) = 1 | \omega_2'; \pi_n(q,\bar{r}), \bar{\pi}) - \frac{1}{2} \| \\ < \quad & \| \lim_{n\to\infty} \Pr(\sigma(s,t) = 1 | \omega_1; \pi_n(q,\bar{r}), \bar{\pi}) - \frac{1}{2} \|, \end{aligned}$$

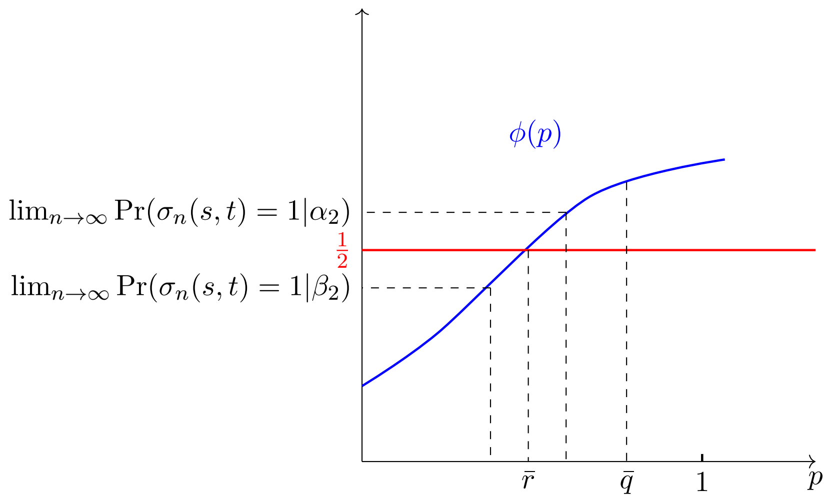
then $\lim_{n\to\infty} \frac{\Pr(\alpha|s,v,piv;\sigma,\pi_n(q,\bar{r}),\bar{\pi})}{\Pr(\beta|s,v,piv;\sigma,\pi_n(p,\bar{r}),\bar{\pi})} = \frac{\Pr(\alpha|s,v,\Omega_2;\sigma,\pi_n(q,\bar{r}),\bar{\pi})}{\Pr(\beta|s,v,\Omega_2;\sigma,\pi_n(q,\bar{r}),\bar{\pi})} \overset{4.3}{=} \frac{q}{1-q} \cdot \frac{\Pr(v|\alpha)}{\Pr(v|\beta)}$ *for $s \in \{a, b\}$ and $v \in \{u, d\}$, and the unique best response to σ in the games $\Gamma_n(q, \bar{r}, 0)$ converges to $\hat{\sigma}_{\Omega_2}(q, \bar{r})$ for $n \to \infty$.*

Equilibrium Construction: We can control the behaviour of agents getting a or b by choosing q appropriately. Recall the first benchmark assumption which assumes that $\phi(p)$ is strictly increasing in p. So, the fraction of voters that prefer A increases with the belief p. There exists $\bar{q} < \bar{r}$ such that for all $v, w \in \{u, d\}$ and $s \in \{a, b\}$,

$$\begin{aligned} (4.5) \qquad \lim_{n\to\infty} \frac{\Pr(\alpha|v,z,\omega_2;\pi_n(\bar{q},\bar{r}),\bar{\pi})}{\Pr(\beta|v,z,\omega_2;\pi_n(\bar{q},\bar{r}),\bar{\pi})} \quad & \overset{4.2}{=} \quad \frac{\bar{r}}{1-\bar{r}} \cdot \frac{\Pr(v|\alpha)}{\Pr(v|\beta)} \\ & \overset{!}{<} \quad \frac{\bar{q}}{1-\bar{q}} \cdot \frac{\Pr(w|\alpha)}{\Pr(w|\beta)} \\ & \overset{4.3}{=} \quad \frac{\Pr(\alpha|w,s,\Omega_2;\pi_n,\bar{\pi})}{\Pr(\beta|w,s,\Omega_2;\pi_n,\bar{\pi})}. \end{aligned}$$

By Step 1, the limit of the expected margin of victory in α_2 is equal to the limit of the expected margin of victory in β_2 in equilibrium. By Step 2, under $\pi_n(\bar{q}, r)$ the limit equilibrium play after z coincides with Ω_2-sincere voting $\hat{\sigma}_{\Omega_2}(\bar{q}, \bar{r})$, for $n \to \infty$. Consider the games $\Gamma_n(\bar{q}, \bar{r}, 0)$: By the first benchmark assumption and 4.1, strictly more voters vote for A in α_2 than in β_2. So, in equilibrium in α_2 a majority of agents votes for A, for $n \to \infty$. But, for large n, by the choice of $\bar{q}$ and 4.5 even more voters vote for A in the states ω_1.
So, when playing a strategy close-by to $\hat{\sigma}_{\Omega_2}(\bar{q}, \bar{r})$ after $s \in \{a, b\}$ and the corresponding partial game equilibrium after z, the limit of the expected margin of victory in the states ω_1 is strictly larger than the limit of the expected margin of victory in the states ω_2, so the condition 4.4 of Lemma 7 is fulfilled, and the unique best response converges to $\hat{\sigma}_{\Omega_2}(\bar{q}, \bar{r})$, for $n \to \infty$. A fixed point argument closes the loop of best replies.
More precisely we invoke Lemma 10 (details in the Appendix). Hence, there exists an equilibrium sequence that converges to $\hat{\sigma}_{\Omega_2}(\bar{q}, \bar{r})$. This shows the

claim of Theorem 2, because under and strategy close-by to $\hat{\sigma}_{\Omega_2}(\bar{q}, \bar{r})$ alternative A gets elected with certainty for $n \to \infty$.



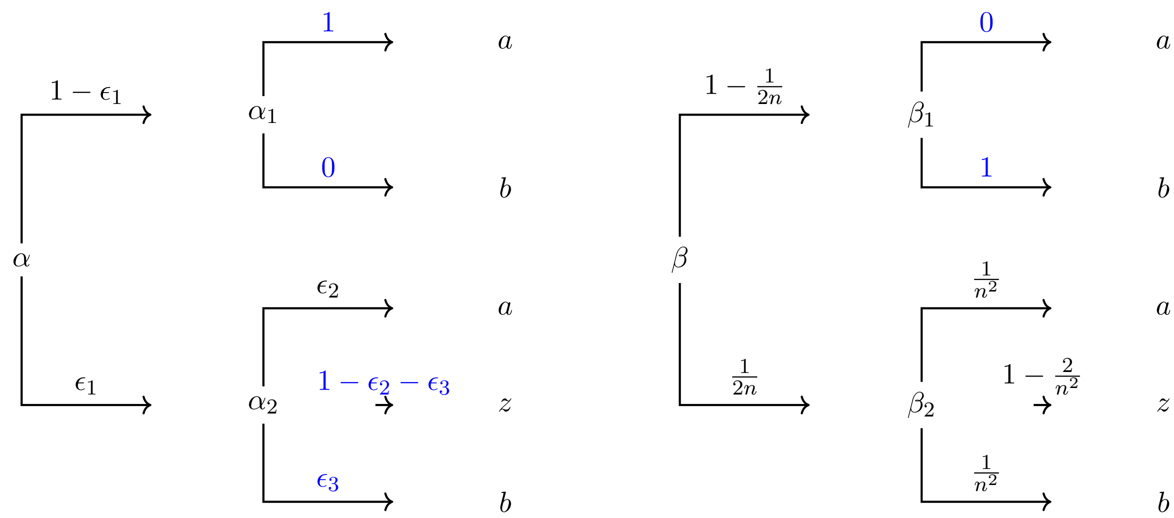
Under $\hat{\sigma}_{\Omega_2}(\bar{q}, \bar{r})$ voters play the unique Bhattacharya [2013]-type equilibrium after z (cf. Figure 2). By choosing $\bar{r}$, we rationalise this equilibrium as Ω_2-sincere voting after z, for $n \to \infty$. Citizens vote A with probability $\phi(\bar{q})$ after a and b. This supports the belief that conditional on being pivotal α_1 and β_1 do not hold, and that voting according to $\Pr(\alpha|\Omega_2, s, v, \pi_n(\bar{q}, \bar{r}))$ is optimal, for $n \to \infty$.

4.1 General Persuasion

We show that the possibility result (Theorem 2) is not limited to implementation of A. Under the benchmark assumptions, we can implement any outcome in α, together with any outcome in β.

We define a class of parametric information structures $\pi_n(q, l, r)$ with $S = \{a, b, z\}$. We illustrate $\pi_n(q, l, r)$ by two diagrams[16]

[16]See the text after the figure for $\pi_n(q, r)$ in Section 1 on how to read these diagrams.



Information structure $\pi_n(q,l,r)$ in α. Information structure $\pi_n(q,l,r)$ in β.

where $\epsilon_1 = \frac{1-p_0}{p_0}\frac{r}{1-r}\frac{1}{2n}$, $\epsilon_2 = \frac{1-r}{r}\frac{q}{1-q}\frac{1}{n^2}$ and $\epsilon_3 = \frac{1-r}{r}\frac{l}{1-l}\frac{1}{n^2}$. The parameter l has an easy interpretation: By definition, the posterior conditional on b and Ω_2 satisfies

$$\Pr(\alpha|b,\Omega_2;q,l,r) \quad = \quad l \tag{4.6}$$

We consider a variation of the voting games from Section 4. The information structure of voters contains signals $S = \{a,b,z\} \times \{u,d\}$ from two sources. We look at the scenario in which voters receive both signals from $\pi_n(q,l,r)$, and from an exogeneous information structure $\bar{\pi}$, which sends binary signals $\{u,d\}$ that are independently, and identically distributed conditional on the state of the world $\omega \in \Omega$. Signals from $\bar{\pi}$ and $\pi_n(q,r)$, are send independently of each other conditional on the state of the world $\omega \in \Omega$. We make the usual assumption on informativeness of signals, $1 > \Pr(u|\alpha) > \Pr(u|\beta) > 0$.

The definitions analogous to the ones in Section 4 are made; in particular we define Ω_2-sincere voting $\hat{\sigma}_{\Omega_2}(q,l,r)$, and denote the voting games by $\Gamma_n(q,l,r,0)$. Note that under the benchmark assumptions, for any $x \in \{A,B\}$ there exists a common belief $p_x \in [0;1]$ such that a strict majority of voters prefers x under p_x.

Proposition 2. *Under the benchmark assumptions: There exist $\bar{r}, p_A, p_B$ such that for any pair of alternatives $(x_\alpha, x_\beta) \in \{A,B\}^2$, the Ω_2-sincere strategy $\hat{\sigma}_{\Omega_2}(p_{x_\alpha}, p_{x_\beta}, \bar{r})$ is the limit of some equilibrium sequence σ in the games $\Gamma_n(p_{x_\alpha}, p_{x_\beta}, \bar{r}, 0)$, and it holds that $\lim_{n\to\infty} \Pr(x_\omega \text{ is elected}|\omega;\sigma_n,\pi_n(p_{x_\alpha}, p_{x_\beta}, \bar{r}), \bar{\pi}) = 1$ for all $\omega \in \Omega$.*

Sketch of the Proof: Lemma 5 and Lemma 6 hold independent of voting behaviour after a and b. The slight change in the signal probability of b in α_2 to $\Pr(b|\alpha_2) = \frac{1-r}{r}\frac{l}{1-l}\frac{1}{n^2} = \epsilon_3$ is immaterial. So, for $n \to \infty$,

equilibrium voting behavior after z is fully captured by the (partial) strategy $\bar{r}$ implicitly defined by 4.4. Also, the analogue of Lemma 7 holds by the same line of proof as before such that we can control the behaviour of agents getting a by choosing q, the posterior conditional on Ω_2 and conditional on a, appropriately. We can control the behaviour after b by choosing l, the posterior conditional on Ω_2 and conditional on b, appropriately.
We set $q = p_{x_\alpha}$, and $l = p_{x_\beta}$, where p_A and p_B are defined as follows. We choose $p_A > \bar{r}_z$ close to 1 such that for any $v, w \in \{u, d\}$ and $s \in \{a, b\}$,

$$\begin{aligned}\lim_{n\to\infty} \frac{\Pr(\alpha|v, z, \omega_2; \pi_n, \bar{\pi})}{\Pr(\beta|v, z, \omega_2; \pi_n, \bar{\pi})} &\overset{4.2}{=} \frac{\bar{r}}{1-\bar{r}} \cdot \frac{\Pr(v|\alpha)}{\Pr(v|\beta)} \\ &\overset{!}{<} \frac{\bar{p_A}}{1-p_A} \cdot \frac{\Pr(w|\alpha)}{\Pr(w|\beta)} \\ &\overset{4.3}{=} \frac{\Pr(\alpha|w, a, \Omega_2; \pi_n, \bar{\pi})}{\Pr(\beta|w, s, \Omega_2; \pi_n, \bar{\pi})}.\end{aligned} \tag{4.7}$$

Under the benchmark assumptions this implies that when agents play $\hat{\sigma}_{\Omega_2}(p_A, l, \bar{r})$ more voters vote for A after getting a than after getting z, independent of the signal v received from $\bar{\pi}$.
We choose $p_B > \bar{r}$ close to 0 such that for $v, w \in \{u, d\}$ and any $s \in \{a, b\}$,

$$\begin{aligned}\lim_{n\to\infty} \frac{\Pr(\alpha|v, z, \omega_2; \pi_n, \bar{\pi})}{\Pr(\beta|v, z, \omega_2; \pi_n, \bar{\pi})} &\overset{4.2}{=} \frac{\bar{r}}{1-\bar{r}} \cdot \frac{\Pr(v|\alpha)}{\Pr(v|\beta)} \\ &\overset{!}{>} \frac{p_B}{1-p_B} \cdot \frac{\Pr(w|\alpha)}{\Pr(w|\beta)} \\ &\overset{4.3}{=} \frac{\Pr(\alpha|w, b, \Omega_2; \pi_n, \bar{\pi})}{\Pr(\beta|w, s, \Omega_2; \pi_n, \bar{\pi})}.\end{aligned} \tag{4.8}$$

Under the benchmark assumptions this implies that when agents play $\hat{\sigma}_{\Omega_2}(q, p_B, \bar{r})$, less voters vote for A after getting b than after getting z, independent of the signal v received from $\bar{\pi}$.

Note that, as before[17], under $\pi_n(p_{x_\alpha}, p_{x_\beta}, \bar{r})$, the limit equilibrium play in Ω_2, coincides with Ω_2-sincere voting $\hat{\sigma}_{\Omega_2}(p_{x_\alpha}, p_{x_\beta}, \bar{r})$ after z, for $n \to \infty$. Recall that all voters receive a in α_1, and all voters receive b in β_2. So, when playing a strategy close-by to $\hat{\sigma}_{\Omega_2}(p_{x_\alpha}, p_{x_\beta}, \bar{r})$ and the corresponding partial game equilibrium after z, the limit of the expected margin of victory in α_1 or β_1 is strictly larger than in both α_2 and β_2, by 4.7 and 4.8. So the condition 4.4 of the analogue of Lemma 7 is fulfilled, and the unique best response converges to Ω_2-sincere voting $\hat{\sigma}_{\Omega_2}(p_{x_\alpha}, p_{x_\beta}, \bar{r})$, for $n \to \infty$. A fixed point argument closes the loop of best replies. More precisely we apply Lemma 10 to a modified best response; hereby we proceed completely analogous to the proof of Theorem 2. Hence, there exists an equilibrium sequence that converges to $\hat{\sigma}_{\Omega_2}(p_{x_\alpha}, p_{x_\beta}, \bar{r})$.

[17] See Step 2 of the proof of Theorem 2.

Under any strategy close-by to $\hat{\sigma}_{\Omega_2}(p_{x_\alpha}, p_{x_\beta}, \bar{r})$, a random citizen votes for x_α with probability $\phi(p_{x_\alpha}) > \frac{1}{2}$ in α_1 by 4.7, thereby electing x_α with certainty, for $n \to \infty$. A random citizen votes for x_α with probability $\phi(p_{x_\beta}) < \frac{1}{2}$ in β_1 by 4.8, thereby electing x_β with certainty, for $n \to \infty$. This shows the claim of Proposition 2.

4.2 Bayes-Correlated Equilibria

An important branch in the information design literature, next to *persuasion*, characterises *robust predictions* that hold under various information structures, potentially for all Bayes-Correlated equilibria (Bergemann and Morris [2017]). In the benchmark voting game (see section 2), the Condorcet Jury Theorem suggests robustness of elections. However, if an outside observer only knows that voters have the preferences as in Feddersen and Pesendorfer [1997] and access to information that is at least as fine as theirs, then it is not certain that information is aggregated in equilibrium. With this knowledge, no robust prediction (Bergemann and Morris [2017]) can be made. In fact, by Proposition 2, we have that

Corollary 1. *For all combinations of outcomes and states $(x_\omega, \omega)_{\omega \in \Omega}$ with $x_\omega \in \{A, B\}$, there exists a sequence of Bayes-correlated equilibria σ_n of the benchmark voting game such that $\lim_{n\to\infty} \Pr(x_\omega \text{ is elected} | \omega, \sigma_n) = 1$ for all $\omega \in \Omega$.*

Proof. The proof follows directly by Proposition 2, and Theorem 1 in Bergemann and Morris [2016].

5 Extensions

5.1 Persuasion With Public Signals

Alonso and Câmara [2015] study persuasion of voters through the release of public signals (and when voters do not receive signals from another exogeneous source). Relative to Alonso and Câmara [2015] we made two departures in the baseline model in Section 3: We allowed for the release of private signals, and we allowed for private preferences. This section 5 illustrates that, for large n, the departure to private preferences only becomes substantial when also allowing for private signals.

For public signals, the unique equilibrium in undominated strategies is sincere voting. In the limit, for $n \to \infty$, the sender is therefore perfectly informed about the aggregate voter behaviour as a function of induced beliefs, no matter if we assume private preferences or not. The posteriors conditional on the public signal have the martingale property: The expected value of the posteriors is equal to the prior.
More formally, by Kamenica and Gentzkow [2011] (Proposition 1), the set

of feasible posteriors for an information structure with m public signals $(s_1, \ldots, s_m)$ is given by

$$\{(p(s_1), \ldots, p(s_m)) : \exists (x_1, \ldots x_m) : \sum_{i=1}^{m} x_i = 1 \text{ and } \sum_{i=1}^{m} x_i \cdot p(s_i) = p_0\}$$

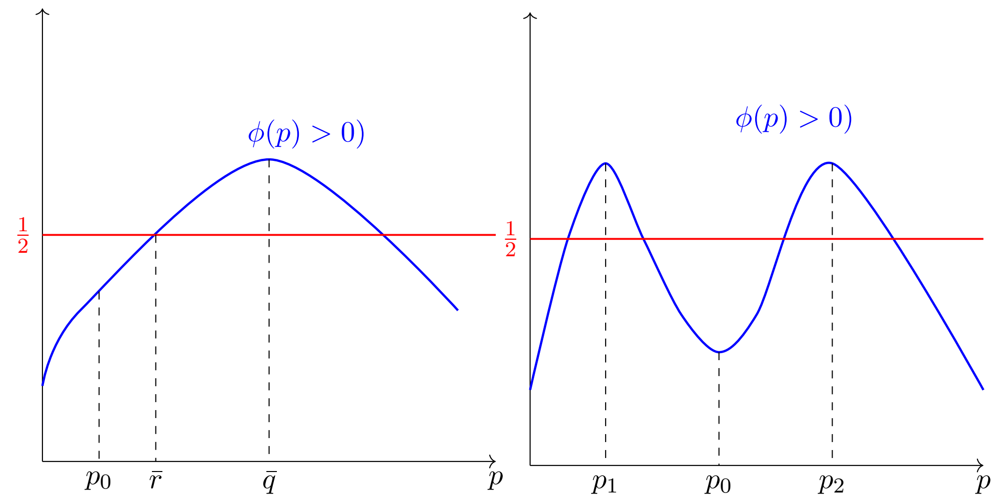
For any public information structure, the (not necessarily convex) set W of posteriors $p \in [0,1]$ that lead to election of A with certainty in the limit $n \to \infty$ is given by all p for which $\phi(p) > \frac{1}{2}$[18]. The set of posteriors p that lead to election of B with certainty in the limit $n \to \infty$ is given by all p for which $\phi(p) < \frac{1}{2}$. The following possibility result holds for public signals:

Proposition 3. *If there exist $0 \leq p_1 \leq p_0 \leq p_2 \leq 1$ such that for $i = 1, 2$, we have $\phi(p_i) > \frac{1}{2}$, then there exists a public information structure π such that, for $n \to \infty$, the probability that A gets elected converges to 1 in the sequence of sincere voting equilibria. The converse also holds.*

Proof. If there exists a belief $p_1 \in W$ weakly lower than the prior belief, $p_1 \leq p_0$, and a belief $p_2 \in W$ weakly larger than the prior belief, $p_2 \geq p_0$, then there exists a binary, public information structure with signals a and b which realises these beliefs as posteriors, $p(a) = p_1$, and $p(b) = p_2$.[19] In the induced sincere voting equilibrium sequence, A is elected with certainty in the limit $n \to \infty$.
If there does not exist $p_1 \in W$ with $p_1 \leq p_0$ or if there does not exist $p_2 \in W$ with $p_2 \geq p_0$, then for every public information structure there exists at least one signal s such that after s alternative A does not get elected with certainty for $n \to \infty$. This is because by the martingale property there exists at least one signal s which induces a posterior larger or equal to the prior, $p(s) \geq p_0$, and at least one signal s which induces a posterior smaller or equal to the prior, $p(s) \leq p_0$.

The following picture illustrates the difference between persuasion with private signals and persuasion with public signals:

[18]Recall the definition of ϕ from Section 1.
[19]This is because we can write $p_0 = p_1 + (p_2 - p_1)x = (1-x)p_1 + xp_2$ for some $x \in [0,1]$.



Persuasion with private signals. Persuasion with public signals.

Different Winning Coalitions: Note that an optimal public information structure induces different winning coalitions, as in Alonso and Câmara [2015]. On the one hand, the winning coalition is inherently random, because preference types of voters are random. On the other hand, when voters hold a common posterior $p_2 \geq p_0$, very different preference types t elect A than when voters hold a common posterior $p_1 \leq p_0$.

Remark 8. In contrast to Theorem 2, when voters receive exogeneous private signals and preferences are monotone, persuasion is not possible with public signals: When adding a public signal to a setting as in Feddersen and Pesendorfer [1997], this is equivalent to a shift in the common prior. However, we know that information is aggregated for all possible non-degenerate priors. A degenerate prior can only be induced by revealing a state, but this only helps information aggregation.

5.2 Pure Common Values

We drop the assumption that G has a strictly positive density, and assume that all voters are of the same type $t = (t_\alpha, t_\beta) = (1, -1)$. We assume that $p_0 < \frac{1}{2}$. We keep the model of Section 1 otherwise. Preferences of each single voter are then common knowledge. This allows for targeted persuasion techniques.[20]

[20]Chan et al. [2016]), Proposition 1 also uses targeted persuasion techniques. Novel is that we show that implementation of A is even possible in all non-degenerate equilibria of the induced game that satisfy a sincerety refinement.

Proposition 4. *For any $n > 1$, there exists a binary information structure π_n ($S = \{a, b\}$) such that* [21] *there exists a strict equilibrium σ_n that satisfies* $\Pr(A \text{ is elected}|\sigma_n) = 1$. *For $n \geq 3$, under π_n all non-degenerate, symmetric equilibria ψ_n for which behaviour after signal b is consistent with sincere voting satisfy* $\Pr(A \text{ is elected}|\psi_n) > 0.9$.

6 Remarks

Section 6.3 and Section 6.1 contains remarks on the voting games $\Gamma_n(q, r, \delta)$ of Section 3 (no exogeneous private signals).

6.1 Feasibility

This section explains that information aggregation is feasible in the situation of Theorem 1, but fails only because of incentives. This is in contrast to several reported failures of information aggregation in the literature: Feddersen and Pesendorfer [1997] (Section 6) show that an invertibility problem arises and information aggregation can fail when there is aggregate uncertainty with respect to the preference distribution. Chan et al. [2016] (Proposition 1) have provided an example of voter persuasion by using signals which are close to the null information structure that always sends the same signal.

Consider for example the benchmark scenario (see Section 2) in which the median voter prefers A in state α, and B in state β: $\phi(1) > \frac{1}{2}$, and $\phi(0) > \frac{1}{2}$. Since under $\pi_n(q, r)$ we have $\Pr(a|\alpha_1) > \frac{1}{2} > \Pr(a|\beta_1)$, any strategy σ which prescribes to vote A after a, and to vote B after b elects the full information outcome with certainty for $n \to \infty$.

6.2 Pivotal Voter Paradigm

The empirical literature has tested the pivotal voter paradigm, and provided correlational and causal evidence for the effect of beliefs about other people's behaviour on political decisions: Cantoni et al. [2017] conduct a field experiment in the context of Hong Kong's pro-democracy movement. They identify a causal effect of beliefs about total turnout of protesters on individual turnout decisions. In a laboratory experiment Guarnaschelli et al. [2000] show that actual behaviour is consistent with the hypothesis that each voter acts optimally against the strategies employed by other voters plus a random error. In another experiment, Duffy and Tavits [2008] observe a positive correlation between the propensity of voting and the beliefs of being pivotal, but subjects systematically overestimate the probability of being pivotal. Coate et al. [2008] provide descriptive evidence and show that field data from small scale-elections on Texas liquor referenda is consistent with strategic voter models in terms of predicted turnout, but not in terms of

[21] Note that equilibria in mixed strategies may exist, since the preference distribution has an atom unlike in the model of Section 1.

margin of victory. Further evidence in favor of strategic voter models has been provided by Ladha et al. [1996].

In this paper the assumption of strategic voting is particularly justified: The probabilities of being pivotal are exceptionally high in the states-of-confusion ω_2 for the information structures $\pi_n(\bar{q}, \bar{r})$ that establish the possibility result Theorem 1. This is because $\bar{r}$ has been chosen to make the election close to being tied in ω_2, such that 3.3 holds. For n sufficiently large, when a random agent votes for A with probability $\frac{1}{2}$ the probability of being pivotal, $\binom{2n}{n}(\frac{1}{2})^{2n}$, is of order $n^{-\frac{1}{2}}$ by Stirling's formula (see Feller [1968], chapter II, formula 9.1).[22]

6.3 Other Equilibria

We further analyze equilibrium sequences of the unperturbed games $\Gamma_n(\bar{q}, \bar{r}, 0)$ with $\bar{r}$ satisfying 3.3, and $\bar{q}$ satisfying 3.5.

Lemma 8. *(Necessary conditions for other equilibria)*
Under the benchmark assumptions[23]*: If $\sigma \neq \hat{\sigma}_{\Omega_2}(\bar{q}, \bar{r})$ is the limit of an equilibrium sequence σ_n in the games $\Gamma_n(\bar{q}, \bar{r}, 0)$, it satisfies*

1. *the limit of the minimum of the margin of victory in the states Ω_1 equals the limit of the minimum of the margin of victory in the states Ω_2, namely* $\lim_{n\to\infty} \min_\omega \|\Pr(\sigma(s,t) = 1|\omega_1) - \frac{1}{2}\| = \lim_{n\to\infty} \min_\omega \|\Pr(\sigma(s,t) = 1|\omega_2) - \frac{1}{2}\| = 0$.
2. *σ is a cut-off strategy with cut-offs $(p(s))_{s\in S}$ that satisfy one of the following conditions: Either $p(s) = \bar{r}$ for all $s \in S$, or $p(z) = \bar{r}$ and $0 < p(b) < \bar{r} < p(a) < 1$.*

Proof. In the Appendix.

Sketch of Proof. Consider σ as in the statement. Property (1) of Lemma 8 must be satisfied since else the margin-of-victory-condition 3.4 of Lemma 3

[22] More precisely, Stirling's formula gives

$$(2n)! \overset{\text{Stirling}}{\approx} (2\pi)^{\frac{1}{2}} 2^{2n+\frac{1}{2}} n^{2n+\frac{1}{2}} e^{-2n},$$
$$(n!)^2 \overset{\text{Stirling}}{\approx} (2\pi) n^{2n+1} e^{-2n}.$$

Consequently

$$\binom{2n}{n} \approx (2\pi)^{-\frac{1}{2}} 2^{2n+\frac{1}{2}} n^{-\frac{1}{2}},$$

which gives

$$\binom{2n}{n}(\frac{1}{2})^{2n} \approx (2\pi)^{-\frac{1}{2}} n^{-\frac{1}{2}} 2^{\frac{1}{2}} = (n\pi)^{-\frac{1}{2}}.$$

[23] See section 2.

is satisfied which implies that σ_n converges to $\hat{\sigma}_{\Omega_2}(\bar{q},\bar{r}) \neq \sigma$. Contradiction. We show that property (2) is implied by property (1) when preferences are monotone (preference monotonicity is part of the benchmark assumptions).

Now, we will construct equilibrium sequences that aggregate information perfectly. For the rest of this section, we assume that signals a and b are informative enough such that sincere voting aggregates information for the given prior (Signal Quality Assumption), meaning that

$$\lim_{n\to\infty} \Pr(\hat{\sigma}(s,t)=1|\alpha) = \sum_{s\in S} \Pr(s|\alpha)\cdot\phi(\Pr(\alpha|s)) > \frac{1}{2}$$
$$> \lim_{n\to\infty} \Pr(\hat{\sigma}(s,t)=1|\beta) = \sum_{s\in S} \Pr(s|\beta)\cdot\phi(\Pr(\alpha|s)).$$

Proposition 5. *Let the benchmark assumptions and the Signal Quality Assumption hold, and let $\bar{r}$ and $\bar{q}$ satisfy 3.3 and 3.5 respectively: Then there exists an equilibrium sequence σ_n in the unperturbed games $\Gamma_n(\bar{q},\bar{r},0)$ which aggregates information, so* $\lim_{n\to\infty}\Pr(A \textit{ is elected}|\alpha)=1$*, and* $\lim_{n\to\infty}\Pr(B \textit{ is elected}|\beta)=1$*.*

7 Literature review

This paper is easily amenable both to the literature on voter persuasion, as well as to the literature on information aggregation in elections. This allows for insightful comparison with and across both streams of literature.

7.1 Voter Persuasion Literature

Alonso and Câmara [2015] (AC) study persuasion of voters with public signals by use of the methodology of Kamenica and Gentzkow [2011] (KG) in an environment with perfect information on preferences. We discussed public persuasion in Section 5 and obtained results reminiscient of (AC) (Proposition 3). Schnakenberg [2015] considers a cheap talk model where a sender publicly persuades voters. Kolotilin et al. [2015] (K) study persuasion of a single, privately informed receiver and show that efficient information structures do not need to screen types. Correspondingly, we showed that persuasion of large electorates with private signals is possible under a weak condition without screening types (Theorem 1).

Chan et al. [2016] (CH) study voter persuasion with private signals. They focus on information structures that induce minimal winning coalitions, because else, in their model, any level of manipulation can be achieved for any population size (see Proposition 1): The sender has perfect knowledge on voter preferences and sends out recommendations to vote for A or for B. In both states, with probability $1-\epsilon$, A is recommended to all voters. Else, in state α, alternative A is recommended to a random minimal winning coalition, in state β, alternative B is recommended to a random minimal winning coalition. So voters with a-signal are only pivotal in α, and voter with b-signal are only pivotal in β. The common preference is A in α, and B in β, so following the recommendation is a strict equilibrium.

Bardhi and Guo [2016a] (BG) study voter persuasion with private signals and focus on the unanimity rule. They allow for heterogeneous, correlated preferences that are only known to the sender. For non-unanimous rules they show that persuasion is possible in a similar way as (CH).

In contrast to (CH) and (BG), we analyse an environment that allows for non-monotone preferences. Some citizens vote for A when believing sufficiently strong in α, and some voters prefer A when believing sufficiently strong in β. When the sender does not screen types, it is a priori unclear which beliefs he should induce with a random receiver. In this paper the informational requirements for persuasion are considerably weak (see discussion at the end of Section 2.2); in particular we allow for private information about preferences. Differently, both (CH) and (BG) assume perfect knowledge on preference realisations by the sender. Also, in contrast to (CH) and (BG), our focus lies on persuasion of large electorates n which makes the results easily amenable to the literature on information aggregation. (CH) and (BG) take an information design approach and study sender-prefered equilibria. In this pape we study other equilibria also, and show that persuasion is robust in manyfold ways (Section 3.4 and Section 4).

Several other papers study persuasion of groups, but are less closely related: Liu [2016] provides results on public persuasion of privately informed voters. Bardhi and Guo [2016b] study sequential persuasion of a group of receivers.

7.2 Information Aggregation Literature

The literature has identified several circumstances in which information may fail to aggregate. We discuss the papers that are most closely related: Feddersen and Pesendorfer [1997] (FP, Section 6) show that an invertibility problem causes a failure when there is aggregate uncertainty with respect to the preference distribution conditional on the state.

Bhattacharya [2013] (BH) shows that failure can happen when preference monotonicity is violated in a model otherwise akin to (FP). With monotone preferences however information is aggregated perfectly. (BH) is the paper that is most closely related to our paper: We showed that a manipulator can create states-of-confusion Ω_2 and thereby ensure election of an intended outcome with almost certainty by a large electorate in a robust equilibrium of a model otherwises identical to (BH) (Theorem 1, Proposition 2). Only a weak condition on voter preferences was needed. In Section 4 we showed that we can add signals to the monotone model and thereby cause failure. This demonstrated that persuasion is possible even when voters receive information from an independent source. The study of Bayes-Nash equilibria of the montone model with refined signals can be understood as an analysis

of the Bayes-Correlated equilibria. Proposition 2 shows that no prediction can be made for election outcomes that is robust across all Bayes-correlated equilibria (Bergemann and Morris [2016]).
In a pure common-values setting, Mandler [2012] (MA) shows that failure can happen when there is aggregate signal uncertainty conditional on the state. The paper does not discuss persuasion, but the results can be understood in terms of it: Signals are send independently and identically distributed conditional on a binary state and conditional on a sub-state, as in this paper. The sub-state captures the signal precision $q = \Pr(a|\omega)$ and is continuous with density $h_\omega(q)$. (MA) shows that, for $n \to \infty$, any limit of an equilibrium sequence can be described by an intersection point q^* of the scaled densities $\Pr(\omega) \cdot h_\omega(q)$, and vice versa (Proposition 1, the discussion before, and Proposition 2). In a q^*-equilibrium sequence alternative A is elected with certainty, for $n \to \infty$, if $q > q^*$ realises, and B is elected with certainty if $q < q^*$ realises, or vice versa (depending on whether $\Pr(\alpha) \cdot h_\alpha(q)$ crosses from below or above). Hence, A can be implemented with arbitrary high probability in an equilibrium sequence, by design of the scaled densities. However, all equilibrium sequences are coequal in terms of robustness, unlike in this paper. Moreover, the continuity of the densities prevents implementation in a unique equilibrium, as an example illustrates: Let $\Pr(\alpha) = \frac{1}{10}$. Consider scaled densities that are single-crossing at $\epsilon > 0$, with $\Pr(\alpha) \cdot h_\alpha(q)$ crossing from below. So, any equilibrium sequence implements A, for $q > \epsilon$ and B for $q < \epsilon$. We must have

$$\begin{aligned} (1 - \Pr(\alpha)) \cdot h_\beta(q) &< \Pr(\alpha) \cdot h_\alpha(q) \quad \text{for all } q > \epsilon, \text{ thus} \\ \frac{9}{10}(1 - \Pr(q < \epsilon|\beta)) &< \frac{1}{10} \cdot (1 - \Pr(q < \epsilon|\alpha)) \quad , \text{thus} \\ (1 - \Pr(q < \epsilon|\beta)) &< \frac{1}{9}. \end{aligned}$$

This implies that alternative B is elected at least with probability $\frac{8}{9}$ in β, the more likely state, for n sufficiently large.
Gerardi et al. [2009] studies aggregation of expert information by an uninformed decision maker. By giving each expert a small change of being a dictator, information can be perfectly extracted at marginal loss, while implementing any intended outcome otherwise. The states-of-confusion ω_2 serve a similar role in our analysis.

8 Appendix

8.1 Appendix A

Theorem 0. *(Bhattacharya [2013])*[24]
Under the benchmark assumptions: When voters receive binary signals that

[24]This theorem is implied by Theorem 1 in Bhattacharya [2013] The notation for the function $\phi(\cdot)$ is $h(\cdot)$ in Bhattacharya [2013].

are independently, and identically distributed conditional on the state ω*, with* $1 > \Pr(a|\alpha) > \Pr(a|\beta) > 0$*, and when preferences are monotone, then the probability that the full information outcome is elected, converges to* 1 *in any sequence of equilibria.*

Proof. Since signal a is indicative of state α, and signal b of β, monotonicity of preferences implies that in any equilibrium σ_n a random citizen votes A with a strictly higher probability after a than after b, $\Pr(\sigma_n(s,t) = 1|a) > \Pr(\sigma_n(s,t) = 1|b)$. We claim that in any equilibrium sequence σ_n it must hold that $\lim_{n\to\infty} \Pr(\sigma_n(s,t) = 1|\alpha) - \frac{1}{2} = \lim_{n\to\infty} \frac{1}{2} - \Pr(\sigma_n(s,t) = 1|\beta)$ $(**)$ and that $\lim_{n\to\infty} \Pr(\sigma_n(s,t) = 1|\alpha) > \frac{1}{2},$ $(**)$. This finishes the proof, because it implies by the law of large numbers that A is elected with probability converging to 1 in α, and B in β. Suppose that $\lim_{n\to\infty} \Pr(\alpha|piv) \in \{0,1\}$. Suppose w.l.o.g. $\lim_{n\to\infty} \Pr(\alpha|piv) = 0$ which implies that $\lim_{n\to\infty} \Pr(\alpha|piv, s) = 0$ for all $s \in S$. Consequently, for all $\omega \in \Omega$ we have

$$\begin{aligned}\lim_{n\to\infty} \Pr(\sigma_n(s,t) = 1|\alpha) &= \sum_{s\in S} \Pr(s|\omega) \cdot \phi(\lim_{n\to\infty} \Pr(\alpha|piv, s)) \\ &= \phi(0) \overset{\text{Benchmark assumptions}}{<} \frac{1}{2}.\end{aligned}$$

For n sufficiently large, this implies $\frac{1}{2} > \Pr(\sigma_n(s,t) = 1|\alpha) > \Pr(\sigma_n(s,t) = 1|\beta)$. So, the election is more likely to being tied in α than in β. Hence the posterior conditional on being pivotal is bounded below by the prior $p_0 \in (0,1)$. Contradiction. Consequently, $\lim_{n\to\infty} \Pr(\alpha|piv) \notin \{0,1\}$. This implies that $1 > \lim_{n\to\infty} \Pr(\alpha|piv, a) > \lim_{n\to\infty} \Pr(\alpha|piv, b) > 0$, and hence, by monotonicity of ϕ, we have

$$\begin{aligned}\lim_{n\to\infty} \Pr(\sigma_n(s,t) = 1|\alpha) &= \sum_{s\in S} \Pr(s|\alpha) \cdot \phi(\lim_{n\to\infty} \Pr(\alpha|piv, s)) \\ > \sum_{s\in S} \Pr(s|\beta) \cdot \phi(\lim_{n\to\infty} \Pr(\alpha|piv, s)) &= \lim_{n\to\infty} \Pr(\sigma_n(s,t) = 1|\beta)\end{aligned}$$

If $(**)$ does not hold, $(*)$ does not hold either. If $(*)$ does not hold, the probability of being pivotal must be infinitely larger in α than in β or vice versa, for $n \to \infty$. Hence $\lim_{n\to\infty} \Pr(\alpha|piv) \in \{0,1\}$, contradiction.

Lemma 1. *Suppose* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *are such that* $\lim_{n\to\infty} x_n \in (0,1)$*,* $\lim_{n\to\infty} y_n \in (0,1)$*.*

(i) *If* $\lim_{n\to\infty} \| \|x_n - \frac{1}{2}\| - \|y_n - \frac{1}{2}\| \| \cdot n^2 < c$ *for some* $c > 0$*, then* $\lim_{n\to\infty} (\frac{x_n(1-x_n)}{y_n(1-y_n)})^n = 1$*.*

(ii) *If* $\lim_{n\to\infty} \|x_n - \frac{1}{2}\| \cdot n^{\frac{1}{2}} = c$ *for some* $c \in \mathbb{R}$*, and* $y_n = \frac{1}{2}$*, then* $\lim_{n\to\infty} (\frac{x_n(1-x_n)}{y_n(1-y_n)})^n = \mu$ *for some* $\mu > 0$*.*

(iii) *If* $\lim_{n\to\infty} (\|x_n - \frac{1}{2}\| - \|y_n - \frac{1}{2}\|) \cdot n^{\frac{1}{2}} = \infty$*, then* $\lim_{n\to\infty} (\frac{x_n(1-x_n)}{y_n(1-y_n)})^n = 0$*.*

Proof. Define $\tilde{x}_n = x_n - \frac{1}{2}$, and $\tilde{y}_n = y_n - \frac{1}{2}$. With this notation it is easier to calculate the following

$$\begin{aligned}
(\frac{x_n(1-x_n)}{y_n(1-y_n)})^n &= \frac{(\tilde{x}_n + \frac{1}{2})(1 - \frac{1}{2} - \tilde{x}_n))^n}{(\tilde{y}_n + \frac{1}{2})(1 - \frac{1}{2} - \tilde{y}_n)^n} \\
&= \frac{(\frac{1}{2}^2 - \tilde{x}_n^2)^n}{(\frac{1}{2}^2 - \tilde{y}_n^2)^n} \\
&= (\frac{\frac{1}{4} - \tilde{y}_n^2 + \tilde{y}_n^2 - \tilde{x}_n^2}{\frac{1}{4} - \tilde{y}_n^2})^n \\
&= (1 - \frac{\tilde{x}_n^2 - \tilde{y}_n^2}{\frac{1}{4} - \tilde{y}_n^2})^n. \qquad (8.1)
\end{aligned}$$

Proof of (i): Split the sequence $\|\tilde{x_n}\| - \|\tilde{y_n}\|$ into maximally two subsequences, consisting of the weakly positive elements, and the negative elements. Consider all $n \in \mathbb{N}$ for which $\|\tilde{x_n}\| - \|\tilde{y_n}\| \geq 0$. Denote $a_n := \tilde{x}_n^2 - \tilde{y}_n^2$ and $b_n := \|\tilde{x_n}\| - \|\tilde{y_n}\|$. It holds that $a_n = 2b_n|\tilde{y}_n\| + b_n^2 < b_n + b_n^2$ $(*)$. The assumption $\lim_{n\to\infty} \|\tilde{x}_n\| - \|\tilde{y}_n\|| \cdot n^2 < c$ and $(*)$ imply that for all n sufficiently large we have

$$a_n \overset{\text{Assumption}}{<} \frac{c}{n^2} + (\frac{c}{n^2})^2 \overset{n \text{ sufficiently large}}{<} \frac{2c}{n^2}. \qquad (8.2)$$

Now, on the one hand

$$\lim_{n\to\infty} (1 - \frac{\tilde{x_n}^2 - \tilde{y_n}^2}{\frac{1}{4} - \tilde{y_n}^2})^n \leq 1. \qquad (8.3)$$

On the other hand,

$$\begin{aligned}
\lim_{n\to\infty} (1 - \frac{\tilde{x_n}^2 - \tilde{y_n}^2}{\frac{1}{4} - \tilde{y_n}^2})^n &\overset{8.2}{\geq} \lim_{n\to\infty} (1 - \frac{1}{n^2} \frac{2c}{\frac{1}{4} - \tilde{y_n}^2})^n \\
&\overset{!}{\geq} \lim_{n\to\infty} (1 - \frac{1}{n} m)^n \quad \text{for all} \quad m \in \mathbb{R}^{>0} \\
&= e^{-m} \quad \text{for all} \quad m \in \mathbb{R}^{>0}. \qquad (8.4)
\end{aligned}$$

For the step to the second last line, recall that $\lim_{n\to\infty} y_n \in (0,1)$. Hence, for n sufficiently large, $\frac{1}{4} - \tilde{y}_n^2$ is bounded above by $\frac{1}{4}$ and below by a constant strictly larger than 0. So $\frac{c}{\frac{1}{4} - \tilde{y}_n^2}$ is bounded. From the last line 8.4, we obtain that

$$\lim_{n\to\infty} (\frac{x_n(1-x_n)}{y_n(1-y_n)})^n \geq 1. \qquad (8.5)$$

For the subsequence of all negative elements $\|\tilde{x_n}\| - \|\tilde{y_n}\| < 0$, we can do the same calculation as above, but we have to replace all greater equal signs with smaller equal signs and vice versa. In any case, 8.1, 8.3 and 8.5 together yield that $\lim_{n\to\infty} (\frac{x_n(1-x_n)}{y_n(1-y_n)})^n = 1$.

Proof of (ii): By assumption, we have that $\lim_{n\to\infty} \|\tilde{x}_n\| \cdot n^{\frac{1}{2}} = c$ with $c \in \mathbb{R}$. So,

$$\begin{aligned} \lim_{n\to\infty} \tilde{x}_n^2 \cdot n &= \lim_{n\to\infty} (\|\tilde{x}_n\| \cdot n^{\frac{1}{2}})^2 \\ &= c^2. \end{aligned} \tag{8.6}$$

We obtain

$$\begin{aligned} & \lim_{n\to\infty} (\frac{x_n(1-x_n)}{y_n(1-y_n)})^n \\ \overset{8.1;\,(y_n=\frac{1}{2})}{=} & \lim_{n\to\infty} (1 - \frac{\tilde{x}_n^2}{\frac{1}{4}})^n \\ = & \lim_{n\to\infty} (1 - \frac{1}{n} \cdot \frac{n \cdot \tilde{x}_n^2}{\frac{1}{4}})^n \\ = & e^{-\lim_{n\to\infty} \frac{n\cdot\tilde{x}_n^2}{\frac{1}{4}}} \overset{8.6}{>} 0. \end{aligned}$$

Proof of (iii): By assumption, we have that $\lim_{n\to\infty}(\|\tilde{x}_n\| - \|\tilde{y_n}\|) \cdot n^{\frac{1}{2}} = c$ with $c = \infty$. So,

$$\begin{aligned} \lim_{n\to\infty} (\tilde{x}_n^2 - \tilde{y}_n^2) \cdot n &= \lim_{n\to\infty} \|(\tilde{x}_n - \tilde{y}_n)(\tilde{x}_n + \tilde{y}_n) \cdot n\| \\ &\overset{\text{Reverse Triangle Inequality}}{\geq} \lim_{n\to\infty} ((\|\tilde{x}_n\| - \|\tilde{y_n}\|) \cdot n^{\frac{1}{2}})^2 \\ &= c^2 = \infty. \end{aligned} \tag{8.7}$$

We obtain

$$\begin{aligned} & \lim_{n\to\infty} (\frac{x_n(1-x_n)}{y_n(1-y_n)})^n \\ \overset{8.1}{=} & e^{-\lim_{n\to\infty} \frac{n\cdot(\tilde{x}_n^2 - \tilde{y}_n^2)}{\frac{1}{4} - \tilde{y}_n^2}} \overset{8.7}{=} 0. \end{aligned}$$

Lemma 2. *For any sequence $(\sigma_n)_{n\in\mathbb{N}}$ of weakly undominated strategies, it holds that* $\lim_{n\to\infty} \frac{\Pr(piv|\alpha_2;\sigma_n,q,r,\delta)}{\Pr(piv|\beta_2;\sigma_n,q,r,\delta)} = 1$.

Proof. Define $x_n := \Pr^\delta(\sigma_n(s,t) = 1|\alpha_2; \sigma_n, q, r, \delta)$, and $y_n := \Pr^\delta(\sigma_n(s,t) = a|\beta_2; \sigma_n, q, r, \delta)$. Then, by definition

$$\begin{aligned} \frac{\Pr(piv|\alpha_2;\sigma_n,q,r,\delta)}{\Pr(piv|\beta_2;\sigma_n,q,r,\delta)} &= \frac{\binom{2n}{n} x_n^n (1-x_n)^n}{\binom{2n}{n} y_n^n (1-y_n)^n} \\ &= (\frac{x_n(1-x_n)}{y_n(1-y_n)})^n. \end{aligned}$$

We will show that the requirements of Lemma 1, (i) are fulfilled. The probability that a voter is of type $t >> 0$ is strictly positive, since G has a strictly positive density. But types $t >> 0$ have the weakly dominant strategy to vote for A, hence do so in any (voting) equilibrium. So, firstly, the probability that a random citizen votes for A in ω is bounded below by a strictly positive constant. The probability that a voter is of type $t << 0$ is strictly positive. So, secondly, the probability that a random citizen votes for A in ω is bounded above by a constant strictly smaller than 1, because Consequently, $\lim_{n\to\infty} x_n \in (0,1)$, and $\lim_{n\to\infty} y_n \in (0,1)$.

By definition

$$\begin{aligned} x_n &= \lim_{n\to\infty}(1-(2\cdot\frac{1-r}{r}\frac{q}{1-q}\frac{1}{n^2}))[(1-\delta)\cdot\Pr(\sigma_n(z,t)=1|\alpha_2)+\delta\cdot\Pr(\hat{\sigma}_n(z,t)=1|\alpha_2)] \\ &+ (\frac{1-r}{r}\frac{q}{1-q}\frac{1}{n^2})[(1-\delta)(\Pr(\sigma_n(a,t)=1|\alpha_2)+\Pr(\sigma_n(b,t)=1|\alpha_2)) \\ &+ \delta(\Pr(\hat{\sigma}_n(a,t)=1|\alpha_2)+\Pr(\hat{\sigma}_n(b,t)=1|\alpha_2))] \end{aligned}$$

and

$$\begin{aligned} y_n &= \lim_{n\to\infty}(1-\frac{2}{n^2})[(1-\delta)\cdot\Pr(\sigma_n(z,t)=1|\beta_2)+\delta\cdot\Pr(\hat{\sigma}_n(z,t)=1|\beta_2)] \\ &+ \frac{1}{n^2}[(1-\delta)(\Pr(\sigma_n(a,t)=1|\beta_2)+\Pr(\sigma_n(b,t)=1|\beta_2)) \\ &+ \delta(\Pr(\hat{\sigma}_n(a,t)=1|\beta_2)+\Pr(\hat{\sigma}_n(b,t)=1|\beta_2))] \end{aligned}$$

In the states ω_2, almost all voters receive the same signal z. The probability that a voter receives a signal $s \neq z$ is smaller than $\tilde{c}\cdot n^2$ for some $\tilde{c} > 0$. Consequently, for all $n \in \mathbb{N}$ we have

$$\|x_n - y_n\| \leq c\cdot n^{-2}$$

for some constant $c > 0$. Therefore,

$$\begin{aligned} \|x_n - y_n\|\cdot n^2 &< c, \\ \Rightarrow \|\|\|x_n-\frac{1}{2}\| - \|y_n-\frac{1}{2}\|\|\|\cdot n^2 &\leq \|x_n-y_n\|\cdot n^2 < c. \end{aligned}$$

where the step to the last line follows by application of the reverse triangle inequality.

Lemma 3. *If*

$$\begin{aligned} \min_{\omega}\|\lim_{n\to\infty}\Pr^{\delta}(\sigma_n(s,t)=1|\omega_2;q,r,\delta)-\frac{1}{2}\| \\ < \min_{\omega}\|\lim_{n\to\infty}\Pr^{\delta}(\sigma_n(s,t)=1|\omega_1;q,r,\delta)-\frac{1}{2}\|, \end{aligned} \tag{3.4}$$

holds, then $\lim_{n\to\infty}\frac{\Pr(\alpha|s,piv;\sigma_n,q,r,\delta)}{\Pr(\beta|s,piv;\sigma_n,q,r,\delta)} = \frac{\Pr(\alpha|s,\Omega_2;q,r,\delta)}{\Pr(\beta|s,\Omega_2;,q,r,\delta)}$ *for* $s\in\{a,b\}$*, and the unique best response to* σ_n *in the games* $\Gamma_n(q,r,\delta)$ *converges to* $\hat{\sigma}_{\Omega_2}(q,r)$ *for* $n\to\infty$*.*

Proof. Set $x_n := \Pr^\delta(\sigma_n(s,t) = 1|\omega_1;\sigma_n,q,r,\delta)$, and $y_n := \Pr^\delta(\sigma_n(s,t) = 1|\omega_2;\sigma_n,q,r,\delta)$. For $s \in \{a,b\}$, consider

$$\begin{aligned}\frac{\Pr(\omega_1|s,piv;\sigma_n,q,r,\delta)}{\Pr(\omega_2|s,piv;\sigma_n,q,r,\delta)} &= \frac{\Pr(\omega_1|q,r)}{\Pr(\omega_2|q,r)}\cdot\frac{\Pr(s|\omega_1;q,r)}{\Pr(s|\omega_2;q,r)}\cdot\frac{\Pr(piv|\omega_1;\sigma_n,q,r)}{\Pr(piv|\omega_2;\sigma_n,q,r)}\\ &\leq c\cdot n^3\cdot\frac{\Pr(piv|\omega_1;\sigma_n,q,r)}{\Pr(piv|\omega_2;\sigma_n,q,r)}\end{aligned}$$

for n large enough, and some constant $c > 0$ that only depends on q and r. We analyse

$$\begin{aligned}\frac{\Pr(piv|\omega_1;\sigma_n,q,r)}{\Pr(piv|\omega_2;\sigma_n,q,r)} &= \left(\frac{x_n(1-x_n)}{y_n(1-y_n)}\right)^n\\ &= \left(1-\frac{(x_n-\frac{1}{2})^2-(y_n-\frac{1}{2})^2}{\frac{1}{4}-(y_n-\frac{1}{2})^2}\right)^n. \end{aligned}\tag{8.8}$$

where the step to the last line follows by 8.1. By the assumption of Lemma 3, we have $\lim_{n\to\infty}(x_n-\frac{1}{2})^2-(y_n-\frac{1}{2})^2 > 0$. Note that $(y_n-\frac{1}{2})^2$ is bounded above by a constant strictly below $\frac{1}{4}$, since a positive mass of voters is of type $t >> 0$ and votes for A in equilibrium, and since a positive mass of voters is of type $t << 0$ and votes for B in equilibrium. Consequently, $\frac{(x_n-\frac{1}{2})^2-(y_n-\frac{1}{2})^2}{\frac{1}{4}-(y_n-\frac{1}{2})^2}$ converges to a strictly positive number. But then 8.8 implies that

$$\lim_{n\to\infty} c\cdot n^3\cdot\frac{\Pr(piv|\omega_1;\sigma_n,q,r)}{\Pr(piv|\omega_2;\sigma_n,q,r)} = 0. \tag{8.9}$$

Therefore the posterior after signal s (for $s \in \{a,b\}$) and conditional on being pivotal, converges to the posterior conditional on s and Ω_2 with $n \to \infty$, as the following shows:

$$\begin{aligned}&\lim_{n\to\infty}\frac{\Pr(\alpha|s,piv;\sigma_n,q,r,\delta)}{\Pr(\beta|s,piv;\sigma_n,q,r,\delta)}\\ =\ &\lim_{n\to\infty}\frac{\Pr(\alpha)}{\Pr(\beta)}\frac{\sum_{j=1,2}\Pr(\alpha_j|\alpha;q,r)\cdot\Pr(s|\alpha_j;q,r)\cdot\Pr(piv|\alpha_j,s;\sigma_n,q,r,\delta)}{\sum_{j=1,2}\Pr(\beta_j|\beta;q,r)\cdot\Pr(s|\beta_j,q,r)\cdot\Pr(piv|\beta_j,s;\sigma_n,q,r,\delta)}\\ \overset{8.9}{=}\ &\lim_{n\to\infty}\frac{\Pr(\alpha)}{\Pr(\beta)}\frac{\Pr(\alpha_2|\alpha;q,r)\cdot\Pr(s|\alpha_2;q,r)\cdot\Pr(piv|\alpha_2,s;\sigma_n,q,r,\delta)}{\Pr(\beta_2|\beta;q,r)\cdot\Pr(s|\beta_2;q,r)\cdot\Pr(piv|\beta_2,s;\sigma_n,q,r,\delta)}\\ \overset{\text{Lemma 2}}{=}\ &\lim_{n\to\infty}\frac{\Pr(\alpha)}{\Pr(\beta)}\frac{\Pr(\alpha_2|\alpha;q,r)\cdot\Pr(s|\alpha_2;q,r)}{\Pr(\beta_2|\beta;q,r)\cdot\Pr(s|\beta_2;q,r)}\\ =\ &\lim_{n\to\infty}\frac{\Pr(\alpha|s,\Omega_2;\sigma_n,q,r,\delta)}{\Pr(\beta|s,\Omega_2;\sigma_n,q,r,\delta)}\end{aligned}\tag{8.10}$$

For the application of Lemma 2 note that the probability that a random voter $j \in -i$ votes for A is independent of voter i's signal, since signals are independent conditional on ω_2.

The unique undominated best response is fully described by the posteriors conditional on being pivotal and conditional on s by 1.1. Consequently, by 8.10, the unique undominated best response after s for $s \in \{a,b\}$ converges to acting optimally upon the posterior belief conditional on Ω_2 and conditional on s, when $n \to \infty$. This is by definition the strategy $\hat{\sigma}_{\Omega_2}$.

The following Lemma is invoked in the proof of Theorem 1.

Lemma 4. *(Fixed Point Argument for Equilibrium Sequence Construction) If for some strategy σ, there exists $\epsilon > 0$, such that the best reponse converges uniformly to σ for $n \to \infty$, for all $\sigma' \in B_\epsilon(\sigma)$, then there exists a sequence of equilibria that converges to σ.*

Proof. The best reponse function is continuous in all Bayesian games analysed in this paper. By assumption, there exists $\bar{m} \in \mathbb{N}$ such that for all $m \geq \bar{m} \in \mathbb{N}$ we can find $\bar{n}(m) \in \mathbb{N}$ such that the best response function is a self-map on $B_{\frac{1}{m}}(\sigma)$ for all $n \geq n(m)$. The Brouwer Fixed Point Theorem yields fixed points $\sigma_n(m) \in B_{\frac{1}{m}}(\sigma))$ in the games with $n \geq n(m)$ voters. For $n < n(\bar{m})$ select any equilibrium of the game of n voters, and denote it by σ_n. For any $m \geq \bar{m}$, and any $n(m+1) \geq n \geq n(m)$, set $\sigma_n := \sigma_n(m)$. Then $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of equilibria that converges to σ, that is $\lim_{n\to\infty} \sigma_n = \sigma$.

Computational Example. Consider any strategy σ which satisfies $\Pr(A|s) \geq 0.7$ for $s \in \{a,b\}$, and $\Pr(A|z) \in [0.45, 0.54]$. We have the following bounds on voting probabilities conditional on the substates ω_j for $\omega \in \{\alpha,\beta\}$ and $j \in \{1,2\}$:

$$\begin{aligned} \Pr(A|\Omega_1) &\geq 0.7, \\ \Pr(A|\omega_2) &\geq 0.45 \text{ for } \omega \in \{\alpha,\beta\}, \\ \Pr(A|\alpha_2) &\leq 0.54 + \frac{q}{1-q} \cdot \frac{2}{n^2} \overset{n\geq 200}{\leq} 0.55. \end{aligned}$$

This gives us

$$\begin{aligned} & \frac{\Pr(piv|\Omega_2)}{\Pr(piv|\Omega_1)} \\ \overset{\text{Lemma 1, (i), Equation 8.1}}{\geq} \quad & (1 + \min_{\omega\in\{\alpha,\beta\}} \frac{(\Pr(A|\omega_1) - \frac{1}{2})^2 - (\Pr(A|\omega_2) - \frac{1}{2})^2}{\frac{1}{4} - (\Pr(A|\omega_1) - \frac{1}{2})^2}, \\ \geq \quad & (1 + \frac{\frac{9}{100} - \frac{1}{400}}{\frac{1}{4} - \frac{9}{100}})^n \\ \geq \quad & (1 + \frac{35}{64})^n \geq (\frac{3}{2})^n. \end{aligned} \tag{8.11}$$

The posterior belief ratios conditional on being pivotal and conditional on $s \in \{a, b\}$ then satisfy

$$\begin{aligned}
&\frac{\Pr(\alpha|piv, s)}{\Pr(\beta|piv, s)} \\
= \; & \frac{p_0}{1-p_0} \cdot \frac{\sum_{j=1,2} \Pr(\alpha_j|\alpha; q, r) \cdot \Pr(s|\alpha_j; q, r) \cdot \Pr(piv|\alpha_j, s; \sigma_n, q, r, \delta)}{\sum_{j=1,2} \Pr(\beta_j|\beta; q, r) \cdot \Pr(s|\beta_j, q, r) \cdot \Pr(piv|\beta_j, s; \sigma_n, q, r, \delta)}, \\
\geq \; & \frac{1}{3} \cdot \frac{\frac{3}{2n} \cdot \frac{3}{n^2} \cdot \Pr(piv|\alpha_2)}{\frac{1}{2n^3} \cdot \Pr(piv|\beta_2) + (1 - \frac{1}{2n}) \cdot \frac{2}{3} \cdot \Pr(piv|\beta_1)}.
\end{aligned} \tag{8.12}$$

For $n \geq 200$, we have that $2n^3 \frac{3}{2}^{-n}(1 - \frac{1}{2n})\frac{2}{3} \leq 10^{-27}$. Consequently, by 8.11 and 8.12, for $s \in \{a, b\}$ we have

$$\begin{aligned}
&\frac{\Pr(\alpha|piv, s)}{\Pr(\beta|piv, s)} \\
\geq \; & 3 \cdot \frac{\Pr(piv|\alpha_2)}{\Pr(piv|\beta_2)} \cdot \frac{1}{1 + 10^{-27}}.
\end{aligned} \tag{8.13}$$

Note that for any $x, y \in [0, 1]$ it holds that $\|x^2 - y^2\| = \|(x + y)(x - y)\| \leq \|x - y\|$ $(*)$. The ratio of pivotal probabilities in α_2 and β_2 is bounded by

$$\begin{aligned}
&\frac{\Pr(piv|\alpha_2)}{\Pr(piv|\beta_2)} \\
\overset{\text{Lemma 1, (i), Equation 8.1}}{\geq} \; & (1 - \frac{\|(\Pr(A|\alpha_2) - \frac{1}{2})^2 - (\Pr(A|\beta_2) - \frac{1}{2})^2\|}{\frac{1}{4} - (\Pr(A|\beta_2) - \frac{1}{2})^2})^n \\
\overset{(*)}{\geq} \; & (1 - \frac{\|\Pr(A|\alpha_2) - \Pr(A|\beta_2)\|}{\frac{1}{4} - (\Pr(A|\beta_2) - \frac{1}{2})^2})^n \\
\geq \; & (1 - \frac{\frac{q}{1-q} \cdot \frac{2}{n^2}}{\frac{1}{4} - \frac{1}{400}})^n \\
= \; & (1 - \frac{\frac{6}{n^2}}{\frac{1}{4} - \frac{1}{400}})^n \\
\overset{n \geq 200}{\geq} \; & 0.885.
\end{aligned} \tag{8.14}$$

Therefore

$$\frac{\Pr(\alpha|piv, z)}{\Pr(\beta|piv, z)} \geq \frac{1 - \frac{6}{n^2}}{1 - \frac{2}{n^2}} \cdot 0.885 \overset{n \geq 200}{\geq} 0.88.$$

Therefore

$$\Pr(\alpha|piv, z) = \Pr(A|z; BR(\sigma)) \geq \frac{0.88}{1 + 0.88} \geq 0.46,$$

where $BR(\sigma)$ denotes the best response to σ. On the other hand, we have

$$\begin{aligned}
& \frac{\Pr(piv|\alpha_2)}{\Pr(piv|\beta_2)} \\
\overset{\text{Lemma 1, (i), Equation 8.1 + } (*)}{\leq} \quad & (1 + \frac{\|\Pr(A|\alpha_2)\Pr(A|\beta_2)\|}{\frac{1}{4} - (\Pr(A|\beta_2) - \frac{1}{2})^2})^n \\
\leq \quad & (1 + \frac{\frac{q}{1-q} \cdot \frac{2}{n^2}}{\frac{1}{4} - \frac{1}{400}})^n \\
= \quad & (1 + \frac{\frac{6}{n^2}}{\frac{1}{4} - \frac{1}{400}})^n \\
\overset{n \geq 200}{\leq} \quad & 1.13 \qquad (8.15)
\end{aligned}$$

Therefore

$$\frac{\Pr(\alpha|piv, z)}{\Pr(\beta|piv, z)} \leq \frac{1 - \frac{6}{n^2}}{1 - \frac{2}{n^2}} \cdot 1.13 \leq 1.13.$$

Therefore

$$\Pr(\alpha|piv, z) = \Pr(A|z; BR(\sigma)) \leq \frac{1.13}{1 + 1.13} < 0.54.$$

Together 8.13 and 8.14 imply that for $s \in \{a, b\}$,

$$\begin{aligned}
& \frac{\Pr(\alpha|piv, s)}{\Pr(\beta|piv, s)} \\
\geq \quad & 3 \cdot 0.884 = 2.652
\end{aligned}$$

Therefore, for $s \in \{a, b\}$,

$$\Pr(\alpha|piv, s) = \Pr(A|s; BR(\sigma)) \geq \frac{2.652}{1 + 2.652} > 0.7.$$

We conclude, that the best response is a self-map on the set of strategies which satisfy $\Pr(A|s) \geq 0.7$ for $s \in \{a, b\}$, and $\Pr(A|z) \in [0.45, 0.54]$. Evaluation of the binomial distribution show that $\Pr(\mathcal{B}(2n + 1, x)) > n) \geq 0.999999$ if $n \geq 200$ and $x \geq 0.7$. Therefore, the Brouwer fixed point theorem yields an equilibrium which satisfies

$$\Pr(A \text{ is elected}) \geq 0.999999(1 - \frac{3}{2n}) \overset{n \geq 200}{\geq} 99\%.$$

8.2 Appendix B (Robustness)

Proposition 1. *(Global Basin of Attraction)*[25]
Under the benchmark assumptions: For any $\epsilon > 0$, *the measure of* $\Sigma^2(\epsilon, n)$

[25] The result holds more generally. If we consider any random (not necessarily cut-off) strategy as starting point, for any $\epsilon > 0$ the probability that the twice iterated best response lies in an ϵ-neighbourhood of conditional sincere voting converges to 1, for $n \to \infty$.

in the space of cut-off-strategies $[0,1]^3$ *converges to* 1, *for* $n \to \infty$. *In particular, there exists* $n(\epsilon) \in \mathbb{N}$ *such that all cut-off strategies* σ *for which*

$$(3.6)\quad \| \|\Pr(\sigma(s,t)=1|\alpha_1) - \frac{1}{2}\| - \|\Pr(\sigma(s,t)=1|\beta_1) - \frac{1}{2}\| \| > n^{-\frac{1}{4}} \quad \textit{and}$$

$$(3.7)\quad \| \min_{\omega} \|\Pr(\sigma(s,t)=1|\omega_1) - \frac{1}{2}\| - \min_{\omega} \|\Pr(\sigma(s,t)=1|\omega_2) - \frac{1}{2}\| \| > n^{-\frac{1}{4}}$$

hold, are elements of $\Sigma^2(\epsilon, n)$ *for* $n \geq n(\epsilon)$.

Proof. Since the measure of the cut-off strategies σ that satisfy 3.6 and 3.7 converges to 1, it is sufficient to show the second claim.

Case 1: $\min_\omega \|\Pr(\sigma(s,t)=1|\omega_1) - \frac{1}{2}\| - \min_\omega \|\Pr(\sigma(s,t)=1|\omega_2) - \frac{1}{2}\| > n^{-\frac{1}{4}}$
Note that $\lim_{n\to\infty} \frac{n^{-\frac{1}{4}}}{n^{-\frac{1}{2}}} = \lim_{n\to\infty} n^{\frac{1}{4}} = \infty$. By application of Lemma 1, (iii) to $x_n = \Pr(\sigma(s,t)=1|\Omega_1)$ and $y_n = \Pr(\sigma(s,t)=1|\Omega_2)$, we obtain that $\lim_{n\to\infty} \Pr(\Omega_2|s, piv; \sigma, q, r, \delta) = 1$ for $s \in \{a,b\}$. Being pivotal contains the information that Ω_2 holds, for $n \to \infty$, and no information beyond that by Lemma 2. Consequently $\lim_{n\to\infty} \Pr(\alpha|piv, s; \sigma) = \Pr(\alpha|\Omega_2, s; \sigma)$, so the cut-offs of the best response to σ converge to the cut-offs of conditional sincere voting. A more detailed version of this proof can be done in complete analogy to the proof of Lemma 3.
Case 2: $\min_\omega \|\Pr(\sigma(s,t)=1|\omega_1) - \frac{1}{2}\| - \min_\omega \|\Pr(\sigma(s,t)=1|\omega_2) - \frac{1}{2}\| < n^{-\frac{1}{4}}$
By application of Lemma 1, (iii)to $x_n = \Pr(\sigma(s,t)=1|\Omega_1)$ and $y_n = \Pr(\sigma(s,t)=1|\Omega_2)$, we obtain that $\lim_{n\to\infty} \Pr(\Omega_1|s, piv; \sigma, q, r, \delta) = 1$ for $s \in \{a,b\}$, and by application of Lemma 1, (iii) to $x_n = \Pr(\sigma(s,t)=1|\alpha_1)$ and $y_n = \Pr(\sigma(s,t)=1|\beta_1)$, we obtain that $\lim_{n\to\infty} \Pr(\alpha_1|s, piv; \sigma, q, r, \delta) \in \{0,1\}$ for $s \in \{a,b\}$. Being pivotal contains the information that either α_1 or β_1 holds, for $n \to \infty$. Since preferences are monotone by benchmark assumption 1, the margin of victory under the best response is large in α_1 and β_1 for n sufficiently large. However, by 3.3 the margin of victory in Ω_2 is zero, for $n \to \infty$. So, the best response satisfies the margin-of-victory condition 3.4 of Lemma 3, and consequently the twice iterated best response converges to conditional sincere voting.

8.3 Appendix C (Exogeneous Private Signals)

Lemma 5. *(Equal Margins of Victory in* α_2, β_2.*)*
For any given partial game, $\lim_{n\to\infty} \|\Pr(\sigma(s,t)=1|\alpha_2) - \frac{1}{2}\| - \lim_{n\to\infty} \|\Pr(\sigma(s,t)=1|\beta_2) - \frac{1}{2}\|$ *is strictly increasing in* r_z. *Moreover, there exists a unique strategy* $\bar{r}_z \in (0,1)$ *under which the margin of victory in* α_2 *is equal to the margin of victory in* β_2 *for* $n \to \infty$, *that is*

$$\lim_{n\to\infty} \|\Pr(\sigma(s,t)=1|\alpha_2) - \frac{1}{2}\| \quad = \quad \lim_{n\to\infty} \|\Pr(\sigma(s,t)=1|\beta_2) - \frac{1}{2}\|.$$

Proof. Condition 4.4 is equivalent to saying that the limit of the expected margin of victory is the same in α_2 and β_2, for $n \to \infty$. By the benchmark assumption $\phi(p)$ is strictly increasing in p. So, the fraction of voters that prefer A strictly increases with the belief p. So, more voters vote for A after getting u than after getting d by 4.1. Since more voters receive u in α than in β, the expected vote share of A is strictly higher in state α_2 than in state β_2, for n sufficiently large.

For r_z sufficiently close to 0 less than a strict majority of citizens votes A under r_z in both α_2 and β_2, by 4.2 and since $\phi(0) < \frac{1}{2}$ by the benchmark assumptions, that is a random voter votes A with probability smaller $\frac{1}{2}$ when he knows that β holds. This implies that the RHS of 4.4 is larger than the LHS.

For r_z sufficiently close to 1, a strict majority of citizens votes A under r_z in both α_2, and β_2, by 4.2 and since $\phi(1) > \frac{1}{2}$ by the benchmark assumptions, that is a random voter votes A with probability larger $\frac{1}{2}$ when he knows that α holds. This implies that the RHS of 4.4 is smaller than the LHS.

The function

$$\begin{aligned}
& \lim_{n\to\infty} \|\Pr(\sigma(s,t)=1|\alpha_2) - \frac{1}{2}\| - \lim_{n\to\infty} \|\Pr(\sigma(s,t)=1|\beta_2) - \frac{1}{2}\| \\
= \quad & \|(\sum_{v=u,d} \Pr(v|\alpha,\bar{\pi}) \cdot \Pr(\sigma(s,t)=1|v,z;\bar{r_z},\pi_n(q,r),\bar{\pi})) - \frac{1}{2}\| \\
& -\|(\sum_{v=u,d} \Pr(v|\beta,\bar{\pi}) \cdot \Pr((\sigma(s,t)=1|v,z;\bar{r_z},\pi_n(q,r),\bar{\pi})) - \frac{1}{2}\|, \\
= \quad & \|(\sum_{v=u,d} \Pr(v|\alpha,\bar{\pi}) \cdot \phi(\frac{r_z}{1-r_z} \cdot \frac{\Pr(v|\alpha)}{\Pr(v|\beta)})) - \frac{1}{2}\| \\
& -\|(\sum_{v=u,d} \Pr(v|\beta,\bar{\pi}) \cdot \phi(\frac{r_z}{1-r_z} \cdot \frac{\Pr(v|\alpha)}{\Pr(v|\beta)})) - \frac{1}{2}\|.
\end{aligned}$$

is continuous and strictly increasing, and since ϕ is continuous[26] and strictly increasing by the benchmark assumptions. Hence, there exists unique $\bar{r_z}$ such that the equality 4.4 holds.

Lemma 6. *There exists $n_0 \in \mathbb{N}$ such that: Let any partial game be given. There exists a unique equilibrium r_z^* in this partial game if $n \geq n_0$, and it holds that* $\lim_{n\to\infty} r_z^* = \bar{r_z}$.

Proof. Consider any equilibrium sequence $r_{z,n}$. We prove that $\lim_{n\to\infty} r_{z,n} = \bar{r}_z$ by contradiction. By the benchmark assumption $\phi(p)$ is strictly increasing in p. So, the fraction of voters that prefer A strictly increases with the belief p. So, more voters vote for A after getting u than after getting d by 4.1. Since more voters receive u in α than in β, the expected vote share of A is strictly higher in state α_2 than in state β_2, for n sufficiently large under

[26]Recall that G has a density.

an strategy r_z, that is $\Pr(\sigma(s,t)=1|\alpha_2) > \Pr(\sigma(s,t)=1|\beta_2)$.

Case 1: If $\lim_{n\to\infty} r_{z,n} = 0$, we have

$$\lim_{n\to} \Pr(\sigma_n(s,t)=1|\alpha) = \lim_{n\to\infty} \Pr(\sigma_n(s,t)=1|\beta) = \phi(0) < \frac{1}{2}.$$

Since $\Pr(\sigma(s,t)=1|\alpha_2) > \Pr(\sigma(s,t)=1|\beta_2)$ for any n sufficiently large, the election is more likely to being tied in α than in β. Hence the posterior conditional on being pivotal and conditional on z, $r_{z,n}$, is bounded below by the prior $p_0 \in (0,1)$. Contradiction. Consequently, by a similar argument for the case $\lim_{n\to\infty} r_{z,n} = 1$, we see that $\lim_{n\to\infty} r_{z,n} \notin \{0,1\}$.

Case 2: If $0 < \lim_{n\to\infty} r_{z,n} < \bar{r_z}$ (see Lemma 5 for a definition of $\bar{r_z}$), then the limit of the expected margin of victory in α_2 is smaller than in β_2, for n large enough. Conditional on being tied, α_2 is much more likely than β_2 (because the probability of the election being tied is decreasing exponentially faster in β_2 than in α_2). Consequently, $\lim_{n\to\infty} \frac{r_{z,n}}{1-r_{z,n}} = \lim_{n\to\infty} \frac{\Pr(\alpha_2|z,piv;r_{z,n},\pi_n,\bar{\pi})}{\Pr(\beta_2|z,piv;r_{z,n},\pi_n,\bar{\pi})} = \infty$. Contradiction to $\lim_{n\to\infty} r_{z,n} < \bar{r_z} \in (0,1)$.

Case 3: If $1 > \lim_{n\to\infty} r_{z,n} > \bar{r_z}$, then similarly the limit of the expected margin of victory is larger in α_2 than in β_2, for n large enough. Conditional on being tied, α_2 is much less likely than β_2 (because the probability of the election being tied is decreasing exponentially faster in α_2 than in β_2). Consequently, $\lim_{n\to\infty} r_{z,n} = \lim_{n\to\infty} \frac{\Pr(\alpha_2|z,piv;r_{z,n},\pi_n,\bar{\pi})}{\Pr(\beta_2|z,piv;r_{z,n},\pi_n,\bar{\pi})} = 0$. This follows, because, Contradiction, and consequently $\lim_{n\to\infty} r_{z,n} = \bar{r_z}$.

Recall that by Lemma 5 $\lim_{n\to\infty} \|\Pr(\sigma(s,t)=1|\alpha_2) - \frac{1}{2}\| - \lim_{n\to\infty} \|\Pr(\sigma(s,t)=1|\beta_2) - \frac{1}{2}\|$ has a unique zero at $\bar{r}_z$ and is strictly increasing in r_z. This implies that for any fixed voting behaviour after a and b, and n sufficiently large, $\|\Pr(\sigma(s,t)=1|\alpha_2) - \frac{1}{2}\|$ and $\|\Pr(\sigma(s,t)=1|\beta_2) - \frac{1}{2}\|$ are single-crossing at some $r_{crossing} \in (0,1)$ with $\lim_{n\to\infty} r_{crossing} = \bar{r}_z$. For n sufficiently large, when increasing r_z, starting from $r_{crossing}$, the best reponse converges almost exponentially fast to approximately 0. For n sufficiently large, when decreasing r_z, starting from $r_{crossing}$, the best reponse converges almost exponentially fast to approximately 1. So the best response $BR(r_z) \in [0,1]$ is strictly decreasing in an environment around $r_{crossing}$, as well as $BR(r_z) - r_z$. By continuity, $BR(r_z) - r_z$ has a zero in neighbourhood of $r_{crossing}$, and the zero is unique by strict monotonicity of $BR(r_z) - r_z$.

Consider the model with exogeneous private signals $\bar{\pi}$ (Section 4). For any strategy σ, denote $\sigma_{a,b} := \sigma_{|S\setminus\{(z,u),(z,d)\}\times[-1,1]^2}$ to be the strategy restricted to actions after a and b. Denote by σ_z the partial game equilibrium associated to $\sigma_{a,b}$.

Lemma 10. *If for conditional voting $\sigma = \hat{\sigma}_{\Omega_2}(\bar{q}, \bar{r}_z)$ with $\bar{r}_z$ as in Lemma 5 there exists $\epsilon > 0$ such that for all cut-off-strategies $\sigma' \in B_\epsilon(\sigma)$, we have*

$$\lim_{n\to\infty} (BR(\sigma'_{a,b}, \sigma'_z))_{a,b} = \sigma_{a,b},$$

then there exists an equilibrium sequence σ_n in $\Gamma_n(\bar{q}, \bar{r}_z, 0)$ with $lim_{n\to\infty}\sigma_n = \sigma$.

Proof. Define $\psi(\sigma'_{a,b}) = (BR(\sigma'_{a,b}, \sigma'_z))_{a,b}$ where BR denotes the best response correspondence. Note that there exists n_0 such that ψ is continuous for all $n \geq n_0$. This follows by the continuity and uniqueness of the partial game equilibrium σ'_z for $n \geq n_0$ (recall Lemma 6).
Consider $\bar{m}$ with $\frac{1}{\bar{m}} < \epsilon$. Consequently, for all $m \geq \bar{m} \in \mathbb{N}$ we can find $\bar{n}(m) \geq n_0 \in \mathbb{N}$ such that ψ is a self-map on $B_{\frac{1}{m}}(\sigma_{a,b})$ for all $n \geq n(m)$. We can choose $n(m)$ to be strictly increasing in m. The Brouwer Fixed Point Theorem yields fixed points $\sigma_n^{res} \in B_{\frac{1}{m}}(\sigma_{a,b})$ in the games of $n \geq n(m)$ voters.

For $n < n(1)$ select any equilibrium of the game of n voters, and denote it by σ_n. For any $m > \bar{m}$, and any $n(m) \leq n < n(m+1)$, let $\sigma_n := \sigma_n^{res}$ after a and b, and by the (unique) partial game equilibrium $(\sigma_n^{res})_z$ after z. Note that σ_n is an equilibrium by construction. The restricted strategies σ_n^{res} converge to $\sigma_{a,b}$ for $n \to \infty$ by construction, and any sequence of partial game equilibria converges to $\bar{r}_z$ by Lemma 5. By Step 2 of the proof of Theorem 2, $\bar{r}_z$ coincides with conditional sincere voting σ under $\pi_n(\bar{q}, \bar{r}_z)$, for $n \to \infty$, So, $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of equilibria in $\Gamma_n(\bar{q}, \bar{r}_z, 0)$ that converges to σ.

8.4 Appendix D (Extensions and Remarks)

Proposition 4. *For any $n > 1$, there exists a binary information structure π_n ($S = \{a, b\}$) such that* [27] *there exists a strict equilibrium σ_n that satisfies* $\Pr(A$ *is elected*$|\sigma_n) = 1$. *For $n \geq 3$, under π_n all non-degenerate, symmetric equilibria ψ_n for which behaviour after signal b is consistent with sincere voting satisfy* $\Pr(A$ *is elected*$|\psi_n) > 0.9$.

Proof. Define π_n as follows: Let $S = \{a, b\}$. In state α, with probability $1-\epsilon$, all $2n+1$ voters receive a. With probability ϵ, let π_n randomize over all signal profiles in which $n+1$ voters receive a, and n voters receive b. In state β, with probability $1-\epsilon$, all $2n+1$ voters receive a. With probability ϵ, let π_n randomize over all signal profiles in which n voters receive a, and $n+1$ voters receive b. Denote the voting probabilities of a strategy σ by $r := \Pr(A|a;\sigma)$ and $q := \Pr(A|b;\sigma)$.

Pure strategy equilibrium. If $r = 1$, and $q = 1$, voters with an a-signal are only pivotal in α, and voters with a b-signal are only pivotal in β,

[27] Note that equilibria in mixed strategies may exist, since the preference distribution has an atom unlike in the model of Section 1.

$\Pr(\alpha|piv, a; r, q) = 1$, $\Pr(\alpha|piv, b; r, q) = 0$. Hence, voting A after a, and B after b is a strict equilibrium. Note that voting B after b is consistent with sincere voting, since $p_0 < \frac{1}{2}$.

Mixed strategy equilibrium. Consider any non-degenerate equilibrium ψ that is consistent with sincere voting after b, that is which satisfies $q = 1$. Then

$$\begin{aligned}\Pr(piv|\alpha, a; \psi) &= (1-\epsilon)\binom{2n}{n} r^n \cdot (1-r)^n + \epsilon \cdot r^n, \\ \Pr(piv|\beta, a; \psi) &= (1-\epsilon)\binom{2n}{n} r^n \cdot (1-r)^n.\end{aligned}$$

Therefore

$$\frac{\Pr(piv|\alpha, a; \psi)}{\Pr(piv|\beta, a; \psi)} = 1 + \frac{\epsilon}{1-\epsilon}(\binom{2n}{n}(1-r)^n)^{-1}.$$

A Stirling approximation gives[28]

$$\binom{2n}{n}(1-r)^n \approx (2\pi)^{-\frac{1}{2}} 2^{\frac{1}{2}} n^{-\frac{1}{2}} (4(1-r))^n.$$

Hence

$$\lim_{n\to\infty} \binom{2n}{n}(1-r)^n = \begin{cases} 0 & \text{if} \quad (1-r) \leq \frac{1}{4}, \\ \infty & \text{if} \quad (1-r) > \frac{1}{4}. \end{cases} \tag{8.16}$$

Consequently,

$$\lim_{n\to\infty} \frac{\Pr(piv|\alpha, a; \psi)}{\Pr(piv|\beta, a; \psi)} = \begin{cases} \infty & \text{if} \quad r \geq \frac{3}{4}, \\ 1 & \text{if} \quad r < \frac{3}{4}. \end{cases}$$

Recall that $p_0 < \frac{1}{2}$. There exists $\bar{n} \in \mathbb{N}$ such that for all $r < 0.74$ and all $n \geq \bar{n}$, we have

$$\frac{\Pr(piv|\alpha, a; \sigma)}{\Pr(piv|\beta, a; \sigma)} < \frac{1-p_0}{p_0}. \tag{8.17}$$

Choose $\epsilon > 0$ small enough, such that 8.17 also holds for any $n \leq \bar{n}$. Then, for all $n \in \mathbb{N}$ voting A after a is not a best response if $r < 0.74$. Note that $\Pr(\mathcal{B}(2n+1, 0.74)) > 0.91$[29] for $n \geq 3$. There exists $\epsilon > 0$ small enough such that any non-degenerate equilibrium ψ that is consistent with sincere voting after b satisfies Pr(A is elected$|\psi) > 0.9$. This finishes the proof.

We note that by 8.16 for n sufficiently large there exists $r^* > 0.74$ such that 8.17 holds,

$$\frac{\Pr(piv|\alpha, a; \sigma)}{\Pr(piv|\beta, a; \sigma)} = 1 + \frac{\epsilon}{1-\epsilon}(\binom{2n}{n}(1-r)^n)^{-1} \stackrel{!}{=} \frac{1-p_0}{p_0}.$$

[28]For Stirling's formula see Section 6.2.

[29]$\mathcal{B}(y, x)$ denotes the binomial distribution with parameters y and x.

Note that r^* is unique for any n, since $(1-r)^{-n}$ is strictly increasing with r.

Lemma 8. *(Necessary conditions for other equilibria)*
Under the benchmark assumptions[30]*: If* $\sigma \neq \hat{\sigma}_{\Omega_2}(\bar{q}, \bar{r})$ *is the limit of an equilibrium sequence* σ_n *in the games* $\Gamma_n(\bar{q}, \bar{r}, 0)$*, it satisfies*

1. *the limit of the minimum of the margin of victory in the states* Ω_1 *equals the limit of the minimum of the margin of victory in the states* Ω_2*, namely* $\lim_{n\to\infty} \min_\omega \|\Pr(\sigma(s,t)=1|\omega_1) - \frac{1}{2}\| = \lim_{n\to\infty} \min_\omega \|\Pr(\sigma(s,t)=1|\omega_2) - \frac{1}{2}\| = 0$.
2. σ *is a cut-off strategy with cut-offs* $(p(s))_{s\in S}$ *that satisfy one of the following conditions: Either* $p(s) = \bar{r}$ *for all* $s \in S$*, or* $p(z) = \bar{r}$ *and* $0 < p(b) < \bar{r} < p(a) < 1$.

Proof. Consider an equilibrium sequence $(\sigma_n)_{n\in\mathbb{N}}$ with $lim_{n\to\infty}\sigma_n = \sigma$. Recall that in any equilibrium sequence, the margin of victory in Ω_2 converges to zero by Lemma 2. If condition (1.) does not hold, the margin of victory in α_1 and β_1 is strictly larger than the margin of victory in Ω_2 under σ_n, for n sufficiently large. So, the margin-of-victory condition 3.4 of Lemma 3 holds and the best response to σ_n converges to conditional sincere voting and not to σ_n. Contradiction, because $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of equilibria.[31] Condition (2.) is an implication of (1.): First, σ is a cut-off strategy with cut-offs $p(s) = \lim_{n\to\infty} \Pr(\alpha|piv, a; \sigma_n)$ since σ is the limit of the equilibrium sequence σ_n. Since for any equilibrium strategy $\Pr(\alpha|piv, a) > \Pr(\beta|piv, b)$ holds, we must have $p(a) \geq p(b)$. So the margin of victory in α_1 or β_1 can only be zero if the expected vote shares after both signals a and b are $\frac{1}{2}$, which is equivalent to $p(a) = p(b) = \bar{r}$, or if the expected vote share after a is strictly larger $\frac{1}{2}$, and smaller $\frac{1}{2}$ after b, which implies $0 < p(b) < \bar{r} < p(a) < 1$.

The following Lemma is needed in the proof of Proposition 5.

Lemma 11. *For any strategy sequence* σ_n *and any* $\omega \in \Omega$ *denote* $c := \lim_{n\to\infty}(\Pr(\sigma_n(s,t)=1|\omega) - \frac{1}{2}) \cdot n^{\frac{1}{2}}$ *(we allow for* $c = \pm\infty$*). Then*

$$\lim_{n\to\infty} \Pr(A \textit{ gets elected}|\omega; \sigma_n) = \Phi(8^{\frac{1}{2}}c).$$

Proof. Denote $x_n := \Pr(\sigma_n(s,t)=1|\omega)$. By using the normal approximation[32]

$$\mathcal{B}(2n+1, x_n) \simeq \mathcal{N}((2n+1)x_n, (2n+1)x_n(1-x_n)),$$

[30]See section 2.

[31]We omit an alternative proof that uses Proposition 1.

[32]For this normal approximation we cannot rely on the standard central limit theorem, because x_n varies with n. However, the central limit theorem for triangular sequences holds for triangular sequences of Bernoulli distributions $\mathcal{B}(y,x)$ with x bounded away from 0 and 1, by an application of the Berry-Esseen-Theorem.

we see that the probability that A wins the election in ω converges to

$$\Phi(\frac{\frac{1}{2}(2n+1)-(2n+1)\cdot x_n}{((2n+1)x_n(1-x_n))^{\frac{1}{2}}})$$

where $\Phi(-)$ denotes the cumulative distribution of the standard normal distribution. Taking limits $n \to \infty$, gives us

$$\begin{aligned}\lim_{n\to\infty}\Phi(\frac{\frac{1}{2}(2n+1)-(2n+1)\cdot x_n}{(2n+1)x_n(1-x_n))^{\frac{1}{2}}}) &= \lim_{n\to\infty}\Phi(\frac{(2n+1)\frac{1}{2}-(2n+1)(\frac{1}{2}+(x_n-\frac{1}{2}))}{(2n+1)^{\frac{1}{2}}(x_n(1-x_n))^{\frac{1}{2}}}) \\ &= \lim_{n\to\infty}\Phi((2n+1)^{\frac{1}{2}}(x_n-\frac{1}{2})(x_n(1-x_n))^{-\frac{1}{2}}) \\ &= \lim_{n\to\infty}\Phi((x_n(1-x_n))^{-\frac{1}{2}}2^{\frac{1}{2}}c)=\Phi(8^{\frac{1}{2}}c).\end{aligned}$$

Proposition 5. *Let the benchmark assumptions and the Signal Quality Assumption hold, and let $\bar{r}$ and $\bar{q}$ satisfy 3.3 and 3.5 respectively: Then there exists an equilibrium sequence σ_n in the unperturbed games $\Gamma_n(\bar{q},\bar{r},0)$ which aggregates information, so* $\lim_{n\to\infty}\Pr(A \textit{ is elected}|\alpha)=1$*, and* $\lim_{n\to\infty}\Pr(B \textit{ is elected}|\beta)=1$.

Proof. **Step 1 (Equilibrium Construction):** Any strategy σ entails probabilities by which a random citizen votes A in ω_j, for $\omega \in \Omega$ and $j \in \{1,2\}$, denoted by $\Pr(\sigma(s,t)=1|\omega_j)$. These probabilities are a sufficient statistic for the posteriors conditional on being pivotal and having received s, for any $s \in S$, and consequently a sufficient statistic for the unique best response by 1.1. Hence, we can write the best response as a function in these probabilities $\Pr(\sigma(s,t)=1|\omega_j)$. Consider the modified best response function that sets the probability to vote for A in β_1, $\Pr(\sigma(s,t)=1|\beta_1)$, to $\frac{1}{2}$ whenever this probability is weakly larger than $\frac{1}{2}$ under the actual best response. The modified best reponse is continuous, and an endomorphism on the closed and convex set of strategies that imply that B receives in expectation $\frac{1}{2}$ or more of the votes in β_1. The modified best response function has a fixed point by the Brouwer Fixed Point Theorem, and we claim that any fixed point is interior for n sufficiently large. By construction, this is sufficient to show that any fixed point corresponds to an equilibrium.
Suppose otherwise. Then, there exists a (sub)sequence of fixed points for which the vote share of B in β_1 is exactly one half. We lead a slightly more general case to a contradiction. Consider any sequence of fixed points for which the vote share of B in β_1 converges to $\frac{1}{2}$ relatively fast: More precisely, assume that $(\Pr(\sigma(s,t)=1|\beta_1)-\frac{1}{2})\cdot n^{\frac{1}{2}}=c$ for some $c<0$. The probability of being pivotal in ω_2, for $\omega \in \{\alpha,\beta\}$, is maximal when the probability to vote A in ω_2 is exactly $\frac{1}{2}$. Even then, the fraction of the probability of being pivotal in β_1 and the probability of being pivotal in ω_2 does not converge to zero by application of Lemma 1, (ii) to $x_n=Pr(\sigma(s,t)=1|\beta_1)$, and $y_n=\frac{1}{2}$.

This implies that β_1 is infinitely more likely than α_2 and β_2 conditional on being pivotal and having received signal a or b,

$$\lim_{n\to\infty} \frac{\Pr(\omega_2|s, piv; \sigma_n, \pi_n)}{\Pr(\beta_1|s, piv; \sigma_n, \pi_n)} = 0$$

for $s \in \{a, b\}$ (since the signals a and b have probability less than $\frac{1}{n}$ in ω_2). So the posteriors conditional on being pivotal and conditional on a or b vanish on Ω_2. Moreover, conditional on being pivotal, the state is weakly more likely to be β_1 than α_1: To see this, note that the posterior conditional on being pivotal and a is higher than the posterior conditional on being pivotal and b, because a is indicative of α_1, and b of β_1 ($\Pr(a|\alpha_1) > \Pr(a|\beta_1), \Pr(b|\alpha_1) < \Pr(b|\beta_1)$). Since preferences are strictly monotone, in expectation strictly more voters vote A after a than after b. More voters receive a in α_1 than in β_1, so the expected vote share for A is weakly larger in α_1 than in β_1. Since under the given fixed point, the expected vote share for A in β_1 is $\frac{1}{2}$ under the modified best response, the expected vote share for A in β_1 is weakly larger than $\frac{1}{2}$ under the actual best response, and therefore the expected vote share for A is strictly farther away from $\frac{1}{2}$ in α_1 relative to β_1.

So, the limit of the posterior conditional on being pivotal must be strictly smaller than the prior. But we assumed at the start that given the prior, B receives a strict majority in β_1 under sincere voting, for n sufficiently large. So, the best response to the fixed point must be interior for n sufficiently large (even more, it must imply a strictly larger vote share of B in β_1 than under the sincere voting strategy, so a vote share that is bounded away from $\frac{1}{2}$.). Contradiction.

Step 2 (Full information equivalence): So far we showed that there exist equilibria that correspond to (interior) fixed points of a modified best reponse, for n sufficiently large. We claim, that any sequence of such equilibria aggregates information, meaning that A gets elected with certainty in α, and B in β, for $n \to \infty$ (recall the benchmark assumptions). Consider any sequence of interior fixed points.

Information aggregation in β: Firstly, suppose that the probability that B gets elected in β does not converge to 1 (hence it does not converge to 1 in β_1 either). By Lemma 11, this implies that the probability that a random citizen votes for B in β_1 must converge to $\frac{1}{2}$ sufficiently fast, namely that $\lim_{n\to\infty}(\Pr(\sigma(s,t) = 1|\beta_1) - \frac{1}{2}) \cdot n = c$ for some $c < 0$. We showed in the preceding paragraph that this implies that the limit of the probability that a random citizen votes B in β_1 does not converge to $\frac{1}{2}$. Contradiction.

***A* receives more vote shares in α_1 (for large n):** Secondly, note that the limit of the probability that a random citizen votes A is weakly larger in α_1 than in β_1, because preferences are monotone and signals informative.

The latter is weakly smaller than $\frac{1}{2}$ by definition of the fixed points. Suppose that there is a subsequence along which the probability to vote A in α_1 is weakly smaller than $\frac{1}{2}$, too. We consider two cases.
Case 1: $\lim_{n\to\infty} \Pr(\alpha_1|piv, s) = 0$ for $s \in \{a, b\}$
This means that the posterior conditional on being pivotal and conditional on a vanishes on α_1. This implies that it also vanishes on β_1, because the probability of being pivotal in β_1 is weakly smaller than the probability of being pivotal in α_1, since the expected vote share for A is weakly larger in α_1 (so weakly closer to $\frac{1}{2}$). Then the posterior conditional on being pivotal and conditional on a or b converges to the posterior conditional on Ω_2 and conditional on a or b, that is for $s \in \{a, b\}$ we have

$$\lim_{n\to\infty} \Pr(\alpha|piv, s) = \Pr(\alpha|\Omega_2, s) \overset{3.3}{=} \bar{q}.$$

So, by definition of $\bar{q}$, a strict majority votes A after a or b, hence in α_1. Contradiction.
Case 2: $\lim_{n\to\infty} \Pr(\alpha_1|piv, a) \neq 0$ for $s \in \{a, b\}$
This means that the posterior conditional on being pivotal and conditional on a or b does not vanish on α_1. Then the limit of the posterior conditional on being pivotal and conditional on s is strictly larger if $s = a$ than if $s = b$, because signals are strictly informative in α_1 and β_1:

$$\begin{aligned}
& \lim_{n\to\infty} \frac{\Pr(\alpha|piv, a)}{\Pr(\beta|piv, a)} \\
= \quad & \lim_{n\to\infty} p_0 \cdot \frac{\sum_{i=1,2} \Pr(\alpha_i|\alpha) \cdot \Pr(a|\alpha_i) \cdot \Pr(piv|\alpha_i)}{\sum_{i=1,2} \Pr(\beta_i|\beta) \cdot \Pr(a|\beta_i) \cdot \Pr(piv|\beta_i)} \\
\overset{\substack{\Pr(b|\alpha_1)<\Pr(a|\alpha_1) \\ \Pr(b|\beta_1)>\Pr(a|\alpha_1)}}{>} \quad & \lim_{n\to\infty} p_0 \cdot \frac{\sum_{i=1,2} \Pr(\alpha_i|\alpha) \cdot \Pr(b|\alpha_i) \cdot \Pr(piv|\alpha_i)}{\sum_{i=1,2} \Pr(\beta_i|\beta) \cdot \Pr(b|\beta_i) \cdot \Pr(piv|\beta_i)} \\
= \quad & \lim_{n\to\infty} \frac{\Pr(\alpha|piv, b)}{\Pr(\beta|piv, b)}
\end{aligned}$$

Consequently, the limit of the probability to vote A in α_1 is strictly larger than in β_1, since a is received more often in α_1, and b more often in β_1, and since preferences are monotone. Since by assumption the expected vote share of A is weakly smaller than $\frac{1}{2}$, conditional on being pivotal, α_1 is infinitely more likely than β_1 for $n \to \infty$. So, the posterior conditional on being pivotal vanishes on β_1. So, for $s \in \{a, b\}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{\Pr(\alpha|piv, s)}{\Pr(\beta|piv, s)} &= \lim_{n\to\infty} p_0 \cdot \frac{\sum_{i=1,2} \Pr(\alpha_i|\alpha) \cdot \Pr(s|\alpha_i) \cdot \Pr(piv|\alpha_i)}{\sum_{i=1,2} \Pr(\beta_i|\beta) \cdot \Pr(s|\beta_i) \cdot \Pr(piv|\beta_i)} \\
&= \lim_{n\to\infty} p_0 \cdot \frac{\sum_{i=1,2} \Pr(\alpha_i|\alpha) \cdot \Pr(s|\alpha_i) \cdot \Pr(piv|\alpha_i)}{\Pr(\beta_2|\beta) \cdot \Pr(s|\beta_2) \cdot \Pr(piv|\beta_2)} \\
&> \lim_{n\to\infty} p_0 \cdot \frac{\Pr(\alpha_2|\alpha) \cdot \Pr(s|\alpha_2) \cdot \Pr(piv|\alpha_2)}{\Pr(\beta_2|\beta) \cdot \Pr(s|\beta_2) \cdot \Pr(piv|\beta_2)} \\
&\overset{\text{Lemma 2}}{=} \frac{\Pr(\alpha|\Omega_2, s)}{\Pr(\beta|\Omega_2, s)} \overset{3.2}{=} \bar{q}.
\end{aligned}$$

So, by definition of $\bar{q}$, a strict majority of voters votes A after a or b, and hence in α_1. Contradiction. We conlude that we showed that the probability to vote A in α_1 is strictly larger than $\frac{1}{2}$ for n sufficiently large.

Information aggregation in α: Suppose that the probability that A gets elected in α does not converge to 1 (hence it does not converge to 1 in α_1 either). By Lemma 11, this implies that the probability that a random citizen votes for A in α_1 must converge to $\frac{1}{2}$ sufficiently fast, namely that $\|\Pr(\sigma(s,t) = 1|\alpha_1) - \frac{1}{2}\| \cdot n^{\frac{1}{2}} = c$ for some $c > 0$. Since we showed that the probability that a random citizen votes for A does not converge to $\frac{1}{2}$ in β_1 at a similarly high rate, namely that $\|\Pr(\sigma(s,t) = 1|\beta_1) - \frac{1}{2}\| \cdot n^{\frac{1}{2}} = \infty$, Lemma 1, (iii) applied to $x_n = \Pr(\sigma(s,t) = 1|\beta_1)$ and $y_n = \Pr(\sigma(s,t) = 1|\alpha_1)$ shows that conditional on being pivotal state α_1 is infinitely more likely than β_1, for $n \to \infty$. The probability of being pivotal in ω_2, for $\omega \in \{\alpha, \beta\}$, is maximal when the probability to vote A in ω_2 is exactly $\frac{1}{2}$. Consequently, even then, the fraction of the probabilities of being pivotal in α_1 and ω_2 does not converge to zero by application of Lemma 1, (ii) to $x_n = Pr(\sigma(s,t) = 1|\alpha_1)$. This implies that α_1 is infinitely more likely than α_2 and β_2 conditional on being pivotal and having received signal a or b, $\lim_{n\to\infty} \frac{\Pr(\Omega_2|s,piv;\sigma_n,\pi_n)}{\Pr(\alpha_1|s,piv;\sigma_n,\pi_n)} = 0$ for $s \in \{a, b\}$ (since the signals a and b have probability less than $\frac{1}{n}$ in ω_2, and probability $\frac{1}{3}$ or $\frac{2}{3}$ in α_1). Hence, the probability of α_1 (and hence α) conditional on being pivotal and a or b, converges to 1. Since A is the full information outcome when voters know that α holds, A gets elected with certainty, for $n \to \infty$. Contradiction.

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