Tort law and the Nucleolus for liability problems

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Abstract: For joint liability problems concerning tort law, a legal compensation scheme may be based on lower and upper bounds of compensation for injury and on case-system consistency. Introducing several properties inspired from this observation, we analyze compensation schemes axiomatically under the situation where causation of the cumulative injury appears in multiple sequences of wrongful acts. These sequences are described by a rooted-tree graph. We show that there is a unique compensation scheme that satisfies three axioms, one about lower bounds of individual compensations, one about upper bounds of individual compensations, and one about case-system consistency. This unique compensation scheme is the Nucleolus of an associated liability game.

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1 Introduction

Tort Law is a body of rights, obligations, and remedies concerning tort. As stated in Black's Law Dictionary (the tenth edition), tort is a civil wrong, other than breach of contract, for which a remedy may be obtained, usually in the form of damages; a breach of a duty that the law imposes on persons who stand in a particular relation to one another. Tort law is applied by courts, and it often deals with the question how the tortfeasor (i.e. the person or persons who is/are responsible for the damage) should financially compensate the plaintiff (i.e. the person who suffered the pecuniary damage). In this paper, we consider situations where an injured party suffers damages caused by wrongful acts performed subsequently by a sequence of injuring parties. The wrongful acts are causally related in the sense that any wrongful act in the sequence would not have occurred if any of the preceding wrongful acts would not have occurred. So, the second (wrongful) act can only occur after the first (wrongful) act has occurred, the third (wrongful) act can only occur when both the first and the second (wrongful) acts have occurred and so on. Any wrongful act results in an amount of damage to the injured party. The injuring parties are the tortfeasors who can be considered to be jointly liable for the full damage. The problem is how to apportion the full damage among the tortfeasors. In many real-life situations, this problem is brought to court. This sharing problem is referred to as the *joint liability problem* (the *liability problem* for short). Since this problem can be viewed as allocating the damage cost over the tortfeasors, the liability problem is in the intersection of *Law and Economics*.

Historically, common law did not accept any apportionment among the tortfeasors, but evolution of common law in the 19th and 20th centuries led to the *Restatement of Law, Torts* (the *Restatement of Torts* for short), which is formulated by the American Law Institute, providing basic principles and rules to apportion the damages.

A systematic apportionment method for liability problems is an important subject of research in Law and Economics. In the existing literature, it is a central topic to clarify whether or not a legal compensation scheme for liability problems is useful, see for instance Landes and Posner (1980), Shavell (1983), and Parisi and Singh (2010). These authors analyze the functioning of compensation schemes from the viewpoint of incentives.

On the other hand, the viewpoint of *fairness* is also important. In fact, *tort law* prescribes an award of damages to achieve fair compensation for injury, which is pointed out in the literature of Law, see for instance Boston (1995-1996). A few researchers have investigated the normative aspects of compensation schemes. Dehez and Ferey (2013) introduce a certain compensation scheme formalized by causal weights among the injuring parties and the list of certain damages caused by every tortfeasor. They show that for every liability problem this compensation scheme yields the *weighted Shapley value* of a corresponding transferable utility game (TU game for short). Ferey and Dehez (2016) axiomatize the *Shapley value* for the liability problems described in Dehez and Ferey (2013). The (weighted) Shapley value (Shapley 1953; Kalai and Samet 1987) is an established solution for TU games, and it is a game theoretic expression of fairness.[1]

In this paper, we focus on the situation where the causal relation is determined but the determination of causal weights among the injuring parties is difficult.[2] There are two

[1]Several notions of fairness underlying the Shapley value are proposed in the existing literature, for instance see Myerson (1980) and van den Brink (2001).

[2]The assumption under which causal relation is determined is the same as in the existing literature,

reasons for this reasoning. Firstly, since a judge determines causal weights subjectively, fair compensation for injury is difficult in practice. Secondly, if the judge's transaction cost of this determination is high, then the determination is impossible.

The purpose of our study is to analyze the functioning of compensation schemes from the viewpoint of fairness. For this purpose, we take an axiomatic approach which is common in Economics.[3] The axioms proposed in this paper are derived by taking into account tort law and case system in practice. Using these axioms, we characterize the compensation scheme associated with the 'difference principle of social justice' à la Rawls (1971).

The liability problem as described in Dehez and Ferey (2013) and Ferey and Dehez (2016) has a *linear structure* in the sense that the agents are linearly ordered: the wrongful act of agent i can only occur when all agents $j < i$ behave wrongfully. In this paper, we generalize this class of liability problems by considering *rooted-tree structures*. As an example, we consider the case where the injured party suffers an injury caused by four agents, agents 1, 2, 3, and 4. Agent 1 has taken a wrongful act that is the root of the injury. After agent 1's wrongful act, agents 2 and 3 have taken wrongful acts. Without agent 1's wrongful act, agents 2 and 3's wrongful acts would not have occurred. On the other hand, agent 2's wrongful act does not affect agent 3's wrongful act, and reversely. After agent's 1 wrongful act, the wrongful act of agent 3 might occur without the wrongful act of agent 2. Similarly, the wrongful act of agent 2 might occur without the wrongful act of agent 3. Without the wrongful acts of both agents 1 and 2, agent 4's wrongful act would not have occurred. This example can be illustrated by the rooted-tree graph of Figure 1. In this graph, agent 1 is located at the root and has two branches, at one branch agent 1 is succeeded by agent 2 and agent 2 is succeeded by agent 4, at the other branch agent 1 is succeeded by agent 3. Agents 3 and 4 are the *leafs* of the tree. The rooted-tree represents the hierarchical structure of causation of the cumulative injury.

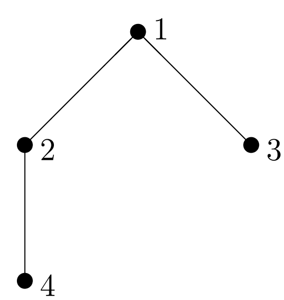

Figure 1: Rooted tree with two branches.

e.g. Dehez and Ferey (2013) and Ferey and Dehez (2016).

[3]Axiomatic approaches to economic allocation problems, where the axioms are based on principles of distributive justice, have been described in, e.g. Moulin (2003).

To illustrate what kind of real-life situation can be described in such a hierarchical structure of causation as in Figure 1, imagine that a natural disaster as a downpour occurs in two cities A and B.[4] These cities belong to a prefecture P. In Figure 1, the prefectural office of P is agent 1. Also, city offices of A and B are agents 2 and 3, respectively. In city A, only the people in a town A' suffer from the downpour. The town office of A' is agent 4. The injured party is the people in the town A' (in city A) or the city B. Let us consider the following scenario: The prefectural office 1 has taken a wrongful act of the evacuation for the injured party. Without the agent 1's wrongful act, the city offices 2 and 3 would not have taken any wrongful act of the evacuation for the injured party. Without the agents 1 and 2's wrongful act, the town office 4 would not have taken any wrongful act of the evacuation for the injured party. Thus, the injured party suffer from the cumulative injury caused by all the agents 1, 2, 3, and 4.

As Dehez and Ferey (2013) and Ferey and Dehez (2016) point out, for every liability problem a legal compensation scheme concerning tort law should satisfy such individual compensation properties as in Property 1.1. In this property, the *additional damage* of tortfeasor i is the sum of all damages that would not have occurred without the wrongful act of i. The *potential damage* of tortfeasor i is the damage that i causes when all the members except for i do not behave wrongfully.

Property 1.1

> (i) *Every injuring party should pay at least the potential damage that he would have caused alone. This principle is supported in the literature of Law (for instance, see Peaslee 1934).*
>
> (ii) *Every injuring party should pay at most the additional damage that he would have caused. This principle is supported by the Restatement of Torts* (*Third*), *Topic 5.*

We propose three axioms, two inspired by Property 1.1, and one by case system observed in the UK and USA.

The first axiom is inspired from Property 1.1 (i). This axiom sets for every tortfeasor a *uniform lower bound*. This lower bound is the same for every tortfeasor, and for an individual tortfeasor the best possible outcome is an outcome where he has to pay this lower bound.

[4] This illustration is inspired from the following case observed in Japan. In 2009, twenty people were killed by the downpour in Sayo town, Hyogo prefecture. Their bereaved family filed a lawsuit against the town office for compensation for injury. They claimed that the cause of their family's death was the Sayo town office's wrongful act of the evacuation.

The second axiom is inspired from Property1.1 (ii). This axiom sets for every tortfeasor an *individual upper bound*. This upper bound differs over the tortfeasors and gives for every tortfeasor his worst possible outcome.

In order to set the lower and upper bound axioms, a *per capita criterion* is employed. This is because in liability problems it is often impossible to determine the causal weights among the injuring parties. In this situation, the per capita criterion might be one of practical methods to determine the lower and upper bounds. For instance, in Japan courts employed the notion of per capita criterion in joint and several liability problems before the 1990's.

The last axiom stems from the stylized fact that in the UK and USA, a legal compensation scheme is based on so-called *case system consistency* (see for instance Ito 1978). This requires that for every liability problem a compensation scheme should provide an outcome that is consistent with the outcome that the same procedure generates for a different, but similar liability problem. In this paper, we propose *leaf consistency*. This type of case system consistency requires that for every liability problem with rooted-tree structure the compensation scheme should be invariant when a leaf agent of the tree pays his compensation and leaves. It guarantees legal stability in the following sense: When one of the tortfeasors, who was the last tortfeasor to do a wrongful act, accepts the payment he has to make, then he need not go to court. When the remaining tortfeasors agree with the departure of the last tortfeasor, a judge faces with the legal situation represented by a reduced liability problem without the leaving tortfeasor. Leaf consistency means that if the leaf agent agrees with the original compensation and leaves, then the remaining agents have no incentive to make the final appeal to the court instead of accepting the recommendations.

We show that there is a unique compensation scheme that satisfies three axioms, one about lower bounds of individual compensations, one about upper bounds of individual compensations, and one about case-system consistency. This scheme assigns to every liability problem with rooted-tree structure the so-called *Nucleolus* of the corresponding TU game derived from liability problems with rooted-tree structure. The Nucleolus (Schmeidler 1969) is another established solution for TU games, and it is also a game theoretic expression of fairness. In this paper, on the class of liability problems with rooted-tree structure the compensation scheme obtained from applying the Nucleolus is referred to as the *Nucleolus compensation scheme*. Besides the axioms discussed above, the Nucleolus compensation scheme has three appealing properties.

Firstly, the Nucleolus (Schmeidler 1969) is an established outcome for TU games. In fact, it is a game-theoretic expression of the 'difference principle of social justice' à la Rawls (1971). So, when it is desirable that a legal compensation scheme for liability problems is to attain a Rawlsian outcome, the three axioms yield a useful compensation scheme.

Secondly, it is well known that the Nucleolus of a game is in the core if the core is non-empty. In this paper, two TU games are derived from Property 1.1, that is, the *lower bound liability game* derived from Property 1.1 (i), and the *upper bound liability game* derived from Property 1.1 (ii). Since the lower bound liability game is shown to be convex, and thus has a nonempty core, the Nucleolus compensation scheme yields a core outcome. Using the fact that the upper bound liability game is the *dual*[5] of the lower bound liability game, it follows that the Nucleolus of the lower bound liability game lexicographically maximizes over all different groups of tortfeasors the cost savings that are the differences between the additional damage caused by the members of every group of tortfeasors and the actual compensation to be paid by them. Equivalently, it lexicographically minimizes the dissatisfactions over the groups of tortfeasors with respect to their 'worst-case' outcomes. This last point also makes the Nucleolus compensation scheme very suitable for real life application. In practice, in liability problems injuring parties sometimes make the final appeal to the court when they feel dissatisfaction of the ruling, which implies that it would take a long time until the injured party can receive compensation. Since the Nucleolus lexicographically minimizes the corresponding dissatisfaction of injuring parties, it is likely that the injured party can receive compensation as soon as possible without facing with injuring parties' final appeal to the court. In this sense, the Nucleolus is an appealing solution for liability problems in practice.

Thirdly, as shown by Aumann and Maschler (1985), the Nucleolus can be seen as one of the first implicitly applied game solutions to legal problems appearing in the Talmud. The Nucleolus of an associated bankruptcy game is the unique solution that supports the Talmudic principle of equal division of a contested amount. On the other hand, in the model appearing in our paper, the Nucleolus of an associated liability game is the unique solution that supports such a modern legal rule as the *Restatement of Torts*. Thus, the Nucleolus is a prominent solution for various real-life problems in Law and Economics.

This paper is organized as follows. In Section 2, preliminaries are given. In Section 3, the liability problem with rooted-tree structure and the corresponding liability games are given. In Section 4, we state and discuss the three axioms to be satisfied by a compensation scheme. In Section 5, we characterize the Nucleolus compensation scheme by using the axioms appearing in Section 4. In Section 6, we discuss a comparison between the Shapley and Nucleolus compensation schemes. Finally, in Section 7, we explore an incentive problem in the situation where the population of the tortfeasors is increasing. The proof of main results are in Appendix A.

[5]For the definition of dual games, see Section 2.

2 Preliminaries

A *cooperative game with transferable utility*, or simply a TU game, is a pair (N, v), where $N \subseteq \mathbb{N}$ is a finite set of players, and $v\colon 2^N \to \mathbb{R}$ is a *characteristic function* that assigns a *worth* $v(S) \in \mathbb{R}$ to every subset (usually called *coalition*) S of N, satisfying $v(\emptyset) = 0$. A TU game (N, v) is convex if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subseteq N$. It is concave if these inequalities are reversed. We denote by $\mathcal{G}$ the class of all TU games. The subclass of all convex TU games is denoted by $\mathcal{G}^{vex}$ and the subclass of all concave TU games by $\mathcal{G}^{cave}$. For a game $(N, v) \in \mathcal{G}$, the dual game, denoted by (N, v^d), assigns to every coalition S what the 'grand coalition' N loses if the players in S stop cooperating, and thus it is defined by $v^d(S) = v(N) - v(N \setminus S)$ for all $S \subseteq N$. Note that $v^d(\emptyset) = 0$ and $v^d(N) = v(N)$. It holds that $(N, v) \in \mathcal{G}^{vex}$ if and only if $(N, v^d) \in \mathcal{G}^{cave}$.

A *payoff vector* of TU game (N, v) is a vector $x \in \mathbb{R}^N$ giving a payoff $x_i \in \mathbb{R}$ to every player $i \in N$. A payoff vector is efficient if $\sum_{i \in N} x_i = v(N)$. Given $(N, v) \in \mathcal{G}$, let $X(N, v)$ be the set of all efficient payoff vectors.[6] The *imputation set*, denoted by $I(N, v)$, is the subset of all vectors in $X(N, v)$ that satisfy $x_i \geq v(\{i\})$ for every $i \in N$ (individual rationality); the *anti-imputation set*, denoted by $AI(N, v)$, is the subset of all vectors in $X(N, v)$ that satisfy $x_i \leq v(\{i\})$ for every $i \in N$. Note that these sets are non-empty if and only if $v(N) \geq \sum_{i \in N} v(\{i\})$, respectively $v(N) \leq \sum_{i \in N} v(\{i\})$. We denote by $\mathcal{G}^I$ the class of all TU-games that satisfy $I(N, v) \neq \emptyset$, and by $\mathcal{G}^{AI}$ the class of all TU-games that satisfy $AI(N, v) \neq \emptyset$. Note that $\mathcal{G}^{vex}$ is a subset of $\mathcal{G}^I$ and $\mathcal{G}^{cave}$ is a subset of $\mathcal{G}^{AI}$.

The *core* of a game (N, v), denoted by $C(N, v)$, is the set of efficient payoff vectors that are group stable, and is given by

$$C(N, v) = \left\{ x \in X(N, v) \,\middle|\, \sum_{i \in S} x_i \geq v(S) \ \text{ for all } S \subseteq N \right\}.$$

Note that $C(N, v)$ is a subset of $I(N, v)$ and that it might be empty. Every game $(N, v) \in \mathcal{G}^{vex}$ has a non-empty core. A vector $x \in C(N, v)$ satisfies the requirement that for every coalition S the total payoff is at least equal to its own worth. This is reasonable when (N, v) is a profit game, i.e., the worth $v(S)$ is the total revenue that the members of S can achieve by cooperating. However, when v is a cost game, i.e., coalition S has costs $v(S)$ when it stands alone, then the worth should be considered as upper bounds on the contributions. For a cost game, it makes sense to apply the *anti-core* of a game (N, v), denoted by $AC(N, v)$. This set of efficient payoff vectors is given by

$$AC(N, v) = \left\{ x \in X(N, v) \,\middle|\, \sum_{i \in S} x_i \leq v(S) \ \text{ for all } S \subseteq N \right\}.$$

[6] In the field of cooperative game theory, $X(N, v)$ is usually referred to as the "preimputation set".

The anti-core $AC(N,v)$ is a subset of $AI(N,v)$ and might be empty, but every game $(N,v) \in \mathcal{G}^{cave}$ has a non-empty anti-core.

For a given subset $\mathcal{G}'$ of the class $\mathcal{G}$ of all TU-games, a (single-valued) *solution* is a function f that assigns to every game (N,v) in $\mathcal{G}'$ a payoff vector $f(N,v) \in X(N,v)$. Note that in this paper, we require that a solution assigns to each game an efficient payoff vector. The best-known solution on the class $\mathcal{G}$ of all TU-games is the *Shapley value* (Shapley 1953), denoted by *Sh*. This solution assigns to every game $(N,v) \in \mathcal{G}$ the payoff vector $Sh(N,v)$ given by[7]

$$Sh_i(N,v) = \sum_{S \subseteq N: i \in S} \frac{(|N|-|S|)!(|S|-1)!}{|N|!}(v(S) - v(S \setminus \{i\})) \text{ for all } i \in N.$$

So, for every player $i \in N$ the payoff is a weighted sum of its marginal contributions $v(S) - v(S \setminus \{i\})$ to the coalitions S containing i. When (N,v) is convex, then $Sh(N,v) \in C(N,v)$. However, in general, on the domain of TU games with non-empty cores it might be that the Shapley value is not in the core. Furthermore, it holds that the Shapley value is *self-dual* (see Kalai and Samet 1987), saying that for every $(N,v) \in \mathcal{G}$ it holds that $Sh(N,v^d) = Sh(N,v)$.[8]

Another well-known solution on the class of all TU-games is the *Nucleolus*. Given a TU game $(N,v) \in \mathcal{G}$, we define for every payoff vector $x \in X(N,v)$ and coalition $S \subseteq N$, the *excess* of S with respect to x as

$$e(S,x,v) \equiv v(S) - \sum_{i \in S} x_i.$$

When the payoffs are revenues, i.e., the payoffs are payments to the players, the excess $e(S,x,v)$ can be seen as a measure of dissatisfaction of coalition S. Let $\theta(x,v) \in \mathbb{R}^{2^N}$ be the vector obtained by arranging all the excesses in non-increasing order, so the first component of $\theta(x,v)$ is the excess of a coalition with the highest excess, the second component is the excess of a coalition with the highest excess under the remaining coalitions, and so on. The *Nucleolus* (Schmeidler 1969) is a solution defined on the class $\mathcal{G}^I$ of games that satisfy $I(N,v) \neq \emptyset$. It assigns to every $(N,v) \in \mathcal{G}^I$ the unique individually rational payoff vector $x \in I(N,v)$ that minimizes lexicographically the dissatisfactions over all vectors in $I(N,v)$. We denote the Nucleolus of a game (N,v) by $Nuc(N,v)$. When the core is non-empty, $Nuc(N,v) \in C(N,v)$.

Similarly, for every game $(N,v) \in \mathcal{G}^{AI}$, we define the *Anti-Nucleolus*, denoted by $ANuc(N,v)$, as the solution that assigns to every $(N,v) \in \mathcal{G}^{AI}$ the unique payoff vector $x \in I(N,v)$ such that $-\theta(v,x)$ is lexicographically smaller than $-\theta(v,y)$ for every $y \in$

[7] For a finite set A, we denote by $|A|$ the number of elements in A (cardinality of A).

[8] The notion of (self-)duality plays an important role in axiomatizing solutions for TU games, see for instance Oishi, Nakayama, Hokari, and Funaki (2016).

$I(N,v)$, i.e. it minimizes lexicographically the vector of negative excesses $\sum_{i\in S} x_i - v(S)$ for all $S \subseteq N$, over all vectors in $AI(N,v)$. When the anti-core is non-empty, $ANuc(N,v) \in AC(N,v)$. So, for a cost game in $\mathcal{G}^{AI}$, the Anti-Nucleolus lexicographically maximizes the cost savings $v(S) - \sum_{i\in S} x_i$ for all $S \subseteq N$.

In the next sections the following proposition, which follows from Oishi and Nakayama (2009), will appear to be useful. Recall that $(N,v) \in \mathcal{G}^{vex}$ if and only if $(N,v^d) \in \mathcal{G}^{cave}$.

Proposition 2.1 *For every $(N,v) \in \mathcal{G}^{vex}$ it holds that*

(i) $AC(N,v^d) = C(N,v)$,

(ii) $ANuc(N,v^d) = Nuc(N,v)$.

Next, we introduce a *rooted-tree graph.* First, a *directed graph* or *digraph* is a pair (N,D), where N is a set of *nodes* and the collection of ordered pairs $D \subseteq \{(i,j)|i,j \in N, i \neq j\}$ is a set of *arcs*. In this paper, the nodes represent the players in a game, and therefore we refer to the nodes as players. For digraph (N,D), a sequence of k different players $(i_1,...,i_k)$ is a (directed) path if $(i_l,i_{l+1}) \in D$ for $l = 1,...,k-1$. For $i \in N$, a player $j \in N$ is a *subordinate* of i if there is a path $(i_1,...,i_k)$ with $i_1 = i$ and $i_k = j$. Player i is a *superior* of j if and only if j is a subordinate of i. We denote by $F_D(i)$ the set of subordinates of i, and by $P_D(i)$ the set of superiors of i in (N,D). We define $F^0_D(i) \equiv F_D(i) \cup \{i\}$ and $P^0_D(i) \equiv P_D(i) \cup \{i\}$. Also for all $S \subseteq N$, we define $F_D(S) \equiv \cup_{i\in S} F_D(i)$ and similarly $P_D(S) \equiv \cup_{i\in S} P_D(i)$, $F^0_D(S) \equiv \cup_{i\in S} F^0_D(i)$ and $P^0_D(S) \equiv \cup_{i\in S} P^0_D(i)$.

A node $i \in N$ is called a *top player* in (N,D) if $P_D(i) = \emptyset$. A digraph (N,D) is a (*directed*) *rooted tree with root* i when (i) player i is the unique top player and (ii) for all $j \neq i$ there is a unique path from i to j.

In the sequel, we denote by $\mathcal{D}$ the class of rooted tree graphs and an element of $\mathcal{D}$ by (N,T). Note that for a rooted tree graph (N,T) with root i it holds that $F_T(i) = N \setminus \{i\}$ and for every node $j \neq i$ there is precisely one player $k \in P_T(j)$ such that $(k,j) \in T$. This player is called the *predecessor* of j and denoted by $p(j)$. A player is called a *leaf* of (N,T) if $F_T(i) = \emptyset$. We denote by $L(T)$ the set of all leafs. We say that a tree (N,T) is *linear* if $|L(T)| = 1$. In this case it holds that for every player $k \notin L(T)$ there is precisely one h such that $(k,h) \in T$. For a rooted-tree graph (N,T) with root i, let $M \subseteq N$ be such that (i) $|M| \geq 2$, (ii) $i \in M$, and (iii) for every $j \in M \setminus \{i\}$ all nodes on the (unique) directed path from i to j are also in M. We denote by $(M,T(M))$ the *subtree* of (N,T) restricted to M, and by $\mathcal{T}$ the collection of all subtrees $(M,T(M))$ of (N,T).

A *permission tree game* is a triple (N,v,T) with $N \subset \mathbb{N}$ a finite set of players, $(N,v) \in \mathcal{G}$ a TU-game and $(N,T) \in \mathcal{D}$ a rooted tree on N. In those games, it is assumed that the tree represents a hierarchy that imposes restrictions on the forming of coalitions. Solutions for permission tree games have been discussed in for instance van den Brink, Herings, van der Laan and Talman (2015) and van den Brink, Dietz, van der Laan and Xu (2017). One of these solutions is the permission value, based on the so-called *conjunctive approach* to permission structures as developed in Gilles, Owen and van den Brink (1992). In this approach, it is assumed that a coalition is feasible if and only if for every player in the coalition all its predecessors are also in the coalition.[9] The set of feasible coalitions is given by

$$\Phi_T = \{S \subseteq N \,|P_T(i) \subseteq S \text{ for all } i \in S\}.$$

In this paper, we only consider triples (N, v, T) where the permission structure $(N, T) \in \mathcal{D}$ is a rooted tree. We denote by $\mathcal{G}_T$ the collection of all permission tree games. A (single-valued) solution f on $\mathcal{G}_T$ assigns a unique payoff vector $f(N, v, T) \in \mathbb{R}^N$ to every $(N, v, T) \in \mathcal{G}_T$. For $S \subseteq N$, let $\sigma_T(S) = \bigcup_{R\in\Phi_T:R\subseteq S} R$ be the largest feasible subset[10] of S. Following Gilles Owen and van den Brink (1992), the induced *permission restricted game* of (N, v, T) is the game $(N, r_{N,v,T}) \in \mathcal{G}$ given by

$$r_{N,v,T}(S) = v(\sigma_T(S)) \text{ for all } S \subseteq N.$$

3 Liability problems with rooted-tree structure

Let N be the set of tortfeasors. Without loss of generality, we assume that for every $(N, T, d) \in \mathcal{L}$, player $1 \in N$ is the *top* of root (N, T).

In the Restatement of Torts (Second), §12A, the word 'damages' is a sum of money awarded to a person injured by the tort of another. We denote by d_i the *direct damage*, which is measured by money, caused by every tortfeasor $i \in N$. Note that d_i is non-negative.

A *joint liability problem with rooted-tree structure* (shortly a *liability problem*) is a triple (N, T, d), where $|N| \geq 2$, $(N, T) \in \mathcal{D}$ is a rooted tree on N, and $d \in \mathbb{R}^N_+$ is a profile of direct damages. We denote by $\mathcal{L}$ the class of all liability problems. We also define the following notions.

The *total damage* of S, denoted by d_S, is the sum of the damages of the players in S.

[9] Other models of allocation on networks are studied by, e.g. Myerson (1977) or Ju (2013), who consider allocation problems where only connected coalitions in an undirected network can form. Since the root of the tree is in every feasible coalition, permission structures can also be seen as centered union structures in Charnes and Littlechild (1975).

[10] Every coalition having a unique largest feasible subset follows from the fact that Φ_T is union closed, i.e. for every $E, F \in \Phi_T$ it holds that $E \cup F \in \Phi_T$

Total damage For $S \subseteq N$, $d_S \equiv \sum_{j \in S} \; d_j$.

The *cumulative damage* of S, denoted by c_S, is the sum of the damages of the players in S and all their superiors.

Cumulative damage For $S \subseteq N$, $c_S \equiv \sum_{j \in P_T^0(S)} \; d_j$.

The *additional damage* of S, denoted by e_S, is the sum of all damages that would have been avoided when none of the members of S exercised a wrongful act. Thus, this damage is represented as the sum of the damages of the players in S and all their subordinates.

Additional damage For $S \subseteq N$, $e_S \equiv \sum_{j \in F_T^0(S)} \; d_j$.

The *potential damage* of S, denoted by b_S, is the sum of all damages that the members of S cause when they do a wrongful act, and the members outside S do not behave wrongfully.

Potential damage For every subset $S \subseteq N$, $b_S \equiv \sum_{j \in S : P_T(j) \subseteq S} \; d_j$.

In the case where the tree is linear and $S = \{j\}$ for some $j \in N$, these four notions of damages coincide with the notions of Dehez and Ferey (2013), respectively. For convenience of notation, we denote $d_S = d_j$ if $S = \{j\}$ and similarly for the other notions. Note that $d_N = c_N = e_N = b_N$, and for the root 1, $d_1 = c_1 = b_1$ and $e_1 = \sum_{k \in F_T^0(1)} \; d_k = d_N$.

The following example illustrates the different notions of damages mentioned above.

Example 3.1 Consider six players and rooted-tree (N, T) with $N = \{1, 2, 3, 4, 5, 6\}$ and $T = \{(1, 2),\ (2, 3),\ (2, 4),\ (1, 5),\ (5, 6)\}$, see Figure 2. Table 1 gives the four notions of damages for two different sets S. Note that $d_N = \sum_{i \in N} d_i$.

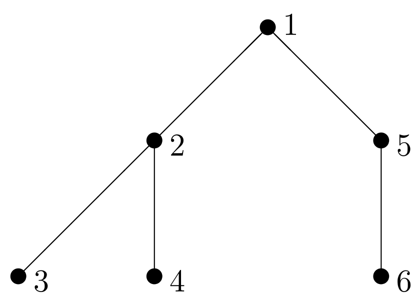

Figure 2: Rooted tree with six players.

Table 1. Four notions of damages for the tree of Figure 2.

	$S = \{1, 4, 6\}$	$S = \{2, 6\}$
Total damage d_S	$d_1 + d_4 + d_6$	$d_2 + d_6$
Cumulative damage c_S	$d_1 + d_2 + d_4 + d_5 + d_6$	$d_1 + d_2 + d_5 + d_6$
Additional damage e_S	d_N	$d_2 + d_3 + d_4 + d_6$
Potential damage b_S	d_1	0

Given $(N, T, d) \in \mathcal{L}$, an *allocation* for (N, T, d) is a non-negative vector $x \in \mathbb{R}^N_+$ such that $\sum_{i \in N} x_i = d_N$. A *compensation scheme* for liability problems is a mapping φ on $\mathcal{L}$ that associates with every problem $(N, T, d) \in \mathcal{L}$ an allocation $\varphi(N, T, d) \in \mathbb{R}^N_+$.

We generalize the class of liability games derived from liability problems with linear structure (Dehez and Ferey 2013; Ferey and Dehez 2016) to the class of liability games derived from liability problems with rooted-tree structure. For every liability problem $(N, T, d) \in \mathcal{L}$, the corresponding *lower-bound liability game* is the game (N, v_L), where the worth of a coalition S is its potential damage, i.e. for all $S \subseteq N$,

$$v_L(S) \equiv b_S.$$

Dehez and Ferey (2013) focus on the Shapley value of the lower-bound liability game. In the sequel we define the Shapley value of a liability problem (N, T, d) as the Shapley value of the corresponding lower-bound liability game. The *Shapley compensation scheme* is the mapping Sh on $\mathcal{L}$ that associates with every problem $(N, T, d) \in \mathcal{L}$, the Shapley value of its corresponding lower-bound liability game (N, v_L):

$$Sh(N, T, d) \equiv Sh(N, v_L).$$

In the literature on game theory, the lower-bound liability game is known as a *peer-group game* introduced by Brânzei, Fragnelli and Tijs (2002). The peer-group game is derived from the *peer-group situation* (N, T, d), where cooperation among agents is hierarchically structured. Since the class of peer-group games is a subclass of convex games (see e.g. Brânzei, Fragnelli and Tijs 2002), the lower-bound liability game (N, v_L) is convex. Furthermore the lower-bound liability game (N, v_L) is also a permission restricted game $(N, r_{N,v,T})$ of the *additive* game (N, v) with the characteristic function defined as $v(S) = d_S$ for all $S \subseteq N$. Therefore, the Shapley value of the lower-bound liability game (N, v_L) is equal to the *permission value* of the permission tree game (N, T, v).[11] Since the lower-bound liability game (N, v_L) is convex, the Shapley value is in its core. Thus, the Shapley compensation scheme satisfies Property 1.1 (i) of the Introduction since it satisfies the requirement that for every $S \subseteq N$ the total compensation paid by its members is at least equal to $v_L(S)$, being the potential damage of S.

While the existing literature (Dehez and Ferey 2013; Ferey and Dehez 2016) focuses on the Shapley value of the lower-bound liability game, we consider the Nucleolus. We define

[11] The *permission value* ψ on $\mathcal{G}_T$ is the solution that assigns to every $(N, v, T) \in \mathcal{G}_T$ the Shapley value of the associated permission restricted game, i.e.,

$$\psi(N, v, T) = \mathit{Sh}(N, r_{N,v,T}) \text{ for all } (N, v, T) \in \mathcal{G}_T.$$

For the details, see van den Brink and Gilles (1996).

the *Nucleolus compensation scheme* as the mapping that assigns to every liability problem $(N, T, d) \in \mathcal{L}$ the Nucleolus of its corresponding lower-bound liability game (N, v_L).

Definition 3.2 *The Nucleolus compensation scheme is the mapping Nuc on $\mathcal{L}$ that associates with every problem $(N, T, d) \in \mathcal{L}$ the Nucleolus of its corresponding lower-bound liability game (N, v_L):*

$$Nuc(N, T, d) \equiv Nuc(N, v_L).$$

Since the Nucleolus of convex games is in its core, the Nucleolus compensation scheme satisfies the requirement that for every $S \subseteq N$ the total compensation paid by its members is at least equal to the potential damage of S. Thus, the Nucleolus compensation scheme satisfies Property 1.1 (i) of the Introduction. Nevertheless, we now run into a difficulty about the interpretation of the core and the Nucleolus. Typically it is considered to be desirable that a payoff vector is in the core of the game. This is called *core-stability*, saying that every coalition S gets at least its own worth $v(S)$ and so no member of S has an incentive to deviate from the grand coalition N. However, this holds for profit games where $v(S)$ is the worth that the members of S can earn by themselves without cooperating with the others, and the entries of x yield payoffs that are paid to the players. In contrast to this usual situation, the Nucleolus of the lower-bound liability game gives a vector of compensations to be paid by the tortfeasors. The tortfeasors are not looking for core stability, on the contrary they want to pay as little as possible. Therefore, the lower-bound liability game should not be considered as a game that is played by the tortfeasors themselves.

An appropriate interpretation of the lower-bound liability game is that this game is a model to help the court to determine the compensations to be paid to the injured party. According to this interpretation, the Nucleolus compensation scheme determines how much every coalition has to pay in addition to its lower bound $v_L(S)$. As for the Shapley value, these additional payments are determined by the marginal contributions of the players.[12] In contrast, the Nucleolus is determined by the excesses of the coalitions. However, for the lower-bound liability game (N, v_L) the excess $e(S, x, v_L)$ is now a measure of 'satisfaction' of S at x, because the bigger the excess is, the lower the total amount of compensation that the members of S have to pay. So, while in a game where the payoff vector yields payments to the players the Nucleolus minimizes lexicographically the vector of dissatisfactions, in the lower-bound liability game the Nucleolus minimizes lexicographically the vector of satisfactions. This is counterintuitive. Even if we consider the lower-bound liability game

[12] Since the marginal contributions of a player do not depend on the damages of his superiors in the tree, this also implies that the Shapley value satisfies the property that the compensation to be paid by a tortfeasor does not depend on the damages of its superiors, see Ferey and Dehez (2016).

as a model used by the court to determine the compensations, there is no a priori reason to do so.

Nevertheless, the Nucleolus compensation scheme is justified as a reasonable solution when we consider that every tortfeasor should pay at most the additional damage that he would have caused as stated in Property 1.1 (ii) of the Introduction.

Next, we will explain how the Nucleolus compensation scheme can be justified. Based on Property 1.1 (ii), we define for a liability problem $(N, T, d) \in \mathcal{L}$ the corresponding *upper-bound liability game*[13] as the game (N, v_U), where the worth of a coalition S is the additional damage that the agents in S might cause, i.e. for all $S \subseteq N$,

$$v_U(S) \equiv e_S.$$

The next lemma states that the upper-bound liability game (N, v_U) is the dual of the lower-bound liability game (N, v_L). The fact that (N, v_L) is convex implies that (N, v_U) is concave.

Lemma 3.3 *For every liability problem $(N, T, d) \in \mathcal{L}$, the upper-bound liability game (N, v_U) is the dual of the lower-bound liability game (N, v_L).*

Proof. For $S \subseteq N$, we have that $v_L(S) = b_S = \sum_{j \in S: P_T(j) \subseteq S} d_j$. Now, note that for every $S \subseteq N$ the set of players in S such that all their predecessors are also in S coincides with the set of players in S that are not subordinates of the players in $N \setminus S$, i.e.

$$\{j \in S : P_T(j) \subseteq S\} = S \setminus F_T(N \setminus S).$$

Since $F_T^0(N \setminus S) = F_T(N \setminus S) \cup (N \setminus S)$, we have $S \setminus F_T(N \setminus S) = N \setminus F_T^0(N \setminus S)$. Hence for every $S \subseteq N$ we obtain that $v_L(S) = \sum_{j \in N: j \notin F_T^0(N \setminus S)} d_j$ and thus

$$\begin{aligned} v_L^d(S) &= v_L(N) - v_L(N \setminus S) = \sum_{j \in N} d_j - \sum_{j \in N: j \notin F_T^0(N \setminus (N \setminus S))} d_j \\ &= \sum_{j \in F_T^0(N \setminus (N \setminus S))} d_j = \sum_{j \in F_T^0(S)} d_j = e_S = v_U(S), \end{aligned}$$

which completes the proof. □

By Lemma 3.3, we have the following proposition. The statement (i) follows from the fact that the Shapley value is *self-dual* and the statements (ii) and (iii) follow from applying Proposition 2.1 to the convex game (N, v_L) and its dual (N, v_U).[14]

[13] For a liability problem with linear structure the upper-bound liability game (N, v_U) is a so-called airport game, see Littlechild and Owen (1973).

[14] On the domain of all TU games, a solution f is self-dual if $f(v) = f(v^d)$, where v and v^d are dual to each other.

Proposition 3.4 *For every liability problem $(N,T,d) \in \mathcal{L}$ the corresponding games (N,v_L) and (N,v_U) satisfy the following statements:*

(i) $Sh(N,v_L) = Sh(N,v_U)$,

(ii) $C(N,v_L) = AC(N,v_U)$,

(iii) $Nuc(N,v_L) = ANuc(N,v_U)$.

The statement (i) and (ii) imply that the Shapley value of the lower-bound liability game belongs to the anti-core of the upper-bound liability game. Thus, for every coalition S $Sh(N,v_L)$ satisfies the upper-bound requirement that $\sum_{i\in S} Sh_i(N,v_L) \leq e_S$. By the same argument, for every coalition S, $Nuc(N,v_L)$ satisfies the upper-bound requirement that $\sum_{i\in S} Nuc_i(N,v_L) \leq e_S$. From the statement (iii) it follows that $Nuc(N,v_L)$ lexicographically maximizes the cost savings $v_U(S) - \sum_{i\in S} x_i$ with respect to the upper-bound liability game. So, when the court decides to implement $Nuc(N,v_L)$, or equivalently $ANuc(N,v_U)$, the smallest cost saving over all coalitions S is made as large as possible, then the second smallest is made as large as possible, then the third smallest, and so on. This gives us a justification of the Nucleolus compensation scheme. This is because, as mentioned in the Introduction, the Nucleolus lexicographically minimizes the corresponding 'dissatisfaction' of injuring parties, and thus it is likely that the injured party can receive compensation as soon as possible without facing with injuring parties final appeal to the court.

We now consider the computation of the compensations for both the Shapley value and the Nucleolus. The Shapley value is easy to compute for liability games. Recall that (N,v_L) is the peer-group game associated to the peer-group situation (N,T,d) and also that it is the permission restricted game $(N, r_{N,v,T})$ of the additive game (N,v) with its characteristic function defined as $v(S) = d_S$ for all $S \subseteq N$. For these games, it is well-known that the Shapley value distributes the damage d_i of a player $i \in N$ equally among player i and all its superiors in (N,T). This gives the following expression for the compensation to be paid by every tortfeasor $j \in N$ according to the Shapley compensation scheme:

$$Sh_j(N,T,d) = \sum_{i \in F_T^0(j)} \frac{d_i}{|P_T^0(i)|}. \tag{3.1}$$

So, when applying the Shapley value the compensation to be paid by tortfeasor j is the sum of all his shares in the damages of himself and his subordinates.[15]

[15] This compensation can also be obtained as the so-called permission value as axiomatized in van den Brink and Gilles (1996) for games with a permission structure, where the game is the above mentioned additive game where the worth of a coalition is its total damage, and the hierarchy (permission structure) is the rooted tee T.

In general, on the other hand, there is no explicit formula available for the Nucleolus, but the Nucleolus compensations can be computed by the algorithm given in Brânzei, Solymosi and Tijs (2005) for peer-group games. For a subtree $(M, T(M)) \in \mathcal{T}$ (see Section 2), let $(M, T(M), a)$, $a \in \mathbb{R}^M_+$, be a *reduced liability problem* on M. We define for every $j \in M \setminus \{1\}$,

$$\tau_j(M, T(M), a) \equiv \frac{\sum_{k \in F^0_{T(M)}(j)} a_k}{|F^0_{T(M)}(j)| + 1}.$$

The Nucleolus compensation scheme is obtained by the following algorithm, where $x = Nuc(N, T, d) \equiv Nuc(N, v_L)$. Recall that $p(j)$ is the *predecessor* of j in (N, T) and so also in every subtree $(M, T(M))$ containing j.

Nucleolus algorithm (Nuc algorithm):
Step 0: Set $M = N$ and $a = d$.

Step 1: Find a $j \in M \setminus \{1\}$ such that $\tau_j(M, T(M), a) = \min_{m \in M \setminus \{1\}} \tau_m(M, T(M), a)$.

Step 2: For every $k \in F^0_{T(M)}(j)$, set $x_k = \tau_j(M, T(M), a)$. If $|M \setminus F^0_{T(M)}(j)| \geq 2$, go to Step 3. Otherwise, set $x_1 = d_N - \sum_{k \in N \setminus \{1\}} x_k$ and stop.

Step 3: Set $M \equiv M \setminus F^0_{T(M)}(j)$ and set $a_{p(j)} \equiv a_{p(j)} + x_j$. Return to Step 1.

Note that in Step 2, if $|M \setminus F^0_{T(M)}(j)| = 1$, then $M \setminus F^0_{T(M)}(j) = \{1\}$ and top agent 1 gets what is left after all other agents $j \neq 1$ received their x_j. In the next Example, we will illustrate the above procedure.

Example 3.5 Let (N, T, d) be the liability problem with rooted tree as given in Figure 2 and with the profile of direct damages given by $d = (0, 12, 40, 36, 12, 30)$.

(1) The Shapley value: Using formula (3.1) we have the following computations for the outcome of the Shapley compensation scheme, starting with the leafs: $Sh_3(N, T, d) = \frac{d_3}{3} = \frac{40}{3}$, $Sh_4(N, T, d) = \frac{d_4}{3} = 12$, $Sh_6(N, T, d) = \frac{d_6}{3} = 10$. Next we obtain $Sh_2(N, T, d) = \frac{d_2}{2} + Sh_3(N, T, d) + Sh_4(N, T, d) = 6 + \frac{40}{3} + 12 = \frac{94}{3}$, $Sh_5(N, T, d) = \frac{d_5}{2} + Sh_6(N, T, d) = 6 + 10 = 16$ and finally $Sh_1(N, T, d) = d_1 + Sh_2(N, T, d) + Sh_5(N, T, d) = 0 + \frac{94}{3} + 16 = \frac{142}{3}$. Thus $Sh(N, T, d) = (\frac{142}{3}, \frac{94}{3}, \frac{40}{3}, 12, 16, 10)$.

(2) The Nucleolus: In the same liability problem mentioned above, let us compute the outcome of the Nucleolus compensation scheme using the algorithm. Let $x = Nuc(N, T, d)$. The Nuc algorithm performs as follows.

Step 0: Set $M = N$ and $a = d$.
Iteration 1:

Step 1: $\min\{\frac{30}{2}, \frac{12+30}{3}, \frac{36}{2}, \frac{40}{2}, \frac{12+40+36}{4}\} = 14 = \tau_5(M, T(M), a)$.
Step 2: $x_5 = x_6 = 14$.
Step 3: Set $M = \{1, 2, 3, 4\}$ and $a = (d_1 + 14, d_2, d_3, d_4) = (14, 12, 40, 36)$.
Iteration 2:
Step 1: $\min\{\frac{36}{2}, \frac{40}{2}, \frac{12+40+36}{4}\} = 18 = \tau_4(M, T(M), a)$.
Step 2: $x_4 = 18$.
Step 3: Set $M = \{1, 2, 3\}$ and $a = (d_1 + 14, d_2 + 18, d_3) = (14, 30, 40)$.
Iteration 3:
Step 1: $\min\{\frac{40}{2}, \frac{30+40}{3}\} = 20 = \tau_3(M, T(M), a)$.
Step 2: $x_3 = 20$.
Step 3: Set $M = \{1, 2\}$ and $a = (d_1 + 14, d_2 + 18 + 20) = (14, 50)$.
Iteration 4:
Step 1: $\tau_2(M, T(M), a) = \frac{50}{2} = 25$.
Step 2: $x_2 = 25$ and $x_1 = d_N - \sum_{k \in N \setminus \{1\}} x_k = 39$. Stop.

We have found that $Nuc(N, T, d) = x = (39, 25, 20, 18, 14, 14)$.

From the formula (3.1) it follows immediately that the Shapley compensation scheme satisfies a strong monotonicity property that every player $j \in N \setminus L(T)$ has to pay at least as much as any of his subordinates, and strictly more when $d_j > 0$. The first part of this property also holds for the Nucleolus compensation scheme, but compensations can be equal when $d_j > 0$.[16]

Lemma 3.6 ***Structural Monotonicity of the Nucleolus compensation scheme***
For every liability problem $(N, T, d) \in \mathcal{L}$, every $j \in N \setminus L(T)$, and every $k \in F_T(j)$,

$$Nuc_j(N, T, d) \geq Nuc_k(N, T, d).$$

Proof. The proof follows from the Nuc algorithm. We have three steps.

Step 1 In the first iteration, the Nuc algorithm starts with (N, T, d). Fix an arbitrary $k \in N \setminus \{1\}$ such that $\tau_k(N, T, d) = \min_{j \in N \setminus \{1\}} \tau_j(N, T, d)$. Then $Nuc_i(N, T, d) = \tau_i(N, T, d)$ for every $i \in F_T^0(k)$ and so the property holds for k and all his subordinates. In the second iteration we have $M = N \setminus F_T^0(k)$ and the reduced liability problem $(M, T(M), a)$ with $a_{p(k)} = d_{p(k)} + \tau_k(N, T, d)$, and $a_q = d_q$ for every $q \in M \setminus \{p(k)\}$. We show that the minimal of $\tau_j(M, T(M), a)$ in the second iteration is at least equal to $\tau_k(N, T, d)$ in the first iteration, that is, $\min_{h \in M \setminus \{1\}} \tau_h(M, T(M), a) \geq \tau_k(N, T, d)$. For simplicity of notation, let $A_l = \tau_l(N, T, d)$ and $n_l = |F_T^0(l)|$ for every $l \in N \setminus \{1\}$. Our target is to show that for every $h \in M \setminus \{1\}$ $\tau_h(M, T(M), a) \geq A_k$. Firstly, consider the case where a player

[16] This property is used in van den Brink and Gilles (1996) to axiomatize the conjunctive (Shapley) permission value for permission restricted games.

$h \in M \setminus \{1\}$ is neither a subordinate of k in (N,T,d) nor a superior of k in (N,T,d). Then $\tau_h(M,T(M),a) = \tau_h(N,T,d) = A_h \geq A_k$ since A_k is minimal in the first iteration. Secondly, consider the case where a player $h \in M \setminus \{1\}$ is a superior of k in (N,T,d). Now note that in the first iteration n_k players (k and its subordinates) have left and that all these players paid compensation A_k, while $a_{p(k)} = d_{p(k)} + A_k$. The player h is either $p(k)$ itself or a superior of $p(k)$. Furthermore, note that in the first iteration $A_h = \frac{1}{n_h+1}\sum_{i \in F_T^0(h)} a_i$. From this observation and the fact that in the first iteration $A_h \geq A_k$ it follows that

$$
\begin{aligned}
\tau_h(M,T(M),a) &= \frac{\sum_{i \in F_T^0(h) \setminus F_T(k)} a_i}{n_h - n_k + 1} \\
&= \frac{(n_h+1)A_h - n_k A_k}{n_h - n_k + 1} \\
&\geq \frac{(n_h+1)A_h - n_k A_h}{n_h - n_k + 1} = A_h \geq A_k.
\end{aligned}
$$

Step 2 For any arbitrary liability problem $(N,T,d) \in \mathcal{L}$ where $d_1 = 0$, we show that $\frac{d_N}{n} \geq \min_{j \in N \setminus \{1\}} \tau_j(N,T,d)$ by induction on $|L(T)|$.

Induction basis. For $|L(T)| = 1$, the claim holds by the Nuc algorithm.

Induction hypothesis. Suppose that the claim holds for $|L(T)| \leq t$ and $t \geq 1$.

Induction step. We show that the claim holds for $|L(T)| = t+1$.

Let N be the set of agents where $|L(T)| = t+1$, and let (N,T,d) be the corresponding liability problem where $d_1 = 0$. Let $R = \{r \in N \mid p(r) = 1\}$, that is, the predecessor of each agent in R is the top player 1. Let (N^r, T^r) be the subtree on which there are the subordinates of r, the top player 1, and r itself. Note that $|L(T^r)| \leq t$. For every $r \in R$, we denote by (N^r, T^r, d^r) where $d_1^r = d_1 = 0$ the reduced liability problem derived from the subtree (N^r, T^r). Fix an arbitrary $k^r \in N^r \setminus \{1\}$ such that $\tau_{k^r}(N^r, T^r, d^r) = \min_{j \in N^r \setminus \{1\}} \tau_j(N^r, T^r, d^r)$. By the *induction hypothesis*, $\sum_{i \in N^r} d_i \geq |N^r| \tau_k(N^r, T^r, d^r)$. Note that $\sum_{r \in R} |N_r| = |N| + r - 1 \geq |N|$. Furthermore, we have $\min_{r \in R}\{\tau_{k^r}(N^r, T^r, d^r)\} = \min_{j \in N \setminus \{1\}} \tau_j(N,T,d)$ since $\{N^r \setminus \{1\}\}_{r \in R}$ is a partition of $N \setminus \{1\}$. Again, by the *induction hypothesis*,

$$
\begin{aligned}
d_N &= \sum_{i \in N} d_i = \sum_{r \in R} \sum_{i \in N^r} d_i \\
&\geq \sum_{r \in R} |N_r| \tau_{k^r}(N^r, T^r, d^r) \geq |N| \min_{r \in R}\{\tau_{k^r}(N^r, T^r, d^r)\} \\
&= n \min_{j \in N \setminus \{1\}} \tau_j(N,T,d),
\end{aligned}
$$

which is the desired claim.

Step 3 Following the Nuc algorithm, continue the same argument as in Step 1 until $|M \backslash F^0_{T(M)}(j)| = 1$. We obtain that the assigned payoffs to players, except for the top player 1, are non-increasing in the iterations. For every liability problem $(N, T, d) \in \mathcal{L}$, let $\tilde{d}_1 = 0$ and $\tilde{d}_j = d_j$ for every $j \neq 1$. It is clear that the lower bound liability game derived from $(N, T, \tilde{d})$ is the zero-normalization of the lower bound liability game derived from (N, T, d). By the *covariance property* of the Nucleolus[17], $Nuc_1(N, T, d) = d_1 + Nuc_1(N, T, \tilde{d})$. By steps 1 and 2 together with the Nuc algorithm,

$$Nuc_1(N, T, d) = d_1 + Nuc_1(N, T, \tilde{d}) \geq d_1 + A_k,$$

which completes the proof. □

4 Axioms

In this section, we propose three axioms of a compensation scheme for liability problems: lower and upper bounds properties, and a consistency property of compensation for the injury. As mentioned in the Introduction, these properties are inspired from the notions of *tort law* and *case system*. We then show that on the class of liability problems the Nucleolus compensation scheme is the unique compensation scheme that satisfies the three properties.

4.1 Bounds

Before our statement on the first two axioms, we explain how the idea underlying these axioms is related to legal observations. The *Restatement of Torts* (*Second*) states that in joint liability problems, whether there is liability in a cause for the injury among two or more tortfeasors may depend upon the effect of an 'intervening agency' as a 'superseding cause' (see Chapter 16, §433A). Here, an *intervening agency* is interpreted as an agency who takes an 'intervening act'. An *intervening act* is an act or event which occurs after the initial act which would have caused injury or damages. In our model, an intervening agency is a subordinate of agent 1. On the other hand, a *superseding cause* is interpreted as an action or event caused by an intervening agency that affects a series of events so significantly that the end result of this action or event is no longer connected with the original direct cause. In our model, we take into consideration a possibility that subordinates of agent 1 may affect a series of events so significantly that the end result of their act may be no longer connected with the original direct action or event caused by agent 1.

[17] A solution f on a subclass $\mathcal{G}'$ of $\mathcal{G}$ satisfies the *covariance property* if for every $(N, v), (N, w) \in \mathcal{G}'$, $\alpha > 0$, $\beta \in \mathbb{R}^N$, and $w = \alpha v + \beta$, we have $f_i(N, w) = \alpha f_i(N, v) + \beta_i$ for all $i \in N$.

From the viewpoint of the *Restatement of Torts* (*Second*), Chapter 16, §433A, we deal with agent i's additional damage. Recall that this damage is the sum of all damages that would have been avoided when agent i did not exercise any wrongful act. Agent i's additional damage e_i is the sum of the damages of i and all his subordinates.

Let us consider two scenarios of the responsibility for *superseding cause* of the additional damage e_i.

Scenario 1: Agent i, his predecessor $p(i)$ and his subordinates are jointly responsible for the superseding cause of the additional damage e_i.

Scenario 2: Both agent i and his predecessor $p(i)$ are jointly responsible for the superseding cause of the additional damage e_i. Here, the subordinates of agent i are not intervening agencies as a superseding cause of e_i. This is because agents i and $p(i)$ affect e_i so significantly that the additional damage e_i is no longer connected with i's subordinates.

Note that in both scenarios, agent i's predecessor $p(i)$ plays a key role of superseding cause of the additional damage e_i. This is because the additional damage e_i would not have occurred without the wrongful acts of $p(i)$.

We assume that causal relation is determined among the tortfeasors but the causal weights among them cannot be determined. Therefore courts need another criterion for compensation. In this paper, we adopt a 'per-capita criterion'. That is, the criterion requires that e_i should be divided *equally* among the intervening agencies associated with superseding cause of the additional damage. This criterion stems from the stylized fact that in Japan courts employed the notion of per capita criterion in joint and several liability problems before 1990's.

A justification of lower bounds of individual compensations stems from a consideration in an article written by Chief Justice Peaslee of New Hampshire (Peaslee 1934). As stated in Prosser et al. (1984), Peaslee pointed out that in the situation where sequentially wrongful acts by tortfeasors occur, a judge may allow for reduction of every tortfeasor's liability. That is, when the cause of an accident by tortfeasor A reduces the blame of tortfeasor B, the judge may allow for reduction of tortfeasor B's liability.[18] This consideration by Peaslee implies that a court may take individual lower bounds of compensations into consideration for determining compensation for the injury. Thus the lower bounds axiom of individual compensations is derived from applying the *per-capita criterion* to Scenario 1.

On the other hand, a justification of the individual upper bounds axiom stems from the stylized fact that in the real world the notion of 'limited liability' is widely employed and

[18] Peaslee (1934) deals with the notion of potential damage in this context. On the other hand, we deal with the notion of additional damage in the same context.

it determines upper bounds of compensation for injury.[19] Thus the upper bounds axiom of individual compensations is derived from applying the *per-capita criterion* to Scenario 2.

Next, we state the lower/upper bounds axioms formally. According to Scenario 1, the first property sets a uniform lower bound on the compensation to be paid by an individual tortfeasor. The axiom is obtained by considering for tortfeasor $i \in N \setminus \{1\}$ the additional damage $e_i = \sum_{k \in F_T^0(i)} d_k$, where the associated superseding cause is an intervening act by agent i, his subordinates, and his predecessor $p(i)$. That is, all agents in $F_T^0(i)$ and $p(i)$ are jointly responsible for the additional damage e_i. We assume the case of torts where the causal weights among the agents in $F_T^0(i)$ and $p(i)$ cannot be determined, and therefore the per-capita criterion is employed.[20]

The *per-capita contribution* $\tau_i(N,T,d)$ is given by $\tau_1(N,T,d) = \frac{e_1}{|N|}(= \frac{d_N}{n})$ and for every $i \in N \setminus \{1\}$

$$\tau_i(N,T,d) = \frac{e_i}{|F_T^0(i)| + 1}.$$

For every $i \in N \setminus \{1\}$, $\tau_i(N,T,d)$ is the equal division of the additional damage e_i among i, all his subordinates and his predecessor. Also, $\tau_1(N,T,d)$ is the equal division of the additional damage e_1 among 1 and all his subordinates. Note that $p(1) = \emptyset$.

Now, the first axiom requires that every tortfeasor $i \in N$ should pay at least the smallest per-capita contribution $\min_{j \in N} \tau_j(N,T,d)$. The smallest per capita contribution is considered as a guarantee of every injuring party's compensation for the total damage. We also stress that this axiom is a very weak lower bound property. Note that it does not require that i contributes at least its own per-capita contribution, but the minimal per capita contribution over *all* tortfeasors.

Uniform Lower Bound (ULB) A compensation scheme φ on the class of liability problems $\mathcal{L}$ satisfies the uniform lower bound if for every $(N,T,d) \in \mathcal{L}$ and every $i \in N$,

$$\varphi_i(N,T,d) \geq \min_{j \in N} \tau_j(N,T,d).$$

In Example 3.5 it requires that every tortfeasor should pay at least the per capita contribution of agent 5, being 14.

[19] Following Black's Law Dictionary (the tenth edition), the *limited liability* is liability restricted by law or contact.

[20] The per-capita criterion is employed more often in the context of economic problems, for instance, Ni and Wang (2007) use a per capita criterion to introduce the upstream equal responsibility sharing method for polluted river problems.

According to Scenario 2, the second axiom sets for every tortfeasor an individual upper bound on his compensation. This axiom is obtained by considering for tortfeasor $i \in N \backslash \{1\}$ the additional damage $e_i = \sum_{k \in F_T^0(i)} d_k$, where the associated superseding cause is an intervening act by agent i, and his predecessor $p(i)$. Assuming again the case of torts, the causal weights between i and $p(i)$ cannot be determined. By the per-capita criterion, an equal division of e_i between i and $p(i)$ is employed. The second axiom requires that every tortfeasor $i \neq 1$ should be held responsible for at most the half of his additional damage e_i. This axiom also requires that agent 1 should be held responsible for at most his full additional damage e_1, which is equal to the total damage d_N, since agent 1 has no predecessor.

Individual Upper Bounds (IUB) A compensation scheme φ on the class of liability problems $\mathcal{L}$ satisfies the individual upper bounds if for every $(N, T, d) \in \mathcal{L}$ and every $i \in N \setminus \{1\}$, $\varphi_1(N, T, d) \leq d_N$, and

$$\varphi_i(N, T, d) \leq \frac{1}{2} e_i.$$

In Example 3.5 the vector of additional damages is given by $e = (e_1, e_2, e_3, e_4, e_5) = (130,$ 88, 40, 36, 42, 30), so the upper bounds for agents 1 to 6 are, respectively, 130, 44, 20, 18, 21 and 15.

4.2 Consistency

The third and last axiom is a consistency property derived from *case system*. This consistency axiom requires that for every liability problem a compensation scheme should provide an outcome that is consistent with the outcome that the same procedure generates for a different, but similar liability problem. A justification of the consistency axiom is the stylized fact that in the UK and USA courts employ case system (see for instance Ito 1978).

In this paper, we introduce a consistency axiom referred to as *leaf consistency*. Before our statement of leaf consistency, we explain its implication in the legal sense by using a simple example. Leaf Consistency implies legal stability. Imagine a line-tree case with three agents $T = \{(1, 2), (2, 3)\}$. Imagine that agent 3 agrees with his payment according to the compensation scheme, and thus agent 3 wants to leave from T since he need not go to court. When the remaining agents 1 and 2 agree with the departure of 'leaf' agent 3, they make the final appeal to the court. As a result, the judge faces the legal situation represented by the reduced liability problem with the subtree $T' = \{(1, 2)\}$. Leaf consistency means that if the leaf agent agrees with the original compensation and the remaining agents agree with the departure of the leaf agent, the remaining agents have no incentive to make the final appeal to the court. In this sense, leaf consistency guarantees legal stability.

Next, we state leaf consistency formally. Given a compensation scheme φ, consider a liability problem $(N, T, d) \in \mathcal{L}$ and a tortfeasor $i \in L(T)$. A leaf $i \in L(T)$ does not have any subordinates. Imagine that leaf i pays his compensation $\varphi_i(N, T, d)$ and the remaining agents agree with the departure of the leaf i. Furthermore, imagine that leaf i's remaining direct-damage $d_i - \varphi_i(N, T, d)$ is added to the direct damage caused by leaf i's predecessor $p(i)$. As a result, the (modified) direct damage caused by agent $p(i)$ is adjusted to $d^i_{p(i)} = d_{p(i)} + d_i - \varphi_i(N, T, d)$. The *reduced liability problem* is given by $(N \setminus \{i\}, T(N \setminus \{i\}), d^i)$, where (i) $N \setminus \{i\}$ is the set of the remaining tortfeasors, (ii) $T(N \setminus \{i\})$ is the subtree of T on $N \setminus \{i\}$, and (iii) $d^i \in \mathbb{R}^N$ is the vector of damages given by $d^i_{p(i)} = d_{p(i)} + d_i - \varphi_i(N, T, d)$ and $d^i_k = d_k$ for every $k \neq p(i)$ in $N \setminus \{i\}$. Roughly speaking, the reduced liability problem is a liability problem where leaf i leaves, paying his assigned compensation, and the remaining agents make the final appeal to the court.

The consistency axiom requires that the outcome chosen by a compensation scheme for every agent $j \in N \setminus \{i\}$ should be invariant under the departure of a leaf $i \in L(T)$ of the tree T. Therefore, the remaining agents have no incentive to make the final appeal to the court.

Leaf Consistency (Leaf Cons) A compensation scheme φ on $\mathcal{L}$ satisfies leaf consistency if for every $(N, T, d) \in \mathcal{L}$, every $i \in L(T)$ and every $j \in N \setminus \{i\}$

$$\varphi_j(N \setminus \{i\}, T(N \setminus \{i\}), d^i) = \varphi_j(N, T, d).$$

In Example 3.5, let us consider that agent 6 leaves the game and the structure, and he pays his Nucleolus contribution 14. The direct-damage vector of the reduced liability problem is the vector where the direct damage of agent 5 is $d_5 + d_6 - 14 = 42 - 14 = 28$, and the other direct-damages do not change. Leaf consistency requires that the Nucleolus payoffs to the agents 1 to 5 should be invariant under the departure of leaf 6.

5 Characterization of the Nucleolus compensation scheme

In this section, we show that the Nucleolus compensation scheme is the unique compensation scheme on the class $\mathcal{L}$ of liability problems that satisfies **ULB**, **IUB**, and **Leaf Cons**.

Firstly, let us consider a *weak* version of **IUB** in the sense that the individual upper bounds requirement holds for only the leafs.

Weak Individual Upper Bounds (WIUB) A compensation scheme φ on the class of liability problems $\mathcal{L}$ satisfies weak individual upper bounds if for every $(N,T,d) \in \mathcal{L}$ and every $i \in L(T)$,

$$\varphi_i(N,T,d) \leq \frac{1}{2}e_i.$$

In the next lemma, we show that when a compensation scheme satisfies **WIUB** and **Leaf Cons**, then it also satisfies **IUB**.

Lemma 5.1 *Let φ on $\mathcal{L}$ be a compensation scheme that satisfies weak individual upper bounds, and leaf consistency. Then φ satisfies individual upper bounds.*

Proof. We prove that the individual upper bounds requirement holds for every other agent not being a leaf. Let j be a player such that $j \in N \setminus (\{1\} \cup L(T))$. Let $d' \in \mathbb{R}^N$ such that $d'_i = d_i$ for every $i \in N \setminus F_j^0(T)$ and $d'_j = e_j - \sum_{k \in F_T(j)} \varphi_k(N,T,d) \leq e_j$. By subsequently applying **Leaf Cons** for all players in $F_T(j)$, it follows that $(N \setminus F_T(j), T(N \setminus F_T(j)), d') \in \mathcal{L}$ and

$$\varphi_j(N,T,d) = \varphi_j(N \setminus F_T(j), T(N \setminus F_T(j)), d').$$

Since j is a leaf on the subtree $(N \setminus F_T(j), T(N \setminus F_T(j))$ and **WIUB** is satisfied, it follows that

$$\varphi_j(N \setminus F_T(j), T(N \setminus F_T(j)), d') \leq \frac{1}{2}d'_j \leq \frac{1}{2}e_j.$$

Clearly, the individual upper bounds requirement holds for agent 1. $\square$

Next, the following lemma shows that in **ULB** the smallest per-capita contribution is reduced to $\min_{j \in N \setminus \{1\}} \tau_j(N,T,d)$. By the argument appearing in Step 2 of the proof of Lemma 3.6, it is clear that the smallest per-capita contribution $\min_{j \in N} \tau_j(N,T,d)$ is replaced by $\min_{j \in N \setminus \{1\}} \tau_j(N,T,d)$. Therefore we omit the proof.

Lemma 5.2 *A compensation scheme φ on the class of liability problems $\mathcal{L}$ satisfies the uniform lower bound if and only if for every $(N,T,d) \in \mathcal{L}$ and every $i \in N$,*

$$\varphi_i(N,T,d) \geq \min_{j \in N \setminus \{1\}} \tau_j(N,T,d).$$

The next theorem shows that if a compensation scheme satisfies **ULB**, **WIUB**, and **Leaf Cons**, then it must be the Nucleolus compensation scheme. It is the main step to reach our main conclusion.

Theorem 5.3 *Let φ on the class $\mathcal{L}$ of liability problems be a compensation scheme that satisfies uniform lower bound, weak individual upper bounds, and leaf consistency. Then $\varphi(N, T, d) = Nuc(N, T, d)$.*

Proof. See Appendix A.

The following main conclusion says that there is a unique compensation scheme supported by *tort law* bounds on the compensations (**ULB** and **IUB**) and a type of case-system consistency (**Leaf Cons**). This compensation scheme assigns to every liability problem the Rawlsian outcome given by the Nucleolus.

Theorem 5.4 *A compensation scheme φ on the class $\mathcal{L}$ of liability problems satisfies uniform lower bound, individual upper bounds, and leaf consistency if and only if $\varphi(N, T, d) = Nuc(N, T, d)$.*

Proof. By Theorem 5.3, it is clear that if a compensation scheme satisfies **ULB**, **IUB**, and **Leaf Cons**, then the compensation scheme is the Nucleolus compensation scheme.

Conversely, we show that the Nucleolus compensation scheme satisfies **ULB**, **IUB** and **Leaf Cons**. Firstly, from Lemma 3.6 and Step 1 of the Nuc algorithm, it follows that the Nucleolus compensation scheme satisfies that for every $(N, T, d) \in \mathcal{L}$ and every $i \in N$, $\varphi_i(N, T, d) \geq \min_{j \in N \setminus \{1\}} \tau_j(N, T, d)$, which is equivalent to **ULB** by Lemma 5.2. For $|N| = 2$, it follows by calculation that the Nucleolus compensation scheme satisfies **Leaf Cons**. For $|N| \geq 3$ and every $i \in L(T)$, let $N' = N \setminus \{i\}$, $T(N')$ be the subtree of T on N', and $d' \in \mathbb{R}^{N'}$ be such that $d'_j = d_j$ for $j \in N' \setminus \{p(i)\}$ and $d'_{p(i)} = d_{p(i)} + d_i - Nuc_i(N, T, d)$. By the same argument as in the proof of Theorem 5.3, the lower-bound liability game (N', v'_L) derived from the reduced liability problem $(N', T(N'), d')$ is the *Davis-Maschler reduced game*[21] of (N, v_L) on N' with respect to $Nuc(N, T, d)$. It follows by this observation that for $|N| \geq 3$ the Nucleolus compensation scheme satisfies **Leaf Cons**. By the Nuc algorithm, it is clear that the Nucleolus compensation scheme satisfies **WIUB**. From the fact that the Nucleolus compensation scheme satisfies **Leaf Cons** and **WIUB**, it follows by Lemma 5.1 that **IUB** is satisfied. □

Finally, we check logical independence of the three axioms.

[21] For the definition of the *Davis-Maschler reduced game*, see Appendix A.

- Let φ^1 be the compensation scheme given by $\varphi_i^1(N,T,d) = \min_{j\in N} \tau_j(N,T,d)$ and $\varphi_1^1(N,T,d) = d_N - \sum_{i\neq 1} \varphi_i^1(N,T,d)$. Then φ^1 satisfies **ULB** and **IUB**, but not **Leaf Cons**.

- Let φ^2 be the compensation scheme obtained by applying the *equal division solution*, which is axiomatized for TU games (see for instance van den Brink 2007, and Casajus and Huettner 2014), given by $\varphi_i^2(N,T,d) = \frac{v_L(N)}{|N|} = \frac{d_N}{|N|}$ for every $i \in N$ and every $(N,T,d) \in \mathcal{L}$. Then φ^2 satisfies **ULB** and **Leaf Cons**, but not **IUB**.

- Let φ^3 be the *Shapley* compensation scheme. Then φ^3 satisfies **IUB** and **Leaf Cons** (see Katsev 2009, Theorem 6.3.1), but not **ULB**.

6 Comparison with the Shapley compensation scheme

As mentioned in Section 3, the class of lower-bound liability games is equivalent to the class of peer-group games. On the class of peer-group games, Katsev (2009, Theorem 6.3.5) provides a characterization of the Shapley value, namely the Shapley value is the unique solution that satisfies *efficiency*, *leaf consistency*, *weak veto property*, *top monotonicity* and *independence of non-subordinates*.[22] All these five axioms are logically independent. The weak veto property states that player i pays at least the same as any other player when $d_j = 0$ for every $j \neq i$. Top monotonicity states that top player 1 pays at least the same amount as any other player. Independence of non-subordinates states that if the damage of only one player i changes, then the compensations to be paid by the subordinates of i do not change. From Katsev's axiomatization of the Shapley value on the class of peer-group games, it follows immediately that on the class of liability problems the Shapley compensation scheme is the unique compensation scheme that satisfies leaf consistency, the weak veto property, top monotonicity and independence of non-subordinates.

We are now ready to compare the Shapley compensation scheme and the Nucleolus compensation scheme by axioms involved in their axiomatizations. Let us consider the list of properties involved in axiomatization of the Shapley and Nucleolus compensation schemes appearing in Katsev (2009, Theorem 6.3.5) and our paper. The Shapley compensation scheme satisfies all the properties appearing in the list except for *uniform lower bound*, and the Nucleolus compensation scheme satisfies all the properties appearing in the list except for *independence of non-subordinates*. Thus, by comparing axioms involved

[22] Ferey and Dehez (2016, Proposition 1) axiomatized the Shapley value on the class of lower-bound liability games derived from liability problems with line-tree structure. In this axiomatization, leaf consistency, weak veto property and top monotonicity are replaced by the single axiom of *zero immediate damage*, saying that a player i pays the same as its predecessor $p(i)$ if the damage of the predecessor is equal to zero.

in the axiomatization of the Shapley and Nucleolus compensation schemes, the difference between the two compensation is uncovered. Furthermore, the difference between the outcomes chosen by the two compensation schemes might be big even in liability problems with line-tree structure: Let n be a leaf of a line-tree with n tortfeasors. By the equation (3.1), leaf n pays $\frac{d_n}{n}$ as the outcome chosen by the Shapley compensation scheme. On the other hand, when d_n is relatively small compared to the damages of leaf n's superiors, leaf n might pay $\frac{d_n}{2}$ as the outcome chosen by the Nucleolus compensation scheme. When d_n is relatively large compared to the damages of leaf n's superiors, leaf n might pay $\frac{d_n}{n}$ as the outcome chosen by the Nucleolus compensation scheme. Therefore, while leaf n is always held responsible for a share $\frac{1}{n}$ of its own damage as the outcome chosen by the Shapley compensation scheme, he is held responsible for a share, which might be taken from $\frac{1}{n}$ to $\frac{1}{2}$, of its own damage as the outcome chosen by the Nucleolus compensation scheme.

7 Concluding remarks

In this paper, we axiomatized the Nucleolus compensation scheme as a compensation scheme for liability problems, where causation of the cumulative injury results from multiple sequences of wrongful acts taken by different injuring-parties. It appears that the Nucleolus compensation scheme of a liability problem can be simply computed by using an algorithm for the Nucleolus of a corresponding liability game. Three axioms involved in the axiomatization of the Nucleolus compensation scheme are introduced: a uniform lower bound, an individual upper bound and an axiom on case-system consistency. The three axioms are derived from stylized facts concerning tort law and case system in practice. The outcome chosen by the Nucleolus compensation scheme lexicographically minimizes the corresponding dissatisfaction of injuring parties in the sense of cost savings. So, it is likely that the injured party can receive compensation as soon as possible without facing with injuring parties' final appeal to the court. This is a very appealing property of the Nucleolus compensation scheme.

We conclude by discussing what happens when the population of the tortfeasors is increasing. Suppose that a new tortfeasor arrives at the end of a branch of the tree for a liability problem, so he is added as a new leaf to one of the leafs of the existing tree. One may wonder whether every original tortfeasor pays in the new situation at least the same as in the original situation. If the answer for this question is negative, then it might be that some of the original tortfeasors have an incentive to increase the population of the tortfeasors, which leads to an increase of the total damage. From this aspect, it is appropriate to require that a compensation scheme satisfies *leaf population monotonicity*, stating that when a new tortfeasor arrives, in the new situation every original tortfeasor pays at least the same as in the original situation. If this property is satisfied, then no original tortfeasor has an incentive to increase the population of the tortfeasors.

From equation (3.1) it follows immediately that the Shapley compensation scheme satisfies the leaf population property. When a new tortfeasor arrives, then his (additional) damage is equally shared among himself and his superiors with no effect on how the other damages are shared. The Nucleolus compensation scheme satisfying leaf population monotonicity follows from a result in Katsev (2009, Theorem 6.4.4), who shows that when d and d' are such that $d'_i \geq d_i$ for every $i \in N$, then for the Nucleolus compensation scheme every tortfeasor pays at d' at least the same as at d. So, for a given tree with a fixed set of tortfeasors, the compensations to be paid by the tortfeasors are non-decreasing in the damages.[23] Now, suppose that a new tortfeasor is added. Then, according to the Nucleolus compensation scheme this tortfeasor pays at most half of his damage and leaf consistency says that the others have to pay according to the original situation, but with the damage of the predecessor of the new leaf replaced by the sum of his own damage and the remaining part of the damage of the new tortfeasor. So, the new situation reduces to the old situation, but with higher damage for the predecessor of the new leaf. So, by the result of Katsev (2009, Theorem 6.4.4) it follows that every original tortfeasor pays at least the same as before. Therefore, the Nucleolus compensation scheme satisfies leaf population monotonicity. It might be interesting to study the class of compensation schemes that satisfy leaf population monotonicity, uniform lower bound, and individual upper bounds, which we leave for future research.

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[23] In the literature, this monotonicity is known as *resource monotonicity*, but here the resources are the damages. For the details of resource monotonicity, see for instance Thomson (2011).

Appendix A

Proof of Theorem 5.3

We show that for every liability problem, **ULB**, **WIUB**, and **Leaf Cons** yield the outcome computed by the Nuc algorithm. For notational simplicity, let $x = \varphi(N, T, d)$. We consider liability problems with $|N| = 2$ and $|N| \geq 3$, respectively

Case 1 $|N| = 2$. Consider a liability problem $(N, T, d) \in \mathcal{L}$ with $|N| = 2$. Then, by **ULB** it follows that $x_2 \geq \tau_2(N, T, d) = \frac{1}{2}d_2$ and by the **WIUB** that $x_2 \leq \frac{1}{2}e_2 = \frac{1}{2}d_2$. Hence $x_2 = \frac{1}{2}d_2$, and thus $x_1 = d_1 + \frac{1}{2}d_2$. This is equal to the outcome computed by the Nuc algorithm.

Case 2 $|N| \geq 3$. We show that the claim holds by induction on $|N|$.

Induction basis. We show that the claim holds for $|N| = 3$.

For a liability problem $(N, T, d) \in \mathcal{L}$ with $|N| = 3$, there are two possibilities: either (N, T) is a line-tree (one branch) or (N, T) is a tree with two branches.

Firstly, we consider a line-tree (N, T) with $T = \{(1, 2), (2, 3)\}$. We have two possibilities **(i)** and **(ii)**: either $\frac{d_3}{2} \leq \frac{d_2+d_3}{3}$ or not.

(i) $\frac{d_3}{2} \leq \frac{d_2+d_3}{3}$. By **ULB** and **WIUB**, it follows that $x_3 = \frac{d_3}{2}$. Next, it follows by **ULB**, **WIUB** for leaf 2, and **Leaf Cons** in the subtree after removing agent 3 that $x_2 = \frac{d_2+d_3-x_3}{2}$, and finally $x_1 = d_N - x_2 - x_3$. So x is equal to the outcome computed by the Nuc algorithm.

(ii) $\frac{d_3}{2} > \frac{d_2+d_3}{3}$. By **ULB**, we must have that $x_j \geq \frac{d_2+d_3}{3}$ for $j = 1, 2, 3$. **WIUB** for leaf 3 requires that $x_3 = \frac{d_3}{2} - c$ for some $c \geq 0$, and thus $\frac{d_3}{2} - c \geq \frac{d_2+d_3}{3}$. This yields $c \leq \frac{d_3}{6} - \frac{d_2}{3}$. By **ULB**, **WIUB** for leaf 2 and **Leaf Cons** in the subtree after removing agent 3, it follows that $x_2 = \frac{d_2+d_3-x_3}{2}$, and thus $x_2 = \frac{d_2+d_3-x_3}{2} \geq \frac{d_2+d_3}{3}$. Substituting $x_3 = \frac{d_3}{2} - c$ in this inequality gives $c \geq \frac{d_3}{6} - \frac{d_2}{3}$ and thus $c = \frac{d_3}{6} - \frac{d_2}{3}$. This implies that $x_2 = x_3 = \frac{d_2+d_3}{3}$ and thus $x_1 = d_1 + \frac{d_2+d_3}{3}$. Again x is equal to the outcome computed by the Nuc algorithm.

Next, we consider that (N, T) is a tree with two branches, that is, $T = \{(1, 2), (1, 3)\}$. Without loss of generality, let $d_2 \geq d_3$. By **ULB** and **WIUB** for leaf 3 it follows that $x_3 = \frac{d_3}{2}$. Consider the reduced liability problem $(N', T(N'), d')$, where $N' = \{1, 2\}$, $T(N')$ is a line-tree $\{(1, 2)\}$, and $d' = (d_1 + d_2 - x_3, d_2)$. Again, by **ULB** and **WIUB** for leaf 2 together with **Leaf Cons**, $x_2 = \frac{d_2}{2}$, and $x_1 = d_1 + \frac{d_2}{2} + \frac{d_3}{2}$. Also in this case we have that x is equal to the outcome computed by the Nuc algorithm.

Induction hypothesis. Fix an arbitrary n such that $n > 3$. Suppose that for any liability problem $(N', T', d') \in \mathcal{L}$ with $3 \leq |N'| \leq n - 1$, $\varphi(N', T', d') = Nuc(N', T', d')$.

Induction step. We show that for any liability problem $(N', T', d') \in \mathcal{L}$ with $|N'| = n$, $\varphi(N', T', d') = Nuc(N', T', d')$.

We consider two possibilities **(I)** and **(II)**: player j with minimal $\tau_j(N,T,d)$ over N, or equivalently minimal $\tau_j(N,T,d)$ over $N \setminus \{1\}$ (by Lemma 5.2), is either a leaf or not.

(I) There exists $i \in L(T)$ such that $\tau_i(N,T,d) = \min_{j\in N} \tau_j(N,T,d)$, which is equal to $\min_{j\in N\setminus\{1\}} \tau_j(N,T,d)$ by Lemma 5.2. Fix this leaf i (if there are multiple, fix an arbitrary one). By **ULB** and **WIUB** for leaf i, it follows that $x_i = \frac{d_i}{2}$. This is also the outcome for i as computed by the Nuc algorithm, and thus $x_i = Nuc_i(N,T,d)$. Let $N' = N \setminus \{i\}$, $T(N')$ be the subtree of T on N', and $d' \in \mathbb{R}^{N'}$ be such that $d'_j = d_j$ for $j \in N' \setminus \{p(i)\}$ and $d'_{p(i)} = d_{p(i)} + d_i - x_i = d_{p(i)} + d_i - Nuc_i(N,T,d)$.

Consider the lower-bound liability game (N, v_L) derived from the liability problem (N,T,d). Then the lower-bound liability game derived from the reduced liability problem $(N', T(N'), d')$ is the game (N', v'_L), where v'_L is given by setting for $S \subseteq N'$,

$$v'_L(S) = \begin{cases} v_L(S \cup \{i\}) - Nuc_i(N,T,d) & \text{if } p(i) \in S, \\ v_L(S) & \text{otherwise.} \end{cases}$$

It can be shown that (N', v'_L) is the *Davis-Maschler reduced game* of (N, v_L) on N' with respect to $Nuc(N,T,d)$.[24] By this observation and the fact that the Nucleolus satisfies the *Davis-Maschler consistency*[25] , it holds that for every $j \in N'$ that $Nuc_j(N,T,d) = Nuc_j(N', T(N'), d')$. Furthermore, for every $j \in N'$, we have by *induction hypothesis* that $\varphi_j(N', T(N'), d') = Nuc_j(N', T(N'), d')$. Hence, for every $j \in N'$ it holds that $Nuc_j(N,T,d) = \varphi_j(N', T(N'), d')$. From **Leaf Cons**, it now follows that for every $j \in N'$, $x_j = \varphi_j(N', T(N'), d') = Nuc_j(N,T,d)$.

(II) For every $i \in L(T)$, it holds that $\min_{j\in N} \tau_j(N,T,d) < \tau_i(N,T,d)$, or equivalently $\min_{j\in N\setminus\{1\}} \tau_j(N,T,d) < \tau_i(N,T,d)$ by Lemma 5.2. Let k be such that $\tau_k(N,T,d) = \min_{j\in N\setminus\{1\}} \tau_j(N,T,d)$ and take some $i \in L(T) \cap F^0_T(k)$. Thus i is subordinate of k**.** Since $i \neq k$, we have that also $p(i) \in F^0_T(k)$. From the Nuc algorithm we obtain that

$$Nuc_i(N,T,d) = Nuc_{p(i)}(N,T,d) = \tau_k(N,T,d). \tag{7.2}$$

[24] For a game $(N,v) \in \mathcal{G}$, a vector $x \in \mathbb{R}^N$ and non-empty subset N' of N, the *Davis-Maschler reduced game* (Davis and Maschler 1965) on N' with respect to (N,v) and x is the game $(N', w^x) \in \mathcal{G}$ defined by setting for all $S \subseteq N'$,

$$w^x(S) = \begin{cases} v(N) - \sum_{i\in N\setminus N'} x_i & \text{if } S = N', \\ \max_{T\subseteq N\setminus N'} \left[v(S\cup T) - \sum_{i\in T} x_i\right] & \text{if } S \neq N', \emptyset, \\ 0 & \text{if } S = \emptyset. \end{cases}$$

[25] A solution f on a subclass $\mathcal{G}'$ of $\mathcal{G}$ satisfies the *Davis-Maschler consistency* (Davis and Maschler 1965) if for all $(N,v) \in \mathcal{G}'$ and every non-empty $N' \subset N$ it holds that $f_i(N,v) = f_i(N', w^x)$ for all $i \in N'$, where $x = f(N,v)$.

By **ULB**, we have that $x_i \geq \tau_k(N,T,d) = Nuc_i(N,T,d)$. Suppose that $x_i > Nuc_i(N,T,d)$. Let $N' = N \setminus \{i\}$, $T(N')$ be the subtree of T on N', and $d' \in \mathbb{R}^{N'}$ be such that $d'_j = d_j$ for $j \in N' \setminus \{p(i)\}$ and $d'_{p(i)} = d_{p(i)} + d_i - x_i = d_{p(i)} + d_i - Nuc_i(N,T,d)$. By **Leaf Cons**, we have that

$$x_{p(i)} = \varphi_{p(i)}(N', T(N'), d') \tag{7.3}$$

and by *induction hypothesis*,

$$\varphi_{p(i)}(N', T(N'), d') = Nuc_{p(i)}(N', T(N'), d'). \tag{7.4}$$

Let $d'' \in \mathbb{R}^{N'}$ be such that $d''_j = d_j$ for $j \in N' \setminus \{p(i)\}$, and $d''_{p(i)} = d_{p(i)} + d_i - Nuc_i(N,T,d)$. Since $d'_{p(i)} < d''_{p(i)}$ by the assumption that $x_i > Nuc_i(N,T,d)$, it follows from applying the Nuc algorithm that

$$Nuc_{p(i)}(N', T(N'), d') < Nuc_{p(i)}(N', T(N'), d''). \tag{7.5}$$

By the fact that the lower bound liability game derived from the reduced liability problem $(N', T(N'), d'')$ is the *Davis-Maschler reduced game* of (N, v_L) on N' with respect to $Nuc(N,T,d)$, it follows with the *Davis-Maschler consistency* (see Footnotes 24 and 25) that

$$Nuc_{p(i)}(N', T(N'), d'') = Nuc_{p(i)}(N,T,d). \tag{7.6}$$

From the (in)equalities (7.2)-(7.6) it follows that

$$x_{p(i)} < Nuc_{p(i)}(N,T,d) = \tau_k(N,T,d),$$

which contradicts **ULB**. Therefore, for every $i \in L(T) \cap F^0_T(k)$, $x_i = Nuc_i(N,T,d)$. For every $i \in L(T) \cap F^0_T(k)$, let $N' = N \setminus \{i\}$. Then it follows from the same argument as mentioned above and from **Leaf Cons** that for every $j \in N'$, $x_j = Nuc_j(N,T,d)$. □

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