Pivotal Persuasion*

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Abstract

We study a persuasion game between a sender and a group of voters. When the sender can access any information structure, he can always persuade the group to make his preferred decision. When the sender is restricted to using only minimal winning coalitions, he sends private and correlated signals to take advantage of the voters' heterogenous preferences. The optimal persuasion structure in the latter case induces multiple winning coalitions for the sender's preferred action. Interestingly, some of the winning coalitions involves voters who are not the easiest to persuade. The insight from pivotal persuasion is then applied to understand (i) the use of non-monotone voting rules, and (ii) the benefit of private persuasion when voters' signals are independent.

Keywords: Bayesian Persuasion, Information Design, Private Persuasion, Strategic Voting.

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1 Introduction

This paper considers an information design problem where an interested party (information designer) provides information to influence the outcome of a vote by N voters. Examples of this type of group persuasion include an interest group lobbying a legislative committee, a CEO trying to convince a board of directors, or a candidate seeking the support of different constituents.

In our model a group of N voters must decide between two actions: a and b. Action b is the default choice; action a is chosen only if it receives at least K votes. There are two states: A and B. Each voter prefers action a if and only if she believes that the probability of state A is above a cutoff belief. An interested party, henceforth the sender, controls the information of the voters by choosing the state-dependent distribution of the voters' signals. We adopt the information design approach: The signal structure is fixed and observed by all voters after it is chosen. Voting occurs after each voter observes her own private signal. The objective of the sender is to maximize the probability that his preferred action is chosen.

In principle the sender can persuade with a public signal. But in many cases the sender may opt to send different information to different voters. Interest groups, for example, typically lobby legislators privately, and political campaigns sometimes target different constituents with different messages. Our primary interest is to understand *how* the sender can do *strictly* better by employing private persuasion. It is well known that under any non-unanimity rule it is a Nash equilibrium for every voter to vote for the sender's preferred action. But relying on non-pivotal events is not the only channel. In this paper we study the case of *pivotal persuasion*, where the sender is restricted to equilibria in which his preferred action is never selected by a non-minimal coalition. We show that in this environment the sender may exploit a limitation on strategic voting: while in equilibrium each voter votes as if she knows that her vote is pivotal, she would not be able to infer who else are also voting for the alternative. By sending private signals, the sender can create minimum multiple winning coalitions to take advantage of the voters' heterogenous preferences. Under the optimal persuasion structure the incentive-compatibility constraint is binding for every voter who votes for the sender's preferred action. By contrast, under public persuasion only the obedient constraint of the most unwilling voter is binding, while everyone else's in the minimum winning coalition is slack. An interesting feature of the optimal information structure is that a voter may nevertheless be persuaded to voter for the preferred action even if she is not among those who are the easiest to persuade. Interestedly, under pivotal persuasion, the sender cannot achieve a persuasion probability close to one if the all voters prefer b ex ante.

Our analysis takes several steps. Following Taneva (2014) we apply the revelation principle to show that we can focus on obedient mechanisms where each voter follows the state-dependent binary recommendation from the sender. Consequently, the information design problem is equivalent to one where the sender chooses a probability distribution over winning coalitions in each state, subject to the incentive compatibility constraint that each voter i's posterior belief, conditional on being pivotal, exceeds her cutoff belief. This problem is hard to solve directly as there are C_K^N minimal winning coalitions for each action in each state. Our main technical contribution is to show that instead of working with the large number of winning coalitions, the sender can solve a much simpler problem that involves choosing the pivotal probability of each voter subject to an overall "budget constraint" that the total probability of voting for the preferred action of the sender cannot exceed an upper bound.

Throughout the paper we assume the sender has perfect commitment power and full access to all information structures. In reality, a sender may face significant restrictions. For example, a lobbyist may engage multiple consulting firms or NGOs and rely on their reputations to produce credible reports to target different legislators. At the commission stage, the lobbyist will have certain influence over the scope and methodology. He can choose a more critical methodology in order to persuade a more skeptical legislator. He may affect the correlation between the conclusions of two reports by having them use similar or different methodologies and scopes of studies. Still, the control is unlikely to be perfect. Incorporating realistic constraints on information structure, however, is hard as the revelation principle may no longer hold, and the resulting constrained information design problem can be messy. Our view is to interpret the sender as a *metaphorical* information designer as in Bergemann and Morris (2017). The commitment solution bounds what could happen under any communication protocol and with any sets of available persuasion technologies.

One may question the relevance of optimal pivotal persuasion given that the sender could apparently achieve more by resorting to non-pivotal events. There are two ways to understand our results. First, given how easily the sender can manipulate majority rules through non-pivotal events, it is of interest to search for decision rules less susceptible to outside influence. Our results show that non-monotone voting rules that adopt an alternative only when it is supported by a specific number of votes are still manipulable through a different channel, although in this case the sender cannot achieve a persuasion probability close to one unless all voters prefer the sender's preferred alternative ex ante. Second, and perhaps more importantly, because many features that are important in reality are left out, it is inappropriate to use the current model to compare the effectiveness of different mechanisms of private persuasion. The purpose of the exercise is to use a tractable model to gain insights about how the sender can exploit private signals. What we learn in turn helps us better understand more complex situations. We illustrate the last point by briefly considering the case where the sender is constrained to using independent signals. We identify a condition under which, despite the restriction, the sender can use private signals to achieve a higher persuasion probability.

Related Literature. Our work belongs to the growing literature about information design and Bayes correlated equilibria (BCE) developed by Bergemann and Morris (2013, 2016a,b, 2017) and Taneva (2014). As in the aforementioned papers, we use a linear programming approach to characterize the sender's optimal implementable information structures. Our contribution is to analyze the benefit of private persuading in a strategic voting environment.

Our paper is also related to the Bayesian persuasion literature, which uses a belief-based approach to analyze information design problems. In a one-sender-one-receiver model, Kamenica and Gentzkow (2011) show that the sender's problem is simply to split the receiver's belief about the state subject to the standard Bayes' rule (or Bayes plausible condition).[1] Therefore, the sender's optimal payoff, as a function of prior distributions of states, must belong to the concavification of the set of payoffs of the sender in the absence of information design. Alonso and Camara (2016) consider the case of public persuasion in a Bayesian persuasion game between one sender and multiple voters. They show that when there are more than two payoff-dependent states, the optimal public signal may give rise to multiple distinct winning coalitions that adopt the sender's preferred action.[2] In our model, because there are only two payoff-dependent states, and voters have common preference conditional on each state, the optimal public signal always targets the K voters who are the easiest to persuade. In our model the sender can select any correlated information structure. Wang (2015) considers private persuasion with independent and identically distributed (i.i.d.) signals. Bardhi and Guo (2017) investigate persuasion under unanimous rule when voters' preferences are correlated. Also see Shimoji (2016) for a persuasion model where voters have heterogenous prior beliefs. However, the belief-based *concavification* approach loses its tractability in our private persuasion context. When multiple voters receive private signals and then interact strategically, voters form private beliefs, and their higher-order beliefs are also payoff-relevant. See Mathevet, Perego, and Taneva (2017) who proposes an epistemic approach to information design.

There is a large body of literature considering information aggregation through strategic voting since the seminal contributions of Austen-Smith and Banks (1996), Feddersen and Pe-

[1] See also Brocas and Carrillo (2007) and Rayo and Segal (2010).

[2] Schnakenberg (2015) considers a cheap talk model where a sender publicly persuades voters. He also finds that in equilibrium, the sender randomizes over multiple minimum winning coalitions.

sendorfer (1997), Feddersen and Pesendorfer (1998), and Li, Rosen, and Suen (2001).[3] The key insight of this literature is that, despite voting simultaneously, each voter votes as if she knows that her vote is pivotal. Here we show that the sender can adjust the probabilities of different winning coalitions to increase the influence on the voters whose preferences are closer to her own.

The rest of the paper is organized as follows. We present the model in section 2. Section 3 provides some preliminary results. Section 4 characterizes the information structure in the optimal pivotal persuasion. In section 5, we discuss information design when the signals are independent across voters. Section 6 concludes. Omitted proofs are left in the appendix.

2 Model

A group of N voters needs to decide between $x \in \{a, b\}$. The collective decision is made according to a K-majority rule where action a is chosen if and only if it receives K or more votes. We focus on non-unanimity rule, so $K < N$. The payoff of each action depends on a binary state

$$\omega \in \{A, B\}.$$

If the final decision is b, each voter i obtains 0; otherwise, her payoff is state-dependent:

$$\begin{cases} 1 & \text{if } \omega = A, \\ -l_i & \text{if } \omega = B, \end{cases}$$

where $l_i > 0$ measures voter i's cutoff belief—the limit beyond which she prefers action a. We assume

$$l_1 < l_2 < ..... < l_N,$$

so that voter 1 is the easiest to persuade and voter N the hardest. Voters are uncertain about the state and share a common prior belief $\mu_0 = \Pr(\omega = A)$. We assume that $\mu_0 = 0.5$ and $l_K > 1$, so that fewer than K voters prefer a at the prior belief.[4]

[3] This literature has been enriched in many dimensions: Gerardi and Yariv (2007) compare various voting rules when voters are allowed to deliberate before casting their votes. Jackson and Tan (2013) allow voters to consult experts before voting and examine how disclosure and voting vary with different voting rules and with the signal precision of the experts. Li (2001), Persico (2004), Gerardi and Yariv (2008) and Cai (2009) assume that voters (or committee members) endogenously collect their information individually.

[4] The assumption of the prior belief being 0.5 is without loss of generality, as what matters is that $l_K > \mu_0/(1-\mu_0)$ and one can always normalize $\hat{l}_i = l_i(1-\mu_0)/\mu_0, \forall i$. Similarly, one can extend our analysis to the settings where voters have heterogenous prior beliefs.

A sender, who prefers a regardless of the state, tries to influence the voting outcome by controlling the information of the voters.

Definition 1. *An* information structure *consists of a set of finite realization spaces* $\{S^i\}_{i=1,2,...N}$ *and a pair of probabilities* $\{\pi(\cdot|\omega)\}_{\omega=A,B} \in \Delta(S)$*, where* S^i *denotes the realization space for voter i and* $S = \times_{i=1,2,..N} S^i$.

An information structure (S, π) specifies a state-contingent distribution over *signal profiles* $s = (s_1, s_2, ..., s_N)$, where

1. voter i can only observe her own signal $s_i \in S^i$, and
2. $\pi(s|\omega)$ denotes the probability that signal profile s is realized in state ω.

The game proceeds as follows. First, the sender chooses an information structure π. Then ω and s are realized. Finally, the voters vote after observing their private signals to maximize their expected utilities respectively.

Formally, the K-majority rule, the state space, the voters's action space, preference profiles and common prior μ_0, and an information structure (S, π) together define a Bayesian voting game. A mixed strategy of voter i is a function $\sigma_i : S^i \to [0,1]$ that maps each of her private signal s_i to the probability of voting for a.

Definition 2. *A strategy profile* σ *is an equilibrium of* (S, π) *if (i)* σ *is a Bayesian Nash equilibrium and (ii) if* $\sum_{s_i \in S^i} \sigma_i(s_i) > 0$*, then voter* i *is pivotal for action a with strictly positive probability.*

The second condition rules out trivial equilibria where more than K voters always vote a. We focus on "the sender-preferred equilibrium" when there are multiple equilibria.[5]

3 The Information Design Problem

We call an information structure (S, π) *binary* if each $S^i = \{a, b\}$. Under a binary information structure, each voter simply receives a recommendation about how she should vote. We call an equilibrium of $(\{a, b\}^N, \pi)$ in which every voter i follows the sender's recommendation (i.e., $\sigma_i(a) = 1 - \sigma_i(b) = 1$) an *obedient equilibrium.*

[5] The equilibrium selection assumption in our setting, requiring coordination among multiple voters, is stronger than the one in Kamenica and Gentzkow (2011), where only one receiver is involved. Goldstein and Huang (2016) and Inostroza and Pavan (2017) study information design in coordination games. In contrast to us, they focus on the designer's lest preferred equilibrium. See Bergemann and Morris (2017) and Mathevet, Perego, and Taneva (2017) for discussion on the equilibrium selection in information design problems.

Lemma 1. *For any information structure* (S,π)*, if there is an equilibrium* σ *of* (S,π) *in which* a *is chosen with probability* Q*, then there exists a binary signal distribution* π' *such that* a *is chosen with probability* Q *in an obedient equilibrium of* $\left(\{a,b\}^N,\pi\right)$.

Lemma 1 follows the revelation principle. Given the assumption that the sender can select his preferred equilibrium, we can without loss of generality consider only obedient equilibria. Let $S_a^* \equiv \{s : |s_i = a| = K\}$ denote the set of minimal a-winning signal profiles, and $S_b^* \equiv \{s : |s_i = a| = K-1\}$ the set of minimal b-winning signal profiles. For $x \in \{a,b\}$, let $S_{i,x}^* \equiv \left\{s \in S_x^* | s_i = x\right\}$ denote the subset of S_x^* that involves i voting for x.

It is well known that in equilibrium each voter votes as if her vote is pivotal, so the relevant belief for a voter is her *pivotal belief*, i.e., the posterior belief conditional on being pivotal. If a voter receives a recommendation to vote for x, it will update her pivotal belief according to Bayes' rule:

$$\mu_i(x) = \frac{\sum_{s\in S_{i,x}^*} \pi(s|A)}{\sum_{s\in S_{i,x}^*} \pi(s|A) + \sum_{s\in S_{i,x}^*} \pi(s|B)}$$

whenever it is well-defined. In an obedient equilibrium, a voter is willing to follow recommendation a if and only if $\mu_i(a) \geq l_i/(1+l_i)$, while she is willing to follow recommendation b if either $\mu_i(b) \leq l_i/(1+l_i)$ or $\mu_i(b)$ is not well-defined (she is never pivotal). The goal of the sender is to choose an information structure to maximize the equilibrium *persuasion probability*, i.e., the probability that a is elected.

A simple strategy for the sender is to *persuade publicly* by allowing voters to observe the realized signal of a public experiment. It is easy to see that it is optimal for the sender to target the first K voters, who are the easiest to persuade. In state A, they receive recommendation a for sure; while in state B, they receive recommendation a with probability $1/l_K$ and recommendation b with probability $1-1/l_K$. Other voters are always given the recommendation to vote for b.

The following proposition shows that when the sender can choose any information structure, there is a strict obedient equilibrium in which a is chosen with probability arbitrarily close to one.[6]

Proposition 1 (Virtually Full Manipulation)**.** *Suppose the sender can select any information structure. Then for any* $\epsilon > 0$ *there exists* (π, S) *such that there is a* strict *obedient equilibrium in which* a *is adopted with probability greater than* $1-\epsilon$.

Proof. When the state is B, with probability $1-\epsilon$ all voters are asked to vote for a, and with probability ϵ the sender randomly selects one of the minimal b-winning coalitions with equal

[6] Bardhi and Guo (2017) independently obtain a similar result.

probability. When the state is A, the sender randomly selects one of the minimal a-winning coalitions with equal probability. □

Proposition 1 says that when the sender commits to *any* signal, he can almost fully manipulate the collective decision regardless of the voters' prior beliefs and preference profiles! The intuition is simple. Non-pivotal events, no matter how likely, do not affect voters' incentives. The sender, therefore, can convince a voter by making her pivotal with an arbitrary small but positive probability. The result in Proposition 1 looks like it is just one step away from the well-known trivial equilibrium in the strategic voting game, but it is more disturbing. Since it is supported by a *strict* equilibrium, it will survive if the game is perturbed or if voters need to pay a small cost to vote. The result, however, depends crucially on the assumption that the sender can commit to any information structure. We will return to this issue in Section 5.

4 Optimal Pivotal Persuasion

In this section, we focus on the case of *pivotal persuasion* where the sender is restricted to binary-information structures that recommend a to at most K voters. The purpose of this exercise is twofold. First, from the proof of Proposition 1, it is clear that any decision rule whereby the sender's preferred action can be selected by a coalition in which no voter is pivotal is almost fully manipulable. Hence, non-monotone rules whereby a is selected when it receives exactly K votes is the first step in the search for less manipulable decision rules. Under such rules, it is optimal for the sender to adopt pivotal persuasion strategies. Second, because the pivotal-persuasion problem is tractable, we can clearly identify alternative channels the sender can exploit. The insights are useful in understanding more complex constrained information design problems.

Let Π^{piv} denote the set of state-contingent signal distributions over $\{a,b\}^N$ such that $\pi(s|A) = \pi(s|B) = 0$ for any s where $|s_i = a| > K$. The sender's optimal persuasion problem is

$$\max_{\pi\in\Pi^{piv}} 0.5 \sum_{s\in S_a^*} \pi(s|A) + 0.5 \sum_{s\in S_a^*} \pi(s|B) \tag{P-0}$$

such that for every voter i:

$$\sum_{s\in S_{i,a}^*} \pi(s|A) \geq l_i \sum_{s\in S_{i,a}^*} \pi(s|B); \tag{1}$$

$$\sum_{s\in S_{i,b}^*} \pi(s|A) \leq l_i \sum_{s\in S_{i,b}^*} \pi(s|B). \tag{2}$$

The first (1) is the obedient constraint for voting a; the second (2) is the obedient constraint for voting b.

Since both the sender and voters prefer a in state A, raising $\pi(s|A)$ for any $s \in S_a^*$ both increases the persuasion probability and relaxes the voters' obedient constraints. It follows that $\sum_{s\in S_a^*}\pi(s|A) = 1$ and (2) is not binding. We can therefore rewrite the sender's problem as:

$$Q_B = \max_{\pi\in\Pi^{piv}} \sum_{s\in S_a^*} \pi(s|B) \tag{P-1}$$

such that $\sum_{s\in S_a^*}\pi(s|A) = 1$ and (1) holds for each voter i. Intuitively, the goal of the sender is to get a adopted in the "wrong" state B. But to induce a voter to vote for a in state B, the sender must also assign the voter to vote for a in state A with a sufficiently high probability. The sender's problem is to choose the probabilities of different minimal a-winning coalitions to maximize the *persuasion probability* in state B.

4.1 A Three-Voter Case

We illustrate the main ideas with a simple example where $N = 3$ and $K = 2$. Suppose $l_2 > 1$ so that the median voter prefers b ex ante. In this case, the sender's problem (P-1) becomes

$$\max_{\pi(\cdot|\omega)\in\Delta(S)} \{\pi(aab|B) + \pi(aba|B) + \pi(baa|B)\} \tag{3}$$

$$\text{s.t. } \pi(aab|B)) + \pi(aba|B) \leq \frac{1}{l_1}[\pi(aab|A) + \pi(aba|A)] \tag{4}$$

$$\pi(aab|B) + \pi(baa|B) \leq \frac{1}{l_2}[\pi(aab|A) + \pi(baa|A)] \tag{5}$$

$$\pi(aba|B) + \pi(baa|B) \leq \frac{1}{l_3}[\pi(aba|A) + \pi(baa|A)] \tag{6}$$

$$\pi(aab|A) + \pi(aba|A) + \pi(baa|A) = 1. \tag{7}$$

Under public persuasion, the sender should target voters 1 and 2. The optimal public signal is:

$$\pi(baa|A) = 0 \quad ; \quad \pi(baa|B) = 0;$$
$$\pi(aba|A) = 0 \quad ; \quad \pi(aba|B) = 0;$$
$$\pi(aab|A) = 1; \qquad \pi(aab|B) = \frac{1}{l_2}.$$

There is a unique a-winning coalition consisting of voters 1 and 2. Note that while voter 2's obedient constraint is binding, voter 1's is slack. The obedient constraint for voter 3, who never votes for a, is trivially satisfied.

The sender can benefit from the following procedure.

1. Decrease $\pi(aab|A)$ by ϵ and increase $\pi(baa|A)$ by ϵ. Inequality (5) is still binding and both (4) and (6) are slack if $\epsilon > 0$ is sufficiently small.

2. Increase $\pi(aba|B)$ by $\epsilon/l_3 > 0$. Now (5) and (6) are both binding. As long as ϵ is sufficiently small, (4) will still be slack. Q_B, increases by ϵ/l_3.

Reassigning probability from aab to baa in state A allows the sender to take full advantage of the slackness in voter 1's obedient constraint. The procedure relies on the fact that a voter's obedient constraint is only required to hold on average. If voter 3 knew that the recommendations are aba (and not baa), she would not vote for a.[7]

The following proposition describes the optimal information structure in the three-voter case.

Proposition 2. *Under the optimal information structure, the persuasion probability is*

$$Q_B = \min\left(\frac{1}{l_2}\frac{l_2+l_3}{l_1+l_3}, 1\right).$$

When $Q_B < 1$, any optimal information structure must satisfy

$$\begin{aligned}
\pi(baa|A) &= \frac{l_3}{l_2}\frac{l_2-l_1}{l_1+l_3} \quad ; \quad \pi(baa|B) = 0;\\
\pi(aba|A) &= 0 \quad ; \quad \pi(aba|B) = \frac{1}{l_2}\frac{l_2-l_1}{l_1+l_3};\\
\pi(aab|A) &= \frac{l_1}{l_2}\frac{l_3+l_2}{l_1+l_3} \quad ; \quad \pi(aab|B) = \frac{1}{l_2}.
\end{aligned}$$

The information structure in Proposition 2 can be obtained by choosing ϵ in the above procedure such that (4) also binds. In this case all three obedient constraints for voting a are binding. Note that voter 3, who is the hardest to persuade, votes for a with positive probability. As voter 2's obedient constraint is already binding under public persuasion, voter 3 must be involved to raise the probability of voter 1 voting for a. It is straightforward to check that a voter who is easier to persuade is more likely to vote for a in state B:

$$\pi(aab|B) + \pi(aba|B) > \pi(aab|B) + \pi(baa|B) > \pi(baa|B) + \pi(aba|B),$$

As common in the literature, the sender finds it optimal to obfuscates the states. By pooling two states under the same recommendation, a pivotal voter is convinced that the recommended action matches the true state sufficiently likely, and therefore is willing to obey the

[7]In fact, aba is a perfect signal of state B in the sense that $\pi(aba|A) = 0$ and $\pi(aba|B) > 0$.

recommendation. A remarkable feature of private persuasion is that the sender also *obfuscates the winning coalitions*: he randomizes over multiple winning coalitions.[8] This additional obfuscation is beneficial for the sender for the following reason. In the absence of obfuscation of winning coalitions, pivotal voters' pivotal beliefs are derived conditional on the realized winning coalition. Information will be fully aggregated among pivotal voters within winning coalition, making pivotal voters' pivotal beliefs identical. As voters have heterogenous preferences, some pivotal voters are "over-convinced,"i.e., their obedient constraints are slack. To maximize the persuasion probability, the sender would like to discriminatorily obfuscate by pooling multiple pivotal events. Such a persuasion strategy is feasible only when the sender can persuade voters privately.

The gain from private over public persuasion is

$$\pi(aba|B) = \frac{l_2 - l_1}{l_1 l_2 + l_2 l_3}, \tag{8}$$

which decreases in l_3 and l_1 and vanishes as either $l_1 \to l_2$ or $l_3 \to \infty$. At the first limit, the slackness in voter 1's obedient constraint disappears. At the second limit, voter 3 becomes almost impossible to persuade, and the slackness in voter 1's obedient constraint has a very small effect. Since we assume $l_2 > 1$, Q_B can equal one only when $l_1 < 1$ and $l_2 \leq 2$. Intuitively, under the optimal information structure, both voters 2 and 3 prefer a when the signal profile is baa, which is a perfect signal for state A. Since voter 1 is supposed to vote b then, the votes of both voters 2 and 3 are critical. The optimal signal structure leverages the strict preferences of voters 2 and 3 for a in the same event baa in state A to get them to vote for a in two different events in state B; namely, aab for voter 2 and aba for voter 3.

4.2 General Case

We saw that the information structure described in Proposition 2 strictly dominates the optimal public signal. But to prove that it is optimal, we need to show that it is not strictly dominated by another private information structure. Even with only three voters, the problem is not trivial as there are three potential minimum a-winning coalitions to deal with. With N voters there are C_K^N minimum a-winning coalitions.

In this section we show that the key properties of the optimal information structure can be

[8] The randomization over multiple winning coalitions also implies that some winning coalitions contain voters who are not among the easiest to persuade, which is also present in an example in Alonso and Camara (2016). Despite the similarity of the results, the mechanisms behind them are very different.

derived without fully solving the sender's problem (P-1). Define

$$\alpha_i \equiv \sum_{s\in S^*_{i,a}} \pi(s|A), \tag{9}$$

$$\beta_i \equiv \sum_{s\in S^*_{i,a}} \pi(s|B), \tag{10}$$

as the total probabilities that i belongs to a minimal a-winning coalition in state A and state B, respectively. Let $\alpha = (\alpha_1,...,\alpha_N)$ and $\beta = (\beta_1,...,\beta_N)$. Since each minimum a-winning coalition includes K voters voting for a, the probabilities that a is chosen in states A and B are, respectively,

$$\frac{1}{K}\sum_{i=1}^{N}\alpha_i = 1, \tag{11}$$

$$\frac{1}{K}\sum_{i=1}^{N}\beta_i \leq 1. \tag{12}$$

For each state the probability that voter i belongs to a minimum a-winning coalition must be less than the probability that a is selected. Hence,

$$\alpha_i \in [0,1], \forall i, \tag{13}$$

$$\beta_i \in \left[0, \frac{1}{K}\sum_{i=1}^{N}\beta_i\right], \forall i. \tag{14}$$

Lemma 2. *For any (α,β) that satisfy (11), (12), (13), and (14), there exists $\pi \in \Pi^{piv}$ such that (9) and (10) hold.*

Lemma 2 means that we can directly work with (α,β). Consider the following problem:

$$Q_B = \max_{\alpha,\beta}\frac{1}{K}\sum_{i=1}^{N}\beta_i \tag{P-2}$$

such that

$$l_i\beta_i \leq \alpha_i, \forall i, \tag{15}$$

and (11), (12), (13) and (14) hold.

Lemma 3 follows immediately.

Lemma 3. *If $\pi \in \Pi^{piv}$ is a solution to problem (P-1), then (α,β) defined by (9) and (10) is a solution to problem (P-2). Conversely, if (α,β) is a solution to problem (P-2), then there exists a solution $\pi \in \Pi^{piv}$ to problem (P-1) such that (9) and (10) hold.*

Lemma 3 shows that the sender can choose each voter's probability of belonging to a minimum a-winning coalition, subject to an overall "budget constraint" that the total probability of voting for a does not exceed K. The next proposition characterizes the optimal solution when $Q_B < 1$.

Proposition 3. *Suppose that $Q_B < 1$ in problem (P-2). Then the optimal solution $\left(\alpha^*, \beta^*\right)$ must satisfy the following conditions:*

1. $l_i \beta_i^* = \alpha_i^* \; \forall i$,

2. $\alpha_i^* = \min(1, l_i Q_B)$ *if* $\alpha_j^* > 0$ *for some* $j > i$.,

3. β_i^* *decreases in* i, $\beta_1^* = Q_B$, *and* $\beta_{K+1}^* > 0$.

Proposition 3 generalizes Proposition 2. Part 1 says that under the optimal information structure, each voter's incentive constraint is binding. If the constraint for some voter i were not binding, the sender could either increase β_i or reduce α_i (and increase some other α_j). Part 2 says that the sender should use the "budget" of probability of voting a on voters who are the easiest to persuade. If he expends part of the budget on voter j – i.e., $\alpha_j^* > 0$ – then it must be that for all $i < j$ either α_i^* or β_i^* hits the upper bound in (13) or (14). Recall that in the three-voter case, all three voters votes for a with strictly positive probability. In general that need not be the case. Since l_i increases in i, part 2 implies that $\alpha_i < \alpha_{i+1}$ whenever $\alpha_{i+1} > 0$. Part 3 says that voters who are easier to persuade are more likely to vote for a in state B than voters who are harder to persuade. Voter 1, who is the easiest to persuade, belongs to every minimum a-winning coalition in state B. This, together with part 1, implies that the persuasion probability must be strictly less than one when $l_1 > 1$. Finally, voter $K+1$ votes for a with strictly positive probability. Since each minimum winning coalition has only K votes for a, there must be multiple minimum winning coalitions.

Proof. Part 1. Since, by definition, $Q_B \equiv \frac{1}{K}\sum_{i=1}^{N} \beta_i^*$, there must exist some $\beta_i^* < Q_B$. If, by way of contradiction, that $l_j \beta_j^* < \alpha_j^*$ for some j, then the sender could increase the objective function by reducing α_j^* by ϵ, and increasing α_i^* by ϵ and β_i^* by ϵ / l_i.

Part 2. Consider the following problem:

$$U(Y) = \frac{1}{K} \max_{\alpha,\beta} \sum_{i=1}^{N} \beta_i \tag{P-3}$$

$$\text{s.t. } l_i \beta_i = \alpha_i, \forall i \tag{16}$$

$$\sum_i \alpha_i = K, \alpha_i \in [0,1]; \beta_i \in [0, Y], \forall i \tag{17}$$

Problem (P-3) differs from problem (P-2) in that the inequality sign in (15) is replaced by an equality sign in (16) and the endogenous variable $\sum_i \beta_i/K$ in (14) is replaced by a parameter Y in (17). From part 1, we know that a solution to problem (P-2) must be feasible in problem (P-3) with $Y = Q_B$. Conversely, if (α, β) is feasible in problem (P-3) for some $Y < 1$ and $\sum_i \beta_i/K \in (Y, 1)$, then it is also feasible in problem (P-2). It follows that if (α^*, β^*) is a solution to problem (P-2), then it is also a solution to problem (P-3) with $Y = Q_B$.

We can therefore derive part 2 by solving problem (P-3). Substituting out β_i, the sender's problem becomes choosing $\{\alpha_i\}_{i=1,\ldots N}$ to maximize $\sum_{i=1}^N \alpha_i/l_i$ subject to (17). The Lagrangian function is

$$\mathscr{L} = \sum_{i=1}^{n}\left(\frac{\alpha_i}{l_i K}\right) - \rho\left(\sum_{i=1}^{n}\alpha_i - K\right) + \sum_{i=1}^{n}\psi_i\alpha_i - \sum_{i=1}^{n}\xi_i\left(\alpha_i - \min\left(l_i Y, 1\right)\right). \tag{18}$$

The Kuhn-Tucker conditions for α_i are that

$$\frac{\partial \mathscr{L}}{\partial \alpha_i} = \frac{1}{l_i K} - \rho + \psi_i - \xi_i = 0 \ \forall i,$$

and ρ, ψ_i, and ξ_i be positive (zero) if the corresponding constraints are binding (non-binding). Part 2 follows from the fact that l_i is increasing in i.

Part 3. That β_i^* decreases in i follows immediately from parts 1 and 2 and the fact that l_i is increasing.

By part 1 and 2, we have

$$\beta_1^* = \frac{\alpha_1^*}{l_1} = \min\{\frac{1}{l_1}, Q_B\}.$$

We want to show that $\beta_1^* = Q_B < 1/l_1$. Suppose not. Then $\beta_1^* = \min\{\frac{1}{l_1}, Q_B\} = \frac{1}{l_1} \le Q_B$. Since l_i is increasing in i, (α_i^*, β_i^*) would be equal to $(1, 1/l_i)$ for $i \le K$ and $(0,0)$ for $i > K$. But then

$$Q_B = \frac{1}{K}\sum_{i=1}^{K}\frac{1}{l_i} < \frac{1}{l_1},$$

contradicting the supposition that $1/l_1 \le Q_B$. This proves that $\beta_1^* = Q_B < 1/l_1$.

Finally, since $\alpha_1^* = l_1 Q_B < 1$, $\sum_{i=1}^K \alpha_i^* < K$. It then follows from the fact that $\sum_{i=1}^N \alpha_i^* = K$ that α_{K+1}^* and β_{K+1}^* are strictly positive.

□

Proposition 3 characterizes the optimal solution when $Q_B < 1$. The next proposition provides a necessary and sufficient condition for $Q_B < 1$.

Proposition 4. *In problem (P-2), $Q_B = 1$ if and only if*

$$\frac{1}{K}\left[\sum_{j=1}^{i^*}\min\left(1,\frac{1}{l_j}\right)+\left(K-\sum_{j=1}^{i^*}\min\left(1,l_j\right)\right)\frac{1}{l_{i^*+1}}\right]\geq 1, \tag{19}$$

where $i^* = \max\left\{i \leq N | \sum_{j=1}^{i}\min\left(1,l_j\right) < K\right\}$ and $l_{N+1} = \infty$. When $Q_B < 1$, Q_B is strictly decreasing in l_i if $\alpha_i^* > 0$, and $Q_B \to 1/l_K$ as either $l_1 \to l_K$ or $l_{K+1} \to \infty$.

The left-hand side of (19) is $U(1)$ as defined in problem (P-3). The first part of Proposition 4 says that $Q_B = 1$ if and only if the value of problem (P-3) is greater than one. The remaining parts of the proposition generalize the discussion of (8) in the three-voter case: Q_B will be higher if any voter who votes a with a strictly positive probability becomes easier to persuade, and the gain from private persuasion disappears if either voters 1 to K have the same preferences or voters $K+1$ to N are impossible to persuade.

4.3 Implications on Institutional Design

In ancient Jewish law (see Epstein, 1978), a suspect cannot be unanimously convicted of a capital crime. It is because the absence of even one dissenting opinion among the judges indicates that there must remain some form of undiscovered exculpatory evidence. See more discussion on non-monotone voting rule in Chwe (2010). Our analysis contributes another rationale of a non-monotone voting rule: preventing information manipulation. When action a is chosen if and only if it receives exactly K votes, it is optimal for the sender to use a pivotal-persuasion strategy. As we have shown, the persuasion probability Q_B is less than one as long as the voter who is the easiest to persuade prefers b ex ante. By contrast, under any non-unanimous majority rule, the sender can manipulate the group to implement a with probability arbitrarily close to one, regardless of the preferences of the voters.

It is well known that a group may choose a large majority requirement to encourage information acquisition when information is costly (see Li, 2001). Our results suggest that doing so may also have the benefit of limiting the influence of interested third parties. In our model a larger K, in addition to forcing the sender to persuade more voters, also reduces the number of minimum winning coalitions, making it harder to manipulate the collective decision.

Proposition 5. *The persuasion probability Q_B is decreasing in K.*

Given the optimal pivotal persuasion outcome, voter i's expected payoff is $\mu_0 - (1-\mu_0)\,Q_B\,l_i$. Therefore, every voter is better off as K increases. It is thus in the voters' best interests to adopt a unanimous voting rule favoring b. But this implication is predicated on the assumption that the sender's bias is known. In our model, if the sender actually prefers b, unanimous voting rule favoring b will make the collective decision extremely vulnerable to manipulation.

5 Independent Signals

Thus far, we have assumed that the sender can *fully* control the information structure. In realty, the sender may face additional informational constraints. For example, the set of available experiments may be limited. The arbitrary correlation between voters' experiments may be hard to obtain. More importantly, voters may have multiple information sources: they may have private information. Insights from the pivotal persuasion problem are relevant to other constrained information design problems. Consider the case where voters' signals are constrained to be conditional independent. Imagine a sender who hires N agents, each of whom produces a signal for a different voter. Whether a signal supports a depends on the state ω, as well as on extraneous factors uncorrelated across agents. Ex ante, the sender can control $q_{i,x,\omega}$, the probability that voter i's signal supports $x \in \{a,b\}$ when the state is $\omega \in \{A,B\}$ (by manipulating the methodology the agent uses to produce the signal). But, ex post, the agents would not falsify or suppress the signal (due to reputation concerns).

Write

$$\mathbf{q} = \left(q_{1,a,A}, q_{1,b,A}, q_{1,a,B}, q_{1,b,B}, ..., q_{N,a,A}, q_{N,b,A}, q_{N,a,B}, q_{N,b,B}\right).$$

for a profile of marginal probabilities that fully characterize the information structure. Given $\mathbf{q}$, the probability of signal profile s in state ω is

$$\pi\left(s|\omega,\mathbf{q}\right) = \Pi_{i=1}^{N} q_{i,s_i,\omega}.$$

We assume that the sender has full control over $\mathbf{q}$. Denote $S_a \equiv \{s : |s_i = a| \geq K\}$ as the set of a-winning signal profiles. Thus, the sender's optimization problem with independent signals is

$$\max_{\mathbf{q}\geq 0} 0.5 \sum_{s\in S_a} \pi\left(s|A,\mathbf{q}\right) + 0.5 \sum_{s\in S_a} \pi\left(s|B,\mathbf{q}\right)$$

such that

$$\begin{aligned} l_i \left[\sum_{s\in S^*_{i,a}} \pi\left(s|B,\mathbf{q}\right)\right] - \left[\sum_{s\in S^*_{i,a}} \pi\left(s|A,\mathbf{q}\right)\right] &\leq 0, \\ -l_i \left[\sum_{s\in S^*_{i,b}} \pi\left(s|B,\mathbf{q}\right)\right] + \left[\sum_{s\in S^*_{i,b}} \pi\left(s|A,\mathbf{q}\right)\right] &\leq 0, \end{aligned}$$

and, for $i = 1, ..., N$ and $\omega = A, B$,

$$q_{i,a,\omega} + q_{i,b,\omega} = 1.$$

Note that the obedient constraints imply that $q_{i,a,A} \geq q_{i,a,B}$ for each voter i.

The restriction to conditionally independent signals complicates the information design problem significantly. Consider the case $N = 3$ and $K = 2$. In addition, assume $l_3 = l_2$. Then the obedient constraints for voting a become

$$\begin{aligned}
\pi\left(aab|A,\mathbf{q}\right)+\pi\left(aba|A,\mathbf{q}\right) &\geq l_1\left(\pi\left(aab|B,\mathbf{q}\right)+\pi\left(aba|B,\mathbf{q}\right)\right);\\
\pi\left(aab|A,\mathbf{q}\right)+\pi\left(baa|A,\mathbf{q}\right) &\geq l_2\left(\pi\left(aab|B,\mathbf{q}\right)+\pi\left(baa|B,\mathbf{q}\right)\right);\\
\pi\left(aba|A,\mathbf{q}\right)+\pi\left(baa|A,\mathbf{q}\right) &\geq l_2\left(\pi\left(aba|B,\mathbf{q}\right)+\pi\left(baa|B,\mathbf{q}\right)\right).
\end{aligned}$$

Set

$$\begin{aligned}
q_{2,a,A} &= q_{3,a,A} = q_{1,b,A} = 1;\\
q_{2,b,A} &= q_{3,b,A} = q_{1,a,A} = 0;\\
q_{2,a,B} &= q_{3,a,B} = \frac{1}{\sqrt{l_2}};\\
q_{2,b,B} &= q_{3,b,B} = 1-\frac{1}{\sqrt{l_2}};\\
q_{1,a,B} &= 1-q_{1,b,B} = 0.
\end{aligned}$$

We have

$$\pi\left(baa|A,\mathbf{q}\right) = 1, \pi\left(baa|B,\mathbf{q}\right) = \frac{1}{l_2},$$

and $\pi\left(s|A,\mathbf{q}\right), \pi\left(s|B,\mathbf{q}\right) = 0\ \forall s \in S_a \setminus \{baa\}$. All three obedient constraints are satisfied.[9] Under this feasible solution, a is chosen only when voters 2 and 3 vote for it. Voter 1 who is the easiest to persuade always votes b. This does not entail a loss, as we assume $l_3 = l_2 > l_1$. The total persuasion probability is $\mu_0 + (1-\mu_0)/l_2$, making the strategy as good as the optimal public persuasion.

Ignore the independent constraint for the moment. Suppose we can directly assign the probability of each minimal a-winning coalition as in the last section. Similar to the argument in Section 4.1, we can increase the persuasion probability in state B by the following steps to take advantage of the fact that $l_1 < l_2$.

1. Decrease $\pi(baa|B)$ by ϵ and increase both $\pi(aba|B)$ and $\pi(aab|B)$ by ϵ. This will keep the obedient constraints for voters 2 and 3 binding. But the obedient constraint for voter 1 will be violated.

[9] It is straightforward to verify that the obedient constraints in state B are also satisfied.

2. Increase $\pi(aba|A)$ and $\pi(aab|A)$ each by $l_1\epsilon$ and decrease $\pi(baa|A)$ by $2l_1\epsilon$ and $\pi(baa|B)$ by $l_1\epsilon/l_2$. All three obedient constraints are satisfied and the persuasion probability increases by $(l_2-l_1)\epsilon/l_2$.

With the independent constraint, we cannot directly transfer probability from one minimum winning coalition to another. To (roughly) reproduce step 1 above, we need to raise $q_{1,a,B}$ from 0 and reduce $q_{2,a,B}$ and $q_{3,a,B}$ from $1/\sqrt{l_2}$. This creates two extra effects. First, since we need $q_{1,a,A} \geq q_{1,a,B}$, we need to increase $q_{1,a,A}$, which reduces $\pi(baa|A)$. Hence, we need to further lower $\pi(baa|B)$, which lowers the persuasion probability, to avoid violating the obedient constraints of voters 2 and 3. Second, increasing $q_{1,a,B}$ also increase $\pi(aaa|B)$, which raises the persuasion probability. It turns out that for small ϵ these two extra effects balance each other out.

Reproducing step 2 is more problematic. To decrease $\pi(baa|A)$ and increase $\pi(aba|A)$ and $\pi(aab|A)$, we must increase $q_{1,a,A}$ from 0 and reduce $q_{2,a,A}$ and $q_{3,a,A}$ from 1, which in turn increases $\pi(abb|A), \pi(bba|A), \pi(bab|A)$ and $\pi(bbb|A)$ from 0. As a result, a is no longer implemented with probability one in state A. This directly lowers the persuasion probability. As the left-hand side of the obedient constraints for voting a is lowered, the right-hand side needs to be lowered as well, which further reduces the persuasion probability. However, note that the extent to which $\pi(aba|A)$ and $\pi(aab|A)$ needs to increase in step 2 is proportional to l_1. When l_1 is small, these extra negative effects would be small as well. Hence, the persuasion probability increases as a result.

We have assumed $l_3 = l_2$. But since the persuasion probability is continuous in l_3, the same conclusion must hold when l_3 is slightly greater than l_2. Hence, we have the following proposition.

Proposition 6. *For any $l_2, ..., l_k$, $l_k > 1$, there exists $\widehat{l}_{k+1}$ and $\widehat{l}_1$ such that the optimal persuasion probability is strictly greater than $1/l_k$ when $l_{k+1} \in (l_k, \widehat{l}_{k+1}]$ and $l_1 \in (0, \widehat{l}_1]$.*

Solving for the constrained optimal information structure is messy as the feasible set is non-convex and the obedient constraints are non-linear in $\mathbf{q}$.[10] Analyzing the "cleaner" pivotal persuasion case helps us to identify conditions under which private signals can outperform the optimal public signal. Wang (2015) argues that the persuasion probability under identical and independent private signals cannot be greater than that under the optimal public signal. Proposition 6 shows that her conclusion depends critically on the assumption of identical signals. As we illustrate in Propositions 1 and 2, the advantage of private pivotal persuasion comes from (1)

[10] See Wang (2015) for a numerical illustration in a three-voter example.

the sender's ability to freely use non-pivotal events, and (2) discriminatory obfuscating states across voters. When the sender is restricted to i.i.d. signals, neither is feasible.

6 Conclusion

How to aggregate diverse and private information is a central question in the design of voting mechanisms. In this paper we consider the information design problem of a sender who is restricted to using minimal winning coalitions. We characterize the optimal information structure and explain why private persuasion strictly outperforms public persuasion. As an application, the insight from the pivotal-persuasion problem is used to investigate the value of using private persuasion when signals are independent across voters. By comparing the persuasion outcome with and without pivotally constraints, we also provide a justification for using a non-monotone voting rule.

We assume that the sender has no private information about the state before choosing the information structure. It is natural to ask what happens alternatively, especially in our voting context where the interested party or politician engaged in persuading has better information about the state than voters. Alonso and Camara (2016) use the results from Alonso and Câmara (2017) to argue that, if the sender knows the state before persuasion, then the sender's favorite equilibrium in this informed-sender game is an equilibrium in which all informed senders pool on the same signal used by an uninformed sender. Their result extends to the current paper. That is, if the information designer knows the true state before choosing the information structure, there is an equilibrium in this informed-designer game where all types of senders pool on the same signal as the uninformed sender. It is also natural to ask what happens if voters have private information, as in Kolotilin, Li, Mylovanov, and Zapechelnyuk (2017), Kolotilin (2017) and Bergemann, Bonatti, and Smolin (2017). We leave this extension for future work.

While it is standard in the persuasion game literature to assume that the sender can select any signal structure, in reality he is likely to face substantial restrictions, particularly when multiple players are involved. In this paper we have restricted the sender to equilibria in which the sender's preferred action is chosen only by minimal coalitions. The assumption allows us to clarify the logic of pivotal persuasion in a tractable way. In future work, it would be worthwhile to explore other plausible restrictions.

A Appendix: Omitted Proofs

Proof of Lemma 1. The proof follows directly from the Revelation Principle as in Bergemann and Morris (2016a) and Taneva (2014) and is omitted. □

Proof of Proposition 2. Suppose $Q_B = 1$. By Proposition 3, $\beta_1^* = Q_B$ and $\alpha_2^* = \min(1, l_2 Q_B) = 1$. It follows that $\pi(baa|B) = \pi(aba|A) = 0$. The solution is the only one that makes (4),(5), (6), and (7) bind. $Q_B = \min(\pi(aba|B) + \pi(aab|B), 1)$. □

Proof of Lemma 2. We show that for any α such that (11) and (13), there exists $\pi(s|A)$ such that (9) holds.

Let $h = C_K^N$ be the number of pivotal signals. It is convenient to state the proposition in matrix form. Let $s^{[1]}, ..., s^{[h]}$ be an order of the pivotal signals for voting a. Let $\theta = (\alpha_1, ..., \alpha_n)$. Let W be a $N \times h$ matrix with

$$W_{ij} = \begin{cases} 1 & if \quad s^{[i]}(j) = a, \\ 0 & if \quad s^{[i]}(j) = b. \end{cases}$$

We need to show that there is an h-th vector $\pi_A = (\pi_{1A}, ..., \pi_{hA})$ such that $\sum_i \pi_{iA} = 1$ and $\pi_{iA} \in [0,1]$, and

$$W\pi_A^T = \theta^T.$$

Suppose by way of contradiction that no such π_A exists. By Farkas' lemma, there exists an n-th vector $\lambda = (\lambda_1, ..., \lambda_N)$ such that

$$\begin{aligned} W_{ij}^T \lambda^T &\geq 0; & (20) \\ \theta\lambda^T &< 0. & (21) \end{aligned}$$

Note that the row of W_{ij}^T that corresponds to the signal profile where players 1 to K observe a begins with $K+1$ ones followed by $N-K$ zeros. Thus, (20) implies that

$$\sum_{i=1}^{K} \lambda_i \geq 0.$$

Since the player ordering is arbitrary, we can assume without loss of generality that λ_i is ascending in i. Hence

$$\begin{aligned} & \min_{x_i} \sum_{i=1}^{N} \lambda_i x_i \text{ s.t. } x_i \in [0,1]\, \forall i,\ \sum_{i=1}^{N} x_i = K\,, \\ = & \sum_{i=1}^{K} \lambda_i \geq 0. \end{aligned}$$

Since $\alpha_i \in [0,1]$ for all i, and $\sum_{i=1}^N x_i = K$,

$$\sum_{i=1}^N \lambda_i \alpha_i \geq 0,$$

which contradicts (21).

Similarly, one can show that for any β such that (12) and (14), there exists $\pi(s|B)$ such that (10) holds. □

Proof of Proposition 4. Define

$$U(Y, l_1, ..., l_N) = \frac{1}{K}\left\{\sum_{j=1}^{i^*} \min\left(Y, \frac{1}{l_j}\right) + \left(K - \sum_{j=1}^{i^*} \min\left(1, l_j\right)\right)\frac{1}{l_{i^*+1}}\right\}, \tag{22}$$

where

$$\begin{aligned} i^* &= \max\left(i \leq N | \sum_{j=1}^{i} \min\left(1, l_j Y\right) < K\right), \\ l_{N+1} &= \infty. \end{aligned}$$

For part 1 we need to show that $Q_B = 1$ if and only if $U(1, l_1, ..., l_N) \geq 1$. Suppose $Q_B = 1$ and $\left(\alpha^*, \beta^*\right)$ is a solution to problem (P-2). Define β such that $\beta_i = \alpha_i^*/l_i$, $i = 1, ..., N$. By construction $\left(\alpha^*, \beta\right)$ is feasible in problem (P-3) with $Y = 1$ and $\sum_i^N \beta_i \geq \sum_i^N \beta_i^* \geq K$. It follows that $U(1, l_1, ..., l_N) \geq 1$. Conversely, suppose $\left(\alpha, \beta\right)$ is a solution to problem (P-3) with $Y = 1$ and $U(1, l_1, ..., l_N) \geq 1$. Then any positive $\beta' \leq \beta$ $\left(\alpha, \beta'\right)$ is feasible in problem (P-2). Hence, we can pick β' such that $\sum_i^N \beta_i' = K$.

Part 2. Let $\left(\alpha^*, \beta^*\right)$ denote a solution to problem (P-2). We have showed in the proof of Proposition 3 that when $Q_B < 1$, $\left(\alpha^*, \beta^*\right)$ is the solution to problem (P-3) with $Y = Q_B$. That is, $U(Q_B) = Q_B$. By the envelop theorem,

$$\left.\frac{dU}{dl_i}\right|_{Y=Q_B} = \left.\frac{\partial \mathscr{L}}{\partial l_i}\right|_{Y=Q_B}.$$

For any i such that $\alpha_i^* > 0$ and $l_i Q_B \leq 1$

$$\begin{aligned} \left.\frac{\partial \mathscr{L}}{\partial l_i}\right|_{Y=Q_B} &= -\frac{Q_B}{l_i^2} + \left(\frac{1}{l_i} - \rho\right) Q_B, \\ &= -\rho Q_B < 0. \end{aligned}$$

The first equality follows from the first-order condition and the last inequality follows from the that $\sum_i \alpha_i = K$ is binding when $Q_B < 1$. For any i such that $\alpha_i^* > 0$ and $l_i Q_B > 1$

$$\left.\frac{\partial \mathscr{L}}{\partial l_i}\right|_{Y=Q_B} = -\frac{1}{l_i^2} < 0.$$

Hence, U is strictly decreasing in l_i for any i such that $\alpha_i^* > 0$. Since any solution to problem (P-3) when $Y < 1$ is feasible in problem (P-2), Q_B must also be strictly decreasing in l_i.

Part 3. Since $1/l_K$ can always be achieved public persuasion, $Q_B \geq 1/l_K$. For any $Y > 1/l_K$, $\left(K - \sum_{j=1}^{K} \min\left(Y, l_j\right)\right) < 0$ when l_1 (and therefore also $l_2, ..., l_{K-1}$) is sufficient close to l_K. Hence,

$$\lim_{l_1 \to l_K} U\left(Y, l_1, ..., l_N\right) < \frac{1}{K} \sum_{i=1}^{K} \min\left(Y, \frac{1}{l_j}\right) < Y.$$

As we argued in part 2, $U\left(Q_B\right) = Q_B$. Since for all $Y > 1/l_K$

$$\lim_{l_1 \to l_K} U\left(Y, l_1, ..., l_N\right) < Y,$$

it follows that

$$\lim_{l_1 \to l_K} Q_B \leq 1/l_K.$$

A similar argument applies when l_{K+1} (and therefore $l_{K+2}, ..., l_N$) becomes sufficiently large, i.e.,

$$\lim_{l_{K+1} \to \infty} Q_B \leq 1/l_K,$$

as for all $Y > 1/l_K$,

$$\begin{aligned} & \lim_{l_{K+1} \to \infty} U\left(Y, l_1, ..., l_N\right), \\ \leq \quad & \frac{1}{K}\left[\sum_{i=1}^{K-1} \min\left(Y, \frac{1}{l_j}\right) + \frac{1}{l_K} + \lim_{l_1 \to l_K}\left(K - \sum_{j=1}^{K} \min\left(1, l_j\right)\right)\frac{1}{l_{K+1}}\right], \\ = \quad & \frac{1}{K}\left(\sum_{i=1}^{K-1} \min\left(Y, \frac{1}{l_j}\right) + \frac{1}{l_K}\right) < Y. \end{aligned}$$

□

Proof of Proposition 5. Suppose $Q_B < 1$. Let k^* denote the last voter who votes a with strictly positive probability. From Proposition 3 we know that β_i is decreasing in i, with $\beta_1 = Q_B$ and $\beta_{k*} < Q_B$. Suppose K increases by 1. Keeping Q_B constant for now. From Proposition 3 we know that we will allocate the new unit of pivotal probability to voter k^* and voter $k^* + 1$. This will increase the sum of all β_i by less than Q_B as $\beta_k^* < Q_B$ and l_i is increasing. This means that the new average (now with denominator $K + 1$) will be strictly less than Q_B. Hence Q_B is no longer the solution. Since $U(Y)$ is monotone in Y, the new solution must be smaller. □

Proof of Proposition 6. We prove Proposition 6 by analyzing a restricted problem that sets

$$q_{i,a,A} = q_{i,a,B} = 0, q_{i,b,A} = q_{i,b,B} = 1 \; \forall i > K + 1; \tag{23}$$

$$q_{i,a,A} = z, q_{i,b,A} = 1 - z, q_{i,a,B} = v, q_{i,b,B} = 1 - v \; \forall i = 2, ..., K + 1; \tag{24}$$

$$q_{1,a,A} = 1 - w + \epsilon, q_{1,b,A} = w - \epsilon, q_{1,a,B} = 1 - w, q_{1,b,B} = w. \tag{25}$$

Substituting (23), (24), and (25) into the sender's problem, we have the restricted sender's problem

$$\max_{z,v,w,\epsilon} 0.5\left(z^K + K(1-w+\epsilon)z^{K-1}(1-z)\right) + 0.5\left(v^K + K(1-w)v^{K-1}(1-v)\right)$$

such that $z, v, w, \epsilon \geq 0$, $z, v \leq 1$, $w + \epsilon \leq 1$, and

$$l_1\left((1-w)Kv^{K-1}(1-v)\right) - (1-w+\epsilon)Kz^{K-1}(1-z) \leq 0, \tag{26}$$

$$-l_1\left(wKv^{K-1}(1-v)\right) + (w-\epsilon)Kz^{K-1}(1-z) \leq 0, \tag{27}$$

and, for all $j = 2, ..., K+1$,

$$l_j\left(v^K w + (K-1)v^{K-1}(1-v)(1-w)\right) - \left(z^K(w-\epsilon) + (K-1)z^{K-1}(1-z)(1-w+\epsilon)\right) \leq 0, \tag{28}$$

$$-l_j\left(v^{K-1}w(1-v) + (K-1)v^{K-2}(1-v)^2(1-w)\right) + \left(z^{K-1}(w-\epsilon)(1-z) + (K-1)z^{K-2}(1-z)^2(1-w+\epsilon)\right) \leq 0, \tag{29}$$

and, for $j > K+1$,

$$-l_j\left(wKv^{K-1}(1-v) + 0.5K(K-1)(1-w)v^{K-2}(1-v)^2\right) + (w-\epsilon)Kz^{K-1}(1-z) + 0.5K(K-1)(1-w+\epsilon)z^{K-2}(1-z)^2 \leq 0. \tag{30}$$

We prove Proposition 6 by showing that the claim holds for the restricted problem. First, consider the case that $l_1 = \epsilon = 1 - z = 0$ and $l_{K+1} = l_K$. In this case, we can ignore (26) and (27), voter 1's obedient constraints, as they are always satisfied. Consider the potential solution

$$w^* = 1, v^* = (1/l_K)^{\frac{1}{K}}. \tag{31}$$

It is straightforward to verify that the solution is feasible with persuasion probability $1/l_K$. The obedient constraint to vote for b for any voter $j > 1$, (29) and (30), are non-binding, so are the obedient constraint to vote for a, (28), for any voter $j \neq K+1$. The only binding constraints, therefore, are the obedient constraint to vote for a for voter $K+1$ and the constraints that $w \leq 1$.

Define the Lagrangian function

$$\mathcal{L} = 0.5\left(v^K + K(1-w)v^{K-1}(1-v)\right) + \lambda_{K+1,a}\left(w - l_{K+1}\left(v^K w + (K-1)v^{K-1}(1-v)(1-w)\right)\right) - \kappa(w-1).$$

Differentiating L with respect to w and v and substituting $1/v^{*K}$ for l_K yield

$$\left.\frac{\partial \mathcal{L}}{\partial w}\right|_{w^*=1, v^*=(1/l_K)^{\frac{1}{K}}} = -0.5K(v^*)^{K-1}\left(1-v^*\right) + \lambda_{K+1,a} l_{K+1}(v^*)^{K-1}\left(1-v^*\right)(K-1) - \kappa; \tag{32}$$

$$\left.\frac{\partial \mathscr{L}}{\partial v}\right|_{w^*=1, v^*=(1/l_{K+1})^{\frac{1}{K}}} = 0.5K(v^*)^{K-1} - \lambda_{K+1} l_{K+1} K(v^*)^{K-1}. \tag{33}$$

It follows that

$$\left.\frac{\partial \mathscr{L}}{\partial w}\right|_{w^*=1, v^*=(1/l_{K+1})^{\frac{1}{K}}} + \left.\frac{\partial \mathscr{L}}{\partial v}\right|_{w^*=1, v^*=(1/l_{K+1})^{\frac{1}{K}}} \left(1-v^*\right)(K-1)/K = -0.5 v^{*K-1}\left(1-v^*\right) - \kappa < 0.$$

This implies that there exist no $\lambda_{K+1,a}, \kappa \geq 0$ such that

$$\left.\frac{\partial \mathscr{L}}{\partial w}\right|_{w^*=1, v^*=(1/l_{K+1})^{\frac{1}{K}}} = \left.\frac{\partial \mathscr{L}}{\partial v}\right|_{w^*=1, v^*=(1/l_{K+1})^{\frac{1}{K}}} = 0.$$

Hence, by the Kuhn-Tucker condition, $w^* = 1, v^* = (1/l_K)^{\frac{1}{K}}$ cannot be a solution to the restricted problem when $l_1 = \epsilon = 1 - z = 0$ and $l_{K+1} = l_K$. This implies that there is some v and w, that together with $\epsilon = 1 - z = 0$, constitute a feasible solution to the restricted problem with persuasion probability $p^* > 1/l_K$ when $l_1 = 0$ and $l_{K+1} = l_K$.

Since persuasion probability is continuous in v, starting with this solution with persuasion probability p^*, we can slightly change the value of v so that the persuasion probability remains greater than $1/l_K$, and (28), (29) for $j = 2, ..., K+1$, and (30) for $j > K+1$ are all non-binding. Since the persuasion probability is continuous in z, there is some $\hat{z}$ such that for any $z \in [\hat{z}, 1)$, the persuasion probability is strictly greater than $1/l_K$ and (28), (29) for $j = 2, ..., K+1$, and (30) for $j > K+1$ all remain non-binding. Since (28) is continuous in l_j, there is some $\hat{l}_{K+1}$ such that for all $l_{K+1} \in (l_k, \hat{l}_{K+1}]$, (28) remains non-binding for l_{K+1}. When $z < 1$, (26) is non-binding. Pick $\hat{l}_1 > 0$ such that there is some $z \in [\hat{z}, 1)$ so that (26) and (27) hold with equality when $l_1 = \hat{l}_1$. Then for all $l_1 \in (0, \hat{l}_1]$, there is some $z(l_1) \in [\hat{z}, 1)$ that makes (26) and (27) holds with equality. Finally, we can make (26) and (27) non-binding by slightly raising ϵ. Since the persuasion probability is continuous in ϵ, the persuasion probability is still strictly higher than $1/l_K$. □

References

ALONSO, R., AND O. CAMARA (2016): "Persuading voters," *American Economic Review*, 106(11), 3590–3605.

ALONSO, R., AND O. CÂMARA (2017): "On the value of persuasion by experts," *Working Paper, London School of Economics.*

AUSTEN-SMITH, D., AND J. S. BANKS (1996): "Information aggregation, rationality, and the Condorcet jury theorem," *American Political Science Review*, 90(01), 34–45.

BARDHI, A., AND Y. GUO (2017): “Modes of Persuasion Toward Unanimous Consent,” *Theoretical Economics*, forthcoming.

BERGEMANN, D., A. BONATTI, AND A. SMOLIN (2017): “The Design and Price of Information,,” *American Economic Review*, forthcoming.

BERGEMANN, D., AND S. MORRIS (2013): “Robust predictions in games with incomplete information,” *Econometrica*, 81(4), 1251–1308.

——— (2016a): “Bayes correlated equilibrium and the comparison of information structures in games,” *Theoretical Economics*, 11(2), 487–522.

——— (2016b): “Information Design and Bayesian Persuasion Information Design, Bayesian Persuasion, and Bayes Correlated Equilibrium,” *The American Economic Review*, 106(5), 586–591.

——— (2017): “Information design: A unified perspective,” *Working Paper, Yale University*.

BROCAS, I., AND J. D. CARRILLO (2007): “Influence through ignorance,” *The RAND Journal of Economics*, 38(4), 931–947.

CAI, H. (2009): “Costly participation and heterogeneous preferences in informational committees,” *The RAND Journal of Economics*, 40(1), 173–189.

CHWE, M. S.-Y. (2010): “Anonymous Procedures for Condorcet’s Model: Robustness, Nonmonotonicity, and Optimality,” *Quarterly Journal of Political Science*, 5(1), 45–70.

EPSTEIN, I. (1978): “The Babylonian Talmud,” *London, England: Soncino Press,*, 18 volumes.

FEDDERSEN, T., AND W. PESENDORFER (1997): “Voting behavior and information aggregation in elections with private information,” *Econometrica*, 65(5), 1029–1058.

——— (1998): “Convicting the innocent: The inferiority of unanimous jury verdicts under strategic voting,” *American Political Science Review*, 92(01), 23–35.

GERARDI, D., AND L. YARIV (2007): “Deliberative voting,” *Journal of Economic Theory*, 134(1), 317–338.

——— (2008): “Information acquisition in committees,” *Games and Economic Behavior*, 62(2), 436–459.

GOLDSTEIN, I., AND C. HUANG (2016): “Bayesian persuasion in coordination games,” *The American Economic Review*, 106(5), 592–596.

INOSTROZA, N., AND A. PAVAN (2017): “Persuasion in Global Games with Application to Stress Testing,” *Working Paper, Northwestern University*.

JACKSON, M. O., AND X. TAN (2013): “Deliberation, disclosure of information, and voting,” *Journal of Economic Theory*, 148(1), 2–30.

KAMENICA, E., AND M. GENTZKOW (2011): “Bayesian persuasion,” *American Economic Review*, 101(6), 2590–2615.

KOLOTILIN, A. (2017): “Optimal information disclosure: A linear programming approach,” *Theoretical Economics*, forthcoming.

KOLOTILIN, A., M. LI, T. MYLOVANOV, AND A. ZAPECHELNYUK (2017): “Persuasion of a privately informed receiver,” *Econometrica*, forthcoming.

LI, H. (2001): “A theory of conservatism,” *Journal of Political Economy*, 109(3), 617–636.

LI, H., S. ROSEN, AND W. SUEN (2001): “Conflicts and common interests in committees,” *The American Economic Review*, 91(5), 1478–1497.

MATHEVET, L., J. PEREGO, AND I. TANEVA (2017): “On Information Design in Games,” *Working Paper, NYU*.

PERSICO, N. (2004): “Committee design with endogenous information,” *The Review of Economic Studies*, 71(1), 165–191.

RAYO, L., AND I. SEGAL (2010): “Optimal information disclosure,” *Journal of Political Economy*, 118(5), 949–987.

SCHNAKENBERG, K. E. (2015): “Expert advice to a voting body,” *Journal of Economic Theory*, 160, 102–113.

SHIMOJI, M. (2016): “Rationalizable Persuasion,” *Working Paper, University of York*.

TANEVA, I. (2014): “Information design,” *Working Paper, University of Edinburgh*.

WANG, Y. (2015): “Bayesian persuasion with multiple receivers,” *Working Paper, Xiamen Univeristy*.