Keeping Your Story Straight: Truthtelling and Liespotting

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Philosophers seemingly agree:

A liar should have a good memory.
—Quintilian

If you tell the truth, you don't have to remember anything.
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Do statisticians and economists agree as well?

A More Concrete Example

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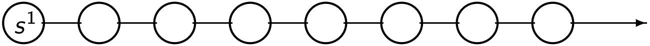
What restrictions should the regulator impose on this sequence?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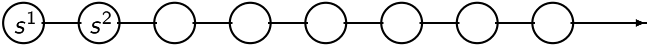
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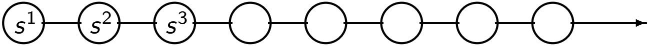
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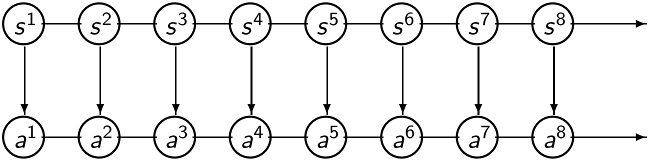
Let λ denote the invariant measure (by extension, the measure on (ss'),...).

Denote the initial distribution $\nu \in \Delta(S)$.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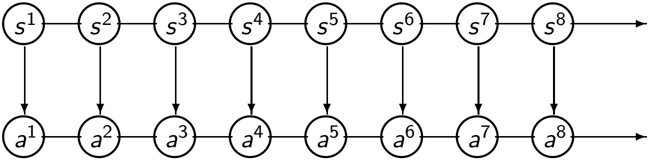

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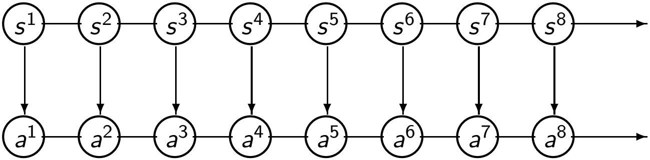

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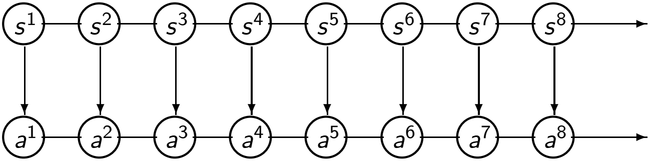
Reduced-form payoff $u(s, a)$ [think of it as $r\left(s, \phi(a)\right)$].



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Total payoff:

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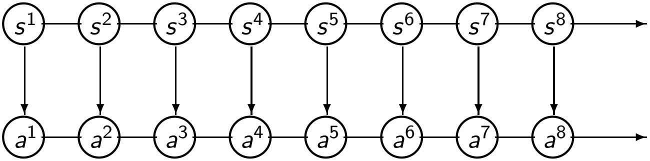

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Strategy: $\sigma = (\sigma^n)$, $\sigma^n : (S \times A)^n \times S \to \Delta(A)$.

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Jackson & Sonnenschein ('07): Force the agent to report in

$$\Sigma_0 := \left\{ (\sigma, \nu) \;\middle|\; \forall s : \lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} \#\{a^n = s\} = \lambda(s),\ \mathbf{P}_{\sigma,\nu}\text{-}a.s. \right\}.$$

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Money is memory: There exists $t : A \to \mathbf{R}$ s.t. truthtelling is optimal in the game with payoff $u(s, a) + t(a)$ iff σ_{tt} is best in Σ_0.

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As an example, consider the process with t.f.

$$
\begin{array}{cc}
 & \begin{array}{ccc} s'_1 & s'_2 & s'_3 \end{array} \\
\begin{array}{c} s_1 \\ s_2 \\ s_3 \end{array} & \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 3/4 & 1/4 \\ 1/2 & 0 & 1/2 \end{pmatrix}
\end{array}
$$

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The invariant distribution is $\lambda = (1/4, 1/2, 1/4)$.

For $x \in [0, 1/4]$, consider the distribution $\mu_x \in \Delta(S \times A)$:

$$\mu_x = \begin{array}{c c} & \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \end{array} & \begin{pmatrix} \frac{1}{4} - x & x & 0 \\ 0 & \frac{1}{2} - x & x \\ x & 0 & \frac{1}{4} - x \end{pmatrix} \end{array}$$

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There exists $\sigma \in \Sigma_0$ such that, for all (s, a),

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} \#\{(s^n, a^n) = (s, a)\} = \mu_x(s, a), \ \mathbf{P}_\sigma - a.s.$$

This includes truthtelling ($\mu_{\mathrm{tt}} := \mu_0$), but also:

$$\mu_{\frac{1}{4}} = \begin{array}{c c} & \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \end{array} & \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 \end{pmatrix} \end{array}$$

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$$\begin{array}{c} s = s_2 \\ \Big\downarrow {\scriptstyle (1/2)} \\ a = a_2 \end{array}$$

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Are pairs the answer? Consider now:

$$\mu_{\frac{1}{6}} = \begin{array}{c c} & \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \end{array} & \begin{pmatrix} 1/12 & 1/6 & 0 \\ 0 & 1/3 & 1/6 \\ 1/6 & 0 & 1/12 \end{pmatrix} \end{array}$$

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$$\begin{tikzcd}
 & \bar{a} = a_3 \arrow[dl, "(1/3)"'] \arrow[dr, "(2/3)"] \arrow[rr] & & a = a_3 \\
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\end{tikzcd}
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To summarize the reporting strategy:

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \end{array} \begin{array}{c} \begin{array}{cc} a_3 & a_1 \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

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This strategy yields the desired frequency $\lambda(s, s')$.

Yet, we can spot such a lie.

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\mu_{\frac{1}{6}} = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \end{array} \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{pmatrix} 1/12 & 1/6 & 0 \\ 0 & 1/3 & 1/6 \\ 1/6 & 0 & 1/12 \end{pmatrix} \end{array}
$$

Suppose:

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\begin{tikzcd}
\bar{a} = a_3 \arrow[r] & a = a_3 \arrow[r, "(3/4)"] \arrow[d] & a' = a_3 \\
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Conditional on $(\bar{a}, a) = (a_3, a_3)$, the next report is a_3 too often.

This generalizes:
For all n, there is $x_n > 0$ s.t. μ_x is undetectable when strings of length n are checked iff $x \leq x_n$. It holds that $\lim_n x_n = 0$.

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But a given t.f. doesn't imply that a chain is Markov (Lévy, Feller).
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Is this example non-generic?

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- Checking k-tuples suffices (fails to suffice), but checking $k-1$-tuples does not, for an open set of P.

Implications

Dynamic interactions allow for richer behavior than with transfers in the one-shot game;
–in fact, there is no *a priori* upper bound on the memory required.

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Implementation (cyclical monotonicity in dynamic contexts).

Extensions and Literature

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Covariates with given laws.

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Verifiable information: Cherrypicking data?

Other Literature

Economics	Cyclical Monotonicity (Rochet 1987).
	Dynamic games with changing types (Athey Bagwell 2001; Renault Solan Vieille 2013; Escobar Toikka 2013; H. Takahashi Vieille 2015).
	Linking Incentives (Jackson Sonnenschein 2007).
	Dynamic implementation and mechanism design (Athey Segal 2013; Mezzetti Renou 2015).

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Sufficiency of a test:

3. When is a distribution detectable/truthtelling optimal for some test iff it is so with a given test? (A property of P)

Testing k-Strings

Define:

$$\Sigma_k := \Bigg\{ (\sigma, \nu) \;\Bigg|\; \forall \underbrace{(s, \ldots, s')}_{k+1 \text{ terms}} :$$

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} \#\{(a^{n-k}, \ldots, a^n) = (s, \ldots, s')\} = \lambda(s, \ldots, s'), \mathbf{P}_{\sigma,\nu} - a.s. \Bigg\}.$$

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Strategies in Σ_k match the frequency of strings of length $k+1$.

Truthtelling is <u>k-optimal</u> if

$$\sup_{(\sigma,\nu) \in \Sigma_k} \liminf_{N \to \infty} \mathbf{E}_{\sigma,\nu} \left[\frac{1}{N} \sum_{n=1}^{N} u(s^n, a^n) \right] \leq \mathbf{E}_{\mu_{\mathrm{tt}}} \left[u(s, a) \right].$$

Undetectability

Define

$$\overline{\Sigma}_k = \left\{ \sigma \;\middle|\; \sigma^n(\vec{s}^n, \vec{a}^n, s^n) = \sigma(\underbrace{a^{n-k}, \ldots, a^{n-1}}_{k \text{ terms}}, s^n) \right\}.$$

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Proposition

For every $\mu \in \mathcal{M}_k$, and $\nu \in \Delta(S)$, there is $\sigma \in \Sigma_k$ s.t.

$$\lim_{N \to \infty} \mu^N_{\sigma,\nu} = \mu.$$

For every $(\sigma, \nu) \in \Sigma_k$ s.t. $\mu := \lim_{N \to \infty} \mu^N_{\sigma,\nu}$ exists, $\mu \in \mathcal{M}_k$.

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Let $\mathcal{M}_\infty = \cap_k \mathcal{M}_k$.

Incentives

Since $\mathcal{M}_k$ is the set of undetectable deviations, it follows:

Proposition

Truthtelling is k-optimal iff $\mu_{\mathrm{tt}} \in \operatorname{argmax}_{\mu \in \mathcal{M}_k} \mathbf{E}_\mu[u(s, a)]$.

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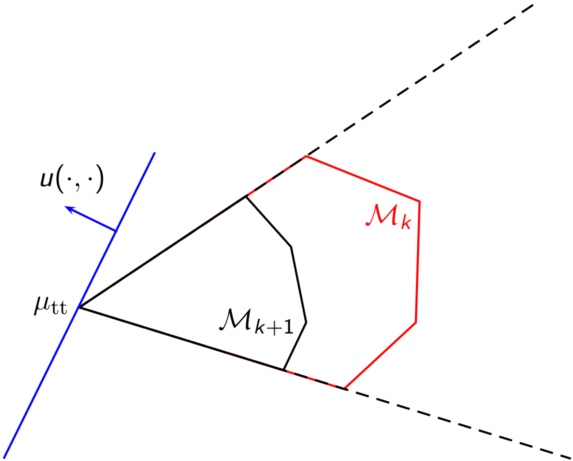
So $\mathcal{M}_k = \mathcal{M}_\infty$ is sufficient, but not necessary, for:

$$\forall u \in \mathbf{R}^{|S| \times |A|} : \mu_{\mathrm{tt}} \in \underset{\mu \in \mathcal{M}_k}{\operatorname{argmax}} \mathbf{E}_\mu[u(s,a)] \Leftrightarrow \mu_{\mathrm{tt}} \in \underset{\mu \in \mathcal{M}_\infty}{\operatorname{argmax}} \mathbf{E}_\mu[u(s,a)].$$



$u(\cdot,\cdot)$
$\mathcal{M}_k$
μ_{tt}
$\mathcal{M}_{k+1}$

What matters to the economist is the cone:

$$\mathcal{C}_k := \{\mu \in \Delta(S \times A) \big| \mu = \mu_{\mathrm{tt}} + \alpha(\mu' - \mu_{\mathrm{tt}}),\ \text{some } \alpha \geq 0, \mu' \in \mathcal{M}_k\}\,.$$

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$\mathcal{C}_k \neq \mathcal{C}_{k+1} \Rightarrow \exists u \in \mathbf{R}^{|S| \times |A|}$:

$$\mathbf{E}_{\mu_{\mathrm{tt}}}[u(s,a)] = \max_{\mu \in \mathcal{M}_{k+1}} \mathbf{E}_\mu[u(s,a)] < \max_{\mu \in \mathcal{M}_k} \mathbf{E}_\mu[u(s,a)].$$

Sufficiency

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These are properties of the Markov matrix P only.

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Proposition

$\mathcal{C}_0 = \mathcal{C}_\infty$ (or: $\mathcal{M}_0 = \mathcal{M}_\infty$) iff (s^n) is pseudo-renewal.

Every two-state Markov chain is pseudo-renewal.

It is a non-generic property for $|S| \geq 3$.

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Proposition

Fix a neighborhood $\mathcal{N}$ of the t.f. for (s^n) i.i.d. and uniform. There exists $\mathcal{N}^{'}, \mathcal{N}^{''} \subseteq \mathcal{N}$ s.t.

- for all $P \in \mathcal{N}^{'}$, $\mathcal{M}_1 = \mathcal{M}_k$ for all k;
- for all $P \in \mathcal{N}^{''}$, $\mathcal{M}_1 \neq \mathcal{M}_k$ for some k.

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$\mathcal{C}_1 = \mathcal{C}_\infty$ if $p_{ss'} \leq \Phi$ for all s, s', where Φ is the golden ratio conjugate ($\Phi = (\sqrt{5} - 1)/2 \simeq 0.618$).

Applications and Implications

Linked Incentives

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Proposition

There exists t with memory k s.t. σ_{tt} is best, given payoff

$$\liminf_{N \to +\infty} \frac{1}{N} \sum_{n \leq N} (u(s^n, a^n) + t(\vec{a}^n)),$$

iff σ_{tt} is best in Σ_k.

Implementation (Rochet '87)

$r(s, y)$, $y \in Y$: utility function.

$\phi : A \to Y$: decision rule.

$u(s, a) := r(s, \phi(a))$: utility given s, a.

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Question: Does there exist $t : A \to \mathbf{R}$ s.t. truthtelling is optimal:

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If so, ϕ is <u>incentive compatible</u> (IC).

Theorem (Afriat, Rochet, Rockafellar)

ϕ is IC iff u is cyclically monotone: for every permutation π of S,

$$\sum_{s \in S} (u(s, \pi(s)) - u(s, s)) \leq 0.$$

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Equivalently, as is easy to show:

$$\phi \text{ is IC iff } \mu_{\mathrm{tt}} \in \operatorname*{argmax}_{\mu \in \mathcal{M}_0} \mathbf{E}_{\mu}[u(s, a)].$$

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Taken together, we obtain the following generalization:

Proposition

The map ϕ is IC using transfers of memory k iff

$$\mu_{\mathrm{tt}} \in \operatorname*{argmax}_{\mu \in \mathcal{M}_k} \mathbf{E}_\mu[u(s, a)].$$

Repeated Agency and Repeated Games

An important literature reduces the analysis of such games to static games with transfers (with side constraints):

Repeated games: Shapley ('53), APS ('90), FLM ('94),...

Agency: Spear-Srivastava ('87), Thomas-Worrall ('90),...

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An implication of our "negative" result is that this is impossible when values are interdependent and types are independent.

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Given $\nu \in \Delta(S)$, $t : A^k \to \mathbf{R}^I$, and a strategy profile σ, i's expected payoff in the finitely repeated game is:

$$v^i(\sigma, \nu) = \mathbf{E}_{\sigma,\nu}\left[\frac{1}{k}\sum_{\kappa=1}^{k} u^i(s_\kappa, a_\kappa) + t^i(a_1, \ldots, a_k)\right].$$

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Let $\mathcal{E}^k(t, \nu)$ be the set of Nash equilibria of this game.

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Let $\mathcal{E}^k(t, \nu)$ be the set of Nash equilibria of this game. Compute

$$M := \sup_{\substack{t:A^k \to \mathbf{R}^I \\ \sigma \in \mathcal{E}^k(t,\nu) \\ \nu \in \Delta(S)}} \sum_i v^i(\sigma, \nu)$$

s.t. $\sum_i t^i(a_1, \ldots, a_k) \leq 0$, $\forall (a_1, \ldots, a_k)$.

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- an infinitely repeated game with incomplete information
- a finitely repeated game with transfers.

Given $\nu \in \Delta(S)$, $t : A^k \to \mathbf{R}^I$, and a strategy profile σ, i's expected payoff in the finitely repeated game is:

$$v^i(\sigma, \nu) = \mathbf{E}_{\sigma,\nu}\left[\frac{1}{k}\sum_{\kappa=1}^{k} u^i(s_\kappa, a_\kappa) + t^i(a_1, \ldots, a_k)\right].$$

Let $\mathcal{E}^k(t, \nu)$ be the set of Nash equilibria of this game. Compute

$$M := \sup_{\substack{t:A^k \to \mathbf{R}^I \\ \sigma \in \mathcal{E}^k(t,\nu) \\ \nu \in \Delta(S)}} \sum_i v^i(\sigma, \nu)$$

s.t. $\sum_i t^i(a_1, \ldots, a_k) \leq 0$, $\forall (a_1, \ldots, a_k)$.

We prove that M is bounded below the highest eq'm surplus in the RPII as $\delta \to 1$.

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Lemma

Suppose the agent is $(\#S + 1)$-prophetic. Then truthtelling is optimal for some test iff it is optimal when testing singleton states.

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