Journal of Mathematical Economics 90 (2020) 1–11

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Randomization under ambiguity: Efficiency and incentive compatibility[☆]

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ARTICLE INFO

Article history:
Received 1 November 2018
Received in revised form 29 February 2020
Accepted 12 May 2020
Available online 20 May 2020

Keywords:
Lottery allocations
Mixed strategy
Efficiency
Incentive compatibility
Wald's maxmin preferences

ABSTRACT

We generalize de Castro and Yannelis (2018) by taking into account the use of randomization. We answer the following questions: Is each efficient allocation of de Castro and Yannelis (2018) still Pareto optimal? Are all efficient allocations still incentive compatible under the Wald's maxmin preferences? We provide positive answers and give applications.

1. Introduction

One of the fundamental problems in mechanism design and equilibrium theory with asymmetric information is the conflict between efficiency and incentive compatibility. As shown in Myerson (1979) and Holmström and Myerson (1983), an efficient allocation may not be incentive compatible in the Bayesian framework.[1] However, when agents have the Wald's maxmin preferences, de Castro and Yannelis (2018) showed that the conflict between efficiency and incentive compatibility no longer exists: all efficient allocations are also incentive compatible if and only if agents have the Wald's maxmin preferences.

We generalize de Castro and Yannelis (2018) by taking into account the use of randomization. As pointed out by Raiffa (1961), and rigorously showed by Saito (2015) and Ke and Zhang (2020), a maxmin agent may achieve a higher payoff through the use of randomization, and hence prefers to randomize her choices.[2] The fact that randomization can improve maxmin agents' payoffs leads to the following questions: Can an efficient allocation of de Castro and Yannelis (2018) be Pareto improved by a *lottery allocation*?[3] Are all efficient allocations still incentive compatible under the Wald's maxmin preferences, when we take into account that each agent may randomize her choices (i.e., use a mixed strategy)? We answer these questions in this paper.

In particular, we explicitly take into account randomization (both lottery allocations and mixed strategies) to study efficiency and incentive compatibility in an *ambiguous exchange economy* with asymmetric information. An ambiguous exchange economy with asymmetric information consists of a finite set of agents, each of whom is characterized by a finite *type set*, an *initial endowment* and an *ex post utility function*. More importantly, the agents have the *maxmin preferences à la* Gilboa and Schmeidler (1989) which include the Wald's maxmin preference as a special case. The main contributions that this paper makes are:

First, we show that when the agents' utility functions satisfy the standard concavity assumption, efficient allocations of

☆ The comments of two referees and an associate editor are gratefully acknowledged. The paper was presented in the 18th SAET Conference, the 9th Shanghai Microeconomics Workshop at SHUFE, the 2018 European Workshop on General Equilibrium Theory, 2018 Nanjing International Conference on Game Theory, 2019 China Meeting of Econometric Society, Beihang University, Central University of Finance and Economics, University of California, Davis. We have benefited from comments, discussions and suggestions by many participants in the above presentations. Special thanks to Shaowei Ke for helpful suggestions and in-depth discussions. Financial support from the National Natural Science Foundation of China [grant No. 71703110] is gratefully acknowledged.

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1 It is well known that to end up with an allocation, the allocation's incentive compatibility plays an important role. There are different ways to define the term "end up with", and the importance of incentive compatibility varies with this definition. We refer interested readers to Pram (2020) for a recent development in this direction.

2 Empirical studies including Dominiak and Schnedler (2011), Dwenger et al. (2016) and Agranov and Ortoleva (2017) found that a non-negligible number of subjects do strictly prefer to randomize.

3 We introduce lotteries over the commodity space to get lottery allocations. An allocation is a special case of a lottery allocation when it is degenerate.

https://doi.org/10.1016/j.jmateco.2020.05.004

de Castro and Yannelis (2018) cannot be Pareto improved by any feasible lottery allocation. Moreover, introducing lotteries over the commodity space enlarges the set of efficient allocations. That is, we show in Section 3 that the efficient lottery allocations set contains the efficient allocations set of de Castro and Yannelis (2018) as a strict subset.

Second, we show that this larger set, i.e., the efficient lottery allocations set, satisfies stronger incentive compatibility notions than the one in de Castro and Yannelis (2018). It follows that all efficient allocations of de Castro and Yannelis (2018) are still incentive compatible under the Wald's maxmin preferences, even if we take into account that an agent may randomize her reports to get higher payoffs. Therefore, we strengthen and generalize the sufficiency part of de Castro and Yannelis (2018), and obtain as a corollary their related theorem. More specifically, there are different ways to define incentive compatibility under mixed strategies, since there are different and equally natural ways for a maxmin agent to evaluate her mixed strategy. Two frequently used ways are: *evaluating a mixed strategy ex ante*, and *evaluating a mixed strategy ex post*. Evaluating a mixed strategy ex ante assumes that an agent learns the realization of her mixed strategy before nature draws a probability law from a set of probability laws (i.e., ambiguity) to minimize the agent's expected utility. It follows that no mixed strategy can eliminate the effect of ambiguity.[4] Evaluating a mixed strategy ex post assumes that an agent learns the realization of her mixed strategy after nature draws a probability law from a set of probability laws to minimize the agent's expected utility. Now, a mixed strategy can fully eliminate the effect of ambiguity.[5] Saito (2015) introduced a more general way: a maxmin agent evaluates her mixed strategy by taking a weighted average of "evaluating a mixed strategy ex ante" and "evaluating a mixed strategy ex post", where the weight captures the agent's subjective belief that her mixed strategy eliminates the effect of ambiguity. We show that when agents have the Wald's maxmin preferences, all efficient lottery allocations are *strongly mixed incentive compatible* (Theorem 1). It follows that when agents' utility functions satisfy the standard concavity assumption, all efficient allocations of de Castro and Yannelis (2018) are strongly mixed incentive compatible (Corollary 3), which is the strongest notion among all the mixed incentive compatibility notions considered in this paper (Proposition 2).

Third, we demonstrate the usefulness of our results through three applications, which are not covered by de Castro and Yannelis (2018). In the first application, we recast the key example of Prescott and Townsend (1984) with Wald's maxmin preferences. Prescott and Townsend (1984) introduced lotteries on the commodity space in an exchange economy with a continuum of agents and the Bayesian preferences. They showed that an efficient allocation x may not be incentive compatible; however, the use of lotteries on the commodity space can achieve both incentive compatibility and the same utility for every agent as the efficient allocation x. We show that in a Bayesian economy with a finite number of agents, Prescott and Townsend (1984)'s method does not always work: when an efficient allocation is not incentive compatible, using lotteries on the commodity space may not be able to achieve both incentive compatibility and the same utility for every agent as the efficient allocation. However, if agents have the Wald's maxmin preferences, every efficient allocation is strongly mixed incentive compatible. In the second and third applications, we show that each *ex ante efficient allocation* of de Castro et al. (2017a,b) and each *interim maxmin value allocation* of Angelopoulos and Koutsougeras (2015) are strongly mixed incentive compatible under the Wald's maxmin preferences.

The paper is organized as follows. Section 2 defines the ambiguous exchange economy with asymmetric information. Sections 3 and 4 discuss efficiency and incentive compatibility respectively, while taking into account the use of randomization. Section 5 shows that each efficient lottery allocation is strongly mixed incentive compatible under the Wald's maxmin preferences. Section 6 discusses the three applications. Finally, we conclude in Section 7. The proofs of our main results are delegated to the Appendix.

2. Ambiguous exchange economy

Let $I = \{1, \dots, N\}$ be the set of N agents. Each agent $i \in I$ observes privately her own type s_i in the interim, where s_i is in a finite set of possible types S_i. That is, S_i is agent i's finite type set. Write $S = \times_{i=1}^{N} S_i$. A vector $s = (s_1, \dots, s_i, \dots, s_N) \in S$ represents agents' types. That is, s is a type profile. Also, write $S_{-i} = \times_{j \neq i} S_j$, and $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_N) \in S_{-i}$. Let $\mathbb{R}_+^{\ell}$ be the ℓ-goods commodity space. Let $e_i : S \to \mathbb{R}_+^{\ell}$ be agent i's *initial endowment*. Each agent receives her endowment in the interim. We assume that e_i is private valued, which means that agent i's endowment depends only on her own type s_i, and not on the types of the other agents s_{-i}, that is, for every s_i, s_{-i} and s'_{-i}, the endowments satisfy $e_i(s_i, s_{-i}) = e_i\left(s_i, s'_{-i}\right)$.[6] Let $c_i \in \mathbb{R}_+^{\ell}$ denote agent i's consumption, and $c = (c_1, \dots, c_N)$ the vector of all agents' consumptions. That is, c is a consumption profile. Clearly, $c \in \mathbb{R}_+^{\ell \times N}$. Agent i's utility of consuming $c_i \in \mathbb{R}_+^{\ell}$ at $s \in S$ is $u_i(c_i, s)$, where u_i is continuous and increasing in consumption. Each agent learns her utility function u_i in the interim. We assume that u_i is private valued, i.e., for every c_i, s_i, s_{-i} and s'_{-i}, the utilities satisfy $u_i(c_i, s_i, s_{-i}) = u_i\left(c_i, s_i, s'_{-i}\right)$.[7] For ease of notation, we write $u_i(c_i, s_i)$ thereafter.

An *allocation* is a mapping $x : S \to \mathbb{R}_+^{\ell \times N}$. For every s, $x(s)$ belongs to the set $\mathbb{R}_+^{\ell \times N}$. That is, after the agents learn s, they receive a consumption profile. Let $L = \left\{x : S \to \mathbb{R}_+^{\ell \times N}\right\}$ be the set of allocations. Let $\Delta\left(\mathbb{R}_+^{\ell \times N}\right)$ denote the set of probability distributions (i.e., lotteries) over $\mathbb{R}_+^{\ell \times N}$ with finite supports, i.e.,

$$\Delta\left(\mathbb{R}_+^{\ell \times N}\right) = \Big\{ \text{probability distribution } \delta : \mathbb{R}_+^{\ell \times N} \to [0, 1] \mid \delta(c) \neq 0 \\ \text{for only finitely many } c\text{'s in } \mathbb{R}_+^{\ell \times N} \Big\}.$$

A *lottery allocation* is a mapping $x : S \to \Delta\left(\mathbb{R}_+^{\ell \times N}\right)$. For every s, $x(s)$ belongs to the set $\Delta\left(\mathbb{R}_+^{\ell \times N}\right)$. That is, after the agents learn s, they receive $x(s)$ which is a lottery over $\mathbb{R}_+^{\ell \times N}$. When the lottery $x(s)$ is realized, the agents receive a consumption profile $c \in \mathbb{R}_+^{\ell \times N}$, i.e., the realization of the lottery $x(s)$ is a consumption profile c. Let $\Delta(L) = \left\{x : S \to \Delta\left(\mathbb{R}_+^{\ell \times N}\right)\right\}$ be the set of lottery allocations. Then, an allocation $x : S \to \mathbb{R}_+^{\ell \times N}$ is a special case of a lottery allocation in which each $x(s)$ assigns probability one to some c in $\mathbb{R}_+^{\ell \times N}$. For notational simplicity, we identify L with the subset $\{x \in \Delta(L) \mid \text{for every } s, x(s)[c] = 1 \text{ for some } c\}$ of $\Delta(L)$, where $x(s)[c]$ denotes the probability of consuming c.

After the agents learn everyone's type s (i.e., at ex post), they receive the lottery $x(s)$. Let $\{c^k\}_{k \in \{1,\dots,K\}}$ denote the K consumption profiles in $\mathbb{R}_+^{\ell \times N}$ that occur with non-zero probabilities. Let

4 This is because regardless of the realization of the agent's mixed strategy, she faces ambiguity. It follows that using mixed strategies cannot improve the agent's payoff.

5 This is because the probability of her mixed strategy may be able to make her payoff constant, which makes ambiguity irrelevant. It follows that a mixed strategy may improve the agent's payoff.

6 This assumption is automatically satisfied, if e_i is constant, i.e., type independent.

7 See Prescott and Townsend (1984) and de Castro and Yannelis (2018) which also adopted private valued utility functions. This private valued assumption is automatically satisfied, if u_i is type independent.

$x(s)\left[c^k\right]$ denote the probability of consuming c^k. That is, after the agents learn s, each agent i's *ex post utility* is

$$v_i(x(s);s_i)=\sum_{k=1}^{K}u_i\left(c_i^k,s_i\right)\cdot x(s)\left[c^k\right],$$

where c_i^k denotes agent i's consumption in the consumption profile c^k.

When agent i privately learns s_i (i.e., in the interim stage), her preference is *maxmin à la* Gilboa and Schmeidler (1989). Given a lottery allocation x, the *interim maxmin expected utility* of agent i with type s_i is

$$\min_{p_i\in P_i(s_i)}\sum_{s_{-i}\in S_{-i}}v_i(x(s_i,s_{-i});s_i)\,p_i(s_{-i}),\qquad (1)$$

where $P_i(s_i)$ is a non-empty, closed and convex set of probabilities over S_{-i}, i.e., the agent's multi-prior set. Let x and y be two lottery allocations. If the interim maxmin expected utility of x is larger than that of y, then agent i with type s_i prefers x to y, $x\succeq^{s_i}y$.[8] Moreover, she strictly prefers x to y, $x\succ^{s_i}y$, if $x\succeq^{s_i}y$ but not $y\succeq^{s_i}x$. This general multi-prior model includes both the Bayesian and the Wald's maxmin preferences in de Castro and Yannelis (2018) as special cases. If, for each i and s_i, the set $P_i(s_i)$ is a singleton set, then the multi-prior preferences become the Bayesian preferences. If, for each i and s_i, the set $P_i(s_i)$ is the set of all probabilities over S_{-i}, then the worst probability should assign the whole weight to the worst type profile. In this case, the multi-prior preferences become the Wald's maxmin preferences in de Castro and Yannelis (2018) and de Castro et al. (2017a), where the following formulation is equivalent to (1),

$$\min_{s_{-i}\in S_{-i}}v_i(x(s_i,s_{-i});s_i).\qquad (2)$$

Remark 1. In Sections 3 and 4, our results hold for any multi-prior set $P_i(s_i)$. In Sections 5 and 6, we focus on the Wald's maxmin preferences, i.e., expression (2), to strengthen and generalize the sufficiency part of de Castro and Yannelis (2018): we take into account the use of randomization (i.e., lottery allocations and mixed strategy deviations) and show that the Wald's maxmin preferences solve the conflict between efficiency and incentive compatibility. It follows that the results of de Castro and Yannelis (2018) are robust in the presence of randomization.

3. Randomization under ambiguity: efficiency

One of the most important notions to evaluate a lottery allocation is efficiency. In this section, we show that when u_i satisfies the standard concavity assumption, every feasible allocation that cannot be Pareto improved by any feasible allocation in L, cannot be Pareto improved by any feasible lottery allocation in $\Delta(L)$ either. Moreover, the efficient lottery allocations set contains the efficient allocations set as a strict subset. In other words, introducing lotteries over $\mathbb{R}_+^{\ell\times N}$ enlarges the set of efficient allocations. As in de Castro and Yannelis (2018), efficiency means interim efficiency, unless stated otherwise.

Definition 1. A lottery allocation x in $\Delta(L)$ is *feasible*, if for every $s\in S$ and $c=(c_1,\ldots,c_N)\in\mathbb{R}_+^{\ell\times N}$ such that $x(s)[c]>0$, then

$$\sum_{i=1}^{N}c_i=\sum_{i=1}^{N}e_i(s).$$

Every agent i knows her own type s_i in the interim, and ranks lottery allocations based on the maxmin preferences, i.e., expression (1). A feasible lottery allocation x is interim efficient, if there does not exist another feasible lottery allocation y such that every type s_i of every agent i prefers y to x, and some type s_i of some agent i strictly prefers y to x under the maxmin preferences.

Definition 2. A feasible lottery allocation $x\in\Delta(L)$ (resp., $\in L$) is said to be interim efficient in $\Delta(L)$ (resp., L), if there does not exist another feasible lottery allocation $y\in\Delta(L)$ (resp., $\in L$), such that $y\succeq^{s_i}x$ for all i and $s_i\in S_i$; furthermore, $y\succ^{s_i}x$ for some i and s_i.

The efficient allocation of de Castro and Yannelis (2018) is a feasible allocation $x\in L$ that is interim efficient in L.

Remark 2. Notice that a feasible allocation $x\in L$ is interim efficient in $\Delta(L)$, if there does not exist another feasible lottery allocation $y\in\Delta(L)$, such that $y\succeq^{s_i}x$ for all i and $s_i\in S_i$; furthermore, $y\succ^{s_i}x$ for some i and s_i. Since $\Delta(L)$ contains L as a strict subset, obviously it is harder for an allocation x to be interim efficient in $\Delta(L)$ than in L.

However, Proposition 1 shows that if agents' utility functions u_i are concave, then every allocation x that is interim efficient in L is also interim efficient in $\Delta(L)$ under the maxmin preferences.[9] Formally, let $X_{\Delta(L)}\subset\Delta(L)$ denote the set of lottery allocations that are interim efficient in $\Delta(L)$, and $X_L\subset L$ denote the set of allocations that are interim efficient in L.

Proposition 1. *In an ambiguous exchange economy, if the utility function u_i is concave for each i, then $X_L\subseteq X_{\Delta(L)}$.*

The proof of Proposition 1 is in the Appendix.[10] The intuition behind Proposition 1 is that at every s, the concavity of u_i implies risk aversion. Then, for every feasible lottery allocation in $\Delta(L)$, there is a feasible allocation in L that gives every agent a higher payoff at every s. Therefore, an allocation that is interim efficient in L cannot be Pareto improved by any feasible lottery allocation in $\Delta(L)$, when the agents' utility functions are concave.

Example 1 shows that $X_L\subset X_{\Delta(L)}$, that is, the efficient allocations set is a strict subset of the efficient lottery allocations set.

Example 1. There is one good and two agents. Agent 1 has one type $S_1=\{s_1\}$, and agent 2 has two types $S_2=\left\{s_2,s_2'\right\}$. Thus, there are two possible type profiles $s=(s_1,s_2)$, $s'=\left(s_1,s_2'\right)$. The aggregate endowment is one unit of the good, regardless of the agents' types. Agents' utility functions are as follows:

$u_1(c,s_1)=c$; $u_2(c,s_2)=c$; $u_2\left(c,s_2'\right)=\sqrt{c}$.

Agent 2 knows agent 1's type, since by construction agent 1 has only one type. Agent 1 has a multi-prior set P_1 which contains all probability distributions over S_2. That is, P_1 is the convex hull of $\{(1,0),(0,1)\}$, where $(1,0)$ denotes the probability distribution that assigns 1 to s_2, and $(0,1)$ denotes the probability distribution that assigns 1 to s_2'.

The set of efficient allocations is

$$X_L=\{x\in L: x\text{ is feasible, and }x_1(s)=x_1\left(s'\right)\},$$

i.e., each allocation in X_L should always assign the same level of consumption to agent 1.

It is straightforward to check that each allocation in X_L is interim efficient in $\Delta(L)$ when all lottery allocations in $\Delta(L)$ are

8 "Larger than" means "greater than or equal to".

9 Concavity of u_i implies risk aversion. Also, concavity imposes restrictions on the agents' preferences over multiple commodities.

10 Proposition 1 holds for any multi-prior set $P_i(s_i)$, i.e., expression (1).

considered. Let $\Delta(L) \setminus L$ denote the set of all elements of $\Delta(L)$ that are not elements of L. Then, there exist lottery allocations in $\Delta(L) \setminus L$ that are interim efficient in $\Delta(L)$ as well. For example, the following lottery allocation x is interim efficient in $\Delta(L)$,

$$x = \begin{cases} (1,0) \text{ with prob } \frac{1}{2}; \ (0,1) \text{ with prob } \frac{1}{2} & s = (s_1, s_2) \\ (0.5, 0.5) & s' = \left(s_1, s_2'\right). \end{cases}$$

That is, at s, agent 1 gets 1 unit of the good with probability $\frac{1}{2}$, and agent 2 gets 1 unit of the good with probability $\frac{1}{2}$. At s', each agent gets 0.5 unit of the good. It follows that the efficient allocations set is a strict subset of the efficient lottery allocations set, i.e., we have $X_L \subset X_{\Delta(L)}$.

4. Randomization under ambiguity: incentive compatibility

When there is ambiguity, Saito (2015) and Ke and Zhang (2020) showed that a maxmin agent may achieve a higher interim payoff through the use of randomization. Furthermore, Saito (2015) pointed out that an agent may randomize her choices in her mind without using observable randomization devices, when she sees fit. This motivates us to take into account that a maxmin agent may adopt a mixed strategy in her decision making.

Given initial endowment e, if agents want to end up with a feasible lottery allocation $x \neq e$, transfers need to take place. Since we allow both e and x to depend on the type profile s, the transfers may depend on s as well. Formally,

Definition 3. Given a lottery allocation x in $\Delta(L)$ and an s in S, define $t(s) \in \Delta\left(\mathbb{R}^{\ell \times N}\right)$ to be the transfer in which the consumption transfer $t^k(s) = c^k - e(s) \in \mathbb{R}^{\ell \times N}$ occurs with probability $x(s)\left[c^k\right]$ for every $k = 1, \ldots, K(s)$.[11]

In the interim, every agent i only knows s_i. Thus, to end up with the correct transfer $t(s)$, it is necessary to pool their private information. Therefore, we assume that each agent i decides which type to report after learning s_i, in order that they may end up with the correct transfer. By doing so, it is possible that agents misreport their true types.

For simplicity, let $e(s_i, s_{-i}) + t\left(\hat{s}_i, s_{-i}\right)$ denote the lottery that the agents get, when their type profile is (s_i, s_{-i}) and their reported type profile is $\left(\hat{s}_i, s_{-i}\right)$. Then, $v_i\left(e(s_i, s_{-i}) + t\left(\hat{s}_i, s_{-i}\right); s_i\right)$ is the utility that agent i gets, when the agents' type profile is (s_i, s_{-i}) and their reported type profile is $\left(\hat{s}_i, s_{-i}\right)$, i.e.,

$$\begin{aligned} & v_i\left(e(s_i, s_{-i}) + t\left(\hat{s}_i, s_{-i}\right); s_i\right) \\ & \quad = \sum_{k=1}^{K\left(\hat{s}_i, s_{-i}\right)} u_i\left(e_i(s_i, s_{-i}) + c_i^k - e_i\left(\hat{s}_i, s_{-i}\right), s_i\right) x\left(\hat{s}_i, s_{-i}\right)\left[c^k\right]. \end{aligned}$$

Since we allow each agent's endowment to vary with her type, we impose the following feasibility condition as in de Castro et al. (2017a,b), Moreno-García and Torres-Martínez (2020): every consumption transfer under x is feasible, i.e., $e_i(s) + c_i^k - e_i(\hat{s}) \in \mathbb{R}_+^{\ell}$, for each $i, s, \hat{s}$ and $k \in \left\{1, \ldots, K(\hat{s})\right\}$. Clearly, if each e_i is constant, then this feasibility condition is automatically satisfied.

When reporting her type, an agent can use a *pure deception*: an agent's pure deception is a pure strategy which specifies a reported type for each true type.

Definition 4. Agent i's pure deception is a mapping $\hat{\alpha}_i : S_i \to S_i$.

The (pure) incentive compatibility notion is standard.

[11] That is, there are $K(s)$ consumption profiles that occur with non-zero probabilities, when the lottery is $x(s)$.

Definition 5. A lottery allocation $x \in \Delta(L)$ is (pure) incentive compatible under the maxmin preferences, if for each agent i, and for each type s_i of agent i,

$$\begin{aligned} & \min_{p_i \in P_i(s_i)} \sum_{s_{-i} \in S_{-i}} v_i\left(x(s_i, s_{-i}); s_i\right) p_i(s_{-i}) \geq \\ & \min_{p_i \in P_i(s_i)} \sum_{s_{-i} \in S_{-i}} v_i\left(e(s_i, s_{-i}) + t\left(\hat{\alpha}_i(s_i), s_{-i}\right); s_i\right) p_i(s_{-i}) \end{aligned}$$

for all $\hat{\alpha}_i(s_i) \in S_i$.

When we take into account randomization, an agent's deception is a mapping from her type set S_i to the set of lotteries over her type set. We call such a deception a *mixed deception*.

Definition 6. Agent i's mixed deception is a mapping $\alpha_i : S_i \to \Delta(S_i)$.

For every s_i, $\alpha_i(s_i)$ is a lottery over her type set. Then, agent i with type s_i reports the realization $\hat{s}_i$ of the lottery $\alpha_i(s_i)$. Let $\alpha_i(s_i)\left[\hat{s}_i\right]$ denote the probability with which agent i with type s_i reports $\hat{s}_i$, where $\hat{s}_i \in S_i$. The *truth telling mixed deception* α_i^T, defined by $\alpha_i^T(s_i)[s_i] = 1$ for every $s_i \in S_i$, will be central to our analysis.

Clearly, an agent's mixed deception corresponds to a mixed strategy.[12] When an agent has maxmin preferences, she may *evaluate her mixed strategy ex ante* or *evaluate her mixed strategy ex post*. Both ways are equally natural. More general ways to evaluate mixed strategies were formulated by Saito (2015) and Ke and Zhang (2020). Different ways to evaluate a mixed strategy lead to different *mixed incentive compatibility* notions. We discuss different notions of mixed incentive compatibility and their relationships below.

4.1. Weak mixed incentive compatibility

Evaluating a mixed strategy ex ante assumes that an agent with type s_i learns the realization $\hat{s}_i$ of her mixed strategy (i.e., $\hat{s}_i$ is the realization of the lottery $\alpha_i(s_i)$) and reports $\hat{s}_i$, before nature draws $p_i \in P_i(s_i)$. More specifically, Fig. 1 illustrates the subjective timing of agent i with type s_i. On the diagram, when we say "Agent i", we mean "Agent i with type s_i".

Clearly, regardless of the realization of the agent's mixed strategy, she faces ambiguity. The utility of agent i with type s_i from playing the mixed strategy α_i is

$$\sum_{\hat{s}_i \in S_i} \left\{ \min_{p_i \in P_i(s_i)} \sum_{s_{-i} \in S_{-i}} v_i\left(e(s_i, s_{-i}) + t\left(\hat{s}_i, s_{-i}\right); s_i\right) p_i(s_{-i}) \right\} \alpha_i(s_i)\left[\hat{s}_i\right], \tag{3}$$

i.e., agent i with type s_i *evaluates her mixed strategy ex ante*. Under the formulation of (3), there does not exist a mixed strategy that can outperform all the pure strategies. Indeed, let s_i^* be her best report among all the choices in S_i, i.e.,

$$\begin{aligned} & \min_{p_i \in P_i(s_i)} \sum_{s_{-i} \in S_{-i}} v_i\left(e(s_i, s_{-i}) + t\left(s_i^*, s_{-i}\right); s_i\right) p_i(s_{-i}) = \qquad (4) \\ & \max_{\hat{s}_i \in S_i} \left\{ \min_{p_i \in P_i(s_i)} \sum_{s_{-i} \in S_{-i}} v_i\left(e(s_i, s_{-i}) + t\left(\hat{s}_i, s_{-i}\right); s_i\right) p_i(s_{-i}) \right\}. \end{aligned}$$

[12] A *mixed strategy* of agent i is a mapping from her type set S_i to the set of lotteries over her action set. An agent's mixed deception is a mixed strategy in which her action set is her type set.

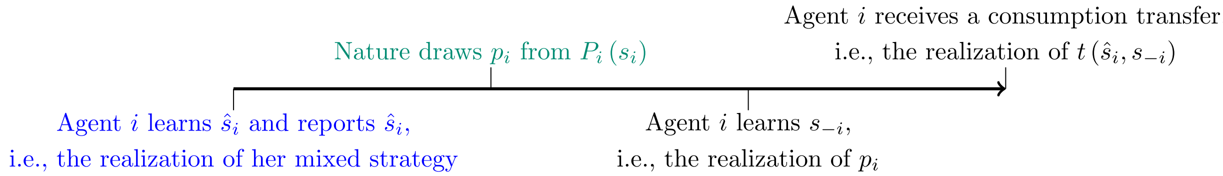

Fig. 1. Subjective timing of Agent i that leads to evaluating a mixed strategy ex ante.

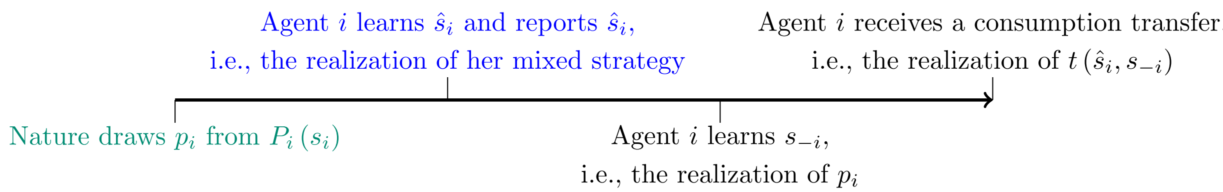

Fig. 2. Subjective timing of Agent i that leads to evaluating a mixed strategy ex post.

Since we have $0 \leq \alpha_i(s_i)\left[\hat{s}_i\right] \leq 1$ for each $\hat{s}_i \in S_i$, and $\sum_{\hat{s}_i \in S_i} \alpha_i(s_i)\left[\hat{s}_i\right] = 1$ by construction, it must be that

$$\sum_{\hat{s}_i \in S_i} \left\{ \min_{p_i \in P_i(s_i)} \sum_{s_{-i} \in S_{-i}} v_i\left(e(s_i, s_{-i}) + t\left(\hat{s}_i, s_{-i}\right); s_i\right) p_i(s_{-i}) \right\} \alpha_i(s_i)\left[\hat{s}_i\right] \leq \tag{5}$$

$$\min_{p_i \in P_i(s_i)} \sum_{s_{-i} \in S_{-i}} v_i\left(e(s_i, s_{-i}) + t\left(s_i^*, s_{-i}\right); s_i\right) p_i(s_{-i}).$$

Thus, no lottery $\alpha_i(s_i)$ can give a higher payoff for (3) than what is obtained in (4). That is, no lottery $\alpha_i(s_i)$ can outperform the best report s_i^*. We can conclude that there does not exist a mixed strategy that can outperform all the pure strategies.

Definition 7. A lottery allocation $x \in \Delta(L)$ is weakly mixed incentive compatible, if for each agent i, and for each type s_i of agent i,

$$\min_{p_i \in P_i(s_i)} \sum_{s_{-i} \in S_{-i}} v_i(x(s_i, s_{-i}); s_i) p_i(s_{-i}) \geq$$

$$\sum_{\hat{s}_i \in S_i} \left\{ \min_{p_i \in P_i(s_i)} \sum_{s_{-i} \in S_{-i}} v_i\left(e(s_i, s_{-i}) + t\left(\hat{s}_i, s_{-i}\right); s_i\right) p_i(s_{-i}) \right\} \alpha_i(s_i)\left[\hat{s}_i\right].$$

for any $\alpha_i(s_i) \in \Delta(S_i)$. That is, every agent prefers truthfully reporting her type to any lottery over her type set S_i.

Since the set $\Delta(S_i)$ includes the lotteries that the agent reports a type with probability one, we know that weak mixed incentive compatibility implies incentive compatibility (Definition 5). Furthermore, we know from (5) that if s_i^* happens to be agent i's true type, then agent i has no incentive to adopt a lottery over her type set. That is, incentive compatibility implies weak mixed incentive compatibility. Therefore, we can conclude this subsection with the following corollary.

Corollary 1. *Let $x \in \Delta(L)$. The lottery allocation x is weakly mixed incentive compatible (Definition 7) if and only if x is incentive compatible (Definition 5).*

4.2. Strong mixed incentive compatibility

Evaluating a mixed strategy ex post assumes that nature draws p_i from $P_i(s_i)$ first, and then agent i with type s_i learns the realization $\hat{s}_i$ of her mixed strategy. Fig. 2 illustrates the subjective timing of agent i with type s_i. On the diagram, when we say "Agent i", we mean "Agent i with type s_i".

Her utility from playing the mixed strategy α_i is

$$\min_{p_i \in P_i(s_i)} \sum_{s_{-i} \in S_{-i}} \sum_{\hat{s}_i \in S_i} v_i\left(e(s_i, s_{-i}) + t\left(\hat{s}_i, s_{-i}\right); s_i\right) \alpha_i(s_i)\left[\hat{s}_i\right] p_i(s_{-i}), \tag{6}$$

i.e., agent i with type s_i *evaluates her mixed strategy ex post.*[13] Since the probability of her mixed strategy enters her utility function (6) in a nonlinear way, randomization may be able to make her payoff across all possible type profiles smooth. Thus, randomization may be able to fully eliminate the effect of ambiguity. Indeed, if there exists a lottery $\alpha_i^*(s_i)$ that gives the agent the same payoff at every possible type profile, i.e.,

$$\sum_{\hat{s}_i \in S_i} v_i\left(e(s_i, s_{-i}) + t\left(\hat{s}_i, s_{-i}\right); s_i\right) \alpha_i^*(s_i)\left[\hat{s}_i\right]$$
$$= \sum_{\hat{s}_i \in S_i} v_i\left(e\left(s_i, s'_{-i}\right) + t\left(\hat{s}_i, s'_{-i}\right); s_i\right) \alpha_i^*(s_i)\left[\hat{s}_i\right]$$

for all s_{-i} and s'_{-i} in S_{-i}, then ambiguity no longer affects her payoff, i.e.,

$$\min_{p_i \in P_i(s_i)} \sum_{s_{-i} \in S_{-i}} \sum_{\hat{s}_i \in S_i} v_i\left(e(s_i, s_{-i}) + t\left(\hat{s}_i, s_{-i}\right); s_i\right) \alpha_i^*(s_i)\left[\hat{s}_i\right] p_i(s_{-i}) =$$
$$\sum_{s_{-i} \in S_{-i}} \sum_{\hat{s}_i \in S_i} v_i\left(e(s_i, s_{-i}) + t\left(\hat{s}_i, s_{-i}\right); s_i\right) \alpha_i^*(s_i)\left[\hat{s}_i\right] p_i(s_{-i}),$$

for all p_i in $P_i(s_i)$. Thus, her mixed strategy fully eliminates the effect of ambiguity.

Definition 8. A lottery allocation $x \in \Delta(L)$ is strongly mixed incentive compatible, if for each agent i, and for each type s_i of agent i,

$$\min_{p_i \in P_i(s_i)} \sum_{s_{-i} \in S_{-i}} v_i(x(s_i, s_{-i}); s_i) p_i(s_{-i}) \geq$$
$$\min_{p_i \in P_i(s_i)} \sum_{s_{-i} \in S_{-i}} \sum_{\hat{s}_i \in S_i} v_i\left(e(s_i, s_{-i}) + t\left(\hat{s}_i, s_{-i}\right); s_i\right) \alpha_i(s_i)\left[\hat{s}_i\right] p_i(s_{-i})$$

for any $\alpha_i(s_i) \in \Delta(S_i)$. That is, every agent prefers truthfully reporting her type to any lottery over her type set.

It can be easily checked that strong mixed incentive compatibility implies incentive compatibility, since the set $\Delta(S_i)$ includes

[13] Now agent i with type s_i evaluates the lottery $\alpha_i(s_i)$ for each type profile s' in the set of all possible type profiles $\{(s_i, s_{-i}) : s_{-i} \in S_{-i}\}$, hence the name "evaluating a mixed strategy ex post".

the lotteries that the agent reports a type with probability one. We illustrate in Example 2 that an incentive compatible lottery allocation may not be strongly mixed incentive compatible under the maxmin preferences.

Example 2. There are two agents, 1 and 2. Agent 1's type set is $S_1 = \{s_1, s_1'\}$. Agent 2's type set is $S_2 = \{s_2, s_2'\}$. There are four possible type profiles,

$$S = \{(s_1, s_2), (s_1, s_2'), (s_1', s_2), (s_1', s_2')\}.$$

For simplicity, we assume that each agent i's utility function u_i and initial endowment e_i is independent of $s \in S$. Let x be a feasible lottery allocation in $\Delta(L)$, such that

$$v_1(x(s_1, s_2); \cdot) = 2, \quad v_1(x(s_1, s_2'); \cdot) = 5,$$
$$v_1(x(s_1', s_2); \cdot) = 5, \quad v_1(x(s_1', s_2'); \cdot) = 2.$$
$$v_2(x(s_1, s_2); \cdot) = 5, \quad v_2(x(s_1, s_2'); \cdot) = 2,$$
$$v_2(x(s_1', s_2); \cdot) = 2, \quad v_2(x(s_1', s_2'); \cdot) = 5.$$

Furthermore, regardless of an agent's type, her multi-prior set P_i contains all probability distributions over S_{-i}, that is, $P_i = \Delta(S_{-i})$.

The lottery allocation x is incentive compatible. Indeed, for agent 1 with type s_1, by reporting the truth, she gets

$$\min_{p_1 \in P_1} v_1(x(s_1, s_2); s_1) p_1(s_2) + v_1(x(s_1, s_2'); s_1) p_1(s_2') =$$
$$\min\{v_1(x(s_1, s_2); s_1), v_1(x(s_1, s_2'); s_1)\} = \min\{2, 5\} = 2;$$

by reporting the lie s_1', she gets

$$\min_{p_1 \in P_1} v_1(x(s_1', s_2); s_1) p_1(s_2) + v_1(x(s_1', s_2'); s_1) p_1(s_2') =$$
$$\min\{v_1(x(s_1', s_2); s_1), v_1(x(s_1', s_2'); s_1)\} = \min\{5, 2\} = 2.$$

Clearly, she has no incentive to lie. The same result holds for the other type and the other agent.

However, the lottery allocation x is not strongly mixed incentive compatible. Indeed, for agent 1 with type s_1, mixing between the truth s_1 and the lie s_1', i.e., $\alpha_1(s_1)[s_1] = 0.5$ and $\alpha_1(s_1)[s_1'] = 0.5$, gives her a strictly higher interim payoff

$$\min_{p_1 \in P_1} \{v_1(x(s_1, s_2); s_1)\alpha_1(s_1)[s_1] + v_1(x(s_1', s_2); s_1)\alpha_1(s_1)[s_1']\} p_1(s_2) +$$
$$\{v_1(x(s_1, s_2'); s_1)\alpha_1(s_1)[s_1] + v_1(x(s_1', s_2'); s_1)\alpha_1(s_1)[s_1']\} p_1(s_2') =$$
$$\min\{v_1(x(s_1, s_2); s_1)\alpha_1(s_1)[s_1] + v_1(x(s_1', s_2); s_1)\alpha_1(s_1)[s_1'],$$
$$v_1(x(s_1, s_2'); s_1)\alpha_1(s_1)[s_1] + v_1(x(s_1', s_2'); s_1)\alpha_1(s_1)[s_1']\} =$$
$$\min\{2 \times 0.5 + 5 \times 0.5, \ 5 \times 0.5 + 2 \times 0.5\} = 3.5.$$

Clearly, x is not strongly mixed incentive compatible under the maxmin preferences.

Remark 3. In Example 2, we assume a type independent u_i, which is a special case of the private valued utility function. One could drop the type independence assumption and reach the same conclusion that incentive compatibility does not imply strong mixed incentive compatibility under the maxmin preferences.

Note that the lottery allocation x in Example 2 is not interim efficient. We discuss the relationship between efficiency and incentive compatibility in Section 5.

We summarize the results of this subsection with the following corollary.

Corollary 2. *Let $x \in \Delta(L)$. If the lottery allocation x is strongly mixed incentive compatible (Definition 8), then x is incentive compatible (Definition 5). However, an incentive compatible lottery allocation may not be strongly mixed incentive compatible.*

4.3. Mixed incentive compatibility under the random uncertainty-averse representation

Saito (2015) axiomatized a utility function that identifies the agent's subjective belief that her randomization eliminates the effects of ambiguity. The utility representation is called a random uncertainty-averse (RUA) representation. According to the random uncertainty-averse (RUA) representation, the utility of agent i with type s_i from playing a mixed strategy α_i is a weighted average of "evaluating a mixed strategy ex ante" (i.e., expression (3)) and "evaluating a mixed strategy ex post" (i.e., expression (6)):

$$\delta \min_{p_i \in P_i(s_i)} \sum_{s_{-i} \in S_{-i}} \sum_{\hat{s}_i \in S_i} v_i\left(e(s_i, s_{-i}) + t(\hat{s}_i, s_{-i}); s_i\right) \alpha_i(s_i)[\hat{s}_i] p_i(s_{-i}) +$$
$$(1-\delta) \sum_{\hat{s}_i \in S_i} \left\{ \min_{p_i \in P_i(s_i)} \sum_{s_{-i} \in S_{-i}} v_i\left(e(s_i, s_{-i}) + t(\hat{s}_i, s_{-i}); s_i\right) p_i(s_{-i}) \right\} \alpha_i(s_i)[\hat{s}_i], \tag{7}$$

where $\delta \in [0, 1]$ is the weight.

That is, the RUA representation (7) assigns a weight $\delta \in [0, 1]$ to evaluating a mixed strategy ex post, and assigns a weight $1-\delta$ to evaluating a mixed strategy ex ante. Hence, the parameter δ captures the agent's subjective belief that her mixed strategy α_i eliminates the effect of ambiguity. Clearly, the RUA representation includes (3), (6) and intermediate cases between (3) and (6) as special cases. Now, the mixed incentive compatibility notion becomes the following.

Definition 9. A lottery allocation $x \in \Delta(L)$ is mixed incentive compatible under the random uncertainty-averse (RUA) representation, if for each agent i, and for each type s_i of agent i,

$$\min_{p_i \in P_i(s_i)} \sum_{s_{-i} \in S_{-i}} v_i(x(s_i, s_{-i}); s_i) p_i(s_{-i}) \geq$$
$$\delta \min_{p_i \in P_i(s_i)} \sum_{s_{-i} \in S_{-i}} \sum_{\hat{s}_i \in S_i} v_i\left(e(s_i, s_{-i}) + t(\hat{s}_i, s_{-i}); s_i\right) \alpha_i(s_i)[\hat{s}_i] p_i(s_{-i}) +$$
$$(1-\delta) \sum_{\hat{s}_i \in S_i} \left\{ \min_{p_i \in P_i(s_i)} \sum_{s_{-i} \in S_{-i}} v_i\left(e(s_i, s_{-i}) + t(\hat{s}_i, s_{-i}); s_i\right) p_i(s_{-i}) \right\} \alpha_i(s_i)[\hat{s}_i],$$

for any $\alpha_i(s_i) \in \Delta(S_i)$. That is, every agent prefers truthfully reporting her type to any lottery over her type set.

When the weight δ is one, Definition 9 becomes strong mixed incentive compatibility (Definition 8). When δ is zero, Definition 9 becomes weak mixed incentive compatibility (Definition 7) which is equivalent to incentive compatibility (Definition 5). We show that the strong mixed incentive compatibility notion is the strongest of all. That is, we have the following proposition, and its proof is in the Appendix.

Proposition 2. *Let $x \in \Delta(L)$. If the lottery allocation x is strongly mixed incentive compatible, then x is mixed incentive compatible under the random uncertainty-averse (RUA) representation for any $\delta \in [0, 1]$ which includes weak mixed incentive compatibility as a special case.*

5. Efficiency and mixed incentive compatibility

In the Bayesian framework, it is known from Myerson (1979) and Holmström and Myerson (1983) that an efficient allocation may not be incentive compatible. Also, in general, an incentive compatible allocation may not be efficient. de Castro and Yannelis (2018) showed that if agents have the Wald's maxmin preferences, any efficient allocation is incentive compatible, that is, the set of efficient allocations is a strict subset of the set of incentive compatible allocations. Furthermore, the Wald's maxmin preference is necessary for this result to hold.

We improve the result of de Castro and Yannelis (2018) by showing that under the Wald's maxmin preferences, each efficient lottery allocation, $x \in X_{\Delta(L)}$, is strongly mixed incentive compatible (Theorem 1). It follows that when u_i satisfies the standard concavity assumption, all efficient allocations of de Castro and Yannelis (2018) are strongly mixed incentive compatible (Corollary 3). The strong mixed incentive compatibility notion is stronger than the incentive compatibility notion (Corollary 2), and it is the strongest among all the mixed incentive compatibility notions considered in this paper (Proposition 2). Therefore, we strengthen and generalize the sufficiency part of de Castro and Yannelis (2018), and obtain as a corollary their related theorem.[14] Formally,

Theorem 1. *In an ambiguous exchange economy, if agents have the Wald's maxmin preferences, then every efficient lottery allocation $x \in X_{\Delta(L)}$ is strongly mixed incentive compatible.*[15]

The proof of Theorem 1 is in the Appendix. The intuition behind Theorem 1 is that under the Wald's maxmin preferences, profitable unilateral deviations lead to Pareto improvements. Indeed, let x be a lottery allocation. Suppose that an agent i with type s_i adopts a mixed deception $\alpha_i(s_i) \in \Delta(S_i)$, and all other agents report their types truthfully. The resulting lottery allocation y is feasible. The mixed deception $\alpha_i(s_i)$ cannot reduce the worst possible payoff of agent $j \neq i$ regardless of the type of agent j. In fact, the mixed deception $\alpha_i(s_i)$ may help agent j to eliminate the effect of ambiguity. Therefore, every agent j with type s_j's interim payoff does not fall under the Wald's maxmin preferences. If the mixed deception $\alpha_i(s_i)$ strictly increases the interim payoff of the agent i with type s_i, then the feasible lottery allocation y Pareto improves x.

Corollary 3 shows that the efficient allocations of de Castro and Yannelis (2018) are not only incentive compatible but also strongly mixed incentive compatible, when each u_i is concave and each agent i has Wald's maxmin preferences.

Corollary 3. *In an ambiguous exchange economy, if agents have the Wald's maxmin preferences, and utility function u_i is concave for each i, then every efficient allocation $x \in X_L$ is strongly mixed incentive compatible.*

Proof. This result follows from Proposition 1 and Theorem 1. □

It follows from Proposition 2 that the efficient allocations of de Castro and Yannelis (2018) are incentive compatible under mixed strategies, regardless of each agent's subjective belief that her mixed strategy eliminates the effect of ambiguity. That is,

Corollary 4. *In an ambiguous exchange economy, if agents have the Wald's maxmin preferences, and utility function u_i is concave for each i, then every efficient allocation $x \in X_L$ is mixed incentive compatible under the random uncertainty-averse (RUA) representation for any $\delta \in [0, 1]$.*

6. Applications

6.1. Comparison with Prescott and Townsend

Prescott and Townsend (1984) showed that in a Bayesian economy with a continuum of agents, an efficient allocation x may not be incentive compatible. However, it is possible to find an incentive compatible lottery allocation that gives every type of agent the same utility as x. Their well chosen lottery allocation is feasible only when there is a continuum of agents. Prescott and Townsend (1984) illustrated their idea with the help of an example (page 15). In their example, there is a continuum of agents, and two types, s_1 and s_2. The agents have increasing and concave utility functions. When an agent's type is s_1, her utility function u_i is risk averse. When an agent's type is s_2, her utility function u_i is risk neutral. There is one good in this economy. Since there are differences in curvatures in the utility functions u_i, an efficient allocation x in L would be for type s_1 to consume $c_1 \in \mathbb{R}_+$ units of the good, and type s_2 to consume $c_2 \in \mathbb{R}_+$ units of the good, where $c_1 < c_2$. However, this efficient allocation x is not incentive compatible. Indeed, type s_1 prefers the larger consumption c_2 to the consumption c_1, thus it is beneficial for type s_1 to pretend to be type s_2. Prescott and Townsend (1984) showed that it is possible to achieve both incentive compatibility and the same utility by replacing c_2 with a lottery over $\mathbb{R}_+$. The new lottery allocation y is in $\Delta(L)$, which gives c_1 units of the good to type s_1 and gives a lottery over $\mathbb{R}_+$ to type s_2. Prescott and Townsend (1984) chose the lottery such that type s_1 strictly prefers her consumption c_1 to the lottery, and type s_2 is indifferent between the lottery and the consumption c_2. Therefore, lotteries can be used to achieve both incentive compatibility and the same utility.

However, we show that Prescott and Townsend (1984)'s method does not always work in a Bayesian economy with a finite number of agents: an efficient allocation x may not be incentive compatible, and there may not exist an incentive compatible lottery allocation that gives every type of agent the same utility as x. If the agents have the Wald's maxmin preferences instead of the Bayesian preferences, then every efficient allocation is strongly mixed incentive compatible, as was shown in Section 5. We illustrate this point with the help of Example 3.

Example 3. There is one good and two agents. Agent 1 has two types $S_1 = \{s_1, s_1'\}$, and agent 2 has two types $S_2 = \{s_2, s_2'\}$. There are four possible type profiles $s = (s_1, s_2)$, $s' = (s_1, s_2')$, $s'' = (s_1', s_2)$, $s''' = (s_1', s_2')$. The aggregate endowment is one unit of the good, regardless of the agents' types. Agents' utility functions are as follows:

$u_1(c, s_1) = \sqrt{c}$; $u_1(c, s_1') = c$; $u_2(c, s_2) = \sqrt{c}$; $u_2(c, s_2') = 10c$.

In the interim, if each agent i thinks that agent $-i's$ two types are equally likely, then an efficient allocation, i.e., x in X_L, of this economy is

$$x = \begin{cases} (0.5, 0.5) & s = (s_1, s_2) \\ (0, 1) & s' = (s_1, s_2') \\ (0.75, 0.25) & s'' = (s_1', s_2) \\ (0, 1) & s''' = (s_1', s_2'). \end{cases}$$

[14] In this paper, we adopt the type model as in de Castro and Yannelis (2018). A more general model is the partition model. Unlike the type model, the partition model can describe situations in which agents' reports do not agree, i.e., the intersection of the reported events is empty. For a detailed discussion on the relationship between partition models and type models, please see de Castro et al. (2017b) and de Castro et al. (2016). Under a partition model, the incentive compatibility notions depend on the punishments that the agents get when their reported events do not agree. Some choices of punishments are: every agent keeps her endowment, impose the worst possible transfer, or randomly assign a transfer, just to name a few, see for example, Glycopantis et al. (2001, 2003), Liu (2016) and de Castro et al. (2020, 2017b). Incentive compatibility should be easier to achieve, if one imposes a severe punishment whenever the agents' reports do not agree.

[15] Theorem 1 implies that direct revelation mechanisms suffice to implement efficient lottery allocations. Therefore, there is no need to consider other mechanisms. The revelation principle under maxmin preferences and mixed strategies is interesting. On this topic, we refer interested readers to Bose and Renou (2014a) and Bose and Renou (2014b).

That is, at s, each agent gets 0.5 unit of the good; at s', agent 2 gets 1 unit of the good, and agent 1 gets nothing; etc. The allocation x is not incentive compatible. Indeed, agent 2 with type s_2 gets $\sqrt{0.5} \times 0.5 + \sqrt{0.25} \times 0.5$ by reporting the truth s_2, and she gets 1 by reporting the lie s_2'. Clearly, lying is better.

Following Prescott and Townsend (1984), we look for an incentive compatible lottery allocation y in $\Delta(L)$ which gives the same interim utility to every type of agent as x. Clearly, if such a y exists, then it is both incentive compatible and efficient. Indeed, since x in L is interim efficient in L, then no feasible lottery allocation z in $\Delta(L)$ can Pareto improve x by Proposition 1. The lottery allocation y gives every type of agent the same interim utility as x, therefore no feasible lottery allocation z in $\Delta(L)$ can Pareto improve y. That is, y is interim efficient in $\Delta(L)$.

The idea is to replace the risk neutral agent's consumption with a lottery, such that the risk neutral agent's interim utility is unchanged. At the same time, the risk averse agent prefers her consumption to the risk neutral agent's lottery. However, in this example, such a lottery allocation does not exist. The risk neutral agent 2 already has the total endowment. It is not possible to replace agent 2's consumption at s' and s''' with lotteries, while keeping the risk neutral agent 2's interim utility unchanged and making the risk averse agent 2 unwilling to lie.

Now, if the agents have the Wald's maxmin preferences, that is, every agent i thinks that any probability distribution over S_{-i} is possible, then every efficient allocation $x \in X_L$ is strongly mixed incentive compatible by Corollary 3.

Remark 4. In order to achieve both efficiency and incentive compatibility, Prescott and Townsend (1984) rely on a continuum of agents. When there is a finite number of agents, one may not be able to achieve both efficiency and incentive compatibility by using lotteries as in Prescott and Townsend (1984) (Example 3). Also, Sun and Yannelis (2008) showed that every efficient allocation is incentive compatible in the Bayesian framework, and this result relies on the law of large numbers. In essence, the continuum of agents with certain assumptions (independence of private information signals) enables the authors to show that the private information becomes negligible/irrelevant and no issue of incentive compatibility arises in their framework. In our model, we do not need a continuum of agents, as we have a fixed finite economy. However, the adaptation of the Wald's maxim preferences enables us to show that efficient allocations are strongly mixed incentive compatible, a result that is false in the Bayesian framework, as Example 3 indicated.

6.2. Ex ante efficient allocations

The efficiency notion we examined is interim efficiency, and an alternative efficiency notion is ex ante efficiency. At ex ante, each agent is able to form a probability assessment μ_i over her types. That is, there is a measure μ_i generating s_i. Assume that for each i and for each type s_i, $\mu_i(s_i) > 0$. Let Δ_i be the set of all prior probability measures on S that agrees with μ_i, i.e.,

$$\Delta_i = \left\{ \text{probability measure } \pi_i : 2^S \to [0,1] \mid \sum_{s_{-i} \in S_{-i}} \pi_i(s_i, s_{-i}) = \mu_i(s_i), \forall s_i \in S_i \right\}.$$

That is, each probability measure π_i in Δ_i assigns the correct probability $\mu_i(s_i)$ to the event $\{(s_i, s_{-i}) : s_{-i} \in S_{-i}\}$ for each $s_i \in S_i$. Clearly, Δ_i is nonempty, closed and convex. If at ex ante, agent i's multi-prior set is Δ_i, then she has the Wald's maxmin preferences $\succeq_i^e$ in de Castro et al. (2017a,b).

Definition 10. For any two allocations x and y in L, agent i prefers x to y at ex ante, $x \succeq_i^e y$, if

$$\min_{\pi_i \in \Delta_i} \sum_{s \in S} v_i(x(s), s_i)\, \pi_i(s) \geq \min_{\pi_i \in \Delta_i} \sum_{s \in S} v_i(y(s), s_i)\, \pi_i(s). \tag{8}$$

Now, agent i uses the worst probability measure in Δ_i to evaluate an allocation. Agent i knows $\mu_i(s_i)$ for each s_i. Therefore, the worst probability measure puts the whole weight $\mu_i(s_i)$ on the worst type profile in the set $\{(s_i, s_{-i}) : s_{-i} \in S_{-i}\}$, for each s_i. Now, we have the following formulation which is equivalent to (8),

$$\sum_{s_i \in S_i} \left(\min_{s_{-i} \in S_{-i}} v_i(x(s_i, s_{-i}), s_i) \right) \mu_i(s_i) \geq \sum_{s_i \in S_i} \left(\min_{s_{-i} \in S_{-i}} v_i(y(s_i, s_{-i}), s_i) \right) \mu_i(s_i).$$

Definition 11. A feasible allocation $x \in L$ is ex ante efficient in L, if there does not exist another feasible allocation $y \in L$, such that at ex ante $y \succeq_i^e x$ for all i, and $y \succ_i^e x$ for at least one i.

We show that each allocation in L that is ex ante efficient in L is strongly mixed incentive compatible under concavity and the Wald's maxmin preferences. Formally,

Proposition 3. *In an ambiguous exchange economy, if agents have the Wald's maxmin preferences, and utility function u_i is concave for each i, then every ex ante efficient allocation $x \in L$ is strongly mixed incentive compatible.*

Proof. By Definitions 2 and 11, every feasible allocation $x \in L$ that is ex ante efficient in L is interim efficient in L. Then by Corollary 3, every feasible allocation $x \in L$ that is ex ante efficient in L is strongly mixed incentive compatible under the Wald's maxmin preferences. □

6.3. Interim maxmin value allocations

We show below that each *interim maxmin value allocation* of Angelopoulos and Koutsougeras (2015) is strongly mixed incentive compatible under the Wald's maxmin preferences. This maxmin value notion has more desirable general equilibrium properties than its Bayesian counterpart. Its Bayesian counterpart restricts the space of allocations to those compatible with the information of individuals (i.e., incentive compatible ones), which excluded many opportunities for trade. As a result, the Bayesian outcomes are only (information) constrained efficient. However, the maxmin value notion abandons the restrictions on the space of allocations, achieving both unconstrained efficiency and incentive compatibility.

Given an economy, let $\underline{u} = (\underline{u}_1, \ldots, \underline{u}_N)$ denote the agents' interim maxmin expected utility, where $\underline{u}_i(x(s)) = \min_{s'_{-i} \in S_{-i}} u_i(x(s_i, s'_{-i}))$, and $x \in L$. For each $s \in S$, an interim maxmin TU game $\Gamma = (I, V_{\lambda,\underline{u},s})$ whose characteristic function $V_{\lambda,\underline{u},s}$ is defined as follows: for each coalition $C \subseteq I$ and each weight λ, define[16]

$$V_{\lambda,\underline{u},s}(C) = \left\{ \max_{x \in L} \sum_{i \in C} \lambda_i(s)\, \underline{u}_i(x(s)) \mid \sum_{i \in C} x_i(s) = \sum_{i \in C} e_i(s), \forall s \in S \right\},$$

16 For $C = \emptyset$, define $V_{\lambda,\underline{u},s}(\emptyset) = 0$.

where $\lambda_i(s)$ is the weight on agent i at s. Given such a TU game, the Shapley value of agent i is defined as

$$Sh_i\left(V_{\lambda,\underline{u},s}\right) = \sum_{C\subseteq I, i\in C} \frac{(|C|-1)!\,(|I|-|C|)!}{|I|!}\left[V_{\lambda,\underline{u},s}(C) - V_{\lambda,\underline{u},s}(C\setminus\{i\})\right],$$

where the summation is over all coalitions C that contain the agent i. The Shapley value can be interpreted as the "worth" or the sum of marginal contributions that an agent made to all the coalitions that she belongs to. The interim maxmin value allocation of Angelopoulos and Koutsougeras (2015) is defined below.

Definition 12. An allocation $x \in L$ is said to be an interim maxmin value allocation if the following two conditions are satisfied for all $s \in S$:

1. $\sum_{i\in I} x_i(s) = \sum_{i\in I} e_i(s)$
2. there exists $\lambda(s) \in \mathbb{R}^I_+\setminus\{0\}$, such that for all $i \in I$,

$$\lambda_i(s)\,\underline{u}_i(x(s)) = Sh_i\left(V_{\lambda,\underline{u},s}\right).$$

The first condition is the feasibility constraint at each s; and the second condition says that the interim maxmin expected utility of each agent multiplied by her weight $\lambda_i(s)$ must be equal to her Shapley value derived from the TU game $\Gamma = \left(I, V_{\lambda,\underline{u},s}\right)$. In other words, the interim maxmin expected utility of each agent multiplied by her weight is equal to her "worth". Thus, fairness is inherent in this concept.

Definition 13. A feasible allocation x in L is said to be strongly interim efficient in L under the maxmin preferences, if there does not exist another feasible allocation y in L and a type profile s, such that $y \succeq_i^{s_i} x$ for all i; furthermore, $y \succ_i^{s_i} x$ for at least one i.

We show that each interim maxmin value allocation is strongly mixed incentive compatible under concavity and the Wald's maxmin preferences. Formally,

Proposition 4. *In an ambiguous exchange economy, if agents have the Wald's maxmin preferences, and utility function u_i is concave for each i, then every interim maxmin value allocation is strongly mixed incentive compatible.*[17]

Proof. Angelopoulos and Koutsougeras (2015) showed that each interim maxmin value allocation is strongly interim efficient in L under the Wald's maxmin preferences. Every allocation in L that is strongly interim efficient in L is interim efficient in L by Definitions 2 and 13.[18] Therefore, it is strongly mixed incentive compatible under the Wald's maxmin preferences by Corollary 3. □

7. Conclusion

We improve the result of de Castro and Yannelis (2018) which says that every efficient allocation is incentive compatible under the Wald's maxmin preferences. By introducing lotteries over the commodity space, we show that the set of efficient allocations becomes larger. By taking into account mixed strategies, we show that strong mixed incentive compatibility is a stronger notion than incentive compatibility. Nevertheless, we show that the larger efficient set satisfies the stronger incentive compatibility notion, i.e., every efficient lottery allocation is strongly mixed incentive compatible under the Wald's maxmin preferences. In our framework, the conflict between efficiency and incentive compatibility is resolved by the use of the Wald's maxmin preferences. Unlike Prescott and Townsend (1984) and Sun and Yannelis (2007, 2008), no continuum of agents or the law of large numbers is used. Furthermore, we are able to show that ex ante efficient allocations and interim maxmin value allocations are all strongly mixed incentive compatible under the Wald's maxmin preferences.

Appendix A. Proof of Proposition 1

Proof. Let $x \in X_L$, i.e., x is interim efficient in L. Let $y \in \Delta(L)$ be an arbitrary feasible lottery allocation. We show below that we cannot have $y \succeq_i^{s_i} x$ for all i and s_i, and $y \succ_i^{s_i} x$ for some i and s_i.

For each agent i with type s_i, her interim maxmin expected utility of y is

$$\min_{p_i\in P_i(s_i)} \sum_{s_{-i}\in S_{-i}} v_i\left(y(s_i,s_{-i});s_i\right) p_i(s_{-i})$$
$$= \min_{p_i\in P_i(s_i)} \sum_{s_{-i}\in S_{-i}} \left\{\sum_{k=1}^{K(s_i,s_{-i})} u_i\left(c_i^k, s_i\right)\cdot y(s_i,s_{-i})\left[c^k\right]\right\} p_i(s_{-i}),$$

where $K(s_i, s_{-i})$ consumption profiles occur with non-zero probabilities under $y(s_i, s_{-i})$. Now, define an allocation $z \in L$, such that for each s and i,

$$z_i(s) = \sum_{k=1}^{K(s)} c_i^k \cdot y(s)\left[c^k\right]. \tag{A.1}$$

We show that the allocation z is feasible. Indeed, since y is feasible in $\Delta(L)$, we have that for each s,

$$\sum_{i=1}^{N} z_i(s) = \sum_{i=1}^{N}\sum_{k=1}^{K(s)} c_i^k\cdot y(s)\left[c^k\right] = \sum_{k=1}^{K(s)} y(s)\left[c^k\right]\sum_{i=1}^{N} c_i^k$$
$$= \sum_{k=1}^{K(s)} y(s)\left[c^k\right]\cdot\sum_{i=1}^{N} e_i(s) = \sum_{i=1}^{N} e_i(s).$$

Since u_i is concave in consumption, we have for each s,

$$v_i(y(s);s_i) = \sum_{k=1}^{K(s)} u_i\left(c_i^k, s_i\right)\cdot y(s)\left[c^k\right]$$
$$\leq u_i\left(\sum_{k=1}^{K(s)} c_i^k\cdot y(s)\left[c^k\right], s_i\right)$$
$$= u_i\left(z_i(s), s_i\right) = v_i(z(s);s_i).$$

Thus, for every feasible lottery allocation y in $\Delta(L)$, we can find a feasible allocation z such that once every agent knows the type profile s, every agent prefers z to y. It follows that every agent prefers z to y in the interim under the maxmin preferences, i.e., $z \succeq_i^{s_i} y$ for all i and s_i.

Now, suppose that there exists a feasible lottery allocation y in $\Delta(L)$, such that $y \succeq_i^{s_i} x$ for all i and s_i; furthermore, $y \succ_i^{s_i} x$ for some i and s_i. Then, we have $z \succeq_i^{s_i} y \succeq_i^{s_i} x$ for all i and s_i; furthermore, $z \succeq_i^{s_i} y \succ_i^{s_i} x$ for some i and s_i, where z is defined by (A.1). Since z is in L and z is feasible, we can conclude that x is not interim efficient in L. This contradiction allows us to conclude that we do not have $y \succeq_i^{s_i} x$ for all i and s_i; furthermore, $y \succ_i^{s_i} x$ for some i and s_i. That is, x is interim efficient in $\Delta(L)$. □

[17] If each agent i's utility function u_i depends on her type, i.e., $u_i(x(s), s_i)$, this proposition works too.

[18] Indeed, if an allocation x in L is not interim efficient in L, then there exists a feasible allocation $y \in L$, such that $y \succeq_i^{s_i} x$ for all i for all s_i; furthermore, $y \succ_i^{s_i} x$ for some i and s_i. Now, clearly, x is not strongly interim efficient by Definition 13. The contrapositive statement is that every allocation in L that is strongly interim efficient in L is interim efficient in L.

Appendix B. Proof of Proposition 2

Proof. By way of contradiction, suppose x is not mixed incentive compatible under the random uncertainty-averse (RUA) representation, then there exists an agent i, a type s_i, and a lottery $\alpha_i(s_i)$ such that

$$\min_{p_i\in P_i(s_i)}\sum_{s_{-i}\in S_{-i}} v_i\left(x(s_i,s_{-i});s_i\right)p_i(s_{-i}) <$$
$$\delta\min_{p_i\in P_i(s_i)}\sum_{s_{-i}\in S_{-i}}\sum_{\hat{s}_i\in S_i} v_i\left(e(s_i,s_{-i})+t(\hat{s}_i,s_{-i});s_i\right)\alpha_i(s_i)\left[\hat{s}_i\right]p_i(s_{-i})+$$
$$(1-\delta)\sum_{\hat{s}_i\in S_i}\left\{\min_{p_i\in P_i(s_i)}\sum_{s_{-i}\in S_{-i}} v_i\left(e(s_i,s_{-i})+t(\hat{s}_i,s_{-i});s_i\right)p_i(s_{-i})\right\}\alpha_i(s_i)\left[\hat{s}_i\right].$$

Since $\delta\in[0,1]$ is the weight, it must be that either

$$\min_{p_i\in P_i(s_i)}\sum_{s_{-i}\in S_{-i}} v_i\left(x(s_i,s_{-i});s_i\right)p_i(s_{-i}) < \min_{p_i\in P_i(s_i)}\sum_{s_{-i}\in S_{-i}}\sum_{\hat{s}_i\in S_i} v_i\left(e(s_i,s_{-i})+t(\hat{s}_i,s_{-i});s_i\right)\alpha_i(s_i)\left[\hat{s}_i\right]p_i(s_{-i}), \tag{B.1}$$

or

$$\min_{p_i\in P_i(s_i)}\sum_{s_{-i}\in S_{-i}} v_i\left(x(s_i,s_{-i});s_i\right)p_i(s_{-i}) < \sum_{\hat{s}_i\in S_i}\left\{\min_{p_i\in P_i(s_i)}\sum_{s_{-i}\in S_{-i}} v_i\left(e(s_i,s_{-i})+t(\hat{s}_i,s_{-i});s_i\right)p_i(s_{-i})\right\}\alpha_i(s_i)\left[\hat{s}_i\right], \tag{B.2}$$

or both. We show below that either case implies that x is not strongly mixed incentive compatible (Definition 8).

Case 1: (B.1) holds. Then, this agent i, type s_i and lottery $\alpha_i(s_i)$ together violate Definition 8. Therefore, x is not strongly mixed incentive compatible.

Case 2: (B.2) holds. Since for each $\hat{s}_i\in S_i$, the $\alpha_i(s_i)\left[\hat{s}_i\right]$ is in the set $[0,1]$, then from (B.2) we have that

$$\min_{p_i\in P_i(s_i)}\sum_{s_{-i}\in S_{-i}} v_i\left(x(s_i,s_{-i});s_i\right)p_i(s_{-i}) < \min_{p_i\in P_i(s_i)}\sum_{s_{-i}\in S_{-i}} v_i\left(e(s_i,s_{-i})+t(\hat{s}_i,s_{-i});s_i\right)p_i(s_{-i})$$

for some $\hat{s}_i\in S_i$. That is, this agent i, type s_i and lie $\hat{s}_i\neq s_i$ together violate Definition 5. Therefore, x is not incentive compatible. We can conclude that x is not strongly mixed incentive compatible, since Definition 8 is a stronger notion than Definition 5 as was shown by Corollary 2. □

Appendix C. Proof of Theorem 1

Proof. Suppose that a feasible lottery allocation $x\in\Delta(L)$ is not strongly mixed incentive compatible. We show below that x cannot be interim efficient in $\Delta(L)$.

Since x is not strongly mixed incentive compatible, there exists an agent i, a type s_i, and a mixed deception $\alpha_i(s_i)$, such that

$$\min_{s_{-i}\in S_{-i}} v_i\left(x(s_i,s_{-i});s_i\right) < \min_{s_{-i}\in S_{-i}}\left\{\sum_{\hat{s}_i\in S_i} v_i\left(e(s_i,s_{-i})+t(\hat{s}_i,s_{-i});s_i\right)\alpha_i(s_i)\left[\hat{s}_i\right]\right\}. \tag{C.1}$$

On the right hand side of (C.1), for every s_{-i}, agent i gets a compound lottery. In particular, she gets the simple lottery $e(s_i,s_{-i})+t(\hat{s}_i,s_{-i})$ with probability $\alpha_i(s_i)\left[\hat{s}_i\right]$, for each $\hat{s}_i\in S_i$. Let $z(s_i,s_{-i})$ denote its corresponding reduced simple lottery. Clearly, from (C.1), we have that

$$\min_{s_{-i}\in S_{-i}} v_i\left(x(s_i,s_{-i});s_i\right) < \min_{s_{-i}\in S_{-i}} v_i\left(z(s_i,s_{-i});s_i\right). \tag{C.2}$$

Now, we define a lottery allocation y that Pareto improves x under the Wald's maxmin preferences. Define a lottery allocation y by

$$y(s')=\begin{cases} z\left(s_i,s'_{-i}\right) & \text{if } s'_i=s_i \text{ and } s'_{-i}\in S_{-i};\\ x\left(s'\right) & \text{otherwise.}\end{cases}$$

To see that y is feasible, it is sufficient to consider what happens at each type profile $\left(s_i,s'_{-i}\right)$, where $s'_{-i}\in S_{-i}$, because y is the same as x at every $\left(s'_i,s'_{-i}\right)$, where $s'_i\neq s_i$. At each type profile $\left(s_i,s'_{-i}\right)$, where $s'_{-i}\in S_{-i}$, if c occurs with a non-zero probability under $y\left(s_i,s'_{-i}\right)$, then there must be a corresponding $\tilde{c}$ which occurs with a non-zero probability under $x\left(\hat{s}_i,s'_{-i}\right)$ for some $\hat{s}_i\in S_i$. Moreover, we have

$$\begin{aligned}\sum_{i=1}^N c_i &= \sum_{i=1}^N\left[e_i\left(s_i,s'_{-i}\right)+\tilde{c}_i-e_i\left(\hat{s}_i,s'_{-i}\right)\right]\\ &= \sum_{i=1}^N e_i\left(s_i,s'_{-i}\right)+\sum_{i=1}^N\tilde{c}_i-\sum_{i=1}^N e_i\left(\hat{s}_i,s'_{-i}\right)\\ &= \sum_{i=1}^N e_i\left(s_i,s'_{-i}\right),\end{aligned}$$

since x is feasible, so $\sum_{i=1}^N\tilde{c}_i=\sum_{i=1}^N e_i\left(\hat{s}_i,s'_{-i}\right)$. We can conclude that y is feasible.

From (C.2) and the definition of y, we have

$$\min_{s_{-i}\in S_{-i}} v_i\left(x(s_i,s_{-i});s_i\right) < \min_{s_{-i}\in S_{-i}} v_i\left(y(s_i,s_{-i});s_i\right),$$

under the type s_i; and for any other type s'_i, we have

$$\min_{s_{-i}\in S_{-i}} v_i\left(x\left(s'_i,s_{-i}\right);s'_i\right) = \min_{s_{-i}\in S_{-i}} v_i\left(y\left(s'_i,s_{-i}\right);s'_i\right).$$

That is, agent i with type s_i strictly prefers y to x, while agent i with type $s'_i\neq s_i$ is indifferent between x and y.

Now, it remains to show that any other agent $j\neq i$ prefers y to x in the interim. Fix an arbitrary agent $j\neq i$, and an arbitrary type s_j of agent j. Define $X_j=\left\{x\left(s_j,s'_{-j}\right):s'_{-j}\in S_{-j}\right\}$ and $Y_j=\left\{y\left(s_j,s'_{-j}\right):s'_{-j}\in S_{-j}\right\}$. If $s'_i=s_i$, then agent j's utility of $y\left(s_j,s_i,s'_{-j-i}\right)$ at the type profile $\left(s_j,s_i,s'_{-j-i}\right)$ is $v_j\left(y\left(s_j,s_i,s'_{-j-i}\right);s_j\right)$ which equals

$$\sum_{\hat{s}_i\in S_i}\left\{\sum_{k=1}^{K\left(s_j,\hat{s}_i,s'_{-j-i}\right)} u_j\left(e_j\left(s_j,s_i,s'_{-j-i}\right)+c_j^k-e_j\left(s_j,\hat{s}_i,s'_{-j-i}\right),s_j\right)\times x\left(s_j,\hat{s}_i,s'_{-j-i}\right)\left[c^k\right]\right\}\alpha_i(s_i)\left[\hat{s}_i\right].$$

Now, since e_j is private valued, we have

$$\begin{aligned}&\sum_{k=1}^{K\left(s_j,\hat{s}_i,s'_{-j-i}\right)} u_j\left(e_j\left(s_j,s_i,s'_{-j-i}\right)+c_j^k-e_j\left(s_j,\hat{s}_i,s'_{-j-i}\right),s_j\right)x\left(s_j,\hat{s}_i,s'_{-j-i}\right)\left[c^k\right]\\ &=\sum_{k=1}^{K\left(s_j,\hat{s}_i,s'_{-j-i}\right)} u_j\left(c_j^k,s_j\right)x\left(s_j,\hat{s}_i,s'_{-j-i}\right)\left[c^k\right]\\ &= v_j\left(x\left(s_j,\hat{s}_i,s'_{-j-i}\right);s_j\right).\end{aligned}$$

It follows that

$$v_j\left(y\left(s_j, s_i, s'_{-j-i}\right); s_j\right)$$
$$= \sum_{\hat{s}_i \in S_i} v_j\left(x\left(s_j, \hat{s}_i, s'_{-j-i}\right); s_j\right) \alpha_i\left(s_i\right)\left[\hat{s}_i\right]$$
$$\geq \min_{\hat{s}_i \in S_i} v_j\left(x\left(s_j, \hat{s}_i, s'_{-j-i}\right); s_j\right).$$

For all $s'_i \neq s_i$, we have $y\left(s_j, s'_i, s'_{-j-i}\right) = x\left(s_j, s'_i, s'_{-j-i}\right) \in X_j$. Therefore,

$$\min_{s_{-j} \in S_{-j}} v_j\left(x\left(s_j, s_{-j}\right); s_j\right) \leq \min_{s_{-j} \in S_{-j}} v_j\left(y\left(s_j, s_{-j}\right); s_j\right).$$

Since the agent j and the type s_j are arbitrary, we have that for every agent $j \neq i$, every type s_j, y is preferred to x under the Wald's maxmin preferences.

Thus, the feasible lottery allocation y Pareto improves the feasible lottery allocation x under the Wald's maxmin preferences, i.e., x is not interim efficient in $\Delta(L)$. □

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