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On monotone approximate and exact equilibria of an asymmetric first-price auction with affiliated private information

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Abstract

This paper develops a new approach to investigating equilibrium existence in first-price auctions with many asymmetric bidders whose types are affiliated and valuations are interdependent and not necessarily strictly increasing in own type. We begin with studying a number of continuity-related properties of the model, which are used, in conjunction with tieless single crossing and H-convexity, to establish the existence of monotone approximate interim equilibria. Then we provide two sets of sufficient conditions for the game to have a sequence of monotone approximate equilibria whose limit points are pure-strategy Bayesian-Nash equilibria.

JEL classification: C65; C72

Keywords: First-price auction; Payoff security; Monotone equilibrium; Generalized convexity

1. Introduction

In Bayesian games where the players' best-reply correspondences are nonempty-valued, upper semicontinuous, and contractible-valued, Eilenberg-Montgomery's (1946) fixed-point theo-

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rem can be of help in establishing the existence of monotone Bayesian-Nash equilibria (Reny, 2011). If the payoff functions are not continuous in actions, values of the best-reply correspondences might be empty, and one has to study either finite-action approximations of the game or its approximate best-reply correspondences.

Finite-action approximations of games have served as an important technique for establishing equilibrium existence for quite a while.[1] In normal-form games, the interest in using such approximations has somewhat faded since the introduction of the notion of a better-reply secure game by Reny (1999).[2] However, the technique has retained its relevance in the Bayesian game framework so far. Using finite-action approximations, Athey (2001) investigated the existence of a monotone Bayesian-Nash equilibrium in games with incomplete information. Within the framework of Athey's (2001) approach, Reny and Zamir (2004) carried out a detailed study of an asymmetric first-price auction with affiliated private information and interdependent values.

The approach we employ to examine equilibrium existence in first-price auctions relies on studying the existence of approximate equilibria. If no pure-strategy equilibrium exists in a Bayesian game, both Athey's (2001) finite-action technique and the lattice-theoretic approach (see, e.g., Vives, 1999, 2005; Amir, 2005; Van Zandt and Vives, 2007) are powerless. A possible way to handle such situations is to turn to approximate equilibria[3] For example, one of the assumptions customarily made to ensure the existence of a pure-strategy Bayesian-Nash equilibrium in first-price auctions is that the bidders' valuations are strictly increasing in own type (see, e.g., Athey, 2001; Maskin and Riley, 2003; and Reny and Zamir, 2004). If it does not hold, a Bayesian-Nash equilibrium can fail to exist (see, e.g., Lebrun, 1996, p. 422). However, this assumption is not needed for the existence of approximate equilibria. Consequently, the problem of existence of a pure-strategy Bayesian-Nash equilibrium is reduced to developing sufficient conditions for the limit point of a convergent sequence of ε-equilibria, when ε tends to 0, to be a Bayesian-Nash equilibrium.

This paper's results concerning approximate equilibrium existence rest on two cornerstones: on a number of continuity-related properties of the game and on the H-convexity of each bidder's set of nondecreasing strategies. Among the continuity-related properties of the auction game studied in this paper are: (i) the transfer lower semicontinuity of each interim payoff function in variables different from own bids (interim payoff security, in other terminology); and (ii) the continuity of the interim value and ex-ante value functions. These properties are employed to establish that, in the class of monotone strategies, every approximate best-reply correspondence has the local intersection property; that is, it has a multivalued selection with open lower sections.

Since, in our setting, values of the approximate best-reply correspondences need be neither convex-valued, nor contractible-valued, nor even closed-valued, it might be problematic to use any Kakutani-type fixed-point theorem. Following Athey (2001), McAdams (2003), and Reny (2011), we employ a kind of generalized convexity as an alternative for convexity; namely, each bidder's set of nondecreasing strategies is interpreted as an H-space, a space comprising contractible families of strategies. Then the approximate best-reply correspondences, whose values consist of nondecreasing approximate interim best replies, have H-convex values. Consequently,

[1] See, e.g., Dasgupta and Maskin (1986), Simon (1987), Maskin and Riley (2000), Athey (2001), McAdams (2003), Reny and Zamir (2004).

[2] See, for some recent generalizations of Reny's (1999) theorem, McLennan et al. (2011), Barelli and Meneghel (2013), Carmona and Podczeck (2016), Prokopovych (2016), Reny (2016).

[3] It is also possible to handle the equilibrium nonexistence problem in auctions by introducing special tie-breaking rules (see, e.g., Maskin and Riley, 2000; Jackson et al., 2002; and Araujo and de Castro, 2009).

as shown in Theorem 2, the existence of monotone approximate interim equilibria in the game follows from Horvath's (1987) fixed-point theorem, a generalization of Browder's (1968) fixed-point theorem.

If, in addition, the bidders' payoffs for every vector of types are aggregate upper semicontinuous in bids, then every limit point of a sequence of monotone interim ε-equilibria, with ε tending to 0, is a Bayesian-Nash equilibrium (Theorem 3). Another condition guaranteeing the existence of a convergent sequence of ex-ante ε-equilibria whose limit point is a Bayesian-Nash equilibrium of the game is that each bidder's valuation is strictly increasing in own type (Theorem 4).

We illustrate the proposed equilibrium existence conditions with a number of examples. Examples 1 and 2 illustrate the fact that no pure-strategy Bayesian-Nash equilibrium might exist in a first-price auction if bidders' valuations are not strictly increasing in own type. In such cases, it is natural to turn to studying approximate Bayesian-Nash equilibria. Example 3 describes another problem that might appear when bidders' valuation are not strictly increasing in own type, namely, the absence of Bayesian-Nash equilibria in strictly increasing strategies, which itself can considerably complicate investigation of such games. Examples 4 and 5 are common-value first-price auctions in which the bidders' valuations are not strictly increasing in own type and the existence of a monotone pure-strategy Bayesian-Nash equilibrium follows from Theorem 3, but not from Athey's (2001) and Reny-Zamir's (2004) results. Example 6 explains some subtleties of choosing an appropriate sequence of monotone approximate equilibria when the bidders' valuations are strictly increasing in own type. Example 7 is a private-value first-price auction with subsidies.

The structure of the paper is as follows. Section 2 contains the model and some theoretical underpinnings necessary for studying equilibrium existence in the Bayesian game. A number of important continuity-related properties of the interim payoff and value functions are investigated in Section 3. In Section 4, we provide sufficient conditions for every approximate interim best-reply correspondence to have a nondecreasing single-valued selection. The existence of monotone interim ε-equilibria and Bayesian-Nash equilibria in the first-price auction is studied in Section 5. The section also contains examples illustrating the equilibrium existence conditions. A number of proofs are relegated to the Appendix.

2. Preliminaries

In this section, we describe the model and provide a number of auxiliary definitions and facts.

2.1. The model

Consider the following n-player Bayesian game, denoted by $\Gamma = (B_i, T_i, f, v_i)_{i\in I}$, where $I = \{1, \dots, n\}$ is the set of bidders (with $n \geq 2$); $B_i = [0, c]$, $c > 0$, is the set of bids available to bidder i; $T_i = [0, 1]$ is player i's set of types. Let $B = \times_{i\in I} B_i$ and $T = \times_{i\in I} T_i$. The joint density of the bidders' types, f, is a continuous function from T to $(0, +\infty)$. Each player i's payoff function $v_i : B \times T \to \mathbb{R}$ can be represented as follows:

$$v_i(b; t) = \begin{cases} \frac{1}{m} u_i(b_i; t) \text{ if } m = \#\{j \in I : b_j = b_i = \max_{k\in I} b_k\} \geqslant 1, \\ 0 \text{ otherwise,} \end{cases}$$

where $u_i : B_i \times T \to \mathbb{R}$ has the following properties:

(i) u_i is jointly continuous in (b_i, t), nondecreasing in t_j for each $j \in I$, and nonincreasing in b_i for all $t \in T$;
(ii) $u_i(0; t) \geq 0$ and $u_i(c; t) < 0$ for all $t \in T$;
(iii) u_i has increasing differences in (b_i, t_i) or (b_i, t).[4]

Each function u_i is assumed to be nondecreasing, not necessarily strictly increasing, in bidder i's own type, which does not suffice for the existence of a pure-strategy Bayesian-Nash equilibrium in first-price auctions with incomplete information (see, e.g., Lebrun, 1996, p. 422; and Examples 1 and 2 below). To ensure the payoff security of the game, it is assumed that, for each $i \in I$, u_i is jointly continuous in its variables, and, moreover, that the joint density function f is continuous, which is a conventional assumption in action theory (see, e.g., Milgrom, 2004; and Krishna, 2010). Another assumption to be mentioned here is that, for each $i \in I$, u_i is nonincreasing in own bid b_i, which is also not counterintuitive, since u_i represents player i's payoff in the case of winning the item. If a bidder wins it with a bid, then a higher bid should not result in a higher payoff to the bidder.

On the other hand, from a theoretical point of view, if the bidders' valuations are nonincreasing in own bids, then the definition of tieless single crossing need not include Reny and Zamir's (2004) individual rationality condition, which is helpful when approximate interim best-reply correspondences are involved.

Denote by L_i the set of Lebesgue measurable functions from T_i to B_i and by S_i the set of nondecreasing functions from T_i to B_i. Clearly, $S_i \subset L_i$. Each set L_i (S_i), equipped with the semimetric $d_1(s_i, s_i') = \int_{T_i} \left| s_i(t_i) - s_i'(t_i) \right| dt_i$ for all $s_i, s_i' \in L_i$ (resp., $s_i, s_i' \in S_i$), can be considered as a metric space (resp., a compact metric space) consisting of equivalent classes of Lebesgue measurable functions, though it is conventional to act as though the elements of the metric space are functions, not equivalence classes of functions. Any Cartesian product of a finite number of metric spaces is assumed to be a metric space endowed with a product metric that induces the product topology on the Cartesian product. Denote $L = \times_{i \in I} L_i$ and $L_{-i} = \times_{j \in I \setminus \{i\}} L_j$. Similarly, define S and S_{-i} to be the Cartesian products of the corresponding S_j's. Though below we confine the bidders' choices to nondecreasing strategies, no bidder will be able to deviate profitably from an approximate or exact monotone equilibrium using a Lebesgue measurable (not necessarily monotone) strategy.

In simple cases (two players or independent types), we will assume increasing differences between the players' bids and their own types only.

Assumption A.1. Each u_i exhibits increasing differences in (b_i, t_i); that is, for every $\overline{b}_i$ and $\underline{b}_i$ in B_i with $\overline{b}_i > \underline{b}_i$ and every $t_{-i} \in T_{-i}$, $(u_i(\overline{b}_i; \cdot, t_{-i}) - u_i(\underline{b}_i; \cdot, t_{-i})) : T_i \to (-\infty, 0]$ is nondecreasing.

In the case of more than two bidders with affiliated types, we will additionally assume increasing differences between b_i and t_{-i}.

Assumption A.2. Each u_i exhibits increasing differences in (b_i, t); that is, for every $\overline{b}_i$ and $\underline{b}_i$ in B_i with $\overline{b}_i > \underline{b}_i$, each $j \in I$ and every $t_{-j} \in T_{-j}$, $(u_i(\overline{b}_i; \cdot, t_{-j}) - u_i(\underline{b}_i; \cdot, t_{-j})) : T_j \to (-\infty, 0]$ is nondecreasing.

[4] These two conditions can be found below as Assumptions A.1 and A.2, respectively.

The next two conditions concern the bidders' beliefs about the distribution of types. The conditional density function of t_{-i} given t_i is denoted by $f_{-i}(t_{-i}|t_i)$. By definition, $f_{-i}(t_{-i}|t_i) = \frac{f(t_i,t_{-i})}{f_i(t_i)}$ where $f_i(t_i) = \int_{T_{-i}} f(t_i, t_{-i})dt_{-i}$ is the marginal density function of t_i. Assumption B.1 naturally arises in applications (see, e.g., Milgrom and Weber, 1982; and Athey, 2002).

Assumption B.1. The function $f : T \to (0, +\infty)$ satisfies the following logsupermodularity condition: $f(t \wedge t')f(t \vee t') \geq f(t)f(t')$ for all $t, t' \in T$, where $\wedge$ and $\vee$ denote the componentwise minimum and maximum of t and t', respectively.

Assumption B.2 is a particular case of Assumption B.1 when the bidders have independent types.

Assumption B.2. The function $f : T \to (0, +\infty)$ satisfies the following condition: $f(t) = f_1(t_1) \times \ldots \times f_n(t_n)$ for all $t \in T$.

Bidder i's interim payoff function $V_i : B_i \times L_{-i} \times T_i \to \mathbb{R}$ is defined by

$$V_i(b_i, s_{-i}; t_i) = \int\limits_{T_{-i}} v_i(b_i, s_{-i}(t_{-i}); t_i, t_{-i}) f_{-i}(t_{-i}|t_i)dt_{-i},$$

and bidder i's interim value function $\overline{V}_i : L_{-i} \times T_i \to \mathbb{R}$ is defined by

$$\overline{V}_i(s_{-i}; t_i) = \sup_{b_i \in B_i} V_i(b_i, s_{-i}; t_i).$$

Given a Lebesgue measurable subset A of T_{-i}, we will also need the following auxiliary function $V_i(\cdot, \cdot; \cdot, A) : B_i \times L_{-i} \times T_i \to \mathbb{R}$ defined by

$$V_i(b_i, s_{-i}; t_i, A) = \int\limits_{A} v_i(b_i, s_{-i}(t_{-i}); t_i, t_{-i}) f_{-i}(t_{-i}|t_i)dt_{-i}.$$

Bidder i's ex-ante payoff function $V_i^* : L \to \mathbb{R}$ is defined by

$$V_i^*(s) = \int\limits_{T_i} V_i(s_i(t_i), s_{-i}; t_i) f_i(t_i)dt_i.$$

Bidder i's ex-ante value function $\overline{V}_i^* : L_{-i} \to \mathbb{R}$ is defined by

$$\overline{V}_i^*(s_{-i}) = \sup_{s_i \in L_i} V_i^*(s_i, s_{-i}).$$

A strategy profile $s \in L$ is an interim ε-equilibrium ($\varepsilon > 0$) of Γ if, for each $i \in I$ and for almost all $t_i \in T_i$,

$$V_i(s_i(t_i), s_{-i}; t_i) > \overline{V}_i(s_{-i}; t_i) - \varepsilon.$$

A strategy profile $s \in L$ is an ex-ante ε-equilibrium of Γ ($\varepsilon > 0$) if $V_i^*(s) > \overline{V}_i^*(s_{-i}) - \varepsilon$.

In general, the notion of an interim ε-equilibrium is stronger than the notion of an ex-ante ε-equilibrium which allows some vanishing set of types of bidders not to be even approximately optimizing (see, for more details, Van Zandt, 2010; and Jackson et al., 2012).

A Bayesian-Nash equilibrium of Γ is a strategy profile $s \in L$ such that $V_i^*(s_i, s_{-i}) = \overline{V}_i^*(s_{-i})$ for each $i \in I$.

We begin with a number of auxiliary facts which will be used throughout the paper.

2.2. Generalized convexity and payoff security

In games with complete information, employing payoff security in conjunction with quasiconcavity might facilitate investigating the existence of approximate Nash equilibria considerably (Prokopovych, 2011). Quasiconcavity turns out to be a too demanding condition which rarely holds in Bayesian games. In some of them, it is possible to proceed completely within a complementarity-based framework (see, e.g., Vives, 1990, 1999; Van Zandt and Vives, 2007: Balbus et al., 2015; and Dekel and Pauzner, 2018). In others, among the tools employed to make the nonquasiconcavity problem more tractable are purification techniques, behavioral strategies, and maximin preferences.[5]

To circumvent the nonquasiconcavity-related problems, Reny (2011) made use of the contractibility of the values of upper semicontinuous best-reply correspondences in a class of continuous Bayesian games. In our model, the approximate best-reply correspondences need be neither upper hemicontinuous nor contractible-valued. Instead, they possess the local intersection property and have H-convex values, if the bidders' strategy sets are confined to nondecreasing functions and approximate interim (not ex-ante) best replies are considered.

We interpret each strategy set S_i as an H-space. The definition of an H-space is as follows. An H-space is a pair $(X, \{F_A\})$, where X is a topological space and $\{F_A\}$ is a family of nonempty contractible subsets of X, indexed by the finite subsets of X, such that $A \subset B$ implies $F_A \subset F_B$ (see, e.g., Horvath, 1987, 1991; Bardaro and Ceppitelli, 1988; and Tarafdar, 1990). If X is compact, then we say that $(X, \{F_A\})$ is a compact H-space. Given an H-space, $(X, \{F_A\})$, a nonempty subset D of X is called H-convex if $F_A \subset D$ for each finite subset A of D.

We now describe an H-space associated with each strategy set S_i. Given $s_i, s_i' \in S_i$, let the function $(s_i \vee s_i') : T_i \to B_i$ be defined by $(s_i \vee s_i')(t_i) = \max(s_i(t_i), s_i'(t_i))$ for all $t_i \in T_i$. For a finite family of strategies $A_i = \{s_{1i}, \ldots, s_{ki}\} \subset S_i$, define F_{A_i} as the minimal subset of S_i which possesses the following properties: (i) it contains A_i; (ii) if $s_i, s_i' \in F_{A_i}$, then $s_i \vee s_i' \in F_{A_i}$; (iii) for every s_i and s_i' are in F_{A_i} and every $\tau \in (0, 1)$, $s_i \mathcal{X}_{[0,1-\tau]} + (s_i \vee s_i')\mathcal{X}_{(1-\tau,1]}$ is in F_{A_i}, where $\mathcal{X}_D$ denotes the indicator function of the set D. We need to check that: (a) the set F_{A_i} is contractible, and (b) $F_{A_i} \subset F_{B_i}$ if $A_i \subset B_i$ and B_i is also a finite subset of S_i. Part (b) is clear. To show (a), we need to find a point $\overline{s}_i \in F_{A_i}$ and a continuous function $h : [0, 1] \times F_{A_i} \to F_{A_i}$ such that $h(0, s_i) = s_i$ and $h(1, s_i) = \overline{s}_i$ for all $s_i \in F_{A_i}$; that is, we need to show the function s_i can be continuously deformed into $\overline{s}_i$ by the homotopy h.

Denote $\overline{s}_i = s_{1i} \vee \ldots \vee s_{ki}$. It is clear that $\overline{s}_i \in F_{A_i}$ and $\overline{s}_i(t_i) \geq s_i(t_i)$ for all $s_i \in F_{A_i}$ and all $t_i \in T_i$. To construct a homotopy that continuously shrinks the entire set F_{A_i} to $\overline{s}_i$, we use a technique due to Reny (2011).[6] For any $\tau \in T_i$ and $s_i \in F_{A_i}$, define the function $h : [0, 1] \times F_{A_i} \to F_{A_i}$ as follows:

$$h(\tau, s_i)(t_i) = \begin{cases} s_i(t_i), \text{ if } t_i \leq 1 - \tau \text{ and } \tau < 1, \\ \overline{s}_i(t_i), \text{ otherwise.} \end{cases}$$

Clearly, h acts continuously from $[0, 1] \times F_{A_i}$ to F_{A_i}, $h(0, s_i) = s_i$ and $h(1, s_i) = \overline{s}_i$. A formal proof of the joint continuity of h in τ and s_i for multidimensional type sets can be found in Section 6 of Reny (2011). Consequently, each pair $(S_i, \{F_{A_i}\})$ is an H-space.

5 See, e.g., He and Sun, 2014; He and Yannelis, 2015; Brookins and Ryvkin, 2016; He and Yannelis, 2017a; and Carbonell-Nicolau and McLean, 2018.

6 The technique is intrinsically related to using decomposability as a substitute for convexity (see, e.g., Olech, 1984; Bressan and Colombo, 1988; Fryszkowski, 2004). Athey (2001) also employed a related kind of generalized convexity.

As shown by Reny (2011), it is possible to use Eilenberg-Montgomery's (1946) theorem for studying the existence of Bayesian-Nash equilibria in a class of continuous Bayesian games. Since the game studied in this paper is discontinuous, another fixed-point result is needed. We first provide a number of basic definitions in the context of normal-form games.

Let each X_i, $i \in I = \{1, \ldots, n\}$, be a topological space, and let $X = \times_{i \in I} X_i$ and $X_{-i} = \times_{j \in I \setminus \{i\}} X_j$. A normal-form game $G = (X_i, u_i)_{i \in I}$ is payoff secure if for each $i \in I$ and every $x \in X$ and $\varepsilon > 0$, there exists $\overline{x}_i \in X_i$ such that $u_i(\overline{x}_i, x'_{-i}) > u_i(x) - \varepsilon$ for all x'_{-i} in some open neighborhood of x_{-i} (Reny, 1999). In other words, in payoff secure games, each player's payoff function is transfer lower semicontinuous in the other players' strategies (see, e.g., Tian, 1992; and Prokopovych, 2011).

If $G = (X_i, u_i)_{i \in I}$ is payoff secure and player i's value function

$$\overline{u}_i(x_{-i}) = \sup_{x'_i \in X_i} u_i(x'_i, x_{-i})$$

is continuous, then player i's ε-best-reply correspondence $M_i^\varepsilon : X_{-i} \twoheadrightarrow X_i$ ($\varepsilon > 0$), defined by

$$M_i^\varepsilon(x_{-i}) = \{x_i \in X_i : u_i(x_i, x_{-i}) > \overline{u}_i(x_{-i}) - \varepsilon\},$$

has the local intersection property; that is, for every $x_{-i} \in X_{-i}$, there exists $\overline{x}_i \in X_i$ such that $\overline{x}_i \in M_i^\varepsilon(x'_{-i})$ for all x'_{-i} in some open neighborhood of x_{-i} in X_i. In compact quasiconcave games, the fact that each M_i^ε is convex-valued makes it possible to apply a generalization of Browder's (1968) fixed-point theorem to the approximate equilibrium existence problem.[7]

In this paper, we make use of another generalization of Browder's fixed-point theorem, due to Horvath (1987), in which convexity is replaced with H-convexity (see Horvath, 1987, Theorem 2′; Tarafdar, 1992, Corollary 2.3; or Tarafdar and Chowdhury, 2008, Theorem 4.69).[8]

Theorem 1 *(Horvath, 1987). Let $(X, \{F_A\})$ be a compact H-space, and let $M : X \twoheadrightarrow X$ be a correspondence with nonempty H-convex values that has the local intersection property. Then there exists $\overline{x} \in X$ such that $\overline{x} \in M(\overline{x})$.*

From an applications' point of view, the following corollary of Theorem 1 is helpful (see, e.g., Tarafdar, 1992, Theorem 2.3). In its statement, $\{F_A^i\}$ denotes a family of nonempty contractible subsets of X_i, indexed by the finite subsets of X_i, such that $A_i \subset B_i$ in X_i implies $F_{A_i}^i \subset F_{B_i}^i$.

Corollary 1. *Let $(X_i, \{F_{A_i}^i\})$, $i = 1, \ldots, n$, be a family of compact H-spaces. If each $M_i : X_{-i} \twoheadrightarrow X_i$ is a correspondence with nonempty H-convex values that has the local intersection property, then there exists $\overline{x} \in X$ such that $\overline{x}_i \in M_i(\overline{x}_{-i})$ for each i.*

Proof. To show Corollary 1, consider the correspondence $M : X \twoheadrightarrow X$ defined by $M(x) = (M_1(x_{-1}), \ldots, M_n(x_{-n}))$ for all $x \in X$. Obviously, it has the local intersection property. By Lemma 2 of Tarafdar (1990) or by Lemma 4.19 of Tarafdar and Chowdhury (2008), the product of any number of H-spaces is an H-space and the product of H-convex subsets is H-convex. Therefore, the hypotheses of Theorem 1 are satisfied. □

[7] See, for more details, Yannelis and Prabhakar (1983); Wu and Shen (1996); Prokopovych (2011); and He and Yannelis (2017b).

[8] Using Browder's fixed-point theorem is a matter of convenience. See, for details, Yannelis (1991).

For these techniques to become applicable to the approximate equilibrium existence problem, we need to provide sufficient conditions for the approximate best-reply correspondences to possess nondecreasing selections and the local intersection property. We begin with a number of important continuity-related properties of the game.

3. Continuity-related properties of the interim payoff and value functions

In this section, we investigate a number of continuity-related properties of the interim payoff and value functions of the first-price auction. Though the interim payoff functions are, obviously, continuous in own type, they need not be continuous in other variables. If the bidders employ nondecreasing strategies, the interim payoff and value functions possess a number of continuity-related properties that might considerably facilitate studying the equilibrium existence problem. Among the propitious properties studied below are the transfer lower semicontinuity of the interim payoff functions in variables different from own bids and the continuity of the interim value and ex-ante value functions.

The next lemma shows that, in the absence of nonzero-measure bid ties, the interim payoff functions are continuous, which is intuitively clear. From now on, for any Lebesgue-measurable subset A of a Euclidean space, $\mu(A)$ denotes its Lebesgue measure.

Lemma 1. *If, in the game Γ, for some player i and some triple $(b_i^0, s_{-i}^0, t_i^0) \in B_i \times L_{-i} \times T_i$, $\mu(t_j \in T_j : b_i^0 = s_j^0(t_j)) = 0$ for all $j \in I \backslash \{i\}$, then the interim payoff function $V_i : B_i \times L_{-i} \times T_i \to \mathbb{R}$ is continuous at (b_i^0, s_{-i}^0, t_i^0).*

Proof. Consider a sequence $\{(b_i^k, s_{-i}^k, t_i^k)\}$, $(b_i^k, s_{-i}^k, t_i^k) \in B_i \times L_{-i} \times T_i$, $k = 1, \ldots$, converging to (b_i^0, s_{-i}^0, t_i^0). We need to show that $\lim_k V_i(b_i^k, s_{-i}^k; t_i^k) = V_i(b_i^0, s_{-i}^0; t_i^0)$.

Assume, by way of contradiction, that $\lim_k V_i(b_i^k, s_{-i}^k; t_i^k)$ exists but is not equal to $V_i(b_i^0, s_{-i}^0; t_i^0)$. For each $k \in \{0, 1, \ldots\}$, define $g_i^k : T_{-i} \to \mathbb{R}$ by $g_i^k(t_{-i}) = v_i(b_i^k, s_{-i}^k(t_{-i}); t_i^k, t_{-i}) f_{-i}(t_{-i}|t_i^k)$. Since each sequence $\{s_j^k\}$, $j \in I \backslash \{i\}$, converges to s_j^0 in mean, it has a subsequence $\{s_j^{k_m}\}$ convergent to s_j^0 almost everywhere (see, e.g., Cohn, 1980, Propositions 3.1.2 and 3.1.4). There is no loss of generality in assuming that the subsequence $\{s_j^{k_m}\}$ tends to s_j^0 almost everywhere for all $j \in I \backslash \{i\}$. Let R_j, $j \in I \backslash \{i\}$, denote a subset of T_j containing all points t_j at which either $\{s_j^{k_m}(t_j)\}$ does not converge to $s_j^0(t_j)$, or $s_j^0(t_j) = b_i^0$, or both of the properties hold. Since each R_j has zero measure, $\lim_{k_m} g_i^{k_m}(t_{-i}) = g_i^0(t_{-i})$ at almost every $t_{-i} \in T_{-i}$. Then, by Lebesgue's dominated convergence theorem, $\lim_{k_m} V_i(b_i^{k_m}, s_{-i}^{k_m}; t_i^{k_m}) = V_i(b_i^0, s_{-i}^0; t_i^0)$, a contradiction. □

We will also need the following corollary of Lemma 1.

Corollary 2. *Let, in the game Γ, $s_i : T_i \to B_i$ be a step function for some $i \in I$, and let $s_{-i} \in L_{-i}$ be such that $\mu(t_j \in T_j : s_i(t_i) = s_j(t_j)) = 0$ for every $t_i \in T_i$ and each $j \in I \backslash \{i\}$. Then, for every $\varepsilon > 0$, there exists an open neighborhood $\mathcal{N}_\varepsilon(s_{-i})$ of s_{-i} in L_{-i} such that $V_i(s_i(t_i), s'_{-i}; t_i) > V_i(s_i(t_i), s_{-i}; t_i) - \varepsilon$ for every $t_i \in T_i$ and every $s'_{-i} \in \mathcal{N}_\varepsilon(s_{-i})$.*

For the reader's convenience, the proof of Corollary 2 is provided in the Appendix.

The next statement shows that it is possible, in a sense, to get rid of bid ties of positive measure. The main difficulty while handling ties at a bidder's bid is that the number of other bidders tied at the bid might change from type subprofile to type subprofile. As a result, if, for some $(b_i, s_{-i}, t_i) \in B_i \times S_{-i} \times T_i$, $V_i(b_i, s_{-i}; t_i) > 0$ and there are ties at b_i, then the aggregate effect of getting rid of ties at b_i by raising the bid a little might look ambiguous. However, it is shown in Lemma 2 that if the other players employ nondecreasing strategies, the number of bidders tied at b_i at the type subprofiles t_{-i} where V_i is negative tends to be smaller than the number of bidders tied at b_i at the type subprofiles t_{-i} where V_i is positive, which follows from the fact that each u_i is nondecreasing in the other bidders' types.

Lemma 2. *If, in the game Γ, for some $i \in I$, a bid $b_i \in B_i$, a subprofile of strategies $s_{-i} \in S_{-i}$, and a type $t_i \in T_i$ are such that $\mu(t_j \in T_j : b_i = s_j(t_j)) > 0$ for some $j \in I\backslash\{i\}$, then for every $\varepsilon > 0$ there exists $\widetilde{b}_i \in B_i$ such that: (i) $V_i(\widetilde{b}_i, s_{-i}; t_i) > V_i(b_i, s_{-i}; t_i) - \varepsilon$; and (ii) $\mu(t_j \in T_j : \widetilde{b}_i = s_j(t_j)) = 0$ for each $j \in I\backslash\{i\}$.*

The proof of Lemma 2 can be found in the Appendix.

Corollary 3. *In the game Γ, for every $i \in I$, $s_{-i} \in S_{-i}$, $t_i \in T_i$, and $\varepsilon > 0$, there exists $\widetilde{b}_i \in B_i$ such that: (i) $V_i(\widetilde{b}_i, s_{-i}; t_i) > \overline{V}_i(s_{-i}; t_i) - \varepsilon$; and (ii) $\mu(t_j \in T_j : \widetilde{b}_i = s_j(t_j)) = 0$ for each $j \in I\backslash\{i\}$.*

The following proposition states an important property of V_i, its transfer lower semicontinuity in (s_{-i}, t_i).

Proposition 1. *In Γ, each interim payoff function $V_i : B_i \times S_{-i} \times T_i \to \mathbb{R}$ is transfer lower semicontinuous in (s_{-i}, t_i).*

Proof. Fix some $i \in I$, $(b_i, s_{-i}, t_i) \in B_i \times S_{-i} \times T_i$ and $\varepsilon > 0$. We need to show that there exist a bid $\widetilde{b}_i \in B_i$ and an open neighborhood $\mathcal{N}(s_{-i}, t_i)$ of (s_{-i}, t_i) in $S_{-i} \times T_i$ such that $V_i(\widetilde{b}_i, s'_{-i}; t'_i) > V_i(b_i, s_{-i}; t_i) - \varepsilon$ for every $(s'_{-i}, t'_i) \in \mathcal{N}(s_{-i}, t_i)$. If $\mu(t_j \in T_j : b_i = s_j(t_j)) = 0$ for all $j \in I\backslash\{i\}$, then the claim follows immediately from Lemma 1. Otherwise, by Lemma 2, there exists $\widetilde{b}_i \in B_i$ such that $V_i(\widetilde{b}_i, s_{-i}; t_i) > V_i(b_i, s_{-i}; t_i) - \varepsilon$ and $\mu(t_j \in T_j : \widetilde{b}_i = s_j(t_j)) = 0$ for all $j \in I\backslash\{i\}$. Then, by Lemma 1, V_i is continuous at $(\widetilde{b}_i, s_{-i}; t_i)$, which completes the proof. □

The transfer lower semicontinuity of V_i in (s_{-i}, t_i) implies the joint lower semicontinuity of $\overline{V}_i$ in (s_{-i}, t_i) (see, e.g., Prokopovych, 2011, Lemma 4).

Corollary 4. *In the game Γ, each interim value function $\overline{V}_i : S_{-i} \times T_i \to \mathbb{R}$ is lower semicontinuous.*

If the players employ nondecreasing strategies, the interim value functions are also continuous.

Proposition 2. *In the game Γ, each interim value function $\overline{V}_i : S_{-i} \times T_i \to \mathbb{R}$ is continuous.*

The proof of Proposition 2 is relegated to the Appendix.

An important corollary of Propositions 1 and 2 is that the interim ε-best-reply correspondences have the local intersection property.

Corollary 5. *In Γ, for each i and every $\varepsilon > 0$, the correspondence $M_i^\varepsilon : S_{-i} \times T_i \twoheadrightarrow B_i$ defined by*

$$M_i^\varepsilon(s_{-i}; t_i) = \{b_i \in B_i : V_i(b_i, s_{-i}; t_i) > \overline{V}_i(s_{-i}; t_i) - \varepsilon\}$$

has the local intersection property.

Proof. Since, by Proposition 1, V_i is transfer lower semicontinuous in $(s_{-i}; t_i)$ and, by Proposition 2, $\overline{V}_i$ is continuous in (s_{-i}, t_i), the function $g : B_i \times S_{-i} \times T_i \to \mathbb{R}$ defined by $g(b_i, s_{-i}, t_i) = V_i(b_i, s_{-i}; t_i) - \overline{V}_i(s_{-i}; t_i)$ is transfer lower semicontinuous in (s_{-i}, t_i), which implies that M_i^ε has the local intersection property. □

Unfortunately, Corollary 5 cannot be applied directly to the equilibrium existence problem since the correspondences act from $S_{-i} \times T_i$ to B_i, not from S_{-i} to S_i.

4. Nondecreasing approximate interim best-reply strategies

In this section, we study sufficient conditions for every approximate interim best-reply correspondence to have a nondecreasing selection. First, we introduce, for the Bayesian game Γ, the tieless single-crossing property (TSCP). Then we study a number of combinations of Assumptions A.1, A.2, B.1, and B.2 that are sufficient for Γ to possess the TSCP. This property, in conjunction with the fact that every approximate interim best-reply correspondence has the local intersection property, makes it possible to choose, in a constructive manner, a nondecreasing single-valued selection from every approximate interim best-reply correspondence of the game.[9]

Definition 1. The Bayesian game $\Gamma = (B_i, T_i, f, u_i)_{i \in I}$ satisfies the upward TSCP (the downward TSCP) if for each $i \in I$, all $b_{i1}, b_{i2} \in B_i$ with $b_{i2} > b_{i1}$ (resp., $b_{i2} < b_{i1}$), and all nondecreasing strategies $s_j : T_j \to B_j$ of the other players $j \in I \backslash \{i\}$ such that $\mu(t_j \in T_j : b_{il} = s_j(t_j)) = 0$, $l = 1, 2$, the following condition holds: If $V_i(b_{i2}, s_{-i}; t_i) - V_i(b_{i1}, s_{-i}; t_i) \geq 0$ for some $t_i \in T_i$, then the inequality holds when t_i rises (resp., falls). If Γ satisfies the upward and downward TSCPs, then we say that it satisfies the TSCP.

Reny and Zamir (2004) showed that, in the first-price auction environment, Milgrom-Shannon's (1994) single-crossing property ought to take the form of the individually rational tieless single-crossing condition (IRT-SCC). The TSCP is akin to Reny and Zamir's (2004) IRT-SCC, with a number of minor distinctions. In the TSCP, (i) all nonzero-measure ties are excluded from consideration, not only such ties at winning bids; (ii) there is no individual rationality condition; and (iii) it is explicitly divided into two directional parts.[10] A number of negative consequences of having to deal with nonzero-measure ties are described in Reny and Zamir (2004) and McAdams (2007).

[9] As shown by Amir and de Castro (2017), replacing monotonicity with quasimonotonicity can be helpful in games with complete information. For the purposes of this paper, we need the existence of monotone approximate interim best-reply strategies.

[10] See Prokopovych and Yannelis (2017) and Kukushkin (2018) for the rationale behind the separation in games with complete information.

Another equivalent definition of the TSCP is the following: The Bayesian game $\Gamma = (B_i, T_i, f, u_i)_{i \in I}$ satisfies the strong upward TSCP (the strong downward TSCP) if for each $i \in I$, all $b_{i1}, b_{i2} \in B_i$ with $b_{i2} > b_{i1}$ (resp., $b_{i2} < b_{i1}$), and all nondecreasing strategies $s_j : T_j \to B_j$ of the other players $j \in I \backslash \{i\}$ such that $\mu(t_j \in T_j : b_{il} = s_j(t_j)) = 0$, $l = 1, 2$, the following condition holds: If $V_i(b_{i2}, s_{-i}; t_i) - V_i(b_{i1}, s_{-i}; t_i) > 0$ for some $t_i \in T_i$, then the inequality holds when t_i rises (resp., falls). If Γ satisfies the strong upward and strong downward TSCPs, then it also satisfies the TSCP. That is, the weak inequality in the definition of the TSCP can be replaced with the strict inequality.

We begin with studying the TSCP in two special cases where increasing differences are needed only between each player's bids and her own types (Assumption A.1). The first one is the case of independent private values.

Proposition 3. *If Assumptions A.1 and B.2 hold in the game Γ, then it satisfies the TSCP.*

The proof of Proposition 3 is relegated to the Appendix.

If the bidders have interdependent values (Assumption B.1), it is also possible to avoid assuming increasing differences between their bids and the other bidders' types (Assumption A.2) in the two-bidder case.

Proposition 4. *If Assumptions A.1 and B.1 hold in the game Γ and $I = \{1, 2\}$, then it satisfies the TSCP.*

Proof. We will now show that the upward TSCP holds for bidder 1. Let $b_{11}, b_{12} \in B_1$ with $b_{11} < b_{12}$, and let s_2 be a nondecreasing strategy of bidder 2 such that $\mu(t_2 \in T_2 : b_{1l} = s_2(t_2)) = 0$, $l = 1, 2$. Denote $\Delta V_1(t_1) = V_1(b_{12}, s_2; t_1) - V_1(b_{11}, s_2; t_1)$ for $t_1 \in T_1$. Pick some $t_{11}, t_{12} \in T_1$ with $t_{11} < t_{12}$. We need to show that $\Delta V_1(t_{11}) \geq 0$ implies that $\Delta V_1(t_{12}) \geq 0$.

Consider the following two subsets of T_2:

$$U = \{t_2 \in T_2 : b_{11} > s_2(t_2)\};$$

$$W = \{t_2 \in T_2 : b_{11} < s_2(t_2) \text{ and } b_{12} > s_2(t_2)\}.$$

Denote $u_1(b_{12}; t_1, t_2) - u_1(b_{11}; t_1, t_2)$ by $h_1(t_1, t_2)$, and $u_1(b_{12}; t_1, t_2)$ by $g_1(t_1, t_2)$, and define $I_U : T_1 \to \mathbb{R}$ and $I_W : T_1 \to \mathbb{R}$ as follows:

$$I_U(t_1) = \int\limits_U h_1(t_1, t_2) f_2(t_2|t_1) dt_2,$$

$$I_W(t_1) = \int\limits_W g_1(t_1, t_2) f_2(t_2|t_1) dt_2.$$

Since u_1 is nonincreasing in b_1 and $b_{12} > b_{11} > s_2(t_2)$ for all $t_2 \in U$, $I_U(t_{11}) \leq 0$. Since $\Delta V_1(t_{11}) = I_U(t_{11}) + I_W(t_{11}) \geq 0$, we have $I_W(t_{11}) \geq 0$.

Put $t_2' = \inf_{t_2 \in T_2}\{t_2 \in W\colon g_1(t_{11}, t_2) \geq 0\}$. Denote $W' = \{t_2 \in W\colon t_2 < t_2'\}$ and $W'' = \{t_2 \in W\colon t_2 \geq t_2'\}$. Clearly, W' might be empty. Denote $k = \frac{f_2(t_2'|t_{12})}{f_2(t_2'|t_{11})}$. It follows from the affiliation of f that $\frac{f_2(t_2|t_{12})}{f_2(t_2|t_{11})} \leq k$ for every $t_2 \in U \cup W'$ and $k \leq \frac{f_2(t_2|t_{12})}{f_2(t_2|t_{11})}$ for every $t_2 \in W''$. Also recall that h_1 takes on only nonpositive values. Then, for all $t_2 \in U$,

$$h_1(t_{12}, t_2) f_2(t_2|t_{12}) \geq h_1(t_{11}, t_2) f_2(t_2|t_{12}) \geq h_1(t_{11}, t_2) f_2(t_2|t_{11}) k;$$

and, for all $t_2 \in W' \cup W''$,

$$g_1(t_{12}, t_2) f_2(t_2|t_{12}) \geq g_1(t_{11}, t_2) f_2(t_2|t_{12}) \geq g_1(t_{11}, t_2) f_2(t_2|t_{11})k.$$

Therefore, $\Delta V_1(t_{12}) = I_U(t_{12}) + I_W(t_{12}) \geq (I_U(t_{11}) + I_W(t_{11}))k \geq 0$.

A similar argument can be used to show that Γ has the downward TSCP. □

If more than two bidders take part in the auction, we also need to assume increasing differences between own bids and the other bidders' types.

Proposition 5. *If Assumptions A.2 and B.1 hold in the game Γ, then it possesses the TSCP.*

The proof of Proposition 5 is relegated to the Appendix.

An important question is under what conditions the approximate interim best-reply correspondences have a nondecreasing single-valued selection.

Proposition 6. *If the game Γ satisfies the TSCP, then, for every $i \in I$, $\varepsilon > 0$, and $s_{-i} \in S_{-i}$, the interim ε-best-reply correspondence $M_i^\varepsilon(s_{-i}; \cdot) : T_i \twoheadrightarrow B_i$ has a single-valued selection $s_i \in S_i$ satisfying the following two properties: (i) it is a nondecreasing step function; and (ii)*

$$\mu(t_j \in T_j : s_i(t_i) = s_j(t_j)) = 0 \text{ for every } t_i \in T_i \text{ and each } j \in I \backslash \{i\}.$$

Proof. Fix some $\varepsilon > 0$, $i \in I$, and $s_{-i} \in S_{-i}$. It follows from Corollary 3, Lemma 1, and Proposition 2 that, for every $\widehat{t_i} \in [0, 1]$, there exist $b_i(\widehat{t_i})$ and an open ball $\mathcal{B}(\widehat{t_i}; \delta_i)$ in T_i with center $\widehat{t_i}$ and radius δ_i such that

$$V_i(b_i(\widehat{t_i}), s_{-i}; t_i') > \overline{V}_i(s_{-i}; t_i') - \varepsilon \text{ for all } t_i' \in \mathcal{B}(\widehat{t_i}; \delta_i)$$
$$\text{and } \mu(t_j \in T_j : b_i(\widehat{t_i}) = s_j(t_j)) = 0 \text{ for each } j \in I \backslash \{i\}.$$

The open cover $\{\mathcal{B}(\widehat{t_i}; \frac{\delta_i}{2})\}_{\widehat{t_i} \in [0,1]}$ of $[0, 1]$ has a finite minimal subcover, denoted by $\{\mathcal{B}(\widehat{t_{il}}; \frac{\delta_{il}}{2})\}_{l=1,\dots,m}$. Without loss of generality, $\widehat{t_{i1}} < \dots < \widehat{t_{im}}$ and $\widehat{t_{i1}} + \frac{\delta_{i1}}{2} < 1$. Denote $T_{i1} = [0, \widehat{t_{i1}} + \frac{\delta_{i1}}{2}]$, $T_{il} = (\widehat{t_{i(l-1)}} + \frac{\delta_{i(l-1)}}{2}, \widehat{t_{il}} + \frac{\delta_{il}}{2}]$ for $l = 2, \dots, m-1$, and $T_{im} = (\widehat{t_{i(m-1)}} + \frac{\delta_{i(m-1)}}{2}, 1]$.

We will construct a nondecreasing single-valued interim ε-best reply s_i using an iterative procedure. Put $s_{i1}^1 = b_i(\widehat{t_{i1}})$, where, for convenience, s_{ij}^k denotes the bid chosen in step k of the iterative procedure on the interval T_{ij}.

If $b_i(\widehat{t_{i2}}) \geq s_{i1}^1$, put $s_{i1}^2 = s_{i1}^1$ and $s_{i2}^2 = b_i(\widehat{t_{i2}})$. If $b_i(\widehat{t_{i2}}) < s_{i1}^1$, then either $V_i(s_{i1}^1, s_{-i}; t_i) \geq V_i(b_i(\widehat{t_{i2}}), s_{-i}; t_i)$ for all $t_i \in T_{i2}$ or $V_i(b_i(\widehat{t_{i2}}), s_{-i}; t_i) > V_i(s_{i1}^1, s_{-i}; t_i)$ for some $t_i \in T_{i2}$. In the former case, $s_{ij}^2 = s_{i1}^1$ for $j = 1, 2$. In the latter case, the downward TSCP implies that $b_i(\widehat{t_{i2}}) \in M_i^\varepsilon(s_{-i}; t_i)$ for all $t_i \in T_{i1}$, and, therefore, put $s_{ij}^2 = b_i(\widehat{t_{i2}})$ for $j = 1, 2$.

Consider the k-th step with $2 < k \leq m$. If $b_i(\widehat{t_{ik}}) \geq s_{i(k-1)}^{k-1}$, put $s_{ij}^k = s_{ij}^{k-1}$ for $j = 1, \dots, k-1$, and $s_{ik}^k = b_i(\widehat{t_{ik}})$. If $b_i(\widehat{t_{ik}}) < s_{i(k-1)}^{k-1}$, then either $V_i(s_{i(k-1)}^{k-1}, s_{-i}; t_i) \geq V_i(b_i(\widehat{t_{ik}}), s_{-i}; t_i)$ for all $t_i \in T_{ik}$ or $V_i(s_{i(k-1)}^{k-1}, s_{-i}; t_i) < V_i(b_i(\widehat{t_{ik}}), s_{-i}; t_i)$ for some $t_i \in T_{ik}$. In the former case, put $s_{ij}^k = s_{ij}^{k-1}$ for $j = 1, \dots, k-1$, and $s_{ik}^k = s_{i(k-1)}^{k-1}$. In the latter case, the downward TSCP implies that $V_i(s_{i(k-1)}^{k-1}, s_{-i}; t_i) < V_i(b_i(\widehat{t_{ik}}), s_{-i}; t_i)$ for all $t_i \in T_{i(k-1)}$; that is, $b_i(\widehat{t_{ik}}) \in M_i^\varepsilon(s_{-i}; t_i)$ for all $t_i \in T_{i(k-1)}$. If $b_i(\widehat{t_{ik}}) \geq s_{i(k-2)}^{k-1}$, then put $s_{ij}^k = s_{ij}^{k-1}$ for $j = 1, \dots, k-2$, and $s_{ij}^k = b_i(\widehat{t_{ik}})$ for $j = k-1, k$. If $b_i(\widehat{t_{ik}}) < s_{i(k-2)}^{k-1}$, then again either $V_i(s_{i(k-2)}^{k-1}, s_{-i}; t_i) \geq V_i(b_i(\widehat{t_{ik}}), s_{-i}; t_i)$ for all $t_i \in T_{i(k-1)}$ or $V_i(s^{k-1}_{i(k-2)}, s_{-i}; t_i) < V_i(b_i(\widehat{t_{ik}}), s_{-i}; t_i)$ for some $t_i \in T_{i(k-1)}$. In the former case, put $s^k_{ij} = s^{k-1}_{ij}$ for $j = 1, \dots, k-2$, and $s^k_{ij} = s^{k-1}_{i(k-2)}$ for $j = k-1, k$, which is suitable, owing to the upward TSCP. In the latter case, we have $b_i(\widehat{t_{ik}}) \in M^{\varepsilon}_i(s_{-i}, ; t_i)$ for all $t_i \in T_{i(k-2)}$ and again we have to consider two cases. If needed, the backward procedure is repeated until a nondecreasing approximate interim best reply is constructed on the first k intervals of the cover.

Since the number of intervals covering $T_i = [0, 1]$ is finite, the process will terminate in a finite number of steps. □

In particular, Proposition 6, along with Corollary 2, implies the ex-ante payoff security of the auction game in the class of nondecreasing strategies, without employing any kind of uniform payoff security.[11]

The next corollary examines the continuity of the ex-ante value functions.

Corollary 6. *If the game Γ satisfies the TSCP, then $\overline{V}^*_i(s_{-i}) = \int_{T_i} \overline{V}_i(s_{-i}; t_i) f_i(t_i) dt_i$ for all $s_{-i} \in S_{-i}$. Therefore, each ex-ante value function $\overline{V}^*_i : S_{-i} \to \mathbb{R}$ is continuous.*

Proof. Assume, by contradiction, that $\int_{T_i} \overline{V}_i(\overline{s}_{-i}; t_i) f_i(t_i) dt_i > \overline{V}^*_i(\overline{s}_{-i})$ for some $\overline{s}_{-i} \in S_{-i}$. Put $\varepsilon = (\int_{T_i} \overline{V}_i(\overline{s}_{-i}; t_i) f_i(t_i) dt_i - \overline{V}^*_i(\overline{s}_{-i}))/2$. By Proposition 6, $M^{\varepsilon}_i(\overline{s}_{-i}, \cdot)$ has a single-valued selection $\overline{s}_i \in S_i$. Then $V_i(\overline{s}_i, \overline{s}_{-i}) > \int_{T_i} \overline{V}_i(\overline{s}_{-i}; t_i) f_i(t_i) dt_i - \varepsilon$, which contradicts the initial premise. Therefore, $\overline{V}^*_i(s_{-i}) = \int_{T_i} \overline{V}_i(s_{-i}; t_i) f_i(t_i) dt_i$ for every $s_{-i} \in S_{-i}$, and the continuity of $\overline{V}_i$ on $S_{-i} \times T_i$ implies the continuity of $\overline{V}^*_i$ on S_{-i}. □

Another useful corollary of Proposition 6 is the following.

Corollary 7. *If the game Γ satisfies the TSCP, then, for every $\varepsilon > 0$ and $s_{-i} \in S_{-i}$, there exists $s_i \in S_i$ such that $V^*_i(s_i . s_{-i}) > \overline{V}^*_i(s_{-i}) - \varepsilon$, and, therefore, $\overline{V}^*_i(s_{-i}) = \sup_{s_i \in S_i} V^*_i(s_i, s_{-i})$ for every $s_{-i} \in S_{-i}$.*

In particular, Corollary 7 implies that the monotone equilibrium strategy profiles described in Theorems 3 and 4 below are indeed Bayesian-Nash equilibria of the game.

5. Equilibria of the first-price auction

In this section, we first apply Horvath's (1987) fixed-point theorem to the problem of existence of monotone approximate interim equilibria in the first-price auction (Theorem 2). Then, the existence of a monotone pure-strategy Bayesian-Nash equilibrium in the first-price auction is established under the additional condition that the auction is aggregate upper semicontinuous in bids, which, in particular, covers the common-value case (Theorem 3). In Theorem 4, we employ another condition guaranteeing the existence of a monotone Bayesian-Nash equilibrium, namely that the bidders' payoff functions are strictly increasing in own type. A number of examples illustrate the equilibrium existence conditions.

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[11] See, for details regarding uniform payoff security, Monteiro and Page (2007), Carbonell-Nicolau and Ok (2007); Allison and Lepore (2014); Prokopovych and Yannelis (2014); He and Yannelis (2016); and Carbonell-Nicolau and McLean (2018).

Fix some $i \in I$ and $\varepsilon > 0$. Consider bidder i's ε-best-reply correspondence $\widetilde{M}_i^{\varepsilon} : S_{-i} \twoheadrightarrow S_i$ whose values consist of bidder i's interim ε-best replies; that is, for every $s_{-i} \in S_{-i}$,

$$\widetilde{M}_i^{\varepsilon}(s_{-i}) = \{s_i \in S_i : V_i(s_i(t_i), s_{-i}; t_i) > \overline{V}_i(s_{-i}; t_i) - \varepsilon \text{ for almost all } t_i \in T_i\}.$$

It is nonempty-valued by Proposition 6. Fix any $s_{-i} \in S_{-i}$. Since any piecewise combination of two interim ε-best replies is also an interim ε-best reply, it is clear that, for any finite family of bidder i's strategies $A_i = \{s_{1i}, \ldots, s_{ki}\}$ in $\widetilde{M}_i^{\varepsilon}(s_{-i})$, $F_{A_i} \subset \widetilde{M}_i^{\varepsilon}(s_{-i})$. Therefore, though $\widetilde{M}_i^{\varepsilon}$ is not necessarily convex-valued, it is H-convex-valued. In order to be able to apply Theorem 1, we also need to show that $\widetilde{M}_i^{\varepsilon}$ possesses the local intersection property.

Proposition 7. *If the game Γ satisfies the TSCP, then, for every $\varepsilon > 0$, each correspondence $\widetilde{M}_i^{\varepsilon} : S_{-i} \twoheadrightarrow S_i$ defined just above has the local intersection property.*

Proof. Fix some $i \in I$, $\varepsilon > 0$, and $s_{-i} \in S_{-i}$. Then, by Proposition 6, there exists $\overline{s}_i \in \widetilde{M}_i^{\frac{\varepsilon}{3}}(s_{-i}) \subset \widetilde{M}_i^{\varepsilon}(s_{-i})$ satisfying the following two properties: (i) $\overline{s}_i$ is a nondecreasing step function from T_i into B_i that can be represented as $\overline{s}_i(t) = \sum_{k=1}^{K} c_k \mathcal{X}_{D_k}(t)$, where D_k are disjoint intervals whose union is T_i and $c_k \in B_i$ for all $k \in \{1, \ldots, K\}$; and (ii) $\mu(t_j \in T_j : \overline{s}_i(t_i) = s_j(t_j)) = 0$ for every $t_i \in T_i$ and each $j \in I \backslash \{i\}$.

By Lemma 2, there exists a neighborhood $\mathcal{N}_1(s_{-i})$ of s_{-i} in S_{-i} such that $V_i(\overline{s}_i(t_i), s'_{-i}; t_i) > V_i(\overline{s}_i(t_i), s_{-i}; t_i) - \frac{\varepsilon}{3}$ for every $s'_{-i} \in \mathcal{N}_1(s_{-i})$ and every $t_i \in T_i$. On the other hand, by Proposition 2, $\overline{V}_i$ is continuous on $S_{-i} \times T_i$, which implies, in particular, its uniform continuity on $S_{-i} \times T_i$. Therefore, there exists a neighborhood $\mathcal{N}_2(s_{-i})$ of s_{-i} in S_{-i} such that $\overline{V}_i(s_{-i}; t_i) > \overline{V}_i(s''_{-i}; t_i) - \frac{\varepsilon}{3}$ for every $s''_{-i} \in \mathcal{N}_2(s_{-i})$ and every $t_i \in T_i$. Then, for every $\widehat{s}_{-i} \in \mathcal{N}(s_{-i}) = \mathcal{N}_1(s_{-i}) \cap \mathcal{N}_2(s_{-i})$ and almost every $t_i \in T_i$, we have $V_i(\overline{s}_i(t_i), \widehat{s}_{-i}; t_i) > V_i(\overline{s}_i(t_i), s_{-i}; t_i) - \frac{\varepsilon}{3} > \overline{V}_i(s_{-i}; t_i) - \frac{2\varepsilon}{3} > \overline{V}_i(\widehat{s}_{-i}; t_i) - \varepsilon$. Therefore, $\overline{s}_i \in \widetilde{M}_i^{\varepsilon}(\widehat{s}_{-i})$ for every $\widehat{s}_{-i} \in \mathcal{N}(s_{-i})$; that is, $\widetilde{M}_i^{\varepsilon}$ has the local intersection property. □

The next statement follows from Corollary 1, since each $\widetilde{M}_i^{\varepsilon}$ is H-convex-valued and has the local intersection property.

Theorem 2. *If the game Γ has the TSCP, then it has a monotone interim ε-equilibrium for every $\varepsilon > 0$.*

If a first-price auction has no pure-strategy Bayesian-Nash equilibrium, it is reasonable to turn to studying whether the game has approximate equilibria.

Example 1. Consider the following two-bidder first-price auction. Let $T_1 = T_2 = [0, 1]$, $B_1 = B_2 = [0, c]$, $c > 1$. The functions u_1 and u_2 are defined as follows: $u_1(b_1; t) = -b_1$ and $u_2(b_2; t) = t_2 - b_2$ for all $(b, t) \in B \times T$, and the bidders' types are independently uniformly distributed on $[0, 1]$.

The auction has no pure-strategy Bayesian-Nash equilibrium (see Lebrun, 1996, p. 422). At the same time, the auction has a pure-strategy monotone interim ε-equilibrium for every $\varepsilon > 0$ by Theorem 2.

A natural question appears whether the nonexistence of pure-strategy Bayesian-Nash equilibria in Example 1 results from the fact that it is degenerate in the sense that the bidders' payoffs do not depend in any way on bidder 1's type. Example 2 shows that it is not so.

Example 2. Consider the game studied in Example 1 with the only difference that $u_1(b_1; t) = -b_1$ for all $(b_1, t) \in B_1 \times T$ with $t_1 \in [0, \frac{1}{2}]$ and $u_1(b_1; t) = 1 - b_1$ for all $(b_1, t) \in B_1 \times T$ with $t_1 \in (\frac{1}{2}, 1]$. The discontinuity of u_1 plays no significant role in the argument below. If needed, u_1 can be easily smoothed in an arbitrarily small ball around $t_1 = \frac{1}{2}$.

To show that the game has no pure-strategy Bayesian-Nash equilibria, assume, by contradiction, that (s_1, s_2) is one of them. Since there are arbitrarily small, positive types of bidder 2 and $s_2(t_2) \leq t_2$ for almost all $t_2 \in [0, 1]$, almost all types of bidder 1 in $[0, \frac{1}{2}]$ bid zero; that is, $\operatorname{ess\,sup}_{t_1 \in [0, \frac{1}{2}]} s_1(t_1) = 0$. Then $s_2(t_2) > 0$ for almost all $t_2 \in [0, 1]$. If $\operatorname{ess\,inf}_{t_1 \in (\frac{1}{2}, 1]} s_1(t_1) > 0$, then bidder 2's types in $(0, \operatorname{ess\,inf}_{t_1 \in (\frac{1}{2}, 1]} s_1(t_1)]$ have no best reply to s_1. If $\operatorname{ess\,inf}_{t_1 \in (\frac{1}{2}, 1]} s_1(t_1) = 0$, then, for every (whatever small) $\varepsilon > 0$, the set of bidder 1's types in $(\frac{1}{2}, 1]$ with $s_1(t_1) < \varepsilon$ is of positive measure. On the other hand, since $\mu(t_2 \in T_2 : s_2(t_2) = 0) = 0$, for every $\delta > 0$ there exists $\varepsilon(\delta) > 0$ such that $\mu(t_2 \in T_2 : s_2(t_2) \leq \varepsilon(\delta)) < \delta$. Consequently, for every $\gamma > 0$ there is a nonzero-measure subset D of $(\frac{1}{2}, 1]$ such that $V_1(s_1(t_1), s_2; t_1) < \gamma$ for every $t_1 \in D$, which contradicts the fact that $V_1(s_1(t_1), s_2; t_1) = \overline{V}_1(s_2; t_1) \geq V_1(\frac{1}{2}, s_2; t_1) \geq \frac{1}{4}$ for almost all $t_1 \in (\frac{1}{2}, 1]$.

If bidders' valuations are not strictly increasing in own type, another major problem can arise, namely, the absence of Bayesian-Nash equilibria in strictly increasing strategies, which precludes using the inverse functions of strategies.[12]

Example 3. Consider another two-bidder first-price auction. Let $T_1 = T_2 = [0, 1]$, $B_1 = B_2 = [0, c]$, $c > 1$. The functions u_1 and u_2 are defined as follows: $u_1(b_1; t) = -b_1$ and $u_2(b_2; t) = \frac{1}{2} - b_2$ for all $(b, t) \in B \times T$, and the bidders' types are independently uniformly distributed on $[0, 1]$.

Despite the fact that the bidders' valuations of the item do not depend on the bidders' types and are distinct, this game has Bayesian-Nash equilibria in nondecreasing strategies. Consider, for example, the strategy profile $(s_1, s_2) \in S$ defined as follows: $s_1(t_1) = \frac{1}{4}t_1$ for all $t_1 \in T_1$ and $s_2(t_2) = \frac{1}{4}$ for all $t_2 \in T_2$. Clearly, $V_2(\frac{1}{4}, s_1; t_2) = \frac{1}{4}$ for all $t_2 \in T_2$ and $V_2(b_2, s_1; t_2) \leq \frac{1}{4}$ for all $(b_2, t_2) \in B_2 \times T_2$. For example, for every $\varepsilon \in (0, \frac{1}{4}]$,

$$V_2(\frac{1}{4} - \varepsilon, s_1; t_2) = (\frac{1}{4} + \varepsilon)(1 - 4\varepsilon) = \frac{1}{4} - 4\varepsilon^2.$$

On the other hand, $V_1(s_1(t_1), s_2; t_1) = 0$ for every $t_1 \in T_1$, and, given bidder 2's strategy s_2, every type of bidder 1 cannot get a positive payoff by choosing a bid different from the bid prescribed by s_1. Therefore, the strategy profile (s_1, s_2) constitutes a Bayesian-Nash equilibrium of the game.

At the same time, the game has no Bayesian-Nash equilibrium in strictly increasing strategies. Assume, by contradiction, that (s_1, s_2) is a Bayesian-Nash equilibrium in strictly increasing strategies. Clearly, $\operatorname{ess\,sup}_{t_2 \in [0,1]} s_2(t_2) \leq \frac{1}{2}$, and it must be the case that $\operatorname{ess\,sup}_{t_1 \in [0,1]} s_1(t_1) = \operatorname{ess\,sup}_{t_2 \in [0,1]} s_2(t_2)$. The fact that s_2 is a strictly increasing strategy implies that $\operatorname{ess\,inf}_{t_2 \in [0,1]} s_2(t_2) < \operatorname{ess\,sup}_{t_2 \in [0,1]} s_2(t_2)$ and, therefore, $V_1(s_1(t_1), s_2; t_1) < 0$ on some nonzero-measure subset of T_1. Consequently, bidder 1's ex-ante payoff at (s_1, s_2) is negative, a contradiction.

[12] The inverse functions of bidders' strategies play an important role in auction theory (see, e.g., Lebrun, 1999; Maskin and Riley, 2000; Krishna, 2010).

In order to ensure the existence of a pure-strategy monotone Bayesian-Nash equilibrium in the first-price action, some additional condition is needed. We begin with the case where the game is aggregate upper semicontinuous in bids; that is, $\sum_{i\in I} v_i(\cdot;t) : B \to \mathbb{R}$ is upper semicontinuous for every $t \in T$.

Lemma 3. *If, in the game Γ, $\sum_{i\in I} v_i(\cdot;t) : B \to \mathbb{R}$ is upper semicontinuous for every $t \in T$, then the sum of the ex-ante payoff functions $\sum_{i\in I} V^* : S \to \mathbb{R}$ is upper semicontinuous as well.*

Proof. By definition, $V_i^*(s) = \int_T v_i(s_i(t_i), s_{-i}(t_{-i}); t) f(t)dt$ for $i \in I$ and $s \in S$. Pick some $s^0 \in S$ and consider a sequence $\{s^k\}$, $s^k \in S$, converging to s^0. Without loss of generality, the sequence $\{s^k\}$ converges to s^0 pointwise by Helly's selection theorem for sequences of monotone functions. We need to show that $\limsup_k \sum_{i\in I} V_i^*(s^k) \le \sum_{i\in I} V_i^*(s^0)$.

It follows from Fatou's lemma that

$$\limsup_k \sum_{i\in I} V_i^*(s^k) \le \int_T \limsup_k \sum_{i\in I} v_i(s_i^k(t_i), s_{-i}^k(t_{-i}); t) f(t)dt.$$

Since $\lim_k s_i^k(t_i) = s_i^0(t_i)$ for every $t_i \in T_i$ and each $i \in I$ and $\sum_{i\in I} v_i(\cdot;t)$ is upper semicontinuous for every $t \in T$, $\limsup_k \sum_{i\in I} v_i(s_i^k(t_i), s_{-i}^k(t_{-i}); t) \le \sum_{i\in I} v_i(s_i^0(t_i), s_{-i}^0(t_{-i}); t)$ for every $t \in T$, and the claim obtains. $\square$

In particular, the aggregate upper semicontinuity of the ex-ante payoff functions implies the reciprocal upper semicontinuity of Γ; that is, if $\{s^k\}$, $s^k \in S$, is a sequence of strategy profiles converging to s^0 such as each sequence $\{V_i^*(s^k)\}$ is convergent and $\lim_k V_j^*(s^k) > V_j^*(s^0)$ for some $j \in I$, then $\lim_k V_l^*(s^k) < V_l^*(s^0)$ for some other $l \in I$ (Simon, 1987).

We now show that If Γ is aggregate upper semicontinuous in bids and has the TSCP, then every limit point of a sequence of monotone interim ε-equilibria, with ε tending to zero, is a Bayesian-Nash equilibrium of the game.

Theorem 3. *If the game Γ has the TSCP and $\sum_{i\in I} v_i(\cdot;t) : B \to \mathbb{R}$ is upper semicontinuous for every $t \in T$, then Γ has a monotone Bayesian-Nash equilibrium.*

Proof. Since Γ satisfies the TSCP, Theorem 2 implies that, for each $k = 1, 2, \ldots$, it possesses a monotone interim $\frac{1}{k}$-equilibrium $s^k = (s_1^k, \ldots, s_n^k) \in S$ such that $V_i(s_i^k(t_i), s_{-i}^k; t_i) > \overline{V}_i(s_{-i}^k; t_i) - \frac{1}{k}$ for almost every $t_i \in T_i$ and each $i \in I$. Then, for each $i \in I$ and each $k = 1, 2, \ldots$,

$$V_i^*(s_i^k, s_{-i}^k) > \overline{V}_i^*(s_{-i}^k) - \frac{1}{k}.$$

Without loss of generality, the sequence $\{s^k\}$ tends pointwise to some s^0 in S and each sequence $\{V_i^*(s^k)\}$ is convergent.

Assume, by contradiction, that s^0 is not a Bayesian-Nash equilibrium of Γ; that is, for some $j \in I$, $V_j^*(s^0) < \overline{V}_j^*(s_{-j}^0)$. Since $\overline{V}_j^*$ is continuous on S_{-j} by Corollary 6, $\lim_k V_j^*(s^k) = \lim_k \overline{V}_j^*(s_{-j}^k) = \overline{V}_j^*(s_{-j}^0) > V_j^*(s^0)$, and, therefore, by Lemma 3, there exists $l \in I\backslash\{j\}$ such that $\lim_k V_l^*(s^k) < V_l^*(s^0)$. On the other hand, we have $\lim_k V_l^*(s^k) = \lim_k \overline{V}_l^*(s_{-l}^k) = \overline{V}_l^*(s_{-l}^0)$, a contradiction. $\square$

Aggregate upper semicontinuity in bids holds in common-value auctions.

Example 4. Consider the following n-bidder first-price common-value auction. Let $T_i = [0,1]$, $B_i = [0,c]$, $c > 1$ for all $i \in I = \{1, \ldots, n\}$. Each function u_i is defined as follows: $u_i(b_i;t) = \max\{\frac{1}{n}\sum_{i\in I} t_i, \frac{1}{2}\} - b_i$ for all $(b_i,t) \in B_i \times T$, and the bidders' types are independently uniformly distributed on $[0,1]$.

The common value of the item being sold is modeled as the average of the bidders' types bounded above by $\frac{1}{2}$.[13] Having the upper bound in the definition of the common value of the item is not counterintuitive. Its presence makes the common value weakly (not strictly) increasing in the bidders' types; otherwise, the existence of pure-strategy Bayesian-Nash equilibria in this game would follow not only from Theorem 3 but also from Athey's (2001) and Reny-Zamir's (2004) results and Theorem 4 below.

If the common value of an item pertains to the resale price of a painting or the market value of a firm, another specification of bidders' payoff functions is possible.

Example 5. Consider the following n-bidder first-price common-value auction. Let $T_i = [0,1]$, $B_i = [0,c]$, $c > 1$ for all $i \in I$. Each function u_i is defined as follows: $u_i(b_i;t) = \max\{t_1, \ldots, t_n\} - b_i$ for all $(b_i,t) \in B_i \times T$. The function $f : [0,1]^n \to (0,+\infty)$ is the density function of an n-variate truncated logsupermodular Gaussian distribution with mean $\mu = 0$ and covariance matrix Σ.[14]

Clearly, the functions u_i's are not strictly increasing in t_i. Equilibrium existence in this game follows from Theorem 3.

In private-value auctions, the condition that each u_i is strictly increasing in own type can be employed to ensure the existence of a monotone Bayesian-Nash equilibrium. This condition is often used in first-price auction literature (see, e.g., Maskin and Riley, 2000; Athey, 2001; Reny and Zamir, 2004). However, it does not guarantee that the limit of every convergent sequence of monotone interim ε-equilibria, with ε tending to 0, is a Bayesian-Nash equilibrium of the game.

Example 6. Consider the following two-bidder first-price auction. Let $T_1 = T_2 = [0,1]$, $B_1 = B_2 = [0,10]$ for $i = 1,2$. The functions u_1 and u_2 are defined as follows: $u_1(b_1;t) = t_1 - b_1$ and $u_2(b_2;t) = t_2 - b_2 + 7$ for all $(b,t) \in B \times T$, and the bidders' types are independently uniformly distributed on $[0,1]$. Clearly, each u_i is strictly increasing in own type t_i.

It follows from Theorem 2 that the game has a monotone interim ε-equilibrium for every $\varepsilon > 0$. However, we now describe a convergent sequence of monotone interim approximate equilibria of the game whose limit is not a Bayesian-Nash equilibrium.

Let, for $k \in \{1,2,\ldots\}$, bidder 1's strategy $s_1^k : T_1 \to B_1$ be $s_1^k(t_1) = \frac{1}{10k}t_1 + (5 - \frac{1}{10k})$ and bidder 2's strategy $s_2^k : T_2 \to B_2$ be $s_2^k(t_2) = 5$. Put $\varepsilon_k = \frac{1}{10k}$ for $k \in \{1,2,\ldots\}$. It is not difficult to see that each strategy profile (s_1^k, s_2^k) is a monotone interim ε_k-equilibrium of the game. However, the limit of the convergent sequence $\{s^k\}$ is not a Bayesian-Nash equilibrium of the game.

However, if each u_i is strictly increasing in own type t_i, it is possible to construct a convergent sequence of approximate ex-ante equilibria whose limit is a Bayesian-Nash equilibrium of the first-price auction. A similar problem was handled by Athey (2001) and Reny and Zamir (2004) for finite-bid approximations of first-price auctions.

[13] See, e.g., Bikhchandani and Riley, 1991; Goeree and Offerman, 2003; Krishna, 2010, for some examples of the use of the average of the bidders' types as the common value.

[14] If the off-diagonal elements of the inverse of Σ are nonpositive, then f is logsupermodular (see, e.g., Karlin and Renott, 1980, Example 3.1).

Theorem 4. *If the game Γ has the TSCP and each u_i is strictly increasing in t_i, then it has a monotone Bayesian-Nash equilibrium.*

The proof of Theorem 4 can be found in the Appendix.

Example 7. Consider the following two-bidder first-price auction with subsidies (see, e.g., Athey et al., 2013). Let $T_1 = T_2 = [0, 1]$, $B_1 = B_2 = [0, 3]$. The functions u_1 and u_2 are defined as follows: $u_1(b_1; t) = t_1 - \frac{b_1}{2}$ and $u_2(b_2; t) = (t_1 + 1)t_2 - b_2$ for all $(b, t) \in B \times T$, and the bidders' types are independently uniformly distributed on $[0, 1]$.

The existence of a monotone pure-strategy Bayesian-Nash equilibrium in this game follows from Theorem 4.

6. Conclusions

This paper develops a new approach to investigating monotone equilibrium existence in asymmetric first-price auctions with interdependent values and affiliated types. Instead of using finite-bid approximations, we study properties of the approximate best-reply correspondences. In particular, it is shown that if the tieless single-crossing property holds and the other bidders employ nondecreasing strategies, then each approximate best-reply correspondence consisting of bidder i's monotone interim ε-best replies has the local intersection property; that is, the correspondence possesses a multivalued selection with open lower sections.

On the other hand, though the values of the approximate best-reply correspondences need not be contractible, they can be described as collections of contractible sets, each of which is associated with a finite number of monotone single-valued approximate best replies; that is, they are H-convex. Consequently, the existence of monotone approximate interim equilibria in the game follows from Horvath's (1987) extension of Browder's (1968) fixed-point theorem. In particular, this approach makes it possible to study approximate monotone equilibria of first-price auctions with no pure-strategy Bayesian-Nash equilibria.

Then we provide two sets of sufficient conditions for an auction to possess a sequence of approximate equilibria that converges to a pure-strategy Bayesian-Nash equilibrium. The first set is designed for common-value auctions where bidders' valuations are not necessarily strictly increasing in own type. The second set of sufficient conditions is akin to Athey's (2001) and Reny-Zamir's (2004) results, with its proof based on approximate equilibria, not finite-action approximations.

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Appendix A

The Appendix contains the proofs of a number of auxiliary results.

A.1. Proof of Corollary 2

Proof. Fix some $\varepsilon > 0$. Player i's strategy s_i can be represented as $s_i(t_i) = \sum_{k=1}^{K} c_k \mathcal{X}_{D_k}(t_i)$ for $t_i \in T_i$, where D_k are disjoint intervals whose union is T_i and $c_k \in B_i$ for all $k \in \{1, \ldots, K\}$.

First, consider the case when s_i is a constant function; that is, there exists $c_1 \in B_i$ such that $s_i(t_i) = c_1$ for all $t_i \in T_i$. It follows from Lemma 1 that for every $t_i \in T_i$, there exist a neighborhood $\mathcal{N}(s_{-i})$ of s_{-i} in L_{-i} and a neighborhood $\mathcal{N}(t_i)$ of t_i in T_i such that

$$V_i(c_1, s'_{-i}; t'_i) > V_i(c_1, s_{-i}; t_i) - \frac{\varepsilon}{2} > V_i(c_1, s_{-i}; t'_i) - \varepsilon$$

for every $(s'_{-i}, t'_i) \in \mathcal{N}(s_{-i}) \times \mathcal{N}(t_i)$. Since T_i is compact, the cover $\{\mathcal{N}(t_i)\}_{t_i \in T_i}$ of T_i has a finite subcover $\{\mathcal{N}(t_i^{11}), \ldots, \mathcal{N}(t_i^{1m})\}$. Consider the corresponding neighborhoods of s_{-i}, $\mathcal{N}_{11}(s_{-i}), \ldots, \mathcal{N}_{1m}(s_{-i})$, associated with the points $t_i^{11}, \ldots, t_i^{1m}$. Let $\mathcal{N}_1(s_{-i}) = \bigcap_{j=1}^{m} \mathcal{N}_{1j}(s_{-i})$. Then $V_i(c_1, s'_{-i}; t_i) > V_i(c_1, s_{-i}; t_i) - \varepsilon$ for every $s'_{-i} \in \mathcal{N}_1(s_{-i})$ and every $t_i \in T_i$.

A similar argument can also be provided for each of $c_2, \ldots, c_K$. Denote the corresponding neighborhoods of s_{-i} in S_{-i} by $\mathcal{N}_2(s_{-i}), \ldots, \mathcal{N}_K(s_{-i})$. Then $\mathcal{N}_\varepsilon(s_{-i}) = \bigcap_{j=1}^{K} \mathcal{N}_j(s_{-i})$ is a neighborhood for which the statement holds. $\square$

A.2. *Proof of Lemma 2*

Before providing the proof of Lemma 2, we state a couple of basic properties of Lebesgue measurable functions.

Lemma 4. *If $h : [0, 1] \to [0, +\infty)$ be a Lebesgue measurable function, then:*

(i) *the set $\{r \in [0, +\infty) : \mu(s \in [0, 1] : h(s) = r) > 0\}$ is not more than countable;*
(ii) *for every $r \in [0, +\infty)$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mu(t \in [0, 1] : h(t) \in [r - \delta, r + \delta] \cap ([0, +\infty) \backslash \{r\})) < \varepsilon$.*

Item (i) of Lemma 4 follows from the finiteness of the Lebesgue measure μ on $[0, 1]$, and item (ii) follows from the fact that if $\{A_k\}$ is a decreasing sequence of measurable subsets of $[0, 1]$, then $\mu(\bigcap_k A_k) = \lim_k \mu(A_k)$. Statements similar to Lemma 4 are often employed in studies concerning the existence of mixed strategy equilibria in normal-form games (see, e.g., Dasgupta and Maskin, 1986; and Prokopovych and Yannelis, 2014).

Proof of Lemma 2. For the simplicity of further notation, let $i = 1$. Fix some $\varepsilon > 0$. Let $b_1 = 0$. Since $u_1(0; t) \geq 0$ for every $t \in T$ and u_1 is jointly continuous in its variables, player 1's interim payoff function cannot jump upward at $b_1 = 0$. Consequently, it is possible, using Lemma 4, to pick $\widetilde{b}_1 > 0$ close enough to 0 such that both of the conditions concerning $\widetilde{b}_1$ are satisfied. If $b_1 = c$, then again it is possible to choose $\widetilde{b}_1 < c$ close enough to c to satisfy both conditions.

Consider the case when $0 < b_1 < c$. Since each s_j, $j \in I \backslash \{1\}$, is a nondecreasing strategy, T_{-1} can be divided into three subsets (some of these might be empty):

$$T_{-1}^1 = \{t_{-1} \in T_{-1} : \max_{j \in I \backslash \{1\}} s_j(t_j) < b_1\};$$
$$T_{-1}^2 = \{t_{-1} \in T_{-1} : \max_{j \in I \backslash \{1\}} s_j(t_j) = b_1\};$$
$$T_{-1}^3 = \{t_{-1} \in T_{-1} : \max_{j \in I \backslash \{1\}} s_j(t_j) > b_1\}.$$

If $V_1(b_1, s_{-1}; t_1, T^2_{-1}) \leq 0$, then, since $V_1(b'_1, s_{-1}; t_1, T^2_{-1}) = 0$ for every $b'_1 \in [0, b_1)$, it is possible to choose $\widetilde{b}_1$ slightly below b_1 and satisfy (i) and (ii).

The case when $V_1(b_1, s_{-1}; t_1, T^2_{-1}) > 0$ needs a more detailed explanation. Obviously, $\mu(T^2_{-1}) > 0$. We need to investigate whether there exists a slight increase in b_1 which is not accompanied by a downward jump in V_1. For each $j \in I\backslash\{1\}$, denote

$$T^1_j = \{t_j \in T_j : s_j(t_j) < b_1\},$$
$$T^2_j = \{t_j \in T_j : s_j(t_j) = b_1\}.$$

Then, up to a set of zero measure,

$$\begin{aligned} T^2_{-1} = {} & (T^2_2 \times T^2_3 \times \dots T^2_{n-1} \times T^2_n) \cup (T^1_2 \times T^2_3 \times \dots \times T^2_{n-1} \times T^2_n) \cup \dots \\ & \cup (T^1_2 \times T^1_3 \times \dots \times T^1_{n-1} \times T^2_n), \end{aligned}$$

where some of the Cartesian products might be empty or of zero measure.

Denote $J(b_1, s_{-1}) = \{j \in I\backslash\{1\} : \mu(T^2_j) > 0\}$, and let $q = |J(b_1, s_{-1})|$ be the cardinality of $J(b_1, s_{-1})$. Denote by $U^q_k(b_1, s_{-1})$, $k \in \{1, \dots, q\}$, a subset of T^2_{-1} which is the union of all Cartesian products in T^2_{-1} with T^2_j for k bidders from $J(b_1, s_{-1})$ exactly. We need to show that, whatever $q \in \{2, \dots, n-1\}$ is, if $V_1(b_1, s_{-1}; t_1, U^q_k(b_1, s_{-1})) > 0$ for some $k \in \{1, \dots, q-1\}$, then $V_1(b_1, s_{-1}; t_1, U^q_{k+1}(b_1, s_{-1})) > 0$. Assume, without loss of generality, $\mu(T^1_j)\mu(T^2_j) > 0$ for $j = 2, \dots, q+1$.

Consider the case when $q = 2$. Without loss of generality, let $J(b_1, s_{-1}) = \{2, 3\}$. For $m \in \{2, \dots, n\}$, denote $T^1_{mn} = T^1_m \times \dots \times T^1_n$. Then, up to a set of zero measure,

$$U^2_1(b_1, s_{-1}) = T^1_2 \times T^2_3 \times T^1_{4n} \cup T^2_2 \times T^1_3 \times T^1_{4n}$$
$$U^2_2(b_1, s_{-1}) = T^2_2 \times T^2_3 \times T^1_{4n}.$$

We need to show that $V_1(b_1, s_{-1}; t_1, U^2_1(b_1, s_{-1})) > 0$ implies $V_1(b_1, s_{-1}; t_1, U^2_2(b_1, s_{-1})) > 0$. Assume, without loss of generality, that $V_1(b_1, s_{-1}; t_1, T^1_2 \times T^2_3 \times T^1_{4n}) > 0$. Then, since u_1 is nondecreasing in t_2, we have $V_1(b_1, s_{-1}; t_1, T^2_2 \times T^2_3 \times T^1_{4n}) > 0$.

In general, whatever $q \in \{1, \dots, n-1\}$ is, the inequality $V_1(b_1, s_{-1}; t_1, T^2_{-1}) > 0$ implies that $V_1(b_1, s_{-1}; t_1, U^q_q(b_1, s_{-1})) > 0$ since u_1 is nondecreasing in the other bidders' types; that is, the case $q = 2$ is trivial, but we will rely on the provided argument later.

For the sake of developing intuition, consider the case when $q = 3$. Without loss of generality, let $J(b_1, s_{-1}) = \{2, 3, 4\}$. We need to show that $V_1(b_1, s_{-1}; t_1, U^3_1(b_1, s_{-1})) > 0$ implies $V_1(b_1, s_{-1}; t_1, U^3_2(b_1, s_{-1})) > 0$. Notice that, up to a set of zero measure,

$$U^3_1(b_1, s_{-1}) = T^2_2 \times T^1_3 \times T^1_4 \times T^1_{5n} \cup T^1_2 \times T^2_3 \times T^1_4 \times T^1_{5n} \cup T^1_2 \times T^1_3 \times T^2_4 \times T^1_{5n}$$

and

$$U^3_2(b_1, s_{-1}) = T^2_2 \times T^2_3 \times T^1_4 \times T^1_{5n} \cup T^1_2 \times T^2_3 \times T^2_3 \times T^1_{5n} \cup T^2_2 \times T^1_3 \times T^2_3 \times T^1_{5n}.$$

Since $V_1(b_1, s_{-1}; t_1, U^3_1(b_1, s_{-1})) > 0$, there exists $j \in \{2, 3, 4\}$, let it be $j = 2$, such that

$$V_1(b_1, s_{-1}; t_1, T^1_2 \times T^2_3 \times T^1_4 \times T^1_{5n} \cup T^1_2 \times T^1_3 \times T^2_4 \times T^1_{5n}) > 0;$$

that is, $V_1(b_1, s_{-1}; t_1, \cdot)$ is positive on the Cartesian products with T^1_2. Then

$$V_1(b_1, s_{-1}; t_1, T^2_2 \times T^2_3 \times T^1_4 \times T^1_{5n} \cup T^2_2 \times T^1_3 \times T^2_4 \times T^1_{5n}) > 0,$$

since u_1 is nondecreasing in t_2. On the other hand, with T_2^1 fixed, an argument similar to the one provided for the case with $q = 2$ can be employed to show that $V_1(b_1, s_{-1}; t_1, T_2^1 \times T_3^2 \times T_4^2 \times T_{5n}^1) > 0$. Another possible explanation of the last inequality is as follows. Consider a nondecreasing $s_2' : T_2 \to B_2$ such that $s_2'(t_2) = s_2(t_2)$ for all $t_2 \in T_2^1$ and $\mu(t_2 \in T_2 : s_2'(t_2) = b_1) = 0$. Then the inequality follows from the case $q = 2$ for the strategy subprofile $(s_2', s_3, \ldots, s_n)$, where $V_1(b_1, s_2', s_3, \ldots, s_n; t_1, U_1^2(b_1, s_2', s_3, \ldots, s_n)) > 0$ implies $V_1(b_1, s_2', s_3, \ldots, s_n; t_1, U_2^2(b_1, s_2', s_3, \ldots, s_n)) > 0$. At the same time, $U_2^2(b_1, s_2'.s_3, \ldots, s_n) = T_2^1 \times T_3^2 \times T_4^2 \times T_{5n}^1$ up to a set of zero measure. Consequently, $V_1(b_1, s_{-1}; t_1, U_2^3(b_1, s_{-1})) > 0$.

Let $q \in \{4, \ldots, n-1\}$. Again we need to show that $V_1(b_1, s_{-1}; t_1, U_k^q(b_1, s_{-1})) > 0$ implies $V_1(b_1, s_{-1}; t_1, U_{k+1}^q(b_1, s_{-1})) > 0$ for each $k \in \{1, \ldots, q-2\}$, provided that the property holds for every $s_{-1}' \in S_{-1}$ with $|J(b_1, s_{-1}')| = q - 1$. Assume that $V_1(b_1, s_{-1}; t_1, U_k^q(b_1, s_{-1})) > 0$ for some $k \in \{1, \ldots, q-2\}$. Then there exists $j \in \{2, \ldots, q\}$ such that, given b_1, s_{-1}, and t_1, V_1 on the Cartesian products in $U_k^q(b_1, s_{-1})$ with T_j^1 is positive. For the sake of the simplicity of notation, let $j = 2$. Therefore, given b_1, s_{-1}, and t_1, V_1 on the Cartesian products in $U_{k+1}^q(b_1, s_{-1})$ with T_2^2 is positive, since u_1 is nondecreasing in t_2. On the other hand, the induction premise implies that V_1 is also positive on the Cartesian products with T_2^1 in U_{k+1}^q. To verify this, consider a nondecreasing $s_2' : T_2 \to B_2$ such that $s_2'(t_2) = s_2(t_2)$ for all $t_2 \in T_2^1$ and $\mu(t_2 \in T_2 : s_2'(t_2) = b_1) = 0$. It is not difficult to see that, up to a set of zero measure, $U_k^{q-1}(b_1, s_2'.s_3, \ldots, s_n)$ $(U_{k+1}^{q-1}(b_1, s_2'.s_3, \ldots, s_n))$ consists of the Cartesian products in $U_k^q(b_1, s_{-1})$ with T_2^1 (resp., $U_{k+1}^q(b_1, s_{-1})$). Therefore, $V_1(b_1, s_{-1}; t_1, U_{k+1}^q(b_1, s_{-1})) > 0$.

Intuitively, given s_{-1}, b_1, and t_1, the number of bidders tied at b_1 at the type subprofiles t_{-i} where V_1 is negative tends to be less than the number of bidders tied at b_1 at the type subprofiles t_{-i} where V_1 is positive; that is, getting rid of nonzero-measure ties at b_1 by raising the bid a little cannot lead to a downward jump in V_1 if $V_1(b_1, s_{-1}; t_1, T_{-1}^2) > 0$. Consequently, it is possible, using Lemma 4, to pick $\widetilde{b}_1$ close enough to b_1 that satisfies (i) and (ii) of Lemma 2. □

A.3. Proof of Proposition 2

Proof. Fix some $i \in I$ and some $(s_{-i}^0, t_i^0) \in S_{-i} \times T_i$. We need to show that $\overline{V}_i$ is upper semicontinuous at (s_{-i}^0, t_i^0). Consider a sequence $\{(s_{-i}^k, t_i^k)\}$, $(s_{-i}^k, t_i^k) \in S_{-i} \times T_i$, $k = 1, 2, \ldots$, converging to (s_{-i}^0, t_i^0). Without loss of generality, the sequence $\{\overline{V}_i(s_{-i}^k; t_i^k)\}$ tends to some K. Assume, by way of contradiction, that $K > \overline{V}_i(s_{-i}^0; t_i^0)$. Since $\overline{V}_i(s_{-i}^0; t_i^0) \geq 0$, we have $K > 0$. Pick some sequence $\{b_i^k\}$, $k = 1, 2, \ldots$, in B_i such that $\lim_k V_i(b_i^k, s_{-i}^k; t_i^k) = K$ and $\sum_{j \in I \setminus \{i\}} \mu(t_j \in T_j : b_i^k = s_j^k(t_j)) = 0$. Let $\varepsilon = \frac{K - \overline{V}_i(s_{-i}^0, t_i^0)}{2}$. Without loss of generality, assume that the sequence $\{b_i^k\}$ converges to some b_i^0 and $V_i(b_i^k, s_{-i}^k; t_i^k) > 0$ for all $k \in \{1, 2, \ldots\}$. Pick some $\delta > 0$ such that $|u_i(b_i'; t) - u_i(b_i''; t)| < \varepsilon$ for every $b_i', b_i'' \in B_i$ with $|b_i' - b_i''| < 2\delta$ and every $t \in T$.

First consider the case when $\sum_{j \in I \setminus \{i\}} \mu(t_j \in T_j : b_i^0 = s_j^0(t_j)) = 0$. Then, by Lemma 1, the interim payoff function V_i is continuous at (b_i^0, s_{-i}^0, t_i^0), and, therefore, $K = \lim_k V_i(b_i^k, s_{-i}^k; t_i^k) = V_i(b_i^0, s_{-i}^0; t_i^0) \leq \overline{V}_i(s_{-i}^0; t_i^0)$, a contradiction.

Now assume that $\sum_{j\in I\backslash\{i\}} \mu(t_j \in T_j : b_i^0 = s_j^0(t_j)) > 0$. Pick $\widehat{b}_i \in (b_i^0, \min(b_i^0 + \delta, c))$ such that $\sum_{j\in I\backslash\{i\}} \mu(t_j \in T_j : \widehat{b}_i = s_j^k(t_j)) = 0$ for all $k = 0, 1, \ldots$. Then $\lim_k V_i(\widehat{b}_i, s_{-i}^k; t_i^k) = V_i(\widehat{b}_i, s_{-i}^0; t_i^0)$ by Lemma 1, and, therefore, $\lim_k V_i(\widehat{b}_i, s_{-i}^k; t_i^k) \le \overline{V}_i(s_{-i}^0; t_i^0)$. Without loss of generality, $b_i^k \in (b_i^0 - \delta, \widehat{b}_i)$ and $V_i(\widehat{b}_i, s_{-i}^k; t_i^k) + \varepsilon < V_i(b_i^k, s_{-i}^k; t_i^k)$ for all $k \in \{1, 2, \ldots\}$.

For $b_i \in B_i$ and $s_{-i} \in S_{-i}$, denote $T_{-i}(b_i, s_{-i}) = \{t_{-i} \in T_{-i} : s_j(t_j) \le b_i$ for all $j \in I\backslash\{i\}\}$. Fix some $\widehat{k} \in \{1, 2, \ldots\}$. It is not difficult to see that $V_i(\widehat{b}_i, s_{-i}^{\widehat{k}}; t_i^{\widehat{k}}, T_{-i}(b_i^{\widehat{k}}, s_{-i}^{\widehat{k}})) > V_i(b_i^{\widehat{k}}, s_{-i}^{\widehat{k}}; t_i^{\widehat{k}}, T_{-i}(b_i^{\widehat{k}}, s_{-i}^{\widehat{k}})) - \varepsilon$, since the two integrals are taken over the same set $T_{-i}(b_i^{\widehat{k}}, s_{-i}^{\widehat{k}})$ and $\widehat{b}_i > b_i^{\widehat{k}}$. On the other hand, since u_i is nondecreasing in the other bidders' types, the inequality $V_i(\widehat{b}_i, s_{-i}^{\widehat{k}}; t_i^{\widehat{k}}, T_{-i}(b_i^{\widehat{k}}, s_{-i}^{\widehat{k}})) > 0$ implies that $V_i(\widehat{b}_i, s_{-i}^{\widehat{k}}; t_i^{\widehat{k}}, T_{-i}(\widehat{b}_i, s_{-i}^{\widehat{k}})\backslash T_{-i}(b_i^{\widehat{k}}, s_{-i}^{\widehat{k}})) \ge 0$, where the last inequality becomes an equality if $\mu(T_{-i}(\widehat{b}_i, s_{-i}^{\widehat{k}})\backslash T_{-i}(b_i^{\widehat{k}}, s_{-i}^{\widehat{k}})) = 0$. Therefore,

$$
\begin{aligned}
&V_i(\widehat{b}_i, s_{-i}^{\widehat{k}}; t_i^{\widehat{k}}, T_{-i}(\widehat{b}_i, s_{-i}^{\widehat{k}})) = V_i(\widehat{b}_i, s_{-i}^{\widehat{k}}; t_i^{\widehat{k}}, T_{-i}(b_i^{\widehat{k}}, s_{-i}^{\widehat{k}})) \\
&+V_i(\widehat{b}_i, s_{-i}^{\widehat{k}}; t_i^{\widehat{k}}, T_{-i}(\widehat{b}_i, s_{-i}^{\widehat{k}})\backslash T_{-i}(b_i^{\widehat{k}}, s_{-i}^{\widehat{k}})) \\
&> V_i(b_i^{\widehat{k}}, s_{-i}^{\widehat{k}}; t_i^{\widehat{k}}, T_{-i}(b_i^{\widehat{k}}, s_{-i}^{\widehat{k}})) - \varepsilon,
\end{aligned}
$$

a contradiction. □

A.4. Proof of Proposition 3

Proof. We now show that the upward TSCP is satisfied for bidder $i = 1$. Fix some nondecreasing strategies $s_j : T_j \to B_j$, $j \in I\backslash\{1\}$, and $b_{11}, b_{12} \in B_1$ with $b_{11} < b_{12}$ such that $\mu(t_j \in T_j : b_{1l} = s_j(t_j)) = 0$, $l = 1, 2$, for all $j \in I\backslash\{1\}$. Denote $\Delta V_1(t_1) = V_1(b_{12}, s_{-1}; t_1) - V_1(b_{11}, s_{-1}; t_1)$ for $t_1 \in T_1$.

Consider the following two subsets of T_{-1}:

$$
\begin{aligned}
&U = \{t_{-1} \in T_{-1} : b_{11} > \max{}_{j\in I\backslash\{1\}} s_j(t_j)\}; \\
&W = \{t_{-1} \in T_{-1} : b_{11} < \max{}_{j\in I\backslash\{1\}} s_j(t_j) \text{ and } b_{12} > \max{}_{j\in I\backslash\{1\}} s_j(t_j)\}.
\end{aligned}
$$

For the sake of convenience, denote $u_1(b_{12}; t_1, t_{-1}) - u_1(b_{11}; t_1, t_{-1})$ by $h_1(t_1, t_{-1})$, and $u_1(b_{12}; t_1, t_{-1})$ by $g_1(t_1, t_{-1})$. Denote $f_{-1}(t_{-1}) = f_2(t_2) \times \ldots \times f_n(t_n)$ for all $t_{-1} \in T_{-1}$. Then

$$
\Delta V_1(t_1) = \int_U h_1(t_1, t_{-1}) f_{-1}(t_{-1}) dt_{-1} + \int_W g_1(t_1, t_{-1}) f_{-1}(t_{-1}) dt_{-1}.
$$

The properties of u_1 imply that both h_1 for all $t_{-1} \in U$ and g_1 for all $t_{-1} \in W$ are nondecreasing in t_1. That is, ΔV_1 is nondecreasing on T_1.

It is not difficult to employ a similar argument to show that Γ satisfies the downward TSCP. □

A.5. Proof of Proposition 5

First we will show several auxiliary basic results concerning affiliated variables.[15] In what follows, $T = T_1 \times \ldots \times T_n = [0,1]^n$ and $n \geq 3$.

Since every truncated distribution resulting from restricting the domain of a distribution satisfying Assumption B.1 is also log-supermodular, the following lemma holds.

Lemma 5. *Let $f : T \to [0, +\infty)$ satisfy Assumption B.1, and let $D \subset T_{-i}$ be a set of nonzero measure. Then, for each $i \in I$, the function $g_i : T_i \times D \to \mathbb{R}$ defined by $g_i(t) = \frac{f_{-i}(t_{-i}|t_i)}{\int_D f_{-i}(t'_{-i}|t_i)dt'_{-i}}$ also satisfies Assumption B.1.*

For subsets A and B of an Euclidean space, $A \leq B$ means that $x \in A$ and $y \in B$ imply that $x \leq y$, where the inequality is understood coordinatewise (see, e.g., Shaked and Shanthikumar, 2007, p. 42). The next statement is reminiscent of the monotone increasing likelihood property.

Lemma 6. *Let $f : T \to (0, +\infty)$ satisfy Assumption B.1. If U and W are subsets of $T_{-1} = T_2 \times \ldots \times T_n$ of nonzero measure such that $U \leq W$, then for every $t_{11}, t_{12} \in T_1$ with $t_{11} < t_{12}$,*

$$\frac{\int_U f_{-1}(t_{-1}|t_{12})dt_{-1}}{\int_U f_{-1}(t_{-1}|t_{11})dt_{-1}} \leq \frac{\int_W f_{-1}(t_{-1}|t_{12})dt_{-1}}{\int_W f_{-1}(t_{-1}|t_{11})dt_{-1}}.$$

Proof. Fix some t_{11} and t_{12} in T_1 such that $t_{11} < t_{12}$. Since $U \leq W$, the affiliation of the players' types implies that $\frac{f_{-1}(t'_{-1}|t_{12})}{f_{-1}(t'_{-1}|t_{11})} \leq \frac{f_{-1}(t''_{-1}|t_{12})}{f_{-1}(t''_{-1}|t_{11})}$ for every $t'_{-1} = (t'_2, \ldots, t'_n) \in U$ and $t''_{-1} = (t''_2, \ldots, t''_n) \in W$. Then $\frac{\int_U \frac{f_{-1}(t_{-1}|t_{12})}{f_{-1}(t_{-1}|t_{11})} f_{-1}(t_{-1}|t_{11})dt_{-1}}{\int_U f_{-1}(t_{-1}|t_{11})dt_{-1}} \leq \frac{\int_W \frac{f_{-1}(t_{-1}|t_{12})}{f_{-1}(t_{-1}|t_{11})} f_{-1}(t_{-1}|t_{11})dt_{-1}}{\int_W f_{-1}(t_{-1}|t_{11})dt_{-1}}$. That is, $\frac{\int_U f_{-1}(t_{-1}|t_{12})dt_{-1}}{\int_U f_{-1}(t_{-1}|t_{11})dt_{-1}} \leq \frac{\int_W f_{-1}(t_{-1}|t_{12})dt_{-1}}{\int_W f_{-1}(t_{-1}|t_{11})dt_{-1}}$. □

It is well-known that logsupermodularity is preserved under integration (see, e.g., Karlin and Renott, 1980).

Lemma 7. *Let $f : T \to (0, +\infty)$ satisfy Assumption B.1, and let $U_j \subset T_j$ be a set of nonzero measure for each $j \in \{1, \ldots, h\}$, $1 \leq h < n$, and $U = U_1 \times \ldots \times U_h$. Then $g : T_{h+1} \times \ldots \times T_n \to (0, +\infty)$ defined by $g(t_{h+1}, \ldots, t_n) = \int_U f(t)dt_1 \ldots dt_h$ also satisfies Assumption B.1.*

A simple proof of Lemma 7 can be found in Quah and Strulovici (2012, Corollary 1).

Corollary 8. *Let $f : T \to (0, +\infty)$ satisfy Assumption B.1. Let $U = U_2 \times \ldots \times U_n \subset T_{-1}$, and $W = U_2 \times \ldots \times U_h \times W_{h+1} \times \ldots \times W_n \subset T_{-1}$, $2 \leq h \leq n-1$, be sets of nonzero measure such that $U_{h+1} \times \ldots \times U_n \leq W_{h+1} \times \ldots \times W_n$. Then, for every $t_{11}, t_{12} \in T_1$ with $t_{11} < t_{12}$,*

$$\frac{\int_U f_{-1}(t_{-1}|t_{12})dt_{-1}}{\int_U f_{-1}(t_{-1}|t_{11})dt_{-1}} \leq \frac{\int_W f_{-1}(t_{-1}|t_{12})dt_{-1}}{\int_W f_{-1}(t_{-1}|t_{11})dt_{-1}}.$$

[15] Most of them in some form or another can be found in Milgrom (2004).

Proof. Fix some $t_{11}, t_{12} \in T_1$ with $t_{11} < t_{12}$. Denote $\widetilde{U} = U_2 \times \ldots \times U_h$. Since $f(t_{-1}|t_1)$ satisfies Assumption B.1, $g(t_{h+1}, \ldots, t_n|t_1) = \int_{\widetilde{U}} f_{-1}(t_{-1}|t_1)dt_2 \ldots t_h$ also satisfies Assumption B.1 by Lemma 7. Then the claim follows from Lemma 6. □

We will need one more auxiliary lemma.

Lemma 8. *Let a_i, b_i, c_i, $i = 1, 2$, be positive real numbers, and let $\frac{a_2}{a_1} \leq \frac{c_2}{c_1}$. If $\frac{a_2}{a_1} \leq \frac{b_2}{b_1}$, then $\frac{a_2}{a_1} \leq \frac{c_2+b_2}{c_1+b_1}$.*

Proof. It suffices to notice that the inequality $\frac{a_2}{a_1} \leq \frac{c_2+b_2}{c_1+b_1}$ is equivalent to $a_2c_1 + a_2b_1 \leq a_1c_2 + a_1b_2$, and the latter is clearly valid. □

Now we are ready to proceed to the proof of Proposition 5.

Proof of Proposition 5. Since the case $n = 2$ is covered by Proposition 4, let $n \geq 3$. We will show Γ that satisfies the strong upward TSCP for $i = 1$. Fix some bidder 1's bids $b_{11} < b_{12}$ and a subprofile of nondecreasing strategies s_j, $j \in I\backslash\{1\}$ such that $\mu(t_j \in T_j : b_{1l} = s_j(t_j)) = 0$, $l = 1, 2$, for all $j \in I\backslash\{i\}$, and fix some bidder 1's types $t_{11} < t_{12}$. As before, denote $\Delta V_1(t_1) = V_1(b_{12}, s_{-1}; t_1) - V_1(b_{11}, s_{-1}; t_1)$ for $t_1 \in T_1$. We will show that $\Delta V_1(t_{11}) > 0$ implies that $\Delta V_1(t_{12}) > 0$. Define the sets U and W as in the proof of Proposition 3. Assume, without loss of generality, that U is of nonzero measure.

For each $j \in I\backslash\{1\}$, consider

$$U_j = \{t_j \in T_j : b_{11} > s_j(t_j)\};$$
$$W_j = \{t_j \in T_j : b_{12} > s_j(t_j) \text{ and } b_{11} < s_j(t_j)\}.$$

Clearly, $U = U_2 \times \ldots \times U_n$. For the sake of convenience, for $l = 1, 2$ and all $t_{-1} \in U$, denote $u_1(b_{12}; t_{1l}, t_{-1}) - u_1(b_{11}; t_{1l}, t_{-1})$ by $h_1(t_{1l}, t_{-1})$, and, for all $t_{-1} \in W$, denote $u_1(b_{12}; t_{1l}, t_{-1})$ by $g_1(t_{1l}, t_{-1})$. Since

$$\int_U h_1(t_{11}, t_{-1}) f_{-1}(t_{-1}|t_{11})dt_{-1} + \int_W g_1(t_{11}, t_{-1}) f_{-1}(t_{-1}|t_{11})dt_{-1} > 0,$$
$$\text{and} \int_U h_1(t_{11}, t_{-1}) f_{-1}(t_{-1}|t_{11})dt_{-1} \leq 0,$$

the set W is of nonzero measure. The structure of W is as follows: It is the union of some zero-measure subset of T_{-1} and a number of Cartesian products from

$$\left(\bigcup_{j=2}^{n}(W_j \times U_{-1j})\right) \cup \left(\bigcup_{j,k=2, j\neq k}^{n} (W_j \times W_k \times U_{-1jk})\right) \cup \ldots \cup (\times_{j=2}^{n} W_j),$$

where $W_j \times U_{-1j}$ denotes the Cartesian product of $U_2, \ldots, U_n$ in which U_j is replaced with W_j. The product $W_j \times W_k \times U_{-1jk}$ is defined in a similar manner, and so on.

Then

$$\Delta V_1(t_{12}) = \int_U h_1(t_{12}, t_{-1}) f_{-1}(t_{-1}|t_{12})dt_{-1} + \int_W g_1(t_{12}, t_{-1}) f_{-1}(t_{-1}|t_{12})dt_{-i}$$

$$= \frac{\int_U h_1(t_{12}, t_{-1}) f_{-1}(t_{-1}|t_{12}) dt_{-1}}{\int_U f_{-1}(t_{-1}|t_{12}) dt_{-1}} \int\limits_U f_{-1}(t_{-1}|t_{12}) dt_{-i}$$
$$+ \frac{\int_W g_1(t_{12}, t_{-1}) f_{-1}(t_{-1}|t_{12}) dt_{-1}}{\int_W f_{-1}(t_{-1}|t_{12}) dt_{-1}} \int\limits_W f_{-1}(t_{-1}|t_{12}) dt_{-1}.$$

Since $h_1 : [0,1] \times U \to \mathbb{R}$ and $g_1 : [0,1] \times W \to \mathbb{R}$ are nondecreasing, Lemma 5 above and Theorem 5 of Milgrom and Weber (1982) imply that

$$\frac{\int_U h_1(t_{12}, t_{-1}) f_{-1}(t_{-1}|t_{12}) dt_{-1}}{\int_U f_{-1}(t_{-1}|t_{12}) dt_{-1}} \geq \frac{\int_U h_1(t_{11}, t_{-1}) f_{-1}(t_{-1}|t_{11}) dt_{-1}}{\int_U f_{-1}(t_{-1}|t_{11}) dt_{-1}}$$

and

$$\frac{\int_W g_1(t_{12}, t_{-1}) f_{-1}(t_{-1}|t_{12}) dt_{-1}}{\int_W f_{-1}(t_{-1}|t_{12}) dt_{-1}} \geq \frac{\int_W g_1(t_{11}, t_{-1}) f_{-1}(t_{-1}|t_{11}) dt_{-1}}{\int_W f_{-1}(t_{-1}|t_{11}) dt_{-1}}.$$

Moreover, taking into account the structure of the set W, it is not difficult to see that Corollary 8 and Lemma 8 imply that

$$\frac{\int_U f_{-1}(t_{-1}|t_{12}) dt_{-1}}{\int_U f_{-1}(t_{-1}|t_{11}) dt_{-1}} \leq \frac{\int_W f_{-1}(t_{-1}|t_{12}) dt_{-1}}{\int_W f_{-1}(t_{-1}|t_{11}) dt_{-1}}$$

Therefore,

$$\Delta V_1(t_{12}) \geq \frac{\int_U h_1(t_{11}, t_{-1}) f_{-1}(t_{-1}|t_{11}) dt_{-1}}{\int_U f_{-1}(t_{-1}|t_{11}) dt_{-1}} \int\limits_U f_{-1}(t_{-1}|t_{11}) dt_{-1} \frac{\int_U f_{-1}(t_{-1}|t_{12}) dt_{-1}}{\int_U f_{-1}(t_{-1}|t_{11}) dt_{-1}}$$
$$+ \frac{\int_W g_1(t_{11}, t_{-1}) f_{-1}(t_{-1}|t_{11}) dt_{-1}}{\int_W f_{-1}(t_{-1}|t_{11}) dt_{-1}} \int\limits_W f_{-1}(t_{-1}|t_{11}) dt_{-i} \frac{\int_W f_{-1}(t_{-1}|t_{12}) dt_{-i}}{\int_W f_{-1}(t_{-1}|t_{11}) dt_{-i}}$$
$$\geq \frac{\int_W f_{-1}(t_{-1}|t_{12}) dt_{-1}}{\int_W f_{-1}(t_{-1}|t_{11}) dt_{-1}} \Delta V_1(t_{11}) > 0.$$

A similar argument can be employed to show that Γ has the strong downward TSCP. □

A.6. Proof of Theorem 4

Proof. Define the auxiliary game Γ^l, $l \in \{1, 2, \ldots\}$, by restricting the bidders' strategy spaces as follows: each L_i^l (S_i^l) consists of the elements of L_i (resp., S_i) whose values are equal to 0 on $[0, \frac{1}{10l})$. Let $S^l = S_1^l \times \ldots \times S_n^l$ for $l \in \{1, 2, \ldots\}$, and denote bidder i's interim payoff, interim value, and ex-ante value functions in Γ^l by V_i^l, $\overline{V}_i^l$, and $\overline{V}_i^{*l}$, respectively. By definition, each V_i^l is the restriction of the corresponding V_i to the set $B_i \times S_{-i}^l \times T_i$. It is worth noticing that $\overline{V}_i^l$ might differ from $\overline{V}_i$ on $S_{-i}^l \times [0, \frac{1}{10l})$ since $\overline{V}_i^l(s_{-i}; t_i) = V_i^l(0, s_{-i}; t_i) = V_i(0, s_{-i}; t_i)$ for all $(s_{-i}, t_i) \in S_{-i}^l \times [0, \frac{1}{10l})$ and all $i \in I$.

Consider the sequence of games $\{\Gamma^l\}$. Each V_i^l is transfer lower semicontinuous in (s_{-i}, t_i) on $S_{-i}^l \times [\frac{1}{10l}, 1]$ and each $\overline{V}_i^l$ is jointly continuous on $S_{-i}^l \times [\frac{1}{10l}, 1]$. Therefore, for every $\varepsilon > 0$, each correspondence $\widetilde{M}_i^\varepsilon : S_{-i}^l \twoheadrightarrow S_i^l$ defined for every $s_{-i} \in S_{-i}^l$ by

$$\widetilde{M}_i^{\varepsilon}(s_{-i}) = \{s_i \in S_i^l : V_i^l(s_i(t_i), s_{-i}; t_i) > \overline{V}_i^l(s_{-i}; t_i) - \varepsilon \text{ for almost all } t_i \in T_i\}$$

has the local intersection property. Consequently, for each $k \in \{1, 2, \ldots\}$, each Γ^l possesses a monotone interim $\frac{1}{k}$-equilibrium $s^{lk} = (s_1^{lk}, \ldots, s_n^{lk}) \in S^l$; that is, $V_i^l(s_i^{lk}(t_i), s_{-i}^{lk}; t_i) > \overline{V}_i^l(s_{-i}^{lk}; t_i) - \frac{1}{k}$ for almost every $t_i \in T_i$ and each $i \in I$.

Fix any $l \in \{1, 2, \ldots\}$. Without loss of generality, the sequence $\{s^{lk}\}_{k=1}^{\infty}$ tends pointwise to some $s^l \in S^l$. The strategy profile s^l need not be a Bayesian-Nash equilibrium of Γ^l, but it possesses some propitious properties. Denote $A_i^l(s_i, b) = \{t_i \in T_i \backslash [0, \frac{1}{10l}) : s_i(t_i) = b\}$ for $(s_i, b) \in S_i^l \times B_i$ and $i \in I$.

Consider the case when $V_j^{*l}(s^l) < \overline{V}_j^{*l}(s_{-j}^l)$ for some $j \in I$. Then there exists $\overline{b} \in [0.c)$ and a nonzero-measure subset $\widehat{A}_j^l(s_j^l, \overline{b})$ of $A_j^l(s_j^l, \overline{b})$ such that $\lim_k V_j^l(s_j^{lk}(t_j), s_{-j}^{lk}; t_j) = \overline{V}_j^l(s_{-j}^l; t_j) > V_j^l(s_j^l(t_j), s_{-j}^l; t_j)$ for all $t_j \in \widehat{A}_j^l(s_j^l, \overline{b})$, which can be explained only by a drop in the probability of winning the item by bidder j at $(s_j^l(t_j), s_{-j}^l)$ for every $t_j \in \widehat{A}_j^l(s_j^l, \overline{b})$. Denote $H^l(s^l, \overline{b}) = \{i \in I : \mu(A_i^l(s_i^l, \overline{b})) > 0\}$ and the cardinality of $H^l(s^l, \overline{b})$ by r. Without loss of generality, $H^l(s^l, \overline{b}) = \{1, \ldots, r\}$.

Assume that $\overline{b} = 0$ and $r \geq 2$. Since the bidders employ nondecreasing strategies, each $A_i^l(s_i^l, 0)$, $i \in H^l(s^l, 0)$, contains a segment $[t_i', t_i'']$ with $0 < t_i' < t_i''$. Then each sequence $\{s_i^{lk}\}$, $i \in H^l(s^l, 0)$, converges uniformly to s_i^l on $[t_i', t_i'']$ (see, e.g., Resnik, 2008, Section 0.1); that is, for every $\varepsilon > 0$ there exists $k(\varepsilon) \in \{1, 2, \ldots\}$ such that $\sup_{t_i \in [t_i', t_i'']} s_i^{lk}(t_i) < \frac{\varepsilon}{2}$ for all $k \geq k(\varepsilon)$ and all $i \in H^l(s^l, 0)$. Moreover, for all $i \in H^l(s^l, 0)$, $\lim_k V_i^l(s_i^{lk}(t_i), s_{-i}^{lk}; t_i) = \overline{V}_i^l(s_{-i}^l; t_i) > \beta_i$ for some $\beta_i > 0$ and almost all $t_i \in [t_i', t_i'']$, since $\min_{t \in [t_i', t_i''] \times T_{-i}} u_i(0; t) > 0$ (recall that u_i is strictly increasing in t_i) and f is positive and continuous on T.

On the other hand, for each $k \in \{1, 2, \ldots\}$, there exists $h(k) \in H^l(s^l, 0)$ such that $\mu((t_1, \ldots, t_r) \in [t_1', t_1''] \times \ldots \times [t_r', t_r''] : s_r^{lk}(t_{h(k)}) \leq \max_{i \in H^l(s^l, \overline{b}) \backslash \{h(k)\}} s_i^{lk}(t_i)) \geq (t_1'' - t_1') \times \ldots \times (t_r'' - t_r')/r$. Since the other bidders employ nondecreasing strategies and each u_i is nondecreasing in t_{-i}, bidder $h(k)$ can gain, when k is large enough, a discrete amount of utility, not depending on k, at an infinitesimal cost by bidding $\frac{\varepsilon}{2}$ instead of $s_{h(k)}^{lk}(t_{h(k)})$ for all $t_{h(k)} \in [t_{h(k)}', t_{h(k)}'']$, which is impossible since $\overline{V}_i^l(s_{-i}^{lk}; t_i) - \frac{1}{k} < V_i^l(s_i^{lk}(t_i), s_{-i}^k; t_i) \leq \overline{V}_i^l(s_{-i}^{lk}; t_i)$ for almost all $t_i \in T_i \backslash [0, \frac{1}{10l})$, all $i \in I$, and all k. That is, if $\overline{b} = 0$, then r has to be equal to 1.

If $\overline{b} > 0$, then $H^l(s^l, \overline{b})$ contains at least two elements. An argument similar to the one just provided can be employed to show this case is impossible.

To sum up, $\lim_k V_i(s_i^{lk}(t_i), s_{-i}^{lk}; t_i) = V_i(s_i^l(t_i), s_{-i}^l; t_i)$ for almost every $t_i \in T_i \backslash [0, \frac{1}{10l})$ and at least $n - 1$ bidders. For at most one bidder j, $\lim_k V_j(s_j^{lk}(t_j), s_{-j}^{lk}; t_j)$ might be larger than $V_j(s_j^l(t_j), s_{-j}^l; t_j)$ for the types in some nonzero-measure subset of $T_j \backslash [0, \frac{1}{10l})$ where s_j^l recommends that they choose the zero bid.

Now consider the sequence of strategy profiles $\{s^l\}$, with each s^l being constructed for Γ^l as described just above. Without loss of generality, the sequence $\{s^l\}$ tends pointwise to some $s^0 \in S$. Since the disturbances, caused by fixing the bidders' strategies to 0 on $[0, \frac{1}{10l})$, tend to become less pronounced as l tends to ∞, we have $\lim_l (V_i^*(s^l) - \overline{V}_i^*(s_{-i}^l)) = 0$ for each $i \in I$. That is, $\{s^l\}$ is a convergent sequence of approximate ex-ante equilibria of Γ. Denote $A_i(s_i, b) = \{t_i \in T_i : s_i(t_i) = b\}$ for $s_i \in S_i$, $i \in I$, and $b \in [0, c]$.

Assume, by contradiction, that $V_j^*(s^0) < \overline{V}_j^*(s_{-j}^0)$ for some $j \in I$. Then there exist $\overline{b} \in [0, c)$ and a nonzero-measure subset $\widehat{A}_j(s_j^0, \overline{b})$ of $A_j(s_j^0, \overline{b})$ such that $\lim_l V_j(s_j^l(t_j), s_{-j}^l; t_j) =$

$\overline{V}_j(s^0_{-j};t_j) > V_j(s^0_j(t_j), s^0_{-j};t_j)$ for all $t_j \in \widehat{A}_j(s^0_j,\overline{b})$. Denote $H(s^0,\overline{b}) = \{i \in I : \mu(A_i(s^0_i,\overline{b})) > 0\}$. Since $V_j(s^l_j(t_j), s^l_{-j};t_j) > 0$ for all l and all $t_j \in (0,1]$, the drop in the value of V_j at $(s^0_j(t_j), s^0_{-j})$ for $t_j \in \widehat{A}_j(s^0_j,\overline{b})$ can be explained only by a drop in the probability of winning the item by bidder j at $(s^0_j(t_j), s^0_{-j})$. Then there exists a bidder h in $H(s^0,\overline{b})$ whose probability of winning the item jumps up at $(s^0_h(t_h), s^0_{-h})$ for all t_h in some nonzero-measure subset $\widehat{A}_h(s^0_h,\overline{b})$ of $A_h(s^0_h,\overline{b})$. Since $\lim_l V_j(s^l_h(t_h), s^l_{-h};t_h) = \overline{V}_h(s^0_{-h};t_h) \geq 0$ for almost all $t_h \in T_h$ and $V_h(s^0_h(t_h), s^0_{-h};t_h)$ cannot be larger than $\overline{V}_h(s^0_{-h};t_h)$, it must be the case that $V_h(s^0_h(t_h), s^0_{-h};t_h) = \overline{V}_h(s^0_{-h};t_h) = 0$ for almost all $t_h \in \widehat{A}_h(s^0_h,\overline{b})$.

Pick some t'_h and t''_h in $\widehat{A}_h(s^0_h,\overline{b})$ with $t'_h < t''_h$. Denote, as in the proof of Lemma 2, $T^1_{-h} = \{t_{-h} \in T_{-h} : \max_{i\in I\setminus\{h\}} s^0_i(t_i) < \overline{b}\}$ and $T^2_{-h} = \{t_{-h} \in T_{-h} : \max_{i\in I\setminus\{h\}} s^0_i(t_i) = \overline{b}\}$. Clearly, $\mu(t_{-h} \in T^1_{-h} \cup T^2_{-h} : u_h(\overline{b};t'_h,t_{-h}) < 0) > 0$; otherwise, it would be the case that $V_h(s^0_h(t''_h), s^0_{-h};t''_h) > 0$ since u_h is strictly increasing in t_h (though V_h is not necessarily strictly increasing in t_h since, given $\overline{b}$ and s^0_{-h}, the probability of winning the item might be negatively related to t_h). Then $\mu(t_{-h} \in T^1_{-h} \cup T^2_{-h} : u_h(\overline{b};t'_h,t_{-h}) > 0) > 0$, and, therefore, $\mu(t_{-h} \in T^2_{-h} : u_h(\overline{b};t'_h,t_{-h}) > 0) > 0$. Notice that $V_h(\overline{b}, s^0_{-h};t'_h, T^2_{-h}) \geq 0$. An argument similar to the one provided in the proof of Lemma 2 can be employed to conclude that there exists a small enough $\varepsilon > 0$ such that $V_h(\overline{b}+\varepsilon, s^0_{-h};t'_h) > 0 = \overline{V}_h(s^0_{-h};t'_h)$, which is impossible. We will now provide some details of the argument.

Denote $T^2_j = \{t_j \in T_j : s_j(t_j) = \overline{b}\}$ for $j \in I\setminus\{h\}$, $J(\overline{b}, s^0_{-h}) = \{j \in I\setminus\{h\} : \mu(T^2_j) > 0\}$, and $q = \left|J(\overline{b}, s^0_{-h})\right|$. Also denote by U^q_k, $k \in \{1,\ldots,q\}$, a subset of T^2_{-h} which is the union of the Cartesian products from T^2_{-h} with T^2_j for k bidders from $J(\overline{b}, s^0_{-h})$ exactly. To begin with, assume that $q = \left|J(\overline{b}, s^0_{-h})\right| = 1$. If $\mu(T^1_{-h}) > 0$, then it must be the case that $V_h(\overline{b}, s^0_{-h};t'_h, U^1_1) > 0$; otherwise, $V_h(\overline{b}, s^0_{-h};t'_h, U^1_1) = 0$ would imply that $V_h(\overline{b}, s^0_{-h};t'_h, T^1_{-h}) < 0$. To understand the last implication, notice that, since $V_h(\overline{b}, s^0_{-h};t'_h, U^1_1) = \int_{T^2_{-h}} \frac{1}{2} u_h(\overline{b};t'_h,t_{-h}) f_{-h}(t_{-h}|t'_h)dt_{-h} = 0$, it must be the case that $u_h(\overline{b};t'_h,t_{-h}) < 0$ for some $t_{-h} \in T^2_{-h}$, which implies that $V_h(\overline{b}, s^0_{-h};t'_h, T^1_{-h}) < 0$. Therefore, $V_h(\overline{b}, s^0_{-h};t'_h, T^1_{-h}) + V_h(\overline{b}, s^0_{-h};t'_h, T^2_{-h}) < 0$, a contradiction. Then there exists a small enough $\varepsilon > 0$ such that $V_h(\overline{b}+\varepsilon, s^0_{-h};t'_h) > 0$. If $\mu(T^1_{-h}) = 0$ and $\mu(T^2_{-h}) > 0$, then $V_h(\overline{b}, s^0_{-h};t'_h, T^1_{-h} \cup T^2_{-h}) = 0$ implies $V_h(\overline{b}, s^0_{-h};t''_h, T^1_{-h} \cup T^2_{-h}) > 0$, which is also impossible.

If $q > 1$, then it must be the case that $V_h(\overline{b}, s^0_{-h};t'_h, U^q_q) > 0$ and, as in Lemma 2, the number of bidders tied at $\overline{b}$ at the type subprofiles t_{-h} where V_h is negative tends to be less than the number of bidders tied at $\overline{b}$ at the type subprofiles t_{-h} where V_h is positive, which explains why a small increase in $\overline{b}$ results in an increase in the value of V_h. □

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