Online Appendix: Equilibria and Incentives in Private Information Economies

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The online appendix contains the omitted proofs of Lemmas 2–4.

Lemma 2. For each $i \in I$, $\bar{u}_i$ is well-defined and continuous. Furthermore, if $u_i(z, q)$ is (strictly) monotone in $z$ for each fixed $q \in T^0$, then $\bar{u}_i$ is (strictly) monotone.

Proof. Since $u_i(\cdot, q)$ is assumed to be continuous on $\mathbb{R}^m_+$ for any fixed $q \in T^0$, $U_i$ is continuous. It is clear that for each $z \in \mathbb{R}^m_+$, $y_z = z$ and $q \in T^0$. Thus, $E_i^{-1}(z)$ is nonempty. Since $\mu_i(q) > 0$ for any $q \in T^0$, $E_i^{-1}(z)$ is compact for any $z \in \mathbb{R}^m_+$. Hence, $U_i$ attains the maximum on $E_i^{-1}(z)$. Thus, $\bar{u}_i$ is well-defined.

Take a sequence \{z^n\}$_{n \in \mathbb{N}}$ in $\mathbb{R}^m_+$ which converges to $z^0 \in \mathbb{R}^m_+$. Pick a $y^n$ from the set $E_i^{-1}(z^n)$ for each $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, we have $z^n = \sum_{q \in T^0_i} \mu_i(q) \cdot y^n(q)$. Since $T^0_i$ is finite, there is a subsequence \{y^{n_j}\}$_{j \in \mathbb{N}}$ of \{y^n\}$_{n \in \mathbb{N}}$ such that for each $q \in T^0_i$, $y^{n_j}(q)$ converges to some point in $\mathbb{R}^m_+$, denoted by $y^0(q)$. Clearly, $z^0 = \sum_{q \in T^0_i} \mu_i(q) \cdot y^0(q)$. That is, $y^0 \in E_i^{-1}(z^0)$. Thus, the correspondence $E_i^{-1}$ is upper hemicontinuous. Since $U_i$ is continuous and $E_i^{-1}$ is nonempty, compact-valued and upper hemicontinuous, Berge’s maximum theorem implies that $\bar{u}_i(z)$ is continuous.

The (strict) monotonicity follows from the fact that $U_i$ is (strictly) monotone if $u_i$ is also (strictly) monotone and the fact that if $z \leq z'$, then for any $y \in E_i^{-1}(z)$ there is $y' \in E_i^{-1}(z')$ such that $y \leq y'$. This completes the proof.

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Lemma 3. Let $\mathcal{E}$ be a private information economy and $\bar{\mathcal{E}}$ the induced large economy. Assume that $f$ is an idiosyncratic signal process. If $\bar{x}$ is an allocation of $\bar{\mathcal{E}}$, then there exists an allocation $x$ of $\mathcal{E}$ with the following properties:

(i) For each $i \in I$, $x_i$ depends only on agent $i$’s private signal;
(ii) For $\lambda$-almost all $i \in I$, $\bar{x}_i = E_i(x_i)$;
(iii) For $\lambda$-almost all $i \in I$, $\bar{u}_i(\bar{x}_i) = U_i(x_i)$;
(iv) For any coalition $W \in \mathcal{I}$, if $\int_W \bar{x}(i) \, d\lambda(i) = \int_W \bar{e}(i) \, d\lambda(i)$, then $\int_W x(i, f(i, t)) \, d\lambda(i) = \int_W e(i, f(i, t)) \, d\lambda(i)$ for $P$-almost all $t \in T$. In particular, if $\bar{x}$ is feasible in $\bar{\mathcal{E}}$, then $x$ is feasible in $\mathcal{E}$.

Proof. For each $i \in I$, we define $C(i) = \{y \in E_i^{-1}(\bar{x}_i) \mid U_i(y) = \bar{u}_i(\bar{x}_i)\}$, i.e., $C(i)$ is the set of all maximal elements of $U_i$ in $E_i^{-1}(\bar{x}_i)$. It can be easily proved that $C$ is a compact-valued measurable correspondence. Hence, we can obtain a measurable selection $x$ such that $x_i \in C(i)$. It is easy to see that $x$ satisfies the first three properties.

We shall verify that $x$ satisfies the fourth property. For any fixed coalition $W \in \mathcal{I}$, suppose that $\int_W \bar{x}(i) \, d\lambda(i) = \int_W \bar{e}(i) \, d\lambda(i)$. Since $\bar{x}(i) = E_i(x_i)$ and $\bar{e}(i) = E_i(e_i)$, we have $\int_W E_i(x_i) \, d\lambda(i) = \int_W E_i(e_i) \, d\lambda(i)$. By the definition of $E_i$, we have

$$\int_W \int_{T^0} x_i(q) \, d\mu_i(q) \, d\lambda(i) = \int_W \int_{T^0} e_i(q) \, d\mu_i(q) \, d\lambda(i).$$

Since $\mu_i = P f_i^{-1}$,

$$\int_W \int_T x(i, f(i, t)) \, dP(t) \, d\lambda(i) = \int_W \int_T e(i, f(i, t)) \, dP(t) \, d\lambda(i).$$

Because $f$ is essentially pairwise independent, we have

$$\int_W \int_T x(i, f(i, t)) \, dP(t) \, d\lambda(i) = \int_W x(i, f(i, t)) \, d\lambda(i) \text{ for } P\text{-almost all } t \in T,$$

and

$$\int_W \int_T e(i, f(i, t)) \, dP(t) \, d\lambda(i) = \int_W e(i, f(i, t)) \, d\lambda(i) \text{ for } P\text{-almost all } t \in T.$$

Thus, we have $\int_W x(i, f(i, t)) \, d\lambda(i) = \int_W e(i, f(i, t)) \, d\lambda(i)$ for $P$-almost all $t \in T$. \qed
Lemma 4. Assume that $f$ is an idiosyncratic signal process.

(i) Suppose that $x$ is a private core allocation of $\mathcal{E}$. Define an allocation $\bar{x}$ for the economy $\bar{\mathcal{E}}$ such that $\bar{x}(i) = E_i(x_i)$ for each $i \in I$. Then $\bar{x}$ is a core allocation of $\bar{\mathcal{E}}$.

(ii) Suppose that $\bar{x}$ is a core allocation of $\bar{\mathcal{E}}$. Then there exists a private core allocation $x$ of $\mathcal{E}$ such that $\bar{x}_i = E_i(x_i)$ and $\bar{u}_i(\bar{x}_i) = U_i(x_i)$ for $\lambda$-almost all $i \in I$.

Proof. Statement (i). To prove $\bar{x}$ to be a core allocation of the economy $\bar{\mathcal{E}}$, we need to show that

(1) $\bar{x}$ is feasible, i.e., $\int_x \bar{x}(i) d\lambda(i) = \int_x e(i) d\lambda(i)$.

(2) there is no coalition $W$ and no feasible allocation $\bar{x}'$ such that $\int_{W'} \bar{x}'(i) d\lambda(i) = \int_{W'} e(i) d\lambda(i)$ and $\bar{u}_i(\bar{x}') > \bar{u}_i(\bar{x})$ for $\lambda$-almost all $i \in W$.

For part (1), since $x$ is feasible, the following equation holds for $P$-almost all $t \in T$,

$$\int_T x(i, f(i, t)) d\lambda(i) = \int_T e(i, f(i, t)) d\lambda(i).$$

Integrating the above equation with respect to $t$ on $T$ under $P$, we get

$$\int_T \int_T x(i, f(i, t)) d\lambda(i) dP(t) = \int_T \int_T e(i, f(i, t)) d\lambda(i) dP(t).$$

Changing the order of integration (since $f$ is $\mathcal{I} \otimes \mathcal{T}$-measurable), we have

$$\int_T \int_T x(i, f(i, t)) dP(t) d\lambda(i) = \int_T \int_T e(i, f(i, t)) dP(t) d\lambda(i).$$

Since the left hand side is $\int T \bar{x}(i) d\lambda(i)$ and the right hand side is $\int T e(i) d\lambda(i)$, we have

$$\int T \bar{x}(i) d\lambda(i) = \int T e(i) d\lambda.$$

Hence, $\bar{x}$ is feasible. Part (1) has thus been proved.

We prove part (2) by contradiction. Suppose that there exists a coalition $W$ and a feasible allocation $\bar{x}'$ such that $\int_W \bar{x}'(i) d\lambda(i) = \int_W e(i) d\lambda(i)$ and $\bar{u}_i(\bar{x}') > \bar{u}_i(\bar{x})$ for $\lambda$-almost all $i \in W$. By Lemma 3, there exists a feasible allocation $x'$ of the private information economy $E$ such that

- For each $i \in I, x'_i$ depends only on agent $i$'s private signal.
- $x'_i = E_i(x'_i)$ for $\lambda$-almost all $i \in I$.
- $\bar{u}_i(x'_i) = U_i(x'_i)$ for $\lambda$-almost all $i \in I$.

We shall show that $x'$ blocks $x$ on $W$. Since $\int_W \bar{x}'(i) d\lambda(i) = \int_W e(i) d\lambda(i)$, Lemma 3 also indicates that

$$\int_W x'(i, f(i, t)) d\lambda(i) = \int_W e(i, f(i, t)) d\lambda(i).$$
for $\mathbf{P}$-almost all $t \in T$. Note that for each $i \in I$, $x_i \in E_i^{-1}(\bar{x}_i)$, and hence $\bar{u}_i(\bar{x}_i) = \max_{y \in E_i^{-1}(\bar{x}_i)} U_i(y) \geq U_i(x_i)$. Since $\bar{u}_i(\bar{x}_i') > \bar{u}_i(\bar{x}_i)$ for $\lambda$-almost all $i \in W$ and $\bar{u}_i(\bar{x}_i') = U_i(x_i')$, we have

$$U_i(x_i') = \bar{u}_i(\bar{x}_i') > \bar{u}_i(\bar{x}_i) \geq U_i(x_i)$$

for $\lambda$-almost all $i \in W$. It follows that $x'$ blocks $x$ on $W$, contradicting the fact that $x$ is a private core allocation. Part (2) is thus proved. This completes our proof of the statement (i).

Statement (ii). Let $\bar{x}$ be a core allocation of the economy $\bar{E}$. By Lemma 3, we have an allocation $x$ of $E$ such that

- For each $i \in I$, $x_i$ depends only on agent $i$’s private signal.
- $\bar{x}_i = E_i(x_i)$ for $\lambda$-almost all $i \in I$.
- $\bar{u}_i(\bar{x}_i) = U_i(x_i)$ for $\lambda$-almost all $i \in I$.
- $x$ is feasible for the economy $E$ (Since $\bar{x}$ is feasible for the economy $\bar{E}$).

Suppose that $x$ is not a private core allocation of $E$. Then we have a coalition $W$ and a feasible and private information measurable allocation $x'$ for the economy $E$ such that

1. $\int_W x'(i, f(i, t)) \, d\lambda(i) = \int_W e(i, f(i, t)) \, d\lambda(i)$ for $\mathbf{P}$-almost all $t \in T$.
2. $U_i(x_i') > U_i(x_i)$ for $\lambda$-almost all $i \in W$.

Define an allocation $\bar{x}'$ for the economy $\bar{E}$ by letting $\bar{x}_i' = E_i(x_i')$. It is easy to verify that $\bar{x}'$ is feasible and $\int_W \bar{x}'(i) \, d\lambda(i) = \int_W e(i) \, d\lambda(i)$.

Since $\bar{u}_i(\bar{x}_i') = \max_{y \in E_i^{-1}(\bar{x}_i)} U_i(y) \geq U_i(x_i')$ and $\bar{u}_i(\bar{x}_i) = U_i(x_i)$, we have $\bar{u}_i(\bar{x}_i') \geq U_i(x_i') > U_i(x_i) = \bar{u}_i(\bar{x}_i)$ for $\lambda$-almost all $i \in W$. Thus, $\bar{x}'$ blocks $\bar{x}$ on $W$, which leads to a contradiction. Hence, $x$ is a private core allocation of $E$. \[\square\]