Non-emptiness of the alpha-core

V. Filipe Martins-da-Rocha* Nicholas C. Yannelis†

December 5, 2010

Abstract

We prove non-emptiness of the α-core for balanced games with non-ordered preferences, extending and generalizing in several aspects the results of Scarf (1971), Border (1984), Florenzano (1989), Yannelis (1991b) and Kajii (1992). In particular we answer an open question in Kajii (1992) regarding the applicability of the non-emptiness results to models with infinite dimensional strategy spaces. We also provide two different models, one with Knightian preferences and one with voting preferences for which the results of Scarf (1971) and Kajii (1992) cannot be applied but our alpha-core existence result does apply.

JEL Classification: C60, C72

Keywords: α-Core, Externalities, Knightian preferences, Infinite dimensional strategy spaces

*Escola de Pós-Graduação em Economia, Fundação Getulio Vargas, Rio de Janeiro, BRASIL. E-mail: `victor.rocha@fgv.br`

†Department of Economics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA and Economics - School of Social Sciences, The University of Manchester, Oxford Road, Manchester M13 9PL, UK. E-mail: `nyanneli@illinois.edu`, `nicholasyannelis@manchester.ac.uk`

1 Introduction

The core is one of the most popular cooperative solution concept. It is adapted to a situation in which players behave cooperatively within each coalition and competitively across coalitions. A feasible allocation for the grand coalition, also called social state, belongs to the core if no other coalition has an incentive to form and deviate from the social state. For economic environments with externalities, the payoff of an agent depends not only on the agent's action but also on the actions of other agents. In this context, an issue arises concerning the definition of deviation or "blocking". When a coalition plans to block a given social state, actions of the agents outside the coalition affect the welfare of the members of the coalition. They should then anticipate how the agents outside the coalition react.

We may assume that outsiders stick to a given strategy while the coalition attempts to improve its welfare. The corresponding solution concept is the strong Cournot–Nash equilibrium. Unfortunately, strong equilibria often fail to exist since it is too easy for a coalition to block a given social state.

Alternatively, it sounds sensible to suppose that the outsiders of a blocking coalition do not stick to a given strategy and try to take revenge or to adapt themselves to the new situation. Aumann (1961) proposed to consider that agents act very conservatively when forming a coalition to block a social state. A coalition is compelled to change its strategies only if, for any possible choice of strategies made by the counter-coalition, each coalition member prefers the resulting joint strategy over the social state. If a coalition is compelled to form and change its strategies, it is said to α_{A}-block the social state. A social state belongs to the α_{A}-core if no coalition can α_{A}-block it. When entering a blocking coalition, each agent considers that the outsiders are allowed considerable freedom to react against the coalition. It is then difficult to α_{A}-block a feasible joint strategy and hence the α_{A}-core is relatively large. Scarf (1971) proved non-emptiness of the α_A-core for games in normal form where agents' preference relations are represented by continuous and quasi-concave utility/payoff functions.

In the spirit of Border (1984), Kajii (1992) proposed to investigate whether transitivity and completeness of preference relations are crucial for the validity of Scarf's result. He proved that generalized games with possibly non-ordered preferences also have a non-empty α_A-core if continuity of preferences and compactness of feasible strategies is satisfied for a topology derived from a norm. This additional requirement is innocuous for finite dimensional strategy spaces.[1] However, as pointed out by Kajii himself, the norm-compactness assumption on the sets of feasible strategies puts a serious limitation on the applicability of his non-emptiness result in the context of games with an infinite dimensional strategy space. Mas-Colell and Zame (1991) contains a detailed discussion on this problem: in particular they show that for generalized games derived from exchange economies, the set of feasible trades is in general just weakly compact but not norm-compact.[2]

It should also be noted that the non-emptiness result in Kajii (1992), does not contain as a particular case the existing results in the literature. Specifically, Florenzano (1989) proved that if preference relations are non-ordered but exhibit no externalities in consumption, then for generalized games derived from exchange economies, non-emptiness of the α_A-core is guaranteed under compactness and continuity assumptions for any Hausdorff linear topology. The same generality in terms of the topology is obtained by Scarf (1971) for generalized games where preferences relations (may) exhibit externalities but are ordered (i.e., represented by a utility/payoff function).[3]

[1] Any Hausdorff linear topology on a vector space with finite dimension coincides with the Euclidean topology.

[2] In Section 5 we provide two additional examples to illustrate this point.

[3] Rigorously, Scarf (1971) only proved non-emptiness for finite dimensional strategy spaces. However, his arguments can be straightforwardly adapted to handle any Hausdorff

Kajii (1992) considers games that are more general than those aforementioned: preferences relations are non-ordered (as in Florenzano (1989)) and have externalities in consumption (as in Scarf (1971)). However, the existence result in Kajii (1992) does not contain as a special case neither the results in Scarf (1971) nor those in Florenzano (1989) since Kajii (1992) assumes that the topology is normable.

One may think that the additional (and restrictive) assumption imposed by Kajii (1992) is the "price to pay" in order to combine non-ordered preferences and externalities. One of the main contributions of this paper is to show that this is not the case. We prove that non-emptiness of the α_{A}-core can be obtained for any Hausdorff linear topology, including weak topologies that play a crucial for compactness of the set of feasible strategies in infinite dimensions. Our theorem generalizes and unifies the results of Scarf (1971), Border (1984), Florenzano (1989) and Kajii (1992). We also propose two possible applications to illustrate the economic relevance of our result.

Actually, we provide an additional contribution to the literature: we introduce a new solution concept where, when reacting to a blocking coalition, outsiders of the coalition are allowed less freedom than what was suggested by Aumann (1961). In particular we prove non-emptiness of a smaller set than the α_{A}-core. There are many alternatives to the solution proposed by Aumann (1961). We already mentioned the strong Cournot-Nash equilibrium for which agents forming a blocking coalition believe the outsiders will not react. A third solution concept is the β-core for which a blocking coalition is no longer required to select a specific strategy independently of the remaining players, but rather is permitted to vary its blocking strategy as a function of the complementary coalition's choice. Since the blocking possibilities are larger the β-core is a smaller set than the α_{A}-core. However, as shown by Scarf (1971), the β-blocking power is so strong that it is easy

topological space. See Section 4.1 for a detailed discussion.

to construct examples of non-existence. There is another solution concept introduced by Yannelis (1991b) in the context of generalized games derived from exchange economies. When agents decide to form a coalition to block a social state in a generalized game, they are restricted to choose a joint strategy that is feasible for the coalition. A priori, there is no reason to consider that agents in the counter-coalition are not similarly restricted and can choose a joint strategy that may not be feasible for the counter-coalition. Following this line, Yannelis (1991b) proposed an alternative definition of the α-core: a coalition is said to α_{Y}-block a social state if there exists a joint strategy feasible for the coalition such that each coalition member prefers the resulting joint strategy over the social state, whatever are the reactions of the agents in the counter-coalition among those that are feasible for the counter-coalition. In the adaptation of Aumann's blocking power to generalized games proposed by Kajii (1992), agents of a blocking coalition do not consider that outsiders should choose feasible strategies. Since agents entering a coalition consider that the outsiders can react with less freedom, it is easier to α_{Y}-block a feasible joint strategy and hence the α_{Y}-core is smaller than the α_{A}-core. Yannelis (1991b) (see also Koutsougeras and Yannelis (1993)) succeeded to prove non-emptiness of the α_{Y}-core for pure exchange economies with at most two agents. For games with more than two agents, the α_{Y}-blocking power is too strong. Indeed, Holly (1994) proposed an example of a pure exchange economy with three agents, satisfying standard assumptions but with an empty α_{Y}-core.

We introduce an alternative definition of the α-core where, when reacting to a blocking coalition, outsiders of the coalition are allowed less freedom than what was suggested by Aumann (1961) but more freedom than what was proposed by Yannelis (1991b). We prove non-emptiness of our α-core under very general conditions, implying non-emptiness of the "standard" α_{A}-core. Moreover, when there are at most two agents, our α-core coincides with the α_{Y}-core. Therefore, we obtain a general non-emptiness result having the additional interesting feature of unifying the results of non-emptiness of the α_{A}-core and the α_{Y}-core.

The paper is organized as follows. Section 2 defines generalized games and core solutions are presented in Section 3. Sufficient conditions for non-emptiness are introduced in Section 4 and the details of the proof are presented in Section 6. The main theorem is applied in Section 5 where we illustrate the economic relevance of our general result by means of two applications. The definition of standard continuity properties of correspondences is postponed to the appendix, where we also provide the proofs of technical results.

2 Generalized Games

We consider a cooperative generalized game (in normal form) with a finite set I of agents. A subset $E \subset I$ represents a coalition and we fix a subset $\mathbb{I}$ of the set of all non-empty subsets of I that represents the family of admissible coalitions.[4] We assume that the grand coalition I and each individual coalition $\{i\}$ belong to $\mathbb{I}$. Each agent $i \in I$ chooses an individual strategy x_i in his strategy set X_i.

Assumption 2.1. For each agent i, the strategy set X_i is a non-empty and convex subset of a Hausdorff linear topological space L.

An element $x = (x_i)_{i \in I}$ of $X \equiv \prod_{i \in I} X_i$ is called a joint strategy (or an allocation) and may be thought as a social state. If E is a subset of I, then the we denote by X^E the product set $\prod_{i \in E} X_i$.[5] Given a subset $S \subset I$, if y

[4]Usually it is assumed that $\mathbb{I}$ coincides with $2^I \setminus \{0\}$ the set of non-empty subsets of I. We allow for the possibility that some coalitions cannot form.

[5]Both notations X^I and X will be used for $\prod_{i \in I} X_i$.

belongs to X^S and w belongs to $X^{I\setminus S}$ then $z = (y, w)$ denotes the allocation defined by $\pi^S(z) = y$ and $\pi^{I\setminus S}(z) = w$, where for any $E \subset I$, the mapping $\pi^E : X \to X^E$ is the natural projection defined by $\pi^E(x) = (x_i)_{i\in E}$ for each $x = (x_i)_{i\in I} \in X$.[6] For each i, we denote by $\mathbb{I}(i)$ the collection of all admissible coalitions $E \in \mathbb{I}$ containing i.[7]

For each admissible coalition $S \in \mathbb{I}$ there is a set $F^S \subset X^S$ which represents the set of feasible joint strategies for the coalition S. The set F^I of feasible social states will also be denoted by F. We make the standard convexity and compacity assumption on the sets of feasible strategies.

Assumption 2.2. For each admissible coalition $S \in \mathbb{I}$, the set F^S is non-empty compact and the set F is additionally assumed to be convex.

Remark 2.1. We impose that for each admissible coalition $S \in \mathbb{I}$ the set F^S is non-empty. For coalitions different from the grand coalition or individuals, this assumption is imposed without any loss of generality. Indeed, it is sufficient to replace the set $\mathbb{I}$ by the set $\{S \in \mathbb{I} : \ F^S \neq \emptyset\}$.[8]

We consider the case where agents have (possibly non-ordered) preferences displaying externalities (also called interdependent preferences). Formally, each agent i has a preference relation on X which is described by a correspondence P_i from X to X. If $x \in X$ is an allocation, then $P_i(x)$ represents the set of allocations $y \in X$ that are strictly preferred to x by agent i. We make the standard assumption that preferences are convex.

Assumption 2.3. For each $x \in X$ and each agent i, we have $x \notin \operatorname{co} P_i(x)$.

Our **generalized game** can be represented by the family

$$\mathcal{G} = \left\{L, (X_i, P_i)_{i\in I}, F, (F^S)_{S\in\mathbb{I}}\right\}.$$

[6]We still denote by π^E the restriction of π^E to any subset X^F where $E \subset F \subset I$ and for each $i \in I$, the projection $\pi^{\{i\}}$ is denoted by π_i.

[7]Observe that the set $\mathbb{I}(i)$ is always non-empty since it contains $\{i\}$ and I.

[8]This is the reason why we do not assume that all coalitions are admissible.

When the feasible sets are degenerate, in the sense that $F^S = X^S$, we obtain the standard definition of a game in normal form. Throughout the paper we will assume that Assumptions 2.1, 2.2 and 2.3 are always satisfied.

An important class of games is constituted by those derived from pure exchange economies, as defined below.

Definition 2.1. A game $\mathcal{G} = \{L, (X_i, P_i)_{i\in I}, F, (F^S)_{S\in\mathbb{I}}\}$ is said to be *derived from a standard pure exchange economy* if the space L is endowed with a linear order $\geqslant$ such that for each agent i, the strategy set X_i coincides with the cone $L_+ = \{z \in L : z \geqslant 0\}$ and an allocation $(y_i)_{i\in S}$ in L_+^S is feasible for coalition S, i.e., belongs to F^S, if

$$\sum_{i\in S} y_i = \sum_{i\in S} e_i$$

where $e_i \in L_+$ is agent i's initial endowment.

Remark 2.2. If we have a finite number g of commodities, the set L coincides with $\mathbb{R}^g$ and the consumption set of each agent coincides the non-negative cone $\mathbb{R}^g_+$. Under uncertainty with infinitely many states of nature represented by a probability space $(S, \mathcal{S}, \sigma)$, one may choose L to be the space $\mathcal{L}^\infty(S, \mathcal{S}, \sigma)$ of essentially bounded functions and L_+ to be the cone of non-negative functions. We have considered *standard* pure exchange economies for simplicity. The strategy (or consumption) set X_i may be a strict subset of L_+ as it is the case in models with differential information considered in Yannelis (1991a) (see also Podczeck and Yannelis (2008)). For instance, one can restrict agent i's strategies to lie in $\mathcal{L}^\infty_+(S, \mathcal{S}^i, \sigma)$ where $\mathcal{S}^i$ is a sub σ-algebra of $\mathcal{S}$ representing the states agent i can discern ex-post.

Remark 2.3. In Border (1984) and Kajii (1992), the set of feasible strategies for a coalition S may depend on the current social state. For simplicity, we have considered the special case where the set of feasible strategies for a coalition is independent of the current social state.[9] However, by suitably adapting our set of assumptions, we can extend the results of this paper to the more general framework considered in Border (1984) and Kajii (1992).

In our modeling of preference relations, each agent i ranks allocations in X. Naturally, this modeling encompasses preferences without externalities where agent i only ranks his own strategies in X_i. Indeed, if agent i's preference relation is described by a correspondence $\widehat{P}_i$ defined from X_i to X_i, then we can construct the correspondence P_i as follows:

$$\forall x \in X, \quad P_i(x) = \{(y_k)_{k\in I} \in X \ : \ y_i \in \widehat{P}_i(x_i)\}.$$

In that case, we say that agent i's preference relation has *no externalities.* Our framework also encompasses the case where agent i ranks individual strategies but his taste is affected by the "current" social state. This happens when the correspondence P_i is derived from a correspondence $\widetilde{P}_i$ defined from X to X_i, as follows

$$\forall x \in X, \quad P_i(x) = \{(y_k)_{k\in I} \in X \ : \ y_i \in \widetilde{P}_i(x)\}.$$

In that case, we say that agent i's preference relation displays *weak externalities.*

If there exists an ordinal utility function u_i defined from X_i to $[-\infty, \infty)$ such that

$$\forall x = (x_k)_{k\in I} \in X, \quad P_i(x) = \{(y_k)_{k\in I} \in X \ : \ u_i(y_i) > u_i(x_i)\}$$

then agent i's preference relation exhibits no externalities. In the majority of non-cooperative games analyzed in the literature, each agent i's preference relation depends on the actions of the other agents through a payoff function u^i defined from the product space X to $[-\infty, \infty)$. If the correspondence P^i

[9] This property is satisfied for generalized games derived from exchange economies.

is given by

$$\forall x \in X, \quad P_i(x) = \{(y_k)_{k \in I} \in X \ : \ u_i(y_i, x_{-i}) > u_i(x_i, x_{-i})\}$$

then agent i's preference relation displays weak externalities. Florenzano (1989) assumed that agents preference relations satisfy the above property. This kind of modeling could make sense if we were considering a framework where deviations are only unilateral. Since we are interested in a cooperative solution, it seems more natural to consider the following definition

$$\forall x \in X, \quad P_i(x) = \{y \in X \ : \ u_i(y) > u_i(x)\}$$

which is consistent with coalitional deviations and corresponds to the model considered in Scarf (1971).

3 Core solutions

A core allocation is a feasible social state that is robust to all possible deviations (or blocking) by coalitions. Since actions of the agents outside a blocking coalition affect the welfare of the members of the coalition, it is necessary to consider the way the agents outside the coalition react in order to define a core solution. More precisely, when a group of agents forms a coalition to block an allocation, one should specify what are the expectations of these agents about the possible reactions of the agents outside the coalition.

3.1 Weak blocking power: α_{A}-core

Aumann (1961) suggested the following blocking power: an admissible coalition $S \in \mathbb{I}$ is said to α_{A}-block a given social state represented by a feasible joint strategy $x \in F$ if there exists a joint strategy $y = (y_i)_{i \in S}$ feasible for the coalition S, i.e., $y \in F^S$, such that the coalition S can ensure a social state preferred by all the agents in it regardless of any strategies the other agents may choose, i.e.,

$$\{(y,w)\ :\ w \in X^{I\setminus S}\} \subset \bigcap_{i\in S} P_i(x).$$

The **$\boldsymbol{\alpha}_{\mathbf{A}}$-core** is then the set of feasible strategies $x \in F$ such that no coalition can α_{A}-block x. When entering a blocking coalition S, each agent $i \in S$ is very conservative and considers that the outsiders are allowed considerable freedom to react against the coalition.[10] It is then difficult to α_{A}-block a feasible strategy and hence the α_{A}-core is relatively large. Scarf (1971) proved the non-emptiness of the α_{A}-core for games where agents' preference relations are represented by continuous and quasi-concave utility functions. This existence result was generalized to possibly non-ordered preferences by Kajii (1992).

3.2 Strong blocking power: $\boldsymbol{\alpha}_{\mathbf{Y}}$-core

For generalized games that are not standard,[11] Yannelis (1991b) proposed to increase the blocking power of coalitions by assuming that blocking agents expect agents outside the blocking coalition to react by choosing feasible strategies. More precisely, an admissible coalition $S \in \mathbb{I}$ is said to α_{Y}-block a given social state represented by a feasible strategy $x \in F$ if there exists a feasible strategy $y \in F^S$ with which the coalition S can ensure a social state strictly preferred by all the agents in it regardless of feasible strategies $w \in F^{I\setminus S}$ that the coalition $I \setminus S$ of outsiders may choose, i.e.[12]

$$\{(y,w)\ :\ w \in F^{I\setminus S}\} \subset \bigcap_{i\in S} P_i(x).$$

[10]In particular outsiders may take revenge by choosing the worst strategies regarding agent i's preferences.

[11]In the sense that for at least one coalition S, the set F^S is different from X^S.

[12]If $I \setminus S = \emptyset$ then by convention we pose $\{(y,w)\ :\ w \in F^{\emptyset}\} = \{y\}$.

The $\boldsymbol{\alpha}_{\mathbf{Y}}$**-core** is then the set of feasible strategies $x \in F$ such that no coalition can α_{Y}-block x.[13] If the game is not generalized (that is, $F^S = X^S$ for each coalition S) then the α_{Y}-core and the α_{A}-core coincides. However, if the game is generalized (that is, F^S is a strict subset of X^S for at least one coalition S) then the α_{Y}-core may be strictly smaller than the α_{A}-core. Indeed, since agents entering a coalition consider that the outsiders can react with less freedom, it is easier to α_{Y}-block a feasible strategy.

This solution concept seems natural for generalized games derived from exchange economies. If a coalition S forms to block a social state x, the agents within the coalition will choose strategies (or equivalently consumption plans) $(y_i)_{i\in S}$ reallocating their own initial endowments, i.e.,

$$\sum_{i\in S} y_i = \sum_{i\in S} e_i.$$

The allocation $(z_j)_{j\notin S}$ chosen by the agents in the counter-coalition has to be consistent with the scarcity of resources available to the counter-coalition $I \setminus S$, i.e.,

$$\sum_{j\in I\setminus S} y_i = \sum_{i\in I\setminus S} e_i.$$

One may imagine that agents in $I \setminus S$ decide to form several coalitions and redistribute their resources within each sub-coalition. More precisely, one may have that the reaction $(z_j)_{j\notin S}$ of the counter-coalition is such that

$$I \setminus S = \bigcup_{k\in K} T^k \quad \text{and} \quad \forall k \in K, \quad \sum_{\ell\in T^k} z_\ell = \sum_{\ell\in T^k} e_\ell$$

where $(T^k)_{k\in K}$ a finite partition of $I \setminus S$. Nonetheless, the reaction $(z_j)_{j\notin S}$ will still belong to $F^{I\setminus S}$.

[13]Observe that for the validity of this concept, we need to assume that for each admissible coalition $S \in \mathbb{I}$, the coalition of outsiders $I \setminus S$ is also admissible, i.e.

$$\forall S \subset I, \quad S \in \mathbb{I} \Longrightarrow I \setminus S \in \mathbb{I}.$$

This is obviously satisfied if all coalitions are admissible, i.e., $\mathbb{I} = 2^I \setminus \{\emptyset\}$.

Yannelis (1991b) (see also Koutsougeras and Yannelis (1993)) proved that the α_{Y}-core is non-empty for economies with at most two agents. For economies with more than two agents, the α_{Y}-blocking power may be too strong. Indeed, Holly (1994) proposed an example of a pure exchange economy (satisfying standard assumptions) with three agents and having an empty α_{Y}-core.

3.3 Intermediate blocking power: our $\boldsymbol{\alpha}$-core concept

When proving the non-emptiness of the α_{A}-core, we realized that actually our arguments enable us to prove the non-emptiness of a smaller set. To this end we introduce a new α-core notion which coincides with the α_{Y}-core in the 2-agents case. We obtain a general non-emptiness result having the additional interesting feature of unifying the results of non-emptiness of the α_{A}-core and the α_{Y}-core.

Before providing the rigorous definition of our solution concept, we need to introduce some notations. Fix an admissible coalition $S \in \mathbb{I}$ and an outsider $i \notin S$. We let F_i be the set of all $z_i \in X_i$ corresponding to agent i's individual strategy in a feasible joint strategy $z = (z_e)_{e \in E} \in F^E$ where $E \in \mathbb{I}(i)$ is an admissible coalition containing i and different from the grand coalition I, i.e.,

$$F_i \equiv \{\pi_i(z) \;:\; z \in F^E,\; E \in \mathbb{I}(i) \text{ and } E \neq I\}.$$

The set F_i contains all strategies agent i may expect to obtain if he joins any coalition different from the grand coalition.

Definition 3.1. A coalition $S \in \mathbb{I}$ is said to **$\boldsymbol{\alpha}$-block** the feasible joint strategy $x \in F$ if there exists a strategy $y \in F^S$ feasible for coalition S such that the social state (y, w) is strictly preferred to x by every agent i in the coalition S whatever is the reaction $w = (w_j)_{j \notin S}$ of the counter-coalition, where each w_j belongs to the set $W_j \equiv \operatorname{co} F_j$.[14] The $\boldsymbol{\alpha}$**-core** of the game $\mathcal{G}$ is the set of all feasible strategies x in F such that no coalition can α-block x.

A feasible strategy x is α-blocked by coalition $S \in \mathbb{I}$ if there exists a feasible strategy $y \in F^S$ satisfying

$$\{(y, w) \ : \ w \in W^{I \setminus S}\} \subset \bigcap_{i \in S} P_i(x) \tag{3.1}$$

where $W^{I \setminus S}$ is the product set $\prod_{j \notin S} W_j$. This leads us to consider the following definition. If S is an admissible coalition then we let P^S be the correspondence from X to X^S defined by

$$P^S(x) \equiv \{y \in X^S \ : \ \{y\} \times W^{I \setminus S} \subset \bigcap_{i \in S} P_i(x)\},$$

where $\{y\} \times W^{I \setminus S} \equiv \{(y, w) \ : \ w \in W^{I \setminus S}\}$. Following this notation, a feasible joint strategy $x \in F$ belongs to the α-core if and only if there does not exist a feasible coalition $S \in \mathbb{I}$ such that

$$F^S \cap P^S(x) \neq \emptyset.$$

Note that for each admissible coalition $S \in \mathbb{I}$, we have

$$F^{I \setminus S} \subset W^{I \setminus S} \subset X^{I \setminus S}.$$

Therefore the α_{Y}-core is a subset of the α-core, and the α-core is a subset of the α_{A}-core. For non-generalized games, the three concepts coincide since $F^E = X^E$ for every coalition $E \in \mathbb{I}$. For generalized games with two players and convex feasible sets, the α-core coincides with the α_{Y}-core.[15] Indeed, the only coalition $E \in \mathbb{I}(i)$ different from the grand coalition is the singleton $\{i\}$.

[14] If $S = I$ then the notation (y, w) represents y.

[15] In a game with two players $\{i, j\}$, we say that feasible sets are convex if $F^{\{i\}}$ and $F^{\{j\}}$ are convex. Observe that we have already assumed that F^I is convex. For generalized games derived from standard pure exchange economies, feasible sets are always convex.

This implies that F_i coincides with $F^{\{i\}}$ and therefore $W^{I\setminus S}$ coincides with $F^{I\setminus S}$ for any possible blocking coalition.

Under the set of assumptions that we will impose to get non-emptiness of the α_{A}-core, we also obtain non-emptiness of the α-core. Therefore, we will get as a direct corollary of the our existence result, both the non-emptiness result of the α_{A}-core in Kajii (1992) and the non-emptiness result of the α_{Y}-core in Yannelis (1991b).

4 Non-emptiness of the α-core

The main contribution of this paper is to provide conditions on primitives (additional to the standard Assumptions 2.1, 2.2 and 2.3) that are sufficient for non-emptiness of the α-core. We will need to make a continuity assumption on preference relations and impose a balancedness condition for feasible correspondences. The notion of a "balanced" n-person game was first discussed by Bondareva (1962) and Shapley (1965) in the context of a game with transferable utility. For the case of non-transferable utility, balancedness was used by Scarf (1967) for a game in characteristic form and Scarf (1971) for a game in normal form.

Let Δ be the set of weights $\lambda = (\lambda_S)_{S\in\mathbb{I}} \in \mathbb{R}^{\mathbb{I}}_+$ satisfying the condition

$$\forall i \in I, \ \sum_{S\in\mathbb{I}(i)} \lambda_S = 1.$$

An element λ in Δ is called a balancing weight and the associated family of coalitions $\{S \in \mathbb{I} : \ \lambda_S > 0\}$ is said balanced.

Remark 4.1. Observe that Δ is always non-empty. Indeed, let $\kappa = (\kappa_S)_{S\in\mathbb{I}}$ be the balancing weight defined as follows: $\kappa_S = 1$ if S is a singleton and $\kappa_S = 0$ elsewhere. Since all singletons $\{i\}$ belong to $\mathbb{I}$, we obtain that $\kappa \in \Delta$.

We recall the definition of a balanced generalized game.[16]

Definition 4.1. The generalized game $\mathcal{G} = \left\{L, (X_i, P_i)_{i\in I}, F, (F^S)_{S\in\mathbb{I}}\right\}$ is **balanced** if for each balancing weight $\lambda \in \Delta$, if $y^S \in F^S$ is a feasible strategy for each coalition $S \in \mathbb{I}$ with $\lambda_S > 0$, then the strategy $z = (z_i)_{i\in I}$ defined by

$$\forall i \in I, \quad z_i \equiv \sum_{S\in\mathbb{I}(i)} \lambda_S y_i^S \tag{4.1}$$

is a feasible social state, i.e., $z \in F$.

Throughout the paper we will assume the game $\mathcal{G}$ is balanced. It is straightforward to check that generalized games derived from pure exchange economies are always balanced.

Remark 4.2. Let $(x_i)_{i\in I}$ be a family of strategies that are individually feasible, i.e., $x_i \in F^{\{i\}}$ for each agent. Since the game $\mathcal{G}$ is balanced, the associated social state $x = (x_i)_{i\in I}$ is feasible for the grand coalition, i.e., $x \in F$. To see this, we can choose the specific balancing weight κ defined in Remark 4.1 and apply (4.1).

In order to motivate the continuity assumption we will impose, we review the existence results in the literature.

4.1 The literature

Agent i's preference relation is said to be ordered if there exists a function $u_i : X \to [-\infty, \infty)$ such that a joint strategy y is strictly preferred to another joint strategy x if and only if we have $u_i(y) > u_i(x)$. When agents have ordered preferences, we can construct the associated α_{A}-game in characteristic

[16] Many generalizations of this concept have been proposed in the literature: π-balancedness of Billera (1970), Π-balancedness of Predtetchinski and Herings (2004), and payoff-dependent balancedness of Bonnisseau and Iehlé (2007).

form $(V_{\text{A}}(S))_{S\in\mathbb{I}}$ defined by

$$V_{\text{A}}(I) \equiv \left\{(v_i)_{i\in I} \in \mathbb{R}^I \;:\; \exists y \in F, \quad \forall i \in I, \quad v_i \leqslant u_i(y)\right\}$$

and for every coalition $S \in \mathbb{I}' = \mathbb{I} \setminus \{I\}$,

$$V_{\text{A}}(S) \equiv \left\{(v_i)_{i\in I} \in \mathbb{R}^I \;:\; \exists y \in F^S, \quad \forall i \in S, \quad v_i \leqslant \inf_{w\in X^{I\setminus S}} u_i(y,w)\right\}.$$

Observe that a social state $x \in F$ belongs to the α_{A}-core of the game in normal form if and only if the associated profile of payoffs $(u_i(x))_{i\in I}$ is a core of the associated α_{A}-game in the sense that

$$(u_i(x))_{i\in I} \in V_{\text{A}}(I) \setminus \bigcup_{S\in\mathbb{I}'} \operatorname{int} V_{\text{A}}(S).$$

Assume now that the game $\mathcal{G}$ is balanced and fix a balancing weight $\lambda \in \Delta$. Denote by $\mathbb{I}(\lambda)$ the subset of coalitions $S \in \mathbb{I}$ satisfying $\lambda_S > 0$. The main contribution in Scarf (1971) consists on showing that

$$\bigcap_{S\in\mathbb{I}(\lambda)} V_{\text{A}}(S) \subset V_{\text{A}}(I)$$

implying that the α_{A}-game in characteristic form is π-balanced[17] where $\pi = (\pi_S)_{S\in\mathbb{I}}$ is given by $\pi_S = 1$. If, moreover, the utility function of each agent is assumed to be continuous and quasi-concave then the α_{A}-game in characteristic form satisfies the following conditions.

($\mathcal{G}$1) For all $S \in \mathbb{I}$, the set $V_{\text{A}}(S)$ is non-empty proper and closed.

($\mathcal{G}$2) For all $S \in \mathbb{I}$, if $x \in V_{\text{A}}(S)$ and $y \in \mathbb{R}^I$ satisfy $y_i \leqslant x_i$ for all $i \in S$, then $y \in V_{\text{A}}(S)$.

($\mathcal{G}$3) The set $V_{\text{A}}(I) \setminus \bigcup_{i\in I} \operatorname{int} V(\{i\})$ is non-empty and compact.[18]

[17] As defined by Billera (1970).

[18] The fact that the set $V_{\text{A}}(I) \setminus \bigcup_{i\in I} \operatorname{int} V_{\text{A}}(\{i\})$ is non-empty follows from the balancedness of the game and Remark 4.2.

It is well-known that under balancedness of the α_{A}-game, the above conditions are sufficient for non-emptiness of the core (see Scarf (1967) and Predtetchinski and Herings (2004)). Using the above arguments Scarf derived the following non-emptiness result.

Theorem 4.1 (Scarf (1971))**.** *If agents' preference relations are represented by continuous and quasi-concave functions then the* α_{A}*-core is non-empty.*

Rigorously, Scarf (1971) proved a less general result than the one stated above since he assumes that strategy sets are subsets of finite-dimensional Euclidean spaces. However, all the arguments in Scarf (1971) can be straightforwardly adapted to handle the general case.

Kajii (1992) proposed to investigate whether the assumption that preference relations are ordered can be relaxed. This question was already addressed by Border (1984) when agents' preference relations have no externalities and by Florenzano (1989) for the case of weak externalities.[19] Both existence results deal with the non-emptiness of the core and not the α-core since they do not allow for externalities.

As far as we know, Kajii (1992) is the only non-emptiness result of the α-core for (NTU) games with non-ordered preferences displaying externalities. Kajii adapted (and generalized) Border's approach by constructing a "pseudo-utility function" $u^i : X \times X \to \mathbb{R}_+$ where $u^i(x, y)$ is the distance between the pair (x, y) and the complement of the graph of the correspondence P^i. If the correspondence P^i has an open graph then we get the following important property: $u^i(x, y) > 0$ if and only if y is strictly preferred to x. In order to apply Scarf (1971), one should prove that the function $y \mapsto u^i(x, y)$ is quasi-concave, which is true when the distance d is derived from a norm. This explains the following result.

[19] Actually Florenzano (1989) proves the non-emptiness of the core of a production economy. However her arguments can be adapted to handle generalized games.

Theorem 4.2 (Kajii (1992))**.** *Assume that the topology of the strategy vector space L (for which strategy sets are compact) is derived from a norm. If the correspondences defining agent's preference relations have open graphs then the* α_{A}*-core is non-empty.*

This is an important contribution but one may not be fully satisfied by this result. As pointed out by Kajii (1992) the main problem concerns the restriction to normed strategy spaces. From a theoretical perspective, Kajii's result does not generalize neither Scarf's nor Florenzano's results since both encompass general Hausdorff topological vector spaces. Regarding possible applications, the assumption that strategy sets are norm compact imposes a strong restriction when the strategy space is infinite dimensional. This issue is well-documented in Mas-Colell and Zame (1991) where it is shown that for exchange economies, the set of feasible trades is in general just weakly compact but not norm compact.[20]

Despite the fact that Kajii (1992) does not provide any counter-example, one may think that the price to pay in order to handle non-ordered preferences and externalities is to restrict attention to normed strategy spaces. This paper shows that this not the case. We prove that non-emptiness can still be obtained for general preference relations (non-ordered with externalities) and general topologies on the strategy vector space. In particular we show that it is possible to generalize and unify the results of Scarf (1971), Florenzano (1989) and Kajii (1992).

4.2 The Main Theorem

In order to prove that the α-core is non-empty, we impose a continuity requirement on preferences that is consistent with the blocking power we consider. Recall that for each coalition S, the correspondence P^S from X to X^S

[20]We also provide examples in Section 5.

is defined by

$$P^S(x) = \left\{ y \in X^S \ : \ \{y\} \times W^{I \setminus S} \subset \bigcap_{i \in S} P_i(x) \right\}$$

where $\{y\} \times W^{I \setminus S}$ is the set of all allocations (y, w) where w belongs to $W^{I \setminus S}$. In other words, an allocation $y = (y_i)_{i \in S}$ in X^S belongs to the set $P^S(x)$ if each agent $i \in S$ strictly prefers (y, w) to x, whatever is w in $W^{I \setminus S}$.

Definition 4.2. The generalized game $\mathcal{G}$ is said to be ***α*-continuous** if for each admissible coalition $S \in \mathbb{I}$, the correspondence P^S has open lower sections in X, that is, for each agent $i \in S$ and each feasible strategy $y \in F^S$ the set $\{x \in X \ : \ \{y\} \times W^{I \setminus S} \subset P_i(x)\}$ is open in X.[21]

This is the only assumption that is not standard. It seems to require a uniform continuity property of preferences but we show in the following remarks that it is weaker than the corresponding assumptions imposed in the literature.

Remark 4.3. Assume that each agent i's preference relation has weak externalities, in the sense that

$$P_i(x) = \{y \in X \ : \ y_i \in \widehat{P}_i(x)\}$$

where $\widehat{P}_i$ is a correspondence from X to X_i. In that case the game is α-continuous if $\widehat{P}_i$ has open lower sections, i.e., for every $y_i \in X_i$ the set $\{x \in X \ : \ y_i \in \widehat{P}_i(x)\}$ is open in X. In particular, α-continuity is satisfied in the framework considered by Florenzano (1989) (see also Lefebvre (2001)).

Remark 4.4. There is another simple framework where α-continuity is satisfied. Assume that each agent i's preference relation is ordered by a function $u_i : X \to [-\infty, \infty)$, i.e.,

$$P_i(x) = \{y \in X \colon u_i(y) > u_i(x)\}.$$

[21]We refer to Appendix A.1 for precise definitions of all continuity properties for correspondences.

In that case, α-continuity is satisfied if each function u_i is continuous as assumed by Scarf (1971). Indeed, fix an agent $i \in I$, a feasible coalition $S \in \mathbb{I}(i)$, a feasible strategy $y \in F^S$ and a social state $x \in X$ such that for every $w \in W^{I \setminus S}$, we have $u_i(y, w) > u_i(x)$. Let $\alpha \equiv \inf\{u_i(y, w) \colon w \in W^{I \setminus S}\}$. Since u_i is continuous and $W^{I \setminus S}$ is compact, the set $\{z \in X \colon u_i(z) < \alpha\}$ is open, contains x and is a subset of $\{z \in X \colon \{y\} \times W^{I \setminus S} \subset P_i(z)\}$.

If the game has two agents, say $I = \{i, j\}$, then α-continuity is satisfied if each correspondence P_i has open lower sections.[22] For the general case (more than two agents) α-continuity is in particular satisfied if each preference correspondence has an open graph as assumed by Kajii (1992).

Proposition 4.1. If the correspondence P_i has an open graph for each i, then the game is α-continuous.

The proof follows from a direct application of Proposition A.1 (see Appendix A.2) which states that if a correspondence has an open graph then it satisfies automatically a uniform continuity property with respect to compact sets. This property is very intuitive and generalizes the well-know result that every continuous function is actually uniformly continuous on every compact set.

We can now state the main result of the paper whose proof is postponed to Section 6.

Main Theorem. *If the generalized game is α-continuous then its α-core is non-empty.*

The above non-emptiness result unifies and generalizes the results in Scarf (1971) (see Remark 4.4), Florenzano (1989) and Lefebvre (2001) (see Remark 4.3), and Kajii (1992) (see Proposition 4.1). More importantly, it answers an open question in (Kajii 1992, Section 4) by allowing for any linear

[22]Indeed, fix a coalition $S \subset \{i, j\}$. If $S = I$ then $P^S(x) = P_i(x) \cap P_j(x)$ and has open lower sections. If $S = \{i\}$, then $P^S(x) = P_i(x) \cap [X_i \times \{e_j\}]$ and has open lower sections.

topology on the underlying strategy vector space. We propose in Section 5 two settings where our result applies while no existing result does.

Our non-emptiness result contributes to the literature in another aspect since we prove the non-emptiness of a smaller core. Our α-core coincides with the α_{Y}-core when there are at most two agents and feasible sets are convex. In particular, we get as a direct corollary of the Main Theorem the non-emptiness result of Yannelis (1991b) (see also Koutsougeras and Yannelis (1993)).

Remark 4.5. Recently, Bonnisseau and Iehlé (2007) proved the non-emptiness of the core of an NTU game satisfying a condition of payoff-dependent balancedness, based on transfer rate mappings (see also Predtetchinski and Herings (2004) for a related result). They applied their result to recover Kajii's non-emptiness theorem. Their proof is also based on the construction of pseudo-utility functions (as in Border (1984) and Kajii (1992)). In particular, to get quasi-concavity, they also need to assume that the strategy space is a normed vector space.

5 Applications

We illustrate the applicability of the Main Theorem, by considering the two following settings.

5.1 Mixed strategies over infinitely many pure actions

We fix a compact set $(\mathbb{A}, \tau)$. Consider the (non-generalized) game where each agent i chooses a mixed strategy σ_i in the space X_i of Borel probability measures over a closed subset $A_i \subset \mathbb{A}$ of pure actions. The strategy space is then $\mathbb{M}$ the vector space of Borel signed measures on $\mathbb{A}$. If $\mathbb{M}$ is endowed with the weak topology $\sigma(\mathbb{M}, \mathbb{C})$ where $\mathbb{C}$ is the vector space of τ-continuous real valued functions defined on $\mathbb{A}$, then X_i is a compact set.[23]

In many models, agent i's payoff of a joint strategy $\sigma = (\sigma_i)_{i \in I}$ is defined by the expected utility

$$\mathbb{E}^\sigma[v_i] \equiv \int_A v_i(a)\sigma(da)$$

for some "felicity" function $v_i : A \to \mathbb{R}$ continuous for the product topology on $A = \prod_{i \in I} A_i$. We consider an alternative way of modeling agent i's preference relation. Assume that agent i is a decision maker representing a set K_i of individuals. Each $k \in K_i$ is endowed with a continuous felicity function $v_i(k, \cdot) : A \to \mathbb{R}$. Individual preferences are aggregated through a family $\mathcal{K}_i$ of non-empty subsets of the (index) set K_i as defined hereafter.

Definition 5.1. Agent i is said to have **voting preferences** represented by $\mathcal{K}_i$ if a profile of mixed strategies η is strictly preferred to another profile σ when there exists a set $\kappa_i \in \mathcal{K}_i$ such that

$$\forall k \in \kappa_i, \quad \mathbb{E}^\eta\left[v_i(k)\right] > \mathbb{E}^\sigma\left[v_i(k)\right].$$

In other words, the set $P_i(\sigma)$ of agent i's strictly preferred mixed strategies is defined by

$$P_i(\sigma) \equiv \bigcup_{\kappa_i \in \mathcal{K}_i} \bigcap_{k \in \kappa_i} P_{i,k}(\sigma)$$

where $P_{i,k}(\sigma)$ is the set of all η satisfying $\mathbb{E}^\eta\left[v_i(k)\right] > \mathbb{E}^\sigma\left[v_i(k)\right]$, representing individual k's preferences. If $\mathcal{K}_i$ is reduced to the singleton $\{K_i\}$, then agent i prefers η to σ if all individuals in K_i prefer η to σ: this is the unanimity rule. We identify several types of "voting rules" for which the α-core is non-empty.

Theorem 5.1. *Assume that for each agent i, the set K_i of represented individuals is finite and the "voting rule" $\mathcal{K}_i$ satisfies*

$$\bigcap_{\kappa_i \in \mathcal{K}_i} \kappa_i \neq \emptyset. \tag{5.1}$$

[23]The topology $\sigma(\mathbb{M}, \mathbb{C})$ also goes by the names of the weak-star topology or possibly the topology of convergence in distribution.

Then the α-core of the game with voting preferences represented by $(\mathcal{K}_i)_{i \in I}$ is non-empty.

Denote by $\pi_i^\star$ the set of individuals that belong to all κ_i in $\mathcal{K}_i$. The interpretation of (5.1) is that $\pi_i^\star$ is a set of individuals whose opinion is mandatory for the decision maker who prefers a profile η to another profile σ only if all individuals in $\kappa_i^\star$ also prefer η to σ.

Example 5.1. An example of voting rule $\mathcal{K}_i$ satisfying (5.1) is as follows. Fix a specific individual $k_i^\star \in K_i$ and a ratio $\alpha \in [0, 1]$: we denote by $\mathcal{K}_i(k_i^\star, \alpha)$ the set of all subsets of agents $\kappa_i \subset K_i$ containing $k_i^\star$ and representing at least the proportion α, i.e., $\#\kappa_i \geqslant \alpha \# K_i$. For this example, decision maker i prefers the profile η if this profile is preferred by a sufficiently large group κ_i of agents in K_i containing $k_i^\star$.[24]

Let η be a profile of mixed strategies strictly preferred by agent i to σ. This means that there exists a set $\kappa_i \in \mathcal{K}_i$ such that every individual $k \in \kappa_i$ strictly prefers η to σ. It is important to observe that the set κ_i may depend on the pair (η, σ). This implies that in general the binary relation defined by the voting preferences is neither transitive nor complete.[25] Therefore we cannot apply Scarf (1971) to conclude that the α-core is non-empty. The topology for which the strategy set X_i is compact is the weak-topology $\sigma(\mathbb{M}, \mathbb{C})$ which is not normable if A_i is infinite. Therefore we cannot apply Kajii (1992) to prove Theorem 5.1.

Proof of Theorem 5.1. We show that we can apply the Main Theorem to conclude that the α-core is non-empty. Assumptions 2.1 and 2.2 are trivially satisfied. Since the game is not generalized (that is, $F^S = X^S$ for each coalition S) it is automatically balanced. In order to prove that it is α-continuous, we show that the graph of each P_i is open for the (product)

[24] One may assume that the decision maker is one of the individuals.

[25] This is the case in Example 5.1 if K_i has three elements and $\alpha = 2/3$.

weak topology $\sigma(\mathbb{M},\mathbb{C})^I$. The graph $\operatorname{gph} P_i$ of P_i satisfies

$$\operatorname{gph} P_i = \bigcup_{\kappa_i \in \mathcal{K}_i} \bigcap_{k \in \kappa_i} \operatorname{gph} P_{i,k}$$

where $\operatorname{gph} P_{i,k} \equiv \{(\eta,\sigma) : \mathbb{E}^{\eta}[u_i(k)] > \mathbb{E}^{\sigma}[u_i(k)]\}$. Since $u_i(k)$ is a continuous function on A it follows that each set $\operatorname{gph} P_{i,k}$ is open for the (product) weak topology $\sigma(\mathbb{M},\mathbb{C})^I$. The fact that P_i has an open graph follows from the fact that κ_i is finite.

The last property we should verify is the convexity condition defined by Assumption 2.3. Let σ be a profile of mixed strategies and assume by way of contradiction that there exists a finite family $(\eta^\ell)_{\ell \in L}$ such that

$$\forall \ell \in L, \quad \eta^\ell \in P_i(\sigma) \quad \text{and} \quad \sigma = \sum_{\ell \in L} x^\ell \eta^\ell$$

where $x = (x^\ell)_{\ell \in L}$ is a probability measure over L. Fix an agent $k_i^\star$ that belongs to the intersection defined by (5.1). Since the taste of agent $k_i^\star$ always matters, we get the following contradiction:

$$\mathbb{E}^{\sigma}\left[v_i(k_i^\star)\right] = \sum_{\ell \in L} x^\ell \mathbb{E}^{\eta^\ell}\left[v_i(k_i^\star)\right] > \mathbb{E}^{\sigma}\left[v_i(k_i^\star)\right].$$

We can thus apply the Main Theorem to conclude that the α-core of this game is non-empty. □

5.2 Uncertainty with infinitely many states of nature

We fix a countable set S of states of nature representing uncertainty. We denote by $B(n)$ the space of bounded functions from S to $\mathbb{R}^n$. For each s, we let

$$\mathcal{G}(s) = \left\{L, (X_i, u_i)_{i \in I}, F(s), (F^E(s))_{E \in \mathbb{I}}\right\}$$

be a generalized balanced game satisfying the following assumptions: the strategy space L is the finite dimension vector space $\mathbb{R}^n$; the set X_i is a non-empty and convex subset of L; the function $u_i : X \to \mathbb{R}$ is a continuous, bounded and concave function defined on $X = \prod_{i \in I} X_i$; for every coalition $E \in \mathbb{I}$ the set $F^E(s)$ is non-empty compact and the set $F(s)$ is additionally assumed to be convex. An example of a game satisfying the above assumptions is the generalized game derived from a pure exchange economy $\mathcal{E}(s)$ where agents trade n commodities and have initial endowments $(e^i(s))_{i \in I}$.

Given the family $(\mathcal{G}(s))_{s \in S}$ of state contingent games, we consider the associated **ex-ante generalized game** where each agent i chooses a state contingent strategy x_i which is assumed to be a bounded function in $B(n)$ such that $x_i(s) \in X_i$ for every s. The associated strategy set is denoted by $\mathcal{X}_i$. It follows that the strategy vector space, denoted by $\mathcal{L}$, coincides with $B(n)$. Given a coalition $E \in \mathbb{I}$, the set of feasible strategies $\mathcal{F}^E$ is the space of contingent strategy profiles $x^E = (x_i)_{i \in E}$ in $\mathcal{L}^E$ such that $x^E(s) \in F^E(s)$ for every s. As in Bewley (2002) and Rigotti and Shannon (2005), we assume that each agent i has a **Knightian preference relation** defined as follows: the state contingent joint strategy y is strictly preferred to x, denoted by $y \in P_i(x)$, if

$$\forall q \in Q_i, \quad \int_S u_i(y(s))q(ds) > \int_S u_i(x(s))q(ds)$$

where Q_i is a non-empty subset of $\text{Prob}(S)$ the space of probability distributions over S. The ex-ante generalized game is then characterized by the family

$$\mathcal{G} = \{\mathcal{L}, (\mathcal{X}_i, P_i)_{i \in I}, \mathcal{F}, (\mathcal{F}^E)_{E \in \mathbb{I}}\}.$$

Theorem 5.2. *Consider the ex-ante generalized game $\mathcal{G}$ defined above where each agent i has a Knightian preference relation defined by a set Q_i of subjective beliefs. Assume that each set Q_i is a non-empty convex and compact for the weak star topology.*[26] *Then the α-core is non-empty.*

[26] The weak star topology on $\text{Prob}(S)$ is the weak topology $\sigma(\text{Prob}(S), B)$ where $B = B(1)$ is the space of bounded functions from S to $\mathbb{R}$.

Proof of Theorem 5.2. We show that the generalized $\mathcal{G}$ satisfies all the conditions of the Main Theorem. First, the ex-ante game $\mathcal{G}$ is balanced since each state game $\mathcal{G}(s)$ is balanced. Second, Assumptions 2.1 and 2.3 are clearly satisfied. To prove that the other conditions are satisfied, we endow the strategy space $B(n)$ with the product topology. The fact that Assumption 2.2 is satisfied follows from Tychonoff Product Theorem. Fix now an agent i. We propose to prove that the graph of the preference correspondence P_i is open for the product topology. Let x and y be two joint strategies in $\mathcal{X}$ such that $y \in P_i(x)$. Let $\varphi : \text{Prob}(S) \to \mathbb{R}$ be the function defined by

$$\varphi(q) \equiv \int_S [u_i(y(s)) - u_i(x(s))] q(ds).$$

By definition of the weak star topology $\sigma(\text{Prob}(S), B)$, the function φ is weakly star continuous. Since Q_i is weakly star compact, we can conclude that there exists $\varepsilon > 0$ such that

$$\inf_{q \in Q_i} \int_S [u_i(y(s)) - u_i(x(s))] q(ds) \geqslant 2\varepsilon.$$

It is well-known that weak star compactness of Q_i implies tightness. It then follows that there exists a finite set $S(\varepsilon) \subset S$ such that

$$\forall q \in Q_i, \quad q(S \setminus S(\varepsilon)) \leqslant \frac{\varepsilon}{5M}$$

where $M > 0$ is an upper bound of u_i. Since the function u_i is continuous, there exist V_y and V_x open neighborhoods of $(y(s))_{s \in S(\varepsilon)}$ and $(x(s))_{s \in S(\varepsilon)}$ respectively such that

$$\sup_{s \in S(\varepsilon)} |u_i(\widetilde{x}(s)) - u_i(x(s))| + |u_i(\widetilde{y}(s)) - u_i(y(s))| \leqslant \varepsilon/2$$

for every $(\widetilde{x}(s))_{s \in S(\varepsilon)} \in V_x$ and every $(\widetilde{y}(s))_{s \in S(\varepsilon)} \in V_y$. Let W_x and W_y be the set of all $\widetilde{x} \in B(n)$ and $\widetilde{y} \in B(n)$ respectively, satisfying

$$(\widetilde{x}(s))_{s \in S(\varepsilon)} \in V_x \quad \text{and} \quad (\widetilde{y}(s))_{s \in S(\varepsilon)} \in V_y.$$

The set $W_x \times W_y$ is an open neighborhood of (x, y) for the product topology satisfying

$$\forall (\widetilde{x}, \widetilde{y}) \in W_x \times W_y, \quad \widetilde{y} \in P_i(\widetilde{x}). \tag{5.2}$$

Indeed, fix $q \in Q_i$, a pair $(\widetilde{x}, \widetilde{y}) \in W_x \times W_y$ and let

$$\forall s \in S, \quad g(s) \equiv [u_i(\widetilde{y}(s)) - u_i(\widetilde{x}(s))] - [u_i(y(s)) - u_i(x(s))].$$

Condition (5.2) is satisfied if we prove that

$$\int_S g(s) q(ds) > -\frac{3\varepsilon}{2}.$$

Since $|g(s)| \leqslant 4M$, the definition of $S(\varepsilon)$ implies

$$\int_S g(s) q(ds) \geqslant -\frac{4\varepsilon}{5} + \int_{S(\varepsilon)} g(s) q(ds).$$

Given the choices of V_x and V_y we have $g(s) \geqslant -\varepsilon/2$. It then follows that

$$\int_S g(s) q(ds) \geqslant -\frac{4\varepsilon}{5} - \frac{\varepsilon}{2} = -\frac{13\varepsilon}{10} \geqslant -\frac{3\varepsilon}{2}.$$

We have thus proved that the correspondence P_i has an open graph, implying that the ex-ante game is α-continuous. □

6 Proof of the Main Theorem

The proof of the Main Theorem is inspired by the proof of Proposition 1 and Proposition 2 in Florenzano (1989) and the proof of Theorem 2.1 in Lefebvre (2001). Our framework is more general that the one in Florenzano (1989) and Lefebvre (2001) since we allow for externalities in preferences. The crucial difference between our proof and the one in Florenzano (1989) is that we make use of a representing result of balanced collections proved by (Scarf 1971, pp. 178–179). The combination of the techniques used by Scarf

(1971) and Florenzano (1989) constitutes our main technical contribution. We split the proof in two steps. We first assume that the strategy space is finite dimensional. The general result is then derived from a Bewley-type limit argument.

6.1 The finite dimensional case

The fixed-point theorem we use (see Lemma A.1 in Appendix A.1) is valid for finite dimensional spaces. This is the reason why we first study the case where the dimension of the strategy space L is finite.

Proposition 6.1. Let $\mathcal{G} = \{L, (X_i, P_i)_{i\in I}, F, (F^S)_{S\in\mathbb{I}}\}$ be a balanced and α-continuous generalized game satisfying Assumptions 2.1–2.3. If L is finite dimensional, then $\mathcal{G}$ has a non-empty α-core.

Proof of Proposition 6.1. We denote by Z the non-empty, compact and convex set defined by

$$Z \equiv \operatorname{co} \prod_{S\in\mathbb{I}} F^S.$$

Recall that Δ denotes the set of balancing weights defined in Section 4. For each (z, λ) in $Z \times \Delta$, where $z = (z^S)_{S\in\mathbb{I}}$ and $z^S = (z_i^S)_{i\in S}$, we let

$$\theta(z, \lambda) \equiv \{(y_i)_{i\in I}\} \quad \text{where for each } i \in I, \quad y_i \equiv \sum_{S\in\mathbb{I}(i)} \lambda_S z_i^S.$$

Since the generalized game is balanced, we have $\theta(Z \times \Delta) \subset \operatorname{co} F = F$, implying that θ is a continuous correspondence from $Z \times \Delta$ to F.[27] For each $x \in F$, we let

$$\varphi(x) \equiv \prod_{S\in\mathbb{I}} \varphi^S(x)$$

where for each $S \in \mathbb{I}$,

$$\varphi^S(x) \equiv \operatorname{co}[F^S \cap P^S(x)].$$

[27] Actually θ is a function.

Recall that for each $S \in \mathbb{I}$,

$$P^S(x) \equiv \left\{ y \in X^S \ : \ \{y\} \times W^{I \setminus S} \subset \bigcap_{i \in S} P_i(x) \right\}.$$

For each $x \in F$, we let

$$\mathbb{I}(x) \equiv \{S \in \mathbb{I} \ : \ \varphi^S(x) \neq \emptyset\}.$$

We let Σ be the subset of $\mathbb{R}_+^{\mathbb{I}}$ defined by

$$\Sigma \equiv \left\{ \mu = (\mu_S)_{S \in \mathbb{I}} \in \mathbb{R}_+^{\mathbb{I}} \ : \ \sum_{S \in \mathbb{I}} \mu_S = 1 \right\}.$$

For each $(x, \mu) \in F \times \Sigma$, we let

$$\psi(\mu) \equiv \operatorname{argmax} \{ \mu \cdot \lambda \ : \ \lambda \in \Delta \}$$

and

$$\xi(x) \equiv \{ \nu \in \Sigma \ : \ \nu_S = 0, \quad \forall S \notin \mathbb{I}(x) \}.$$

Following Assumption 2.2, the set K defined by

$$K \equiv F \times Z \times \Delta \times \Sigma$$

is a non-empty convex compact subset of a finite dimensional vector space. We consider now χ the correspondence from K to K defined by

$$\forall (x, z, \lambda, \mu) \in K, \quad \chi(x, z, \lambda, \mu) \equiv \theta(z, \lambda) \times \varphi(x) \times \psi(\mu) \times \xi(x).$$

In order to apply Lemma A.1 in Appendix A.1, we propose to prove that correspondences in the definition of χ have convex values and are either lower semi-continuous or upper semi-continuous with closed values.

- The correspondence θ is clearly continuous with compact convex and non-empty values.

- Since the game is α-continuous, for each $S \in \mathbb{I}$, the correspondence $\varphi^S : x \mapsto \mathrm{co}[F^S \cap P^S(x)]$ has open lower sections.

- Since φ^S has open lower sections, it follows that for each $x \in F$ there exists a neighborhood W of x such that

  $$\forall x' \in W, \quad \mathbb{I}(x) \subset \mathbb{I}(x').$$

  As a consequence, we get that ξ has open lower sections.

- The correspondence ψ is clearly convex and compact valued. Moreover from Berge Maximum Theorem, it is upper semi-continuous.

It follows from Lemma A.1 that there exists $(\overline{x}, \overline{z}, \overline{\lambda}, \overline{\mu}) \in K$ such that

$$\overline{x} = \theta(\overline{z}, \overline{\lambda}) \tag{6.1}$$

$$\forall S \in \mathbb{I}, \quad \left[\overline{z}^S \in \varphi^S(\overline{x}) \quad \text{or} \quad \varphi^S(\overline{x}) = \emptyset\right] \tag{6.2}$$

$$\overline{\lambda} \in \psi(\overline{\mu}) \tag{6.3}$$

$$\overline{\mu} \in \xi(\overline{x}) \quad \text{or} \quad \mathbb{I}(\overline{x}) = \emptyset. \tag{6.4}$$

We propose to prove that $\overline{x}$ belongs to the α-core of $\mathcal{G}$. If $\mathbb{I}(\overline{x}) = \emptyset$ then $\varphi^S(\overline{x}) = \mathrm{co}[F^S \cap P^S(\overline{x})] = \emptyset$ for each $S \in \mathbb{I}$, implying that $\overline{x}$ belongs to the α-core. To complete the proof, we only have to show that $\mathbb{I}(\overline{x}) = \emptyset$. Assume by way of contradiction that $\mathbb{I}(\overline{x}) \neq \emptyset$. The following claim is a direct consequence of (6.3) and (6.4).[28]

Claim 6.1. There exists $i_0 \in I$ such that for every coalition $S \notin \mathbb{I}(\overline{x})$, if $i_0 \in S$ then $\overline{\lambda}_S = 0$.

[28] The proof of Claim 6.1 follows from standard Kuhn–Tucker arguments and is similar to the proof of Claim 3.1 in Lefebvre (2001). For the sake of completeness, we provide details in Appendix A.3.

It follows from (6.1) that

$$\forall i \in I, \quad \overline{x}_i = \sum_{S \in \mathbb{I}(i)} \overline{\lambda}_S \overline{z}_i^S.$$

The rest of the proof differentiates with Florenzano (1989) and constitutes the main technical contribution of our paper. Following (Scarf 1971, pp. 178-179), we have the following decomposition of $\overline{x}$

$$\overline{x} \equiv (\overline{x}_i)_{i \in I} = \sum_{S \in \mathbb{I}(i_0)} \overline{\lambda}_S y(S)$$

where for each $S \in \mathbb{I}(i_0)$ such that $\overline{\lambda}_S > 0$, the allocation $y(S) = (y_i(S))_{i \in I} \in X$ is defined by

$$\forall i \in I, \quad y_i(S) = \begin{cases} \overline{z}_i^S & \text{if} \quad i \in S \\ \\ \left(\sum_{E \in \mathbb{I}(i,-i_0)} \overline{\lambda}_E \overline{z}_i^E\right) / \left(\sum_{E \in \mathbb{I}(i,-i_0)} \overline{\lambda}_E\right) & \text{if} \quad i \notin S, \end{cases}$$

where $\mathbb{I}(i, -i_0) \equiv \{E \in \mathbb{I} : \ i \in E \quad \text{and} \quad i_0 \notin E\}$.[29] Now Claim 6.1 implies that

$$\overline{x} = \sum_{S \in \mathbb{I}(i_0) \cap \mathbb{I}(\overline{x})} \overline{\lambda}_S y(S).$$

Observe that for each $S \in \mathbb{I}(i_0) \cap \mathbb{I}(\overline{x})$, we have $\pi^S(y(S)) = \overline{z}^S$. In particular from (6.2) we get $y(S) \in \operatorname{co} P_{i_0}(\overline{x})$. Hence $\overline{x} \in \operatorname{co} P_{i_0}(\overline{x})$ which yields a contradiction. $\square$

6.2 The general case

Now as a corollary of Proposition 6.1, we propose to prove the Main Theorem in the general case: L is a Hausdorff topological vector space.

[29]It is important to note that

$$\sum_{S \in \mathbb{I}(i_0, -i)} \overline{\lambda}_S = \sum_{E \in \mathbb{I}(i, -i_0)} \overline{\lambda}_E.$$

Proof of the Main Theorem. Let $\{\underline{x}^S : S \in \mathbb{I}\}$ be any arbitrary set of feasible strategies $\underline{x}^S$ in F^S and let $\mathcal{H}$ be the collection of all finite dimensional subspaces of L containing all these feasible strategies. For each $H \in \mathcal{H}$, we let $\mathcal{G}^H$ be the restriction of $\mathcal{G}$ to the subspace H:

$$\mathcal{G}^H \equiv \left\{H, (X_i^H, P_i^H)_{i\in I}, F_H, (F_H^S)_{S\in\mathbb{I}}\right\},$$

where for each $i \in I$, we let $X_i^H \equiv X_i \cap H$ and for each $x \in X_H \equiv \prod_{i\in I} X_i^H$, we let

$$P_i^H(x) \equiv P_i(x) \cap H^I \quad \text{and} \quad \forall S \in \mathbb{I}, \quad F_H^S \equiv F^S \cap H^S.$$[30]

We can apply Proposition 6.1 to the game $\mathcal{G}^H$. Then there exists x_H in the α-core of $\mathcal{G}^H$. Since $x_H \in F$ it follows from Assumption 2.2 that, passing to a subnet if necessary, $(x_H)_{H\in\mathcal{H}}$ converges to some $\overline{x} \in F$. We propose to prove that $\overline{x}$ belongs to the α-core of the game $\mathcal{G}$. Assume, by way of contradiction, that there exists $S \in \mathbb{I}$ and $y^S \in F^S$ such that $y^S \in F^S \cap P^S(\overline{x})$. Since the game is α-continuous, every set $\{x \in X : y^S \in P^S(x)\}$ is open in X. This implies that there exists $G \in \mathcal{H}$ such that $y^S \in F^S \cap P^S(x_H)$ for every H in $\mathcal{H}$ containing G. Choosing H to be the linear space generated by G and the set $\{y_i : i \in S\}$, we get a contradiction with the fact that x_H belongs to the α-core of the finite dimensional game $\mathcal{G}^H$. □

In his concluding remarks, Kajii (1992) claimed that "the uniformity involved in the definition of the α_{A}-core prevents us from following a Bewley-type limit argument: a subnet of the net of α_{A}-core strategies converges but the convergence is not uniform with respect to potential blocking strategies which may arise in the limit". To clarify why this is not in contradiction with our proof of the Main Theorem, we need to introduce additional notations. Recall that a feasible joint strategy $x \in F$ belongs to the α-core if and only

[30] It follows that $F_H = F \cap H^I$.

if there does not exist a feasible coalition $S \in \mathbb{I}$ such that $F^S \cap P^S(x) \neq \emptyset$ where

$$P^S(x) = \{y \in X^S \ : \ \{y\} \times W^{I \setminus S} \subset \bigcap_{i \in S} P_i(x)\}.$$

The definition of $P^S(x)$ corresponds to the blocking power of our α-core since $\{y\} \times W^{I\setminus S} = \{(y, w) \ : \ w \in W^{I\setminus S}\}$. We can adapt the definition of $P^S(x)$ to capture the blocking power associated to the α_{A}-core. Let $P^S_{\text{A}}(x)$ be the set defined by

$$P^S_{\text{A}}(x) \equiv \{y \in X^S \ : \ \{y\} \times X^{I \setminus S} \subset \bigcap_{i \in S} P_i(x)\}$$

where $\{y\} \times X^{I\setminus S} \equiv \{(y, z) \ : \ z \in X^{I\setminus S}\}$. Given these notations, a feasible joint strategy $x \in F$ belongs to the α_{A}-core if and only if there does not exist a feasible coalition $S \in \mathbb{I}$ such that $F^S \cap P^S_{\text{A}}(x) \neq \emptyset$. Contrary to $W^{I\setminus S}$, the set $X^{I\setminus S}$ may not be compact. This implies that the correspondence P^S_{A} may not have open lower sections even if the graph of each P^i is open. In particular, as suggested by Kajii (1992), our Bewley-type argument could not be applied if we were trying to prove directly that the α_{A}-core is non-empty. In other words, the introduction of our intermediate blocking power plays a crucial role for the validity of our arguments: it enables us to prove a stronger result under weaker assumptions.

7 Concluding remarks

We proved a new result on the non-emptiness of the α-core which encompasses as a special case all the existing results in the literature. Although, the usefulness of our Main Theorem was indicated by providing new α-core existence theorems for games with Knightian and voting preferences (these results cannot be proved by using the theorems of Scarf (1971) or Kajii (1992)), we think that our main result will find additional applications in asymmetric information economies with externalities. In particular, the problem of proving the existence of incentive compatible and efficient contracts with externalities, is wide open. Our new results appear to be promising in solving this problem and we hope to take up the details on a subsequent paper.

A Appendix

A.1 Continuity of correspondences

We provide hereafter definitions and notations about correspondences. Consider X and Y two topological spaces. A correspondence P from X to Y is said: *lower semi-continuous* if the set $\{x \in X : P(x) \cap V \neq \emptyset\}$ is open for every open subset $V \subset Y$; *upper semi-continuous* if the set $\{x \in X : P(x) \subset V\}$ is open for every open subset $V \subset Y$; *continuous* if it is lower and upper semi-continuous; to have *open lower sections* if the set $\{x \in X : y \in P(x)\}$ is open for every $y \in Y$; to have *open (closed) graph* if the graph $\operatorname{gph} P \equiv \{(x, y) \in X \times Y : y \in P(x)\}$ is open (resp. closed).

We state hereafter a fixed-point result due to Gourdel (1995) that is used in our proof of the Main Theorem.

Lemma A.1. *Given $X = \prod_{k=1}^{m+n} X_k$ where each X_k is a non-empty compact convex subset of some finite dimensional Euclidean space, let for each k: $\varphi_k : X \to X_k$ be a convex (possibly empty) valued correspondence. Assume that for each $k = \{1, \ldots, m\}$, φ_k is lower semi-continuous, and that for each $k \in \{m+1, \ldots, m+n\}$, φ_k is upper semi-continuous with closed values. Then there exists $\overline{x} \in X$ such that for each k, either $\varphi_k(\overline{x}) = \emptyset$ or $\overline{x}_k \in \varphi_k(\overline{x})$.*

A.2 Proof of Proposition 4.1

The proof of Proposition 4.1 is a direct consequence of the following uniform continuity result.

Proposition A.1. Let X, Y and Z be topological spaces and K a compact subset of Z. If F is a correspondence from X to $Y \times Z$ with an open graph, then the following set

$$\{(x, y) \in X \times Y \ : \ \{y\} \times K \subset F(x)\}$$

is open in $X \times Y$.

A direct corollary of the above result is that the correspondence $F^K : X \to Y$ defined by

$$F^K(x) \equiv \{y \in Y \ : \ \{y\} \times K \subset F(x)\}$$

has open lower sections.

Proof of Proposition A.1. Let $(x, y) \in X \times Y$ be such that $(y, k) \in F(x)$ for every $k \in K$. Since F has an open graph, for each $k \in K$ there exists an open neighborhood U_k of x, an open neighborhood V_k of y, and an open neighborhood W_k of k such that

$$(x', y', k') \in U_k \times V_k \times W_k \Longrightarrow (y', k') \in F(x').$$

Since K is compact there exists a finite subset $J \subset K$ such that

$$K \subset \bigcup_{j \in J} W_j.$$

We let U and V be defined by

$$U = \bigcap_{j \in J} U_j \quad \text{and} \quad V = \bigcap_{j \in J} V_j.$$

The set U is an open neighborhood of x and V is an open neighborhood of y such that

$$(x', y') \in U \times V \Longrightarrow \{y'\} \times K \subset F(x').$$

□

A.3 Proof of Claim 6.1

Recall that Σ is the simplex of $\mathbb{R}^{\mathbb{I}}$, i.e., the set of all families $\mu = (\mu_S)_{S\in\mathbb{I}}$ such that

$$\forall S \in \mathbb{I}, \quad \mu_S \geqslant 0 \quad \text{and} \quad \sum_{S\in\mathbb{I}} \mu_S = 1,$$

and Δ is the set of balanced weights, i.e., the set of all families $\lambda = (\lambda_S)_{S\in\mathbb{I}}$ such that

$$\forall S \in \mathbb{I}, \quad \lambda_S \geqslant 0 \quad \text{and} \quad \forall i \in I, \quad \sum_{S\in\mathbb{I}(i)} \mu_S = 1.$$

Consider $\overline{\mu} \in \Sigma$ such that $\overline{\mu}_S = 0$ for every $S \notin \mathbb{J}$ where $\mathbb{J}$ is a non-empty subset of $\mathbb{I}$. Let $\overline{\lambda} \in \operatorname{argmax}\{\overline{\mu} \cdot \lambda : \lambda \in \Delta\}$. It follows from Kuhn–Tucker Theorem that there exist $(\alpha_S)_{S\in\mathbb{I}} \in \mathbb{R}^{\mathbb{I}}_+$ and $(\eta_i)_{i\in I} \in \mathbb{R}^I$ such that

$$\forall S \in \mathbb{I}, \quad \overline{\mu}_S + \alpha_S + \sum_{i\in S} \eta_i = 0 \quad \text{and} \quad \alpha_S \overline{\lambda}_S = 0. \tag{A.1}$$

Since singletons are admissible coalitions, we can choose $S = \{i\}$ in (A.1) implying that $\eta_i \leqslant 0$ for every i. Moreover, there exists $i_0 \in I$ such that $\eta_{i_0} < 0$. Indeed, we know that $\overline{\mu} \in \Sigma$. This implies that there exists $S_0 \in \mathbb{I}$ such that $\overline{\mu}_{S_0} > 0$. Choosing $S = S_0$ in (A.1) implies the desired result.

Now let $S \in \mathbb{I}$ such that $S \notin \mathbb{J}$ and $i_0 \in S$. We claim that $\overline{\lambda}_S = 0$. Indeed, recall that $\overline{\mu}_S = 0$ since $S \notin \mathbb{J}$. It then follows from (A.1) that

$$\overline{\lambda}_S \sum_{i\in S} \eta_i = 0.$$

Since $\eta_i \leqslant 0$ for every i and $\eta_{i_0} < 0$, we get the desired result.

References

Aumann, R. J.: 1961, The core of a cooperative game without side payments, *Transactions of the American Mathematical Society* **98**(3), 539–552.

Bewley, T. F.: 2002, Knightian decision theory: Part I, *Decisions in Economics and Finance* **25**(2), 79–110.

Billera, L. J.: 1970, Some theorems on the core of an n-person game without side-payments, *SIAM Journal of Applied Mathematics* **18**(3), 567–579.

Bondareva, O.: 1962, The core of an N person game, *Vestnik Leningrad University* **17**(13), 141–142.

Bonnisseau, J.-M. and Iehlé, V.: 2007, Payoff-dependent balancedness and the core, *Games and Economic Behavior* **61**(1), 1–26.

Border, K. C.: 1984, A core existence theorem for games without ordered preferences, *Econometrica* **52**(6), 1537–1542.

Florenzano, M.: 1989, On the non-emptiness of the core of a coalitional production economy without ordered preferences, *Journal of Mathematical Analysis and Application* **141**(2), 484–490.

Gourdel, P.: 1995, Existence of intransitive equilibria in nonconvex economies, *Set-Valued Analysis* **3**(4), 307–337.

Holly, C.: 1994, An exchange economy can have an empty α-core, *Economic Theory* **4**(3), 453–461.

Kajii, A.: 1992, A generalization of Scarf's theorem: an α-core existence theorem without transitivity or completeness, *Journal of Economic Theory* **56**(1), 194–205.

Koutsougeras, L. and Yannelis, N. C.: 1993, Incentive compatibility and information superiority of the core of an economy with differential information, *Economic Theory* **3**(2), 195–216.

Lefebvre, I.: 2001, An alternative proof of the nonemptiness of the private core, *Economic Theory* **18**(2), 275–291.

Mas-Colell, A. and Zame, W.: 1991, Equilibrium theory in infinite dimensional spaces, *in* W. Hildenbrand and H. Sonnenschein (eds), *Handbook of Mathematical Economics*, Vol. 4, North-Holland: Amsterdam, pp. 1835–1898.

Podczeck, K. and Yannelis, N. C.: 2008, Equilibrium theory with asymmetric information and with infinitely many commodities, *Journal of Economic Theory* **141**(1), 152–183.

Predtetchinski, A. and Herings, J.-J.: 2004, A necessary and sufficient condition for non-emptiness of the core of a non-transferable utility game, *Journal of Economic Theory* **116**(1), 84–92.

Rigotti, L. and Shannon, C.: 2005, Uncertainty and risk in financial markets, *Econometrica* **73**(1), 203–243.

Scarf, H. E.: 1967, The core of an n-person game, *Econometrica* **35**, 50–69.

Scarf, H. E.: 1971, On the existence of a cooperative solution for a general class of n-person game, *Journal of Economic Theory* **3**(2), 169–181.

Shapley, L.: 1965, On balanced sets and cores. RM-4601-PR, Rand Corporation, Santa Monica, California.

Yannelis, N. C.: 1991a, The core of an economy with differential information, *Economic Theory* **1**(2), 183–198.

Yannelis, N. C.: 1991b, The core of an economy without ordered preferences, *in* M. A. Khan and N. C. Yannelis (eds), *Equilibrium theory in infinite dimensional spaces*, New-York: Springer.