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The Riesz–Kantorovich formula and general equilibrium theory

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Abstract

Let L be an ordered topological vector space with topological dual L' and order dual $L^{\sim}$. Also, let f and g be two order-bounded linear functionals on L for which the supremum $f \vee g$ exists in L. We say that $f \vee g$ *satisfies the Riesz–Kantorovich formula* if for any $0 \leq \omega \in L$ we have

$$f \vee g(\omega) = \sup_{0 \leq x \leq \omega} [f(x) + g(\omega - x)].$$

This is always the case when L is a vector lattice and more generally when L has the Riesz Decomposition Property and its cone is generating. The formula has appeared as the crucial step in many recent proofs of the existence of equilibrium in economies with infinite dimensional commodity spaces. It has also been interpreted by the authors in terms of the revenue function of a discriminatory price auction for commodity bundles and has been used to extend the existence of equilibrium results in models beyond the vector lattice settings. This paper addresses the following open mathematical question:

- *Is there an example of a pair of order-bounded linear functionals f and g for which the supremum $f \vee g$ exists but does not satisfy the Riesz–Kantorovich formula?*

We show that if f and g are continuous, then $f \vee g$ must satisfy the Riesz–Kantorovich formula when L has an order unit and has weakly compact order intervals. If in addition L

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is locally convex, $f \vee g$ exists in $L^{\sim}$ for any pair of continuous linear functionals f and g if and only if L has the Riesz Decomposition Property. In particular, if $L^{\sim}$ separates points in L and order intervals are $\sigma(L,L^{\sim})$-compact, then the order dual $L^{\sim}$ is a vector lattice if and only if L has the Riesz Decomposition Property — that is, if and only if commodity bundles are perfectly divisible.

Keywords: Riesz–Kantorovich formula; General equilibrium theory; Riesz Decomposition Property

1. Introduction

It has for sometime been well-understood that one cannot hope to prove the existence of general equilibrium — or establish the validity of the welfare theorems — under the standard finite dimensional assumptions when the commodity space is infinite dimensional and consumption sets lack interior points. In this literature, the commodity space is most often a Riesz space (vector lattice) and primitive data of the economy are supposed to satisfy various assumptions known as ‘‘properness conditions’’ (see Aliprantis et al., 1990, Aliprantis et al., 2000).

A distinctive feature of this literature is the non-trivial use of the lattice structure of the commodity space. Indeed, Aliprantis and Burkinshaw (1991) show that when the commodity space is a vector lattice, the lattice structure of the dual space is basically equivalent to the validity of the welfare theorems. [1] Furthermore, the various proofs in this literature can be delineated by means of the Riesz Decomposition Property of the commodity space. For example, Mas-Colell (1986) and Aliprantis et al. (1987) use the Decomposition Property to facilitate a separating hyperplane argument, while Yannelis and Zame (1986) use the property to show the continuity and extendibility of the equilibrium prices of truncated economies. This is in sharp contrast to the case where consumption sets are assumed to have interior points and where the existence of a continuous quasi-equilibrium price can be proven with little reference to the lattice structure of the commodity space (see for example, Bewley, 1972; Florenzano, 1983).

In the more recent approach of Mas-Colell and Richard (1991) and Richard (1989) (see also Deghdak and Florenzano, 1999; Podczeck, 1996; Tourky, 1998, 1999) the Decomposition Property is used in an indirect manner. Here, the authors consider economies in the more general setting of a Riesz commodity space that need not be locally solid. In this setting a supporting hyperplane argument in the space of allocations furnishes a list of prices and the crucial part of the proof is showing that the supremum of these prices is indeed the required supporting (equilibrium) price. In this second group of papers, the Decomposition Property is used through two of its consequences. First, the fact that the order dual of the

[1] Of course, here we are talking about those welfare theorems that are traditionally proven using a separating hyperplane argument, i.e., the second welfare theorem and the equivalence of Edgeworth and Walrasian equilibria.

commodity space is a vector lattice and second the Riesz–Kantorovich formula for calculating the supremum of any two order-bounded linear functionals. It is also quite clear that the decentralization arguments in these more recent papers go through with little fuss if both of these properties are present, and without regard to whether the commodity space is a Riesz space or has the Riesz Decomposition Property.

This observation was recently made by Aliprantis et al. (1998), who extended the literature on the existence of equilibrium and on the welfare theorems in infinite dimensional spaces to commodity spaces that are not lattice ordered. They were able to drop the requirement that both the commodity space and the price space be lattice ordered by introducing a new class of non-linear prices based on the Riesz–Kantorovich formula. They also provide concrete economic interpretations to the Riesz Decomposition Property and the Riesz–Kantorovich formula. The Riesz–Kantorovich formula coincides with the revenue function of a discriminatory price auction — a generalization of the revenue function of the single commodity US Treasury Bill Auction. They interpreted the Riesz Decomposition Property (and its extension termed ''Consumption Decomposability'') as the perfect divisibility of commodity bundles and showed that in the presence of such perfectly divisible bundles the revenue function of the auctioneer is always linear.

Motivated by these observations, we address in this paper the following long standing open mathematical question.

- *Is there an example of a pair of order-bounded linear functionals for which the supremum exists but does not satisfy the Riesz–Kantorovich formula?*

We consider an arbitrary ordered vector space L with order unit and weakly compact order intervals. We show if f and g are continuous linear functionals on L and $f \vee g$ exists in the order dual $L^{\sim}$, then $f \vee g$ must satisfy the Riesz–Kantorovich formula. Therefore, we provide a negative answer to our question in the important setting of an ordered vector space with order unit and $\sigma(L, L^{\sim})$-compact order intervals.

We also show that if, in addition, L is locally convex with weakly compact intervals, then $f \vee g$ exists in $L^{\sim}$ for any pair of continuous linear functionals f and g if and only if L has the Riesz Decomposition Property (see also Andô, 1962). In particular, if $\tilde{L}$ separates points in L and order intervals are $\sigma(L, L^{\sim})$-compact then $L^{\sim}$ is a vector lattice if and only if L has the Riesz Decomposition Property — hence, if and only if commodity bundles are perfectly divisible.

Commodity spaces with order units often arise in the study of economies with infinitely many commodities even when the underlying commodity space does not have an order unit. Consider an exchange economy with an ordered vector space L as a commodity space, with total endowment $\omega \geq 0$, and consumption sets that coincide with the positive cone of L. Since all economic activity takes place inside the Edgeworth box $[0, \omega]$, it is often useful to restrict the commodity space to the ideal L_ω generated by ω. This ideal consists of all bundles that are dominated by multiples of ω, i.e.,

$$L_\omega = \{ x \in L : \pm x \leq \lambda \omega \text{ for some } \lambda > 0 \} .$$

The space L_ω is the canonical example of a commodity space that satisfies our assumptions. For an extensive analysis of economies truncated to L_ω, see Aliprantis et al. (1987).

The paper is organized as follows. The mathematical preliminaries are outlined in Section 2. Our main results are in Section 3. In Section 4 we show the usefulness of our main results in the theory of value with non-linear prices as developed by Aliprantis et al. (1998). We show that the perfect divisibility of commodity bundles, the linearity of the non-linear prices, and the lattice ordering of the order dual are equivalent.

2. Mathematical preliminaries

Unless otherwise stated in this work, L shall denote an ordered vector space. Recall that a (real) vector space L is called an *ordered vector space* if L is equipped with an order relation $\geq$ such that $x \geq y$ imply $x + z \geq y + z$ for all $z \in L$ and $\alpha x \geq \alpha y$ for each $\alpha \geq 0$. We shall say that x *dominates* y if $x \geq y$ holds. The convex set $L_+ = \{x \in L : x \geq 0\}$ is called the *positive cone* of L and its members are referred to as *positive vectors*. The positive cone L_+ satisfies the following properties:
1. $L_+ + L_+ \subseteq L_+$,
2. $\alpha L_+ \subseteq L_+$ for each $\alpha \geq 0$, and
3. $L_+ \cap (-L_+) = \{0\}$.

Any subset C of a vector space that satisfies the above properties (1), (2), and (3) is called a *convex cone*. Every convex cone C induces a natural order $\geq$ on X by letting $x \geq y$ whenever $x - y \in C$. This order makes X an ordered vector space satisfying $X_+ = C$. In other words, an ordered vector space is completely characterized by its positive cone.

A (non-empty) subset A of L is *bounded from* above (respectively, *from below*) if there exists some $x \in L$ satisfying $a \leq x$ (respectively, $x \leq a$) for all $a \in A$; the vector x is called an *upper bound* (respectively, *lower bound*) of A. A set is *order-bounded* if it is bounded from above and below. Any set of the form $[x,y] = \{z \in L : x \leq z \leq y\}$ is called an *order interval* or simply an *interval*. Clearly, a subset of L is order-bounded if and only if it is included in an interval.

A subset A of L has a *least upper bound* (or a *supremum*), denoted sup A, if $\sup A$ is an upper bound of A and whenever x is an upper bound of A, then sup $A \leq x$. The *greatest lower bound* (or *infimum*) is defined analogously.

Following the standard lattice terminology, the supremum and infimum of the set $\{x, y\}$ will be denoted by $x \vee y$ and $x \wedge y$, respectively. That is,

$$x \vee y = \sup\{x, y\} \text{ and } x \wedge y = \inf\{x, y\}.$$

An ordered vector space L is called a *Riesz space* (or a *vector lattice*) if $x \vee y$ and $x \wedge y$ exist in L for all $x, y \in L$. In a Riesz space, the elements

$$|x| = x \vee (-x), \; x^+ = x \vee 0, \text{ and } x^- = (-x) \vee 0$$

are called the *absolute value*, the *positive part*, and the *negative part* of the vector x, respectively. We have the identities

$$x = x^+ - x^- \text{ and } |x| = x^+ + x^-. \tag{1}$$

For extensive and detail treatments of Riesz spaces, see Luxemburg and Zaanen (1971) and Aliprantis and Burkinshaw (1985).

The cone L_+ is said to be *generating* if for each $x \in L$ there exist $y, z \in L_+$ such that $x = y - z$. Equivalently, L_+ is generating if every vector of L is dominated by some positive vector. If L is a Riesz space, then it follows from (1) that the cone L_+ is generating.

Recall that a vector $a \in A$, where A is a subset of a vector space X, is an *internal point* of A if for each $x \in X$ there exists some $\lambda_0 > 0$ such that $a + \lambda x \in A$ for each $-\lambda_0 \leq \lambda \leq \lambda_0$ (or, equivalently, if for each x there exists some $\lambda_0 > 0$ such that $a + \lambda x \in A$ for each $0 \leq \lambda \leq \lambda_0$. A vector $e \in L_+$ is called an *order unit* if for each $x \in L$ there exists some $\lambda > 0$ such that $x \leq \lambda e$. Clearly, if L has an order unit, then the cone L_+ is automatically generating. Also, if $e \in L_+$ is an order unit, the so are αe for $\alpha > 0$ and $e + x$ for each $x \in L_+$. The next well-known result (see for instance, Theorem 1.3.1 of Jameson, 1970, p. 11) that characterizes the order units will play an important role in this work.

Lemma 2.1. *A vector $e \in L_+$ is an order unit if and only if it is an internal point of L_+. In particular, if e is an order unit, then the zero vector is an internal point of the convex set $e - L_+ = \{x \in L: x \leq e\}$.*

Proof. Assume first that e is an internal point of L_+ and let $x \in L$. Then there exists some $\alpha > 0$ such that $e + \alpha(-x) \geq 0$. This implies $x \leq (1/\alpha)e$, so that e is an order unit.

Next, suppose that e is an order unit and let $x \in L$. Fix some $\lambda_0 > 0$ such that $\lambda_0(-x) \leq e$. Now notice that for each $0 < \lambda \leq \lambda_0$ we have

$$\lambda(-x) = \left(\frac{\lambda}{\lambda_0}\right)\lambda_0(-x) \leq \frac{\lambda}{\lambda_0} e \leq e.$$

So, $e + \lambda x \geq 0$ for all $0 \leq \lambda \leq \lambda_0$, which shows that e is an internal point of L_+. ■

Some known relationships between order units and interior points are included in the next result.

Lemma 2.2. *For an ordered vector space L we have the following.*
1. *If the positive cone L_+ of L has an interior point e with respect to a linear topology on L, then e is an order unit — and so L_+ is also generating.*
2. *When L is finite dimensional, the positive cone L_+ is generating if and only if it has an interior point with respect to the Euclidean topology of L.*
3. *If L is infinite dimensional and L_+ is generating, then L_+ can have no interior points for many Hausdorff locally convex topologies on L.*

Proof. (1) Assume that L_+ has an interior point e with respect to some linear topology τ on L. Pick a symmetric τ-neighborhood V of zero such that $e + V \subseteq L_+$. Now let $x \in V$. Then we have $-x \in V$ and so $e \pm x \in L_+$ or $-e \le x \le e$. That is, $V \subseteq [-e,e]$, which shows that the order interval $[-e,e]$ is a τ-neighborhood of zero. This implies that e is an order unit.

(2) Assume $L = \mathbb{R}^n$ and that L_+ is generating. We claim that L contains a basis consisting of positive vectors. To see this, let $\{e_1, \dots, e_k\}$ be a maximal set of linearly independent vectors lying in L_+. Now, let $x \in L$. Since L_+ is generating, we can choose $y, z \in L_+$ such that $x = y - z$. It follows that both y and z lie in the span of the set $\{e_1, \dots, e_k\}$, and from this we infer that x likewise lies in the span of $\{e_1, \dots, e_k\}$. In other words, $\{e_1, \dots, e_k\}$ is a basis of L and so $k = n$.

Next, put $e = \sum_{i=1}^n e_i$. We claim that e is an interior point of L_+. To prove this, notice first that the set $V = \{x = \sum_{i=1}^n \lambda_i e_i : \sum_{i=1}^n |\lambda_i| < 1\}$ is an open neighborhood of zero for the Euclidean topology. Moreover, if $x = \sum_{i=1}^n \lambda_i e_i \in V$, then $e + x = \sum_{i=1}^n (1 + \lambda_i) e_i \in L_+$. This shows that $e + V \subseteq L_+$ so that e is an interior point of L_+.

(3) If $L = l_1$, then its standard cone l_1^+ is generating while its interior is empty with respect to the topology induced by the l_1-norm. ■

The symbol L^* denotes the vector space of all linear functionals on L (the *algebraic dual* of L). A linear functional $f \in L^*$ is said to be *positive* if $f(x) \ge 0$ holds for all $x \in L_+$. The collection of positive linear functionals on L is denoted L_+^* and is known as the *cone of positive linear functionals*. If the cone L_+ is generating, then the cone L_+^* makes L^* an ordered vector space by letting $f \ge g$ if $f - g \in L_+^*$, or $f(x) \ge g(x)$ for each $x \in L_+$.

A linear functional f is said to be *order-bounded* if f carries order-bounded sets of L to bounded subsets of $\mathbb{R}$. The collection of all order-bounded linear functionals on L is a vector subspace of L^* called the *order dual* of L and denoted $L^\sim$. Clearly, every positive linear functional is order-bounded, and so every *regular linear functional* (i.e., every linear functional that can be written as a difference of two positive linear functionals) is likewise order-bounded. If L^τ

denotes the vector space of all regular linear functionals, then we have the following vector subspace inclusions:

$$L^r \subseteq L^{\sim} \subseteq L^* .$$

It should be noticed that if $L^{\sim}$ is a Riesz space, then $L^r = L$.

An ordered vector space L is said to have the *Riesz Decomposition Property* (or simply the *Decomposition Property*) whenever $0 \le y \le x_1 + x_2$ with $x_1, x_2 \in L_+$ guarantees the existence of two vectors $y_1, y_2 \in L$ satisfying $0 \le y_1 \le x_1$, $0 \le y_2 \le x_2$, and $y_1 + y_2 = y$. Every Riesz space has the Decomposition Property (see Aliprantis and Burkinshaw, 1985, Theorems 1.9 and 1.15). An ordered vector space with the Decomposition Property need not be a vector lattice (see for instance, Peressini, 1967, p. 14). For a simpler example, let $L = \mathbb{R}^2$ and consider the cone $L_+ = \{0\} \cup \{(x,y): x > 0 \text{ and } y > 0\}$. Then the ordered vector space (L, L_+) is not a Riesz space but it satisfies the Decomposition Property.

The ''ice cream'' cones do not satisfy the Decomposition Property. The following example clarifies the situation.

Example 2.3. Consider the ''ice cream''convex cone C in $\mathbb{R}^3$ defined by

$$C = \left\{ \lambda(x,y,2) : \lambda \ge 0 \text{ and } x^2 + y^2 \le 1 \right\}$$
$$= \left\{ (x,y,z) : \in \mathbb{R}^3 : z \ge 0 \text{ and } z^2 \ge 4\left(x^2 + y^2\right) \right\}.$$

That is, C is the convex cone with vertex zero generated by $\{(x,y,2): x^2 + y^2 \le 1\}$. Its graph is shown in Fig. 1.

We denote the order induced by C on $\mathbb{R}^3$ by $\ge_C$ or $\le_C$, i.e., $x \ge_C y$ if and only if $x - y \in C$. Some straightforward verifications show that the cone C and the ordered vector space $(\mathbb{R}^3, C)$ equipped with its Euclidean topology satisfy the following properties.

(1) C is a closed cone.

(2) $(\mathbb{R}^3, C)$ has order units — and hence, C is also a generating cone. For instance, the vector $e = (0,0,2)$ is an order unit. To see this, fix $u = (x,y,z) \in \mathbb{R}^3$. Choose some $\alpha > 0$ such that the real number $\lambda = (-z + 2\alpha)/2$ satisfies $\lambda > \sqrt{x^2 + y^2}$ and note that

$$-u + \alpha e = (-x, -y, -z) + \alpha(0,0,2) = \lambda\left(\frac{-x}{\lambda}, \frac{-y}{\lambda}, 2 \right) \in C.$$

This implies $u \le \alpha e$.

(3) C has a non-empty interior. For instance, the open ball centered at $(0,0,2)$ with radius $1/2$ lies entirely in C. This can also be derived from Lemma 2.2 via part (2) above.

(4) The order intervals of $(\mathbb{R}^3, C)$ are compact.

(5) If $u = \lambda(x,y,2) \in C$ and $x^2 + y^2 = 1$, then a vector $v \in \mathbb{R}^3$ satisfies $0 \le_C v \le_C u$ if and only if there exists some $0 \le \mu \le 1$ such that $v = \mu u$. (The half-rays $\{\alpha(x,y,2): \alpha \ge 0\}$, where $x^2 + y^2 = 1$, are the extreme half-rays of this cone.)

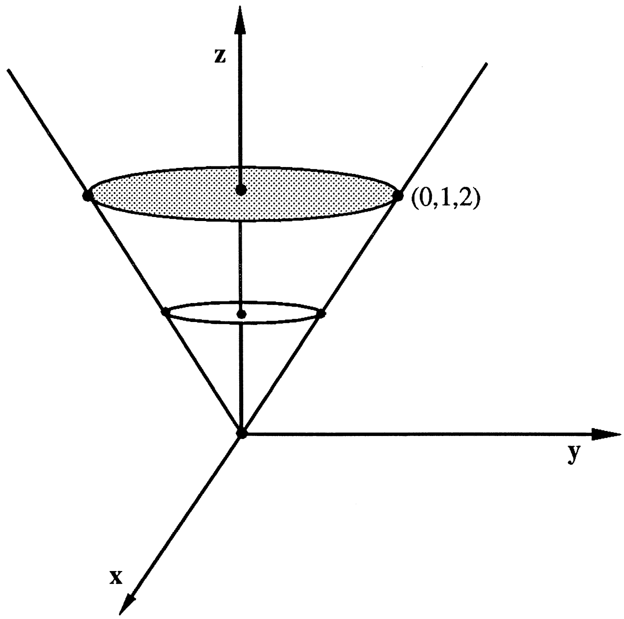

Fig. 1.

To see this, assume that $0 \leq_C \lambda(a,b,2) \leq_C (x,y,2)$ with $x^2 + y^2 = 1$. Clearly, $0 \leq \lambda \leq 1$ and if $\lambda = 0$ or $\lambda = 1$, the conclusion should be obvious. So, let $0 < \lambda < 1$. Since $(x,y,2) - \lambda(a,b,2) \in C$, there exist some pair (α,β) with $\alpha^2 + \beta^2 \leq 1$ and some $\mu \geq 0$ such that $(x,y,2) - \lambda(a,b,2) = \mu(\alpha,\beta,2)$. Hence, $(x,y) = \lambda(a,b) + \mu(\alpha,\beta)$. Since $x^2 + y^2 = 1$, the point (x,y) is an extreme point of the unit disk, and from this we see that $(x,y) = (a,b)$. Hence, $\lambda(a,b,2) = \lambda(x,y,2)$.

(6) The ordered vector space $(\mathbb{R}^3,C)$ does not have the Decomposition Property. To see this, consider the vectors $(1,0,2),(-1,0,2),(1,0,2) \in C$ and note that

$$0 \leq_C (0,1,2) \leq_C (0,0,4) = (1,0,2) + (-1,0,2).$$

If $(\mathbb{R}^3,C)$ has the Decomposition Property, we can find vectors $u,v \in \mathbb{R}^3$ satisfying $0 \leq_C u \leq_C (1,0,2)$, $0 \leq_C v \leq_C (-1,0,2)$, and $u + v = (0,1,2)$. Now by part (5), there exist $0 \leq \lambda,\mu \leq 1$ such that $u = \lambda(1,0,2)$ and $v = \mu(-1,0,2)$. But then,

$$(0,1,2) = u + v = \lambda(1,0,2) + \mu(-1,0,2) = (\lambda + \mu,0,2(\lambda + \mu)),$$

which is impossible. So $(\mathbb{R}^3,C)$ does not satisfy the Decomposition Property. ∎

Regarding ordered vector spaces with the Decomposition Property we have the following basic result.

Theorem 2.4 (Riesz–Kantorovich). *If an ordered vector space L has the Decomposition Property and its cone is generating, then its order dual $L^{\sim}$ is a Riesz space. Moreover, if $f,g \in L^{\sim}$, then for each $x \in L_+$ we have:*

$$f \vee g(x) = \sup\{f(y) + g(z): y,z \in L_+ \text{ and } y+z=x\}$$
$$= \sup_{0 \le y \le x} [f(y) + g(x-y)]$$
$$f \wedge g(x) = \inf\{f(y) + g(z): y,z \in L_+ \text{ and } y+z=x\}$$
$$= \inf_{0 \le y \le x} [f(y) + g(x-y)]$$
$$|f|(x) = \sup\{f(y): |y| \le x\} = \sup_{|y| \le x} f(y).$$

Proof. See Peressini (1967) (Proposition 2.4, p. 23). ∎

The preceding theorem is due to the founders of the theory of Riesz spaces, F. Riesz and L.V. Kantorovich, and lies in the heart of the theory of positive operators. Theorem 2.4, as stated above, was proven first by Riesz (1940) and was generalized to arbitrary order-bounded operators from a Riesz space to an order complete Riesz space by Kantorovich (1936). The formulas describing the lattice operations in the order dual of a Riesz space are known as the *Riesz–Kantorovich* formulas. The following remarkable mathematical problem regarding the Riesz–Kantorovich formulas is still open.

- *Assume that for two regular operators $S,T: L \to M$ between two Riesz spaces the supremum (least upper bound) $S \vee T$ of the operators exists in the ordered vector space of regular operators $\mathscr{L}^r(L,M)$. Does then the supremum $S \vee T$ satisfy the Riesz–Kantorovich formula*

$$S \vee T(x) = \sup_{0 \le y \le x} [S(y) + T(x-y)]$$

for each $x \in L_+$?

For more about this problem and related material we refer the reader to Andô (1962), van Rooij (1985) and Abramovich and Wickstead (1991; 1993). Before moving on to our main results, let us settle on some further notation. For any positive integer $m > 0$ and $x \in L_+$ define the following non-empty convex sets:

$$\mathscr{A}_x^m = \left\{(y_1,\dots,y_m) \in L_+^m : \sum_{i=1}^m y_i \le x\right\} \text{ and}$$

$$\mathscr{F}_x^m = \left\{(y_1,\dots,y_m) \in L_+^m : \sum_{i=1}^m y_i = x\right\}.$$

Also, for a finite number $f_1,\dots,f_m$ of order-bounded linear functionals on L we let

$$\bigvee_{i=1}^{m} f_i = \sup\{f_1,f_2\dots,f_m\} \text{ and } \left(\bigvee_{i=1}^{m} f_i\right)^{+} = \sup\{0,f_1,f_2,\dots,f_m\},$$

where the suprema are taken in $L^{\sim}$.[2] Now notice that if L has the Riesz Decomposition Property then the Riesz–Kantorovich formulas can be rewritten as follows:

$$\left(\bigvee_{i=1}^{m} f_i\right)(x) = \sup\left\{\sum_{i=1}^{m} f_i(y_i):(y_1,y_2,\dots,y_m)\in\mathscr{F}_x^m\right\}$$

$$\left(\bigvee_{i=1}^{m} f_i\right)^{+}(x) = \sup\left\{\sum_{i=1}^{m} f_i(y_i):(y_1,y_2,\dots,y_m)\in\mathscr{A}_x^m\right\}$$

for all $x\in L_+$.

Regarding the convex sets $\mathscr{A}_x^m$ and $\mathscr{F}_x^m$ we have the following result — whose easy proof is omitted.

Lemma 2.5. *Assume that τ is a Hausdorff linear topology on an ordered vector space L such that for some $x\in L_+$ the order interval $[0,x]$ is τ-compact. Then:*

(1) *For each m the convex sets $\mathscr{A}_x^m$ and $\mathscr{F}_x^m$ are both compact subsets of $(L,\tau)^m$.*

(2) *If $f_1,\dots,f_m$ are τ-continuous linear functionals, then the suprema*

$$\sup\left\{\sum_{i=1}^{m} f_i(y_i):(y_1,\dots,y_m)\in\mathscr{F}_x^m\right\} \text{ and } \sup\left\{\sum_{i=1}^{m} f_i(z_i):(z_1,\dots,z_m)\in\mathscr{A}_x^m\right\}$$

are both maxima. That is, there exist $(y_1^*,\dots,y_m^*)\in\mathscr{F}_x^m$ and $(z_1^*,\dots,z_m^*)\in\mathscr{A}_x^m$ such that

$$\sum_{i=1}^{m} f_i(y_i^*) \ge \sum_{i=1}^{m} f_i(y_i) \text{ and } \sum_{i=1}^{m} f_i(z_i^*) \ge \sum_{i=1}^{m} f_i(z_i)$$

hold for all $(y_1,\dots,y_m)\in\mathscr{F}_x^m$ and all $(z_1,\dots,z_m)\in\mathscr{A}_x^m$.

3. The Riesz–Kantorovich formula

We begin with our first important result concerning the Riesz–Kantorovich formula.

[2] It should be observed that $\vee_{i=1}^{m} f_i$ and $(\vee_{i=1}^{m} f_i)^{+}$ are in general different linear functionals. For an example, consider a linear functional $f<0$ and note that $f\vee f=f<0=f^{+}$.

Lemma 3.1. *Let L be an ordered vector space with an order unit e and assume that for the order-bounded linear functionals $f_1, f_2, \ldots, f_m$ the supremum $(\vee_{i=1}^{m} f_i)^+$ exists in $L^\sim$. If there exists some $(x_1^*, x_2^*, \ldots, x_{m^*}) \in \mathscr{A}_e^m$ satisfying*

$$\sum_{i=1}^{m} f_i(x_i^*) \geq \sum_{i=1}^{m} f_i(x_i)$$

for each $(x_1, x_2, \ldots, x_m) \in \mathscr{A}_e^m$, then

$$\left(\bigvee_{i=1}^{m} f_i\right)^+ (e) = \sum_{i=1}^{m} f_i(x_i^*).$$

Proof. Let $g = (\vee_{i=1}^{m} f_i)^+$ and notice first that

$$g(e) \geq g\left(\sum_{i=1}^{m} x_i^*\right) = \sum_{i=1}^{m} g(x_i^*) \geq \sum_{i=1}^{m} f_i(x_i^*) \geq 0.$$

To finish the proof, we must verify that $g(e) \leq \sum_{i=1}^{m} f_i(x_i^*)$ is also true.

If $\sum_{i=1}^{m} f_i(x_i^*) = 0$, then it should be clear (since e is an order unit) that $f_i \leq 0$ (in fact, we have $f_i = 0$) for all $i = 1, \ldots, m$. In this case, $g = 0$ from which we get $g(e) = \sum_{i=1}^{m} f_i(x_i^*) = 0$. So, we can assume that $\sum_{i=1}^{m} f_i(x_i) > 0$.

Let

$$Y = \left\{(y_1, y_2, \ldots, y_m) \in L_+^m : \sum_{i=1}^{m} f_i(y_i) > \sum_{i=1}^{m} f_i(x_i^*)\right\}.$$

Notice that $Y \subseteq L_+^m$ is non-empty (for instance, $2(x_1^*, x_2^*, \ldots, x_m^*) \in Y$) and convex. Furthermore, Y is disjoint from $\mathscr{A}_e^m$ and is therefore disjoint from the convex set

$$Z = \left\{(y_1, y_2, \ldots, y_m) \in L^m : \sum_{i-1}^{m} y_i \leq e\right\},$$

which in turn (in view of Lemma 2.1) has an internal point in L^m. By the Separation Theorem (see for instance, Aliprantis and Border, 1999, Theorem 5.46, p. 188) there exists a non-zero linear functional $(h_1, h_2, \ldots, h_m) \in (L^*)^m$ that separates Z and Y. That is,

$$\sum_{i=1}^{m} h_i(y_i) \geq \sum_{i=1}^{m} h_i(z_i) \text{ for all } (y_1, y_2, \ldots, y_m) \in Y \text{ and } (z_1, z_2, \ldots, z_m) \in Z. \tag{2}$$

Since $(x_1^*, x_2^*, \ldots, x_m^*) \in Z$, it follows from (2) that

$$\sum_{i=1}^{m} h_i(y_i) \geq \sum_{i=1}^{m} h_i(x_i^*) \text{ for all } (y_1, y_2, \ldots, y_m) \in Y. \tag{3}$$

Furthermore, it follows from (2) that

$$\sum_{i=1}^{m} h_i(x_i^*) = \lim_{\alpha \downarrow 1} \sum_{i=1}^{m} h_i(\alpha x_i^*) \geq \sum_{i=1}^{m} h_i(z_i) \text{ for all } (z_1, z_2, \dots, z_m) \in Z. \quad (4)$$

Next, we show that $h_1 = h_2 = \cdots = h_m = h$. Suppose, by way of contradiction, that there exists some $z \in L$ such that $h_1(z) > h_2(z)$. Then for some $\alpha > 1$ we have

$$h_1(x_1^* + z) + h_2(x_2^* - z) + \sum_{i=3}^{m} h_i(x_i^*) > \sum_{i=1}^{m} h_i(\alpha x_i^*).$$

But $(\alpha x_1^*, \alpha x_2^*, \dots, \alpha x_m^*) \in Y$ and $(x_1^* + z, x_2^* - z, x_3^*, \dots, x_m^*) \in Z$, which contradicts (2). By the symmetry of the situation we see that $h_1 = h_2 = \cdots = h_m = h$.

Now we show that $h \geq 0$. Let $x \in L_+$ and note that $(e - \alpha x, 0, \dots, 0) \in Z$ for all $\alpha > 0$. Therefore, from (4) we get $\sum_{i=1}^{m} h(x_i^*) \geq h(e) - \alpha h(x)$ or

$$h(x) \geq \frac{h(e) - \sum_{i=1}^{m} h(x_i^*)}{\alpha},$$

for each $\alpha > 0$. Letting $\alpha \to \infty$ yields $h(x) \geq 0$. That is, $h \geq 0$ and so $h \in L^{\sim}$.

Furthermore, since $h \neq 0$ it must be the case that $h(e) > 0$ and since $(e, 0, \dots, 0) \in Z$, (4) implies that $\sum_{i=1}^{m} h(x_i^*) > 0$. So, we can let

$$\delta = \frac{\sum_{i=1}^{m} f_i(x_i^*)}{\sum_{i=1}^{m} h(x_i^*)} > 0.$$

We claim that $\delta h \geq f_i$ for $i = 1, 2, \dots, m$. To see this, fix i and let $x \in L_+$. If $f_i(x) \leq 0$, then $\delta h(x) \geq 0 \geq f_i(x)$ is trivially true. Assume, therefore, that $f_i(x) > 0$ and let

$$\gamma = \frac{\sum_{i=1}^{m} f_i(x_i^*)}{f_i(x)} > 0.$$

It is clear that the vector $(0, \dots, \alpha\gamma x, 0, \dots, 0)$, where $\alpha\gamma x$ occupies the i-th position, satisfies $(0 \dots, \alpha\gamma x, 0, \dots, 0) \in Y$
for any $\alpha > 1$, and from (3) we see that

$$\delta\gamma h(x) = \delta \lim_{\alpha \downarrow 1} h(\alpha\gamma x) \geq \delta \sum_{i=1}^{m} h(x_i^*) = \sum_{i=1}^{m} f_i(x_i^*) = \gamma f_i(x),$$

or $\delta h(x) \geq f_i(x)$ for each $x \geq 0$. Thus, $\delta h \geq f_i$ for $i = 1, 2, \dots, m$.

Consequently, $g \le \delta h$. In particular, from (4) it follows that

$$g(e) \le \delta h(e) \le \delta \sum_{i=1}^{m} h(x_i^*) = \sum_{i=1}^{m} f_i(x_i^*),$$

and the proof is finished. ■

Corollary 3.2. *Let L be an ordered vector space with an order unit e and assume that for the order-bounded linear functionals $f_1, f_2, \ldots, f_m$ the supremum $\vee_{i=1}^{m} f_i$ exists in $L^\sim$*. If there exists some $(x_1^*, x_2^*, \ldots, x_m^*) \in \mathscr{F}_e^m$, satisfying

$$\sum_{i=1}^{m} f_i(x_i^*) \ge \sum_{i=1}^{m} f_i(x_i)$$

for each $(x_1, x_2, \ldots, x_m) \in \mathscr{F}_e^m$, then

$$\left(\bigvee_{i=1}^{m} f_i\right)(e) = \sum_{i=1}^{m} f_i(x_i^*).$$

Proof. Let $g = \vee_{i=1}^{m} f_i$ and start by observing that $g - f_1 = [\vee_{i=2}^{m}(f_i - f_1)]^+$ and

$$\begin{aligned}\sum_{i=2}^{m}(f_i - f_1)(x_i^*) &= \sum_{i=1}^{m} f_i(x_i^*) - f_1(e)\\ &\ge \sum_{i=2}^{m} f_i(x_i) + f_1\left(e - \sum_{i=2}^{m} x_i\right) - f_1(e)\\ &= \sum_{i=2}^{m}(f_i - f_1)(x_i),\end{aligned}$$

for any $(x_2, \ldots, x_m) \in \mathscr{A}_e^{m-1}$. Therefore, by Lemma 3.1, we have

$$(g - f_1)(e) = \sum_{i=2}^{m}(f_i - f_1)(x_i^*).$$

In particular,

$$g(e) = f_1\left(e - \sum_{i=2}^{m} x_i^*\right) + \sum_{i=2}^{m} f_i(x_i^*) = \sum_{i=1}^{m} f_i(x_i^*),$$

which is the desired formula. ■

From now on we shall assume that L is equipped with a Hausdorff linear topology τ for which the order intervals are compact — in which case the topological dual L' of (L,τ) is a vector subspace of the order dual $L^\sim$, i.e, $L' \subseteq L^\sim$.

We are now ready to state and prove our main theorem.

Theorem 3.3. *Assume that L is an ordered vector space with order units and that it is equipped with a Hausdorff linear topology for which the order intervals of L are compact. If for some continuous linear functionals $f_1, f_2, \dots, f_m$ the supremum $g = \vee_{i=1}^m f_i$ exists in $L^\sim$, then g satisfies the Riesz–Kantorovich formula, i.e., for each $x \in L_+$ we have*

$$\left(\bigvee_{i=1}^m f_i\right)(x) = \sup\left\{\sum_{i=1}^m f_i(y_i) : (y_1, y_2, \dots, y_m) \in \mathscr{F}_x^m\right\}.$$

Proof. Assume that τ is a Hausdorff linear topology on L for which the intervals are τ-compact and notice that for each $x \in L_+$ the set $\mathscr{F}_x^m$ is compact in $(L,\tau)^m$. Now fix an arbitrary $x \in L_+$, and then select an order unit e such that $x \le e$.

For each $0 < \alpha < 1$ let $e_\alpha = \alpha x + (1-\alpha)e$. Clearly, each e_α is an order unit and $0 \le e_\alpha \le e$ holds for each $0 < \alpha < 1$. We consider the index set $(0,1)$ directed by the increasing order relation $\succeq$, i.e, $\alpha \succeq \beta$ in $(0,1)$ if and only if $\alpha \ge \beta$. Clearly, $x_\alpha \to x$. By Lemma 3.1, we know that g satisfies the Riesz–Kantorovich formula for each order unit. Therefore, for each $0 < \alpha < 1$ there exists some $(z_1^\alpha, z_2^\alpha, \dots, z_m^\alpha) \in L_+^m$ such that $\sum_{i=1}^m z_i^\alpha = x_\alpha$ and

$$\alpha g(x) + (1-\alpha) g(e) = g(x_\alpha) = \sum_{i=1}^m f_i(z_i^\alpha). \tag{5}$$

Since $z_i^\alpha \in [0,e]$ for each $\alpha \in (0,1)$ and each $i = 1,2,\dots,m$ and the order interval $[0,e]$ is τ-compact, there exists a subnet of $\{(z_1^\alpha, z_2^\alpha, \dots, z_m^\alpha)\}$ (which without loss of generality we also denote it by $\{(z_1^\alpha, z_2^\alpha, \dots, z_m^\alpha)\}$ such that $z_i^\alpha \to z_i \in [0,e]$, for $i = 1,2,\dots,m$. From $x_\alpha - z_i^\alpha \in [0,e]$ for each α and the closedness of $[0,e]$, we see that $0 \le x - z_i \le e$. So $0 \le z_i \le x$ and $\sum_{i=1}^m z_i = x$. Finally, letting $\alpha \to 1$ in Eq. (5), it follows from the continuity of each f_i that

$$g(x) = \sum_{i=1}^m f_i(z_i) \le \sup\left\{\sum_{i=1}^m f_i(y_i) : (y_1, y_2, \dots, y_m) \in \mathscr{F}_x^m\right\} \le g(x),$$

and the proof is finished. ■

It should be noticed that the hypotheses of Theorem 3.3 do not imply that the partially ordered vector space L satisfies the Decomposition Property. For instance, $\mathbb{R}^3$ with the ''ice cream'' cone of Example 2.3 satisfies all the assumptions of Theorem 3.3 but it does not have the Riesz Decomposition Property.

The next result characterizes the Riesz Decomposition Property in terms of a lattice property of the topological dual.

Corollary 3.4. *Assume that L is an ordered vector space with order units and that it is equipped with a Hausdorff locally convex topology for which the order intervals of L are weakly compact. Then the following statements are equivalent.*
1. *For every pair $f,g \in L'$ the supremum $f \vee g$ exists in $L^{\sim}$.*
2. *L has the Riesz Decomposition Property.*

Moreover, if this is the case, then $L^{\sim}$ is a Riesz space whose lattice operations are given by the Riesz–Kantorovich formulas.

Proof. If (2) is true, then the validity of (1) follows from Theorem 2.4. So, assume that (1) is true and fix $x,y \in L_+$. Clearly, $[0,x]+[0,y] \subseteq [0,x+y]$.

Now suppose, by way of contradiction, that there exists some $z \in [0,x+y]$ such that $z \neq [0,x]+[0,y]$. Since $[0,x]+[0,y]$ is a weakly compact convex set, it follows from the Separation Theorem (see for instance, Aliprantis and Border, 1999, Corollary 5.59, p. 194) that there exist some non-zero $f \in L'$ and some $\varepsilon > 0$ such that

$$f(z) > \varepsilon + f(u) + f(v)$$

for all $u \in [0,x]$ and all $v \in [0,y]$. Now a glance at Theorem 3.3 shows that

$$\begin{aligned}\sup_{0 \le w \le x+y} f(w) = f^+(x+y) = f^+(x) + f^+(y)\\ &= \sup_{0\le u\le x} f(u) + \sup_{0\le v\le y} f(v)\\ &\le f(z) - \varepsilon\\ &< f(z),\end{aligned}$$

which is a contradiction. Hence, $[0,x]+[0,y]=[0,x+y]$ holds true, and so L has the Decomposition Property. ■

Recall that $L^{\sim}$ *separates points* in L if $x \in L$ and $f(x)=0$ for all $f \in L^{\sim}$ implies $x=0$. In this case, the weak topology $\sigma(L,L^{\sim})$ is a Hausdorff locally convex topology on L. We, therefore, obtain the following simple consequence of Corollary 3.4.

Corollary 3.5. *Assume that L is an ordered vector space with order units and that $L^{\sim}$ separates points in L. If the order intervals of L are $\sigma(L,L^{\sim})$-compact, then the following statements are equivalent.*
1. *$L^{\sim}$ is lattice ordered, i.e., $L^{\sim}$ is a Riesz space.*
2. *L has the Riesz Decomposition Property.*

Let us now move to the case where L is an ordered vector space without an order unit. For any $x \in L_+$ denoted by L_x, the linear subspace $\cup_{n=1}^{\infty} n[-x,x]$ and note that L_x under its canonical ordering is an ordered vector space with order unit x. The linear subspace $L_x = \cup_{n=1}^{\infty} n[-x,x]$ is referred to as the *ideal* generated by x. As usual, the restriction of the positive cone L_+ to L_x is denoted by L_x^+, i.e., $L_x^+ = L_x \cap L_+$. Notice that for any $y \in L_x$ the L-order interval $[0,y]$ is contained in L_x^+, i.e, $[0,y] \subseteq L_x^+$.

The next lemma characterizes the Riesz–Kantorovich problem for continuous linear functionals on certain ordered vector spaces (with or without order units). We highlight this technical observation since perhaps it may lead to a more general solution of the Riesz–Kantorovich problem than the one presented in this paper.

Lemma 3.6. *If L is an ordered vector space that is equipped with a Hausdorff linear topology for which the order intervals of L are compact, then the following two statements are equivalent.*

1. *There is a pair of continuous linear functionals f and g such that $f \vee g$ exists in $L^{\sim}$ but does not satisfy the Riesz–Kantorovich formula for all $x \in L_+$.*
2. *There is a pair of continuous functionals f and g and a point $x \in L_+$ such that $f \vee g$ exists in $L^{\sim}$ but the restriction $(f \vee g)|_{L_x}$ is not the supremum of $f|_{L_x}$ and $g|_{L_x}$ in the order dual $L_x^{\sim}$ of L_x.*

Proof. Assume first that L is an ordered vector space that is equipped with a Hausdorff linear topology for which the order intervals of L are compact.

Assume that (1) holds and let $f, g \in L'$ be a pair of continuous functionals for which $f \vee g$ exists in $L^{\sim}$ but does not satisfy the Riesz–Kantorovich formula. That is there is some $x \in L_+$ such that

$$f \vee g(x) > \sup_{0 \leq y \leq x} [f(y) + g(x-y)]. \tag{6}$$

Consider the space L_x for which x is an order unit. Assume, by way of contradiction, that

$$(f \vee g)|_{L_x} = f|_{L_x} \vee g|_{L_x}, \tag{7}$$

where the supremum on the right-hand side of the equation is in the order dual of L_x. By Lemma 3.1, (6) and (7), we have

$$\begin{aligned}
\left(f|_{L_x} \vee g|_{L_x}\right)(x) &= \sup_{0 \leq y \leq x} \left[f|_{L_x}(y) + g|_{L_x}(x-y)\right] \\
&= \sup_{0 \leq y \leq x} [f(y + g(x-y)] \\
&= f \vee g(x) \\
&= \left(f|_{L_x} \vee g|_{L_x}\right)(x),
\end{aligned}$$

which is a contradiction. Therefore, (2) holds and (1) $\Rightarrow$ (2).

For the converse, assume that (2) is true. Let $f,g \in L'$ be a pair of continuous functionals for which $f \vee g$ exists in $L^{\sim}$ and let $x \in L_+$ be a positive vector such that the restriction $(f \vee g)|_{L_x}$ is not the supremum of $f|_{L_x}$ and $g|_{L_x}$ in the order dual of L_x. We claim that $f \vee g$ does not satisfy the Riesz–Kantorovich formula.

To see this, assume, by way of contradiction, that $f \vee g$ satisfies the Riesz–Kantorovich formula at every $z \in L_+$, that is, for each $z \in L_+$ we have

$$f \vee g(z) = \sup_{0 \le y \le z} \left[f(y) + g(z-y) \right]. \tag{8}$$

Let h be a linear functional in the order dual of L_x such that $h \ge f$ and $h \ge g$. We see from Eq. (8) that for any $z \in L_x^+$ we have

$$h(z) \ge \sup_{0 \le y \le z} \left[f|_{L_x}(y) + g|_{L_x}(z-y) \right] = (f \vee g)|_{L_x}(z),$$

which implies $h \ge (f \vee g)|_{L_x}$. In particular, we get that $(f \vee g)|_{L_x} = f|_{L_x} \vee g|_{L_x}$ holds in $L_x^{\sim}$, which contradicts our assumption. Therefore, (1) holds true and (2) $\Rightarrow$ (1). ■

4. Commodity decomposition

In Aliprantis et al. (1998), a theory of value with non-linear prices was developed. The formula for this price is a generalization of the Riesz–Kantorovich formula. The authors also provided concrete economic meaning to the Riesz–Kantorovich formula and the Riesz Decomposition Property in terms of the revenue function of a discriminatory price auction and the perfect decomposability of commodity bundles, respectively. With this in mind, we show the implications of the results of Section 3 to the theory of value with non-linear prices.

Following Aliprantis et al. (1998), let (L,τ) be an ordered Hausdorff locally convex space whose order intervals are τ-bounded. Consider an exchange economy with m consumers and designate the arbitrary consumer by the index i. The bundle $\omega_i \in L$ is the i-th consumer's *initial endowment*. As usual, $\omega = \sum_{i=1}^{m} \omega_i$ is the *total endowment*. The *consumption set* of consumer i is X_i. Throughout this section, for each consumer i, we assume that:

- *the consumption set* X_i *is a convex* τ*-closed subcone of* L_+, *and*
- $0 < \omega_i \in X_i$.

We call an arbitrary linear functional $p = (p_1, p_2, \dots, p_m)$ on L^m a *list of price bids*. Now, each list of price bids defines a value functional that generalizes the revenue function of the single commodity US Treasury Bill Auction. In a short while, we shall call this value functional an auction price, which can be non-linear. The domain C of this value functional will be the convex cone generated in L by $\cup_{i=1}^{m} X_i$, i.e., $C = X_1 + X_2 + \cdots + X_m$.

Clearly, C is a convex subcone of L_+ and $\omega \in C$. The vector space generated by C is denoted by M, i.e., $M = C - C$.

Now for each commodity bundle $x \in L_+$, we let $\mathscr{A}_x$ denote the set of all consumable allocations when the total endowment is x, i.e.,

$$\mathscr{A}_x = \left\{ (y_1, y_2, \ldots, y_m) \in \prod_{i=1}^{m} X_i : \sum_{i=1}^{m} y_i \le x \right\}.$$

Clearly, each $\mathscr{A}_x$ is a non-empty, convex and closed subset of $(L,\tau)^m$. Notice also that for each $x, y \in L_+$ and all $\alpha \ge 0$, we have

$$\mathscr{A}_x + \mathscr{A}_y \subseteq \mathscr{A}_{x+y} \text{ and } \mathscr{A}_{\alpha x} = \alpha \mathscr{A}_x.$$

We move to the definition of our value functional.

Definition 4.1. The *auction price* of a list of price bids $p = (p_1, p_2, \ldots, p_m)$ is the function $\psi_p : C \to [0,\infty]$ defined by

$$\psi_p(x) = \sup_{z \in \mathscr{A}_x} [\, p_1(z_1) + p_2(z_2) + \cdots + p_m(z_m)] .$$

One can think of these non-linear functions as the auctioneer's revenue function in a discriminatory price auction. Consumers participate in a discriminatory price auction for commodity bundles. Their bids comprise a consumption set and a personalized price, which is a linear functional on L. The auctioneer divides commodity bundles into consumable allocations that maximize revenue. Each consumer pays the price she bids. Clearly, if $p \in (L^{\sim})^m$ (in particular, if $p \in (L')^m$), then ψ_p is a real-valued function. The value $\psi_p(x)$ represents the maximum revenue that can be obtained for the commodity bundle x by the auctioneer when p is the list of personalized price bids by the m players.

We are interested in the conditions under which an auction price is additive and can be extended to a linear functional on all of L, in which case the notions of an auction price and a Walrasian price coincide. Before addressing this question, we list the basic properties of the auction prices. This result can be found in Aliprantis et al. (1998).

Lemma 4.2. *If $p = (p_1, p_2, \ldots, p_m) \in (L^{\sim})^m$ is a list of order-bounded price bids, then its auction price $\psi_p : C \to [0,\infty)$ is a non-negative real-valued function such that:*

1. *ψ_p is monotone, i.e., $x,y \in C$ with $x \le y$ implies $\psi_p(x) \le \psi_p(y)$,*
2. *ψ_p is super-additive, i.e., $\psi_p(x) + \psi_p(y) \le \psi_p(x+y)$ for all $x,y \in C$,*
3. *ψ_p is positively homogeneous, i.e., $\psi_p(\alpha x) = \alpha \psi_p(x)$ for all $\alpha \ge 0$ and $x \in C$,*
4. *if $p_1 = p_2 = \cdots = p_m = q \ge 0$, then $\psi_p(x) = q(x)$ for all $x \in C$, i.e., $\psi_p = q$, and*
5. *if $x \in X_i$, then $p_i(x) \le \psi_p(x)$.*

It should be noticed immediately that properties (2) and (3) of Lemma 4.2 guarantee that the auction prices (which are, in general, non-linear) are always concave functions.

We now introduce the decomposability property of the consumption sets. This notion captures the idea of the perfect decomposability of commodity bundles into consumable allocations.

Definition 4.3 (Consumption Decomposability). The economy has the *Consumption Decomposability Property* if for each $x, y \in C$ we have $\mathscr{A}_x + \mathscr{A}_y = \mathscr{A}_{x+y}$.

Regarding the Consumption Decomposability Property and extension of auction prices, we have the following result which is taken from Aliprantis et al. (1998).

Lemma 4.4. *For a vector subspace $\mathscr{P}$ of $L^{\sim}$ the following statements are equivalent.*

1. *For each non-zero $p = (p_1, p_2, \ldots, p_m) \in \mathscr{P}^m$ the auction price $\psi_p: C \to [0,\infty)$ is additive — and hence it has a unique linear extension to $M = C - C$.*
2. *For each $x,y \in C$ we have $\mathscr{A}_{x+y} \subseteq \overline{\mathscr{A}_x + \mathscr{A}_y}$, where the bar denotes $\sigma(L^m, \mathscr{P}^m)$-closure.*

Proof. (1) $\Rightarrow$ (2) Suppose, by way of contradiction, that there exist $x, y \in C$ and some $z = (z_1, z_2, \ldots, z_m) \in \mathscr{A}_{x+y}$ such that $z \notin \overline{\mathscr{A}_x + \mathscr{A}_y}$. Since $\sigma(L^m, \mathscr{P}^m)$ is a locally convex topology, there exists some non-zero $p = (p_1, p_2, \ldots, p_m) \in \mathscr{P}^m$ which strongly separates z and $\overline{\mathscr{A}_x + \mathscr{A}_y}$. That is, there exists some $\varepsilon > 0$ satisfying

$$\sum_{i=1}^{m} p_i(z_i) \geq \varepsilon + \sum_{i=1}^{m} p_i(u_i)$$

for all $(u_1, u_2, \ldots, u_m) \in \overline{\mathscr{A}_x + \mathscr{A}_y}$. This easily implies

$$\psi_p(x+y) \geq \sum_{i=1}^{m} p_i(z_i) \geq \varepsilon + \psi_p(x) + \psi_p(y) > \psi_p(x) + \psi_p(y),$$

which contradicts the additivity of ψ_p.

(2) $\Rightarrow$ (1) Let $p = (p_1, p_2, \ldots, p_m) \in \mathscr{P}^m$. Then the auction price $\psi_p: C \to [0,\infty]$, defined by

$$\psi_p(x) = \sup_{z \in \mathscr{A}_x} [p_1(z_1) + p_2(z_2) + \cdots + p_m(z_m)],$$

is real-valued, positively homogeneous and super-additive. To see that ψ_p is additive, let $x, y \in C$, and fix $z = (z_1, z_2, \ldots, z_m) \in \mathscr{A}_{x+y} \subseteq \overline{\mathscr{A}_x + \mathscr{A}_y}$. Then there exist two nets $\{(u_1^\alpha, \ldots, u_m^\alpha)\} \subseteq \mathscr{A}_x$ and $\{(v_1^\alpha, \ldots, v_m^\alpha)\} \subseteq \mathscr{A}_y$ such that

$$(u_1^\alpha + v_1^\alpha, \ldots, u_m^\alpha + v_m^\alpha) \xrightarrow{\sigma(L^m, \mathscr{P}^m)} z.$$

In particular, we have $\lim_\alpha \sum_{i=1}^m p_i(u_i^\alpha + v_i^\alpha) = \sum_{i=1}^m p_i(z_i)$. Now taking into account that

$$\sum_{i=1}^m p_i(u_i^\alpha + v_i^\alpha) = \sum_{i=1}^m p_i(u_i^\alpha) + \sum_{i=1}^m p_i(v_i^\alpha) \le \psi_p(x) + \psi_p(y),$$

we see that $\sum_{i=1}^m p_i(z_i) \le \psi_p(x) + \psi_p(y)$. Since $z = (z_1, z_2, \dots, z_m) \in \mathscr{A}_{x+y}$ is arbitrary, we conclude that $\psi_p(x+y) \le \psi_p(x) + \psi_p(y)$. Consequently, $\psi_p(x+y) = \psi_p(x) + \psi_p(y)$, so that ψ_p is additive on C.

Now, we leave it to the reader to verify that if for each $x \in M$ we write $x = a - b$ with $a, b \in C$, then the formula

$$\psi_p(x) = \psi_p(a) - \psi_p(b)$$

is independent of the representation of x as a difference of two vectors of C and defines a (unique) linear extension of ψ_p to all of M. (Notice also that ψ_p is a positive linear functional on M with respect to the generating cone C.) ■

We are now ready to present two connections between the preceding results and the results in Section 3.

Theorem 4.5. *Assume that L has order units and that the order intervals of L are weakly compact. Assume further that $X_i = L_+$ for each i and let $p = (p_1, p_2, \dots, p_m) \in (L')^m$ be a list of continuous price bids. The following statements are equivalent:*

1. *The auction price $\psi_p : C \to [0, \infty)$ is additive — and hence it has a unique linear extension to $L = L_+ - L_+$.*
2. *The supremum linear functional $(\vee_{i=1}^m p_i)^+$ exists in $L^\sim$ (and is given by the Riesz–Kantorovich formula).*

Proof. Theorem 3.3 says that (2) ⇒ (1). Assume that (1) holds and let $q \ge 0$ be the linear extension of ψ_p to all of L. Notice that q is in $L^\sim$. Now suppose that a positive linear functional f satisfies $f \ge p_i$ for each $i = 1, \dots, m$. Then it is clear that $f(x) \ge q(x)$ for all $x \in L_+$. Therefore, $f \ge q$ and so $q = (\vee_{i=1}^m p_i)^+$ holds in $L^\sim$. Thus, (1) ⇒ (2). ■

Theorem 4.6. *Assume that L has order units and that the order intervals of L are weakly compact. If $X_i = L_+$ for each i, then the following statements are equivalent.*

1. *For every list of price bids $p = (p_1, p_2, \dots, p_m) \in (L')^m$ the auction price ψ_p is additive — and hence it has a unique linear extension to $L = L_+ - L_+$.*
2. *The economy has the Consumption Decomposability Property.*
3. *The commodity space has the Riesz Decomposition Property.*
4. *For every list of price bids $(p_1, p_2, \dots, p_m) \in (L')^m$ the linear functional $(\vee_{i=1}^m p_i)^+$ exists in $L^\sim$.*

Proof. We show that $(1) \Rightarrow (2)$. Consider Lemma 4.4 and let $\mathscr{P} = L'$. Denote by σ^m the topology $\sigma(L^m,(L')^m)$ on L^m. Since order intervals are weakly compact, we see that for each $x,y \in L_+$ the convex sets $\mathscr{A}_x$ and $\mathscr{A}_y$ are σ^m-compact. Therefore, $\mathscr{A}_x + \mathscr{A}_y = \overline{\mathscr{A}_x + \mathscr{A}_y}$, where the bar denotes σ^m-closure. Notice therefore that this economy has the Consumption Decomposability Property if and only if $\mathscr{A}_x + \mathscr{A}_y \subseteq \overline{\mathscr{A}_x + \mathscr{A}_y}$ for each $x,y \in L_+$. Therefore, by Lemma 4.4 statements (1) and (2) are equivalent.

Now, we show that $(2) \Rightarrow (3)$. So assume that the economy has the Consumption Decomposability Property and let three vectors $x,y,z \in L_+$ satisfy $0 \leq z \leq x + y$. Then $u = (z,0,0, \dots ,0) \in \mathscr{A}_{x+y} = \mathscr{A}_x + \mathscr{A}_y$. So, there exist $v = (v_1,v_2,\dots,v_m) \in \mathscr{A}_x$ and $w = (w_1,w_2,\dots,w_m) \in \mathscr{A}_y$ satisfying $u = v + w$. The latter implies $v_i = w_i = 0$ for $i = 2,3,\dots,m$, and so the vectors $v_1,w_1 \in L_+$ satisfy $0 \leq v_1 \leq x$, $0 \leq w_1 \leq y$ and $v_1 + w_1 = z$. This shows that L has the Riesz Decomposition Property.

Finally, it is well-known that $(3) \Rightarrow (4)$. Furthermore, we know from Theorem 4.5 that $(4) \Rightarrow (1)$, which proves the theorem. ■

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