Available online at www.sciencedirect.com
ScienceDirect
Journal of Economic Theory 134 (2007) 175 – 194
JOURNAL OF Economic Theory
www.elsevier.com/locate/jet

Perfect competition in asymmetric information economies: compatibility of efficiency and incentives☆

Yeneng Sun[a, b], Nicholas C. Yannelis[c, *]

[a]Department of Economics, National University of Singapore, 1 Arts Link, Singapore 117570, Singapore
[b]Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543, Singapore
[c]Department of Economics, University of Illinois at Urbana-Champaign, Champaign, IL 61820, USA

Received 18 November 2004; final version received 12 March 2006
Available online 19 April 2006

Abstract

The idea of perfect competition for an economy with asymmetric information is formalized via an idiosyncratic signal process in which the private signals of almost every individual agent can influence only a negligible group of agents, and the individual agents' relevant signals are essentially pairwise independent conditioned on the true states of nature. Thus, there is no incentive for an individual agent to manipulate her private information. The existence of incentive compatible, ex post Walrasian allocations is shown for such a perfectly competitive asymmetric information economy with or without "common values". Consequently, the conflict between incentive compatibility and Pareto efficiency is resolved exactly, and its asymptotic version is derived for a sequence of large, but finite private information economies.

JEL classification: C70; D50; D82

Keywords: Asymmetric information; Perfect competition; Pareto efficiency; Walrasian allocation; Incentive compatibility; Negligible private information

☆ This work was initiated in October 2003 while Yeneng Sun was visiting the University of Illinois at Urbana-Champaign. The final version of this paper was completed in March 2006 when Nicholas Yannelis was visiting the Department of Economics at the National University of Singapore. The authors thank an associate editor and two referees for their useful comments and suggestions.

* Corresponding author. Fax: +1 217 244 6678.
E-mail addresses: matsuny@nus.edu.sg (Y. Sun), nyanneli@uiuc.edu (N.C. Yannelis).

0022-0531/$ - see front matter
doi:10.1016/j.jet.2006.03.001

1. Introduction

The classical Arrow–Debreu–McKenzie model of perfect competition is obviously at odds with itself as the finitude of economy size implies that individuals can exercise some influence on the prices at which goods are either sold or bought in the economy. Aumann [2] resolves this issue by introducing an economy with an atomless measure space of agents. In such an economy, each individual agent has non-negligible consumption in general, but with negligible impact on the aggregate demand, and therefore takes prices as given. Thus, the formulation of an atomless measure space of agents captures precisely the meaning of perfect competition. [1]

The Aumann model is deterministic as each agent's characteristics are non-random. Thus, in this model, contracts (trades) are made under complete information. It is not an exaggeration to say that all economic activities or all contacts among individuals in an economy are made under conditions of uncertainty or incomplete information. To this end, it is of interest to know whether or not one can introduce asymmetric or private information [2] on the Aumann economy, and still be able to capture the meaning of perfect competition. Notice that once private information is introduced in the Aumann model, an agent may have monopoly power on her information, and thus may have an incentive to manipulate her information to become better off. This poses the following question: can one model the idea of perfect competition in an economy with asymmetric information? To put differently, can one model the concept of negligible private information?

It is well-known that there is a conflict between incentive compatibility and efficiency in a finite-agent asymmetric information economy (see, for example, [3, p. vi, Example 0.1]). However, intuition suggests that a perfectly competitive market should still perform efficiently since no single agent has monopoly power on information. In an important paper [9], McLean and Postlewaite showed the consistency of exact incentive compatibility and approximate ex post efficiency by using independent replicas of a fixed asymmetric information economy with finitely many agents. The key point in this approach is that though an individual agent's information is not becoming more accurate, its overall influence is diminishing when the number of agents goes to infinity. In a way, the model considered in [9] can be viewed as capturing the idea of approximate perfect competition in an asymmetric information economy. [3]

The main purpose of this paper is to formulate precisely the idea of perfect competition for an asymmetric information economy so that the incentive for an individual agent to manipulate her private information is negligible. A heuristic way to capture the idea of perfect competition for such an economy is that the private signal of an individual agent can only influence a negligible corner of the market, and the signals associated with the individual agents (for example, used in their utility functions) are essentially independent of each other conditioned on the true states

[1] See [6] for a systematic development of large economies and extensive references.

[2] When it is appropriate, we shall use the terminologies of private information, differential information, incomplete information and asymmetric information interchangeably.

[3] For an economy with a fixed finite number of agents, McLean and Postlewaite also showed in [9] that the conflict between incentive compatibility and efficiency can be made arbitrarily small when the agents are able to predict the true states of nature with sufficient accuracy in terms of small information size. Krasa and Shafer [7] considered similar questions in terms of convergence of equilibria in a sequence of incomplete information economies of fixed size to equilibria of a complete information economy when the noise in the signals converges to zero. In addition, Prescott–Townsend [11] introduced a lottery model and showed the existence of incentive compatible, ex ante efficient lottery allocations. By restricting allocations to be privately measurable, it was shown in Yannelis [16] (see also [5]) that private core allocations in a finite-agent asymmetric information economy exist and are always incentive compatible. However, the private measurability restrictions result in only a second best efficient outcome.

of nature. This paper shows that such a heuristic idea does work well in a suitable mathematical framework, which enables us to resolve the conflict between efficiency and incentive compatibility exactly. Thus, we provide an answer to the open question posed in [1] and [3, p. ix].

The set-up of the general information structure in this paper is partly motivated by a suggestion in the second paragraph [9, p. 2440]. While the main technical tool behind the "informational smallness" as used in [9] is the classical law of large numbers for a sequence of iid random variables, we need to use an exact law of large numbers for a continuum of (conditionally) independent random variables, developed in [14], [4] for handling the corresponding idea of "negligible information". Based on our exact results, we also obtain an asymptotic efficiency result for a general sequence of large but finite differential information economies.

The paper is organized as follows. After presenting the basic measure-theoretic framework in Section 2, a perfectly competitive differential information economy is considered in Section 3 in the setting of a "common value" model, where agents' types are purely informational in the sense that they do not enter the utility functions. In particular, this is the case, when uncertainty only stems from characteristics of the objects being traded. It is proved in Theorem 1 that any allocation in the corresponding complete information economy can be implemented as an incentive compatible allocation in the private information economy that transforms exactly the usual Pareto efficiency and Walrasian equilibrium to their ex post versions via very simple measure-theoretic methods. The existence of incentive compatible and ex post Walrasian (and hence ex post individually rational and ex post efficient) allocations follows easily (part (3) of Theorem 1), and thus the conflict between incentive compatibility and Pareto efficiency is resolved exactly in this setting, contrary to the fixed finite economy setting.

The same type of existence result as in part (3) of Theorem 1 is shown in Theorem 2 of Section 4 for the more general case that agents' types are allowed to enter the utility functions. Theorem 3 of Section 5 presents the asymptotic version of Theorem 2 for a general sequence of large, but finite private information economies. In Section 6, we discuss the related literature. Section 7 contains some concluding remarks. Section A is an appendix that includes the proofs of Theorems 1 and 2 plus a statement of the exact law of large numbers and associated definitions. The proof of Theorem 3 and a construction of the information structure that satisfies the required conditions use non-standard analysis, which can be found in [15].

2. Some basic definitions

We fix an atomless probability space $(I, \mathcal{I}, \lambda)$ representing the space of economic agents, [5] and $S = \{s_1, s_2, \dots, s_K\}$ the space of true states of nature (its power set denoted by $\mathcal{S}$), which are not known to the agents.

[4] A key point in the economic model of this paper is that the agents observe the realized distribution of an idiosyncratic signal process and use it to deduce the true states of nature. This follows from the exact law of large numbers.

[5] We use the convention that all probability spaces are countably additive. For those interested in the case of a literal continuum of agents, it is noted in [14] that one can indeed take I to be the unit interval with some atomless probability measure. Corollary 4.3 in [14] shows, however, that under a Fubini extension, almost all sample functions of a non-trivial independent process are proven to be non-Lebesgue measurable. A simple reason for the failure of the Lebesgue measure in the setting is that it is based essentially on a countably generated σ-algebra while the sample functions of a non-trivial independent process are very diverse and need a very rich σ-algebra to support them. The Loeb measures introduced in [8] are rich enough for the purpose. Also, a non-trivial independent process is never jointly measurable in the usual sense, and thus we need to work with an extension of the usual measure-theoretic product with the Fubini property to resolve the measurability problem as in [14].

Let $T^0 = \{q_1, q_2, \dots, q_L\}$ be the space of all the possible signals (types) for individual agents, $(T, \mathcal{T})$ a measurable space that model the private signal profiles for all the agents, and thus T is a space of functions from I to T^0.[6] Thus, $t \in T$, as a function from I to T^0, represents a private signal profile for all agents in I. For agent $i \in I$, $t(i)$ (also denoted by t_i) is the *private signal* of agent i while t_{-i} the restriction of the signal profile t to the set $I \setminus \{i\}$ of agents different from i; let T_{-i} be the set of all such t_{-i}. For simplicity, we shall assume that $(T, \mathcal{T})$ has a product structure so that T is a product of T_{-i} and T^0, while $\mathcal{T}$ is the product algebra of the power set $\mathcal{T}^0$ on T^0 with a σ-algebra $\mathcal{T}_{-i}$ on T_{-i}. We shall adopt the usual notation (t_{-i}, t_i') to denote the signal profile whose value is t_i' for agent i ($t_i' \in T^0$), and the same as t for other agents.

Let $(\Omega, \mathcal{F}, P)$ be a probability space representing all the uncertainty on the true states as well as on the signals for all the agents, where $(\Omega, \mathcal{F})$ is the product measurable space $(S \times T, \mathcal{S} \otimes \mathcal{T})$. Let P^S and P^T be the marginal probability measures of P, respectively, on $(S, \mathcal{S})$ and on $(T, \mathcal{T})$. Let $\tilde{s}$ and $\tilde{t}_i$, $i \in I$ be the respective projection mappings from Ω to S and from Ω to T^0 with $\tilde{t}_i(s, t) = t_i$.[7] For each true state $s \in S$, we assume without loss of generality that the state is essential in the sense that $\pi_s = P^S(\{s\}) > 0$; let P_s^T be the conditional probability measure on $(T, \mathcal{T})$ when the random variable $\tilde{s}$ takes value s. Thus, for each $B \in \mathcal{T}$, $P_s^T(B) = P(\{s\} \times B)/\pi_s$. It is obvious that $P^T = \sum_{s \in S} \pi_s P_s^T$. Note that the conditional probability measure P_s^T is often denoted as $P(\cdot|s)$ in the literature.

One can also introduce the conditional probability measure[8] $P^S(\cdot|t)$ on S such that $P^S(\{s\}|t)$ forms a probability weights in $s \in S$ for a fixed $t \in T$, is $\mathcal{T}$-measurable in $t \in T$ for a fixed $s \in S$, and for each $B \in \mathcal{T}$, $P(\{s\} \times B) = \int_B P^S(\{s\}|t)\, dP^T(t)$. Let $p_s(\cdot)$ be the density function of P_s^T with respect to P^T; it is easy to see that $P^S(\{s\}|t) = \pi_s p_s(t)$ for P^T-almost all $t \in T$.

For $i \in I$, let τ_i be the signal distribution of agent i on the space T^0,[9] and $P^{S \times T_{-i}}(\cdot|t_i)$ the conditional probability measure on the product measurable space $(S \times T_{-i}, \mathcal{S} \otimes \mathcal{T}_{-i})$ when the signal of agent i is $t_i \in T^0$. If $\tau_i(\{t_i\}) > 0$, then it is clear that for $D \in \mathcal{S} \otimes \mathcal{T}_{-i}$, $P^{S \times T_{-i}}(D|t_i) = P(D \times \{t_i\})/\tau_i(\{t_i\})$.

In this paper, we need to work with a signal process that is independent conditioned on the true states $s \in S$. However, an immediate technical difficulty arises, which is the so-called measurability problem of independent processes (see, for example, [14] and its references). In our context, a signal process that is essentially independent, conditioned on the true states of nature may not be measurable at all; in fact, it follows from Proposition 1 of [13] that it is never jointly measurable in the usual sense except for trivial cases. Hence, we need to work with a joint agent-probability space $(I \times T, \mathcal{I} \boxtimes \mathcal{T}, \lambda \boxtimes P_s^T)$ that extends the usual measure-theoretic product $(I \times T, \mathcal{I} \otimes \mathcal{T}, \lambda \otimes P_s^T)$ of the agent space $(I, \mathcal{I}, \lambda)$ and the probability space $(T, \mathcal{T}, P_s^T)$, and retains the Fubini property.[10] Its formal definition is given in Definition 6 of the Appendix.

Let $\mathcal{I} \boxtimes \mathcal{F}$ be the collection of all subsets E of $I \times \Omega$ such that there are sets $A \in \mathcal{I} \boxtimes \mathcal{T}$, $C \in \mathcal{S}$ such that $E = \{(i, s, t) \in I \times \Omega : (i, t) \in A,\ s \in C\}$. By abusing the notation, we can denote E by $A \times C$ and $\mathcal{I} \boxtimes \mathcal{F}$ by $(\mathcal{I} \boxtimes \mathcal{T}) \otimes \mathcal{S}$. Define $\lambda \boxtimes P$ on $\mathcal{I} \boxtimes \mathcal{F}$ by letting $\lambda \boxtimes P(A \times C) =$

[6] In the literature, one usually assumes that different agents have possibly different sets of signals and require that the agents take all their own signals with positive probability. For notational simplicity, we choose to work with a common set T^0 of signals, but allow zero probability for some of the signals. There is no loss of generality in this latter approach.

[7] $\tilde{t}_i$ can also be viewed as a projection from T to T^0.

[8] Note that a conditional probability measure is uniquely defined up to a null set.

[9] For $q \in T^0$, $\tau_i(\{q\})$ is the probability $P(\tilde{t}_i = q)$.

[10] $\mathcal{I} \boxtimes \mathcal{T}$ is a σ-algebra that contains the usual product σ-algebra $\mathcal{I} \otimes \mathcal{T}$, and the restriction of the countably additive probability measure $\lambda \boxtimes P_s^T$ to $\mathcal{I} \otimes \mathcal{T}$ is $\lambda \otimes P_s^T$.

$\sum_{s\in C} \pi_s \lambda \boxtimes P_s^T(A)$. Thus, one can view $\lambda \boxtimes P_s^T$ as the conditional probability measure on $I \times T$, given $\tilde{s} = s$.

3. Economies with common values

3.1. The economic model

We shall now follow the definition and notation in Section 2. We consider an atomless economy with asymmetric information, which corresponds to the asymptotic replica economies considered in [9]. The common consumption set is the positive orthant $\mathbb{R}_+^m$. Let u be a function from $I \times \mathbb{R}_+^m \times S$ to $\mathbb{R}_+$ such that for any given $i \in I$, $u(i, x, s)$ is the utility of agent i at consumption bundle $x \in \mathbb{R}_+^m$ and true state $s \in S$.[11] For any given $s \in S$, assume that $u(i, x, s)$, (also denoted by $u_s(i, x)$),[12] is $\mathcal{I}$-measurable in $i \in I$, continuous and monotonic[13] in $x \in \mathbb{R}_+^m$. The utility of agent i does not depend on her or any other agents' signals. Let e be a λ-integrable function from I to $\mathbb{R}_+^m$ such that $\int_I e(i)\,d\lambda$ is in the strictly positive cone[14] $\mathbb{R}_{++}^m$, where $e(i)$ is the initial endowment of agent i. Let Δ_m be the unit simplex in $\mathbb{R}_+^m$.

For each $s \in S$, $\mathcal{E}_s^c = \{(I, \mathcal{I}, \lambda), u_s, e\}$ is a large deterministic economy. The collection $\mathcal{E}^c = \{\mathcal{E}_s^c : s \in S\}$ is called a *Complete Information Economy* (CIE). The following is a basic definition for the CIE.

Definition 1. 1. An *allocation* for the CIE is a function x^c from $I \times S$ to $\mathbb{R}_+^m$ such that for each $s \in S$, x_s^c is λ-integrable. Let $\mathcal{A}^c$ be the collection of all the allocations for the CIE.

2. A CIE allocation x^c is said to be *individually rational* if for each $s \in S$, x_s^c is individually rational in $\mathcal{E}_s^c$, i.e., for λ-almost $i \in I$, $u_s(i, x_s^c(i)) \geqslant u_s(i, e(i))$.

3. A CIE allocation x^c is *feasible* if for each $s \in S$, $\int_I x_s^c(i)\,d\lambda(i) = \int_I e(i)\,d\lambda(i)$, i.e., x_s^c is feasible in $\mathcal{E}_s^c$.

4. A feasible CIE allocation x^c is said to be *efficient* if for each $s \in S$, x_s^c is efficient in $\mathcal{E}_s^c$.[15]

5. A feasible CIE allocation x^c is said to be a *Walrasian allocation* (competitive equilibrium allocation) if for each $s \in S$, there is a price system $p_s \in \Delta_m$ such that (x_s^c, p_s) is a competitive equilibrium in $\mathcal{E}_s^c$.

6. A feasible CIE allocation x^c is said to be in the *core* of the CIE if for each $s \in S$, x_s^c is in the core of $\mathcal{E}_s^c$.

[11] We assume that the utility functions take non-negative values to avoid stating various integrability conditions explicitly. In fact, one can impose the condition of linear growth on the utilities to guarantee that the relevant expected utilities as used in this paper are finite. A real-valued function v on $\mathbb{R}_+^m$ is said to satisfy the condition of linear growth if there exist positive numbers α and β such that $v(x) \leqslant \alpha\|x\| + \beta$ for all $x \in \mathbb{R}_+^m$. When a continuous function v satisfies that condition, $v(y(\cdot))$ is integrable on $(T, \mathcal{T}, P^T)$ whenever $y(\cdot)$ is so. It is obvious that any concave function on $\mathbb{R}_+^m$ always satisfies the condition of linear growth.

[12] In the sequel, we shall often use subscripts to denote some variable of a function that is viewed as a parameter in a particular context.

[13] This means that if $x, y \in \mathbb{R}_+^m$, $x \geqslant y$ with $x \neq y$, then $u_s(i, x) > u_s(i, y)$.

[14] A vector x is in $\mathbb{R}_{++}^m$ if and only if all its components are positive.

[15] That is, there does not exist any other feasible allocation y_s in $\mathcal{E}_s^c$ such that $u_s(i, y_s(i)) > u_s(i, x_s^c(i))$ for λ-almost all $i \in I$. Note that the monotonicity assumption implies that the efficiency of x_s^c in $\mathcal{E}_s^c$ is equivalent to the non-existence of a feasible allocation y_s such that $u_s(i, y_s(i)) \geqslant u_s(i, x_s^c(i))$ for λ-almost all $i \in I$ with a strict inequality for a set of agents i with λ-positive measure.

In the CIE, the agents are informed with the true state. We shall now consider a corresponding Private Information Economy, where the agents are informed with their signals but not the true state. In this case, the agents will use the conditional probability measure $P^S(\cdot|t)$ on S to compute their expected utilities. For $t \in T$, the ex post utility $U_i(x|t)$ of agent i (also denoted by $U(i,x,t)$) for her consumption bundle $x \in \mathbb{R}^m_+$ with the given signal profile t is $\sum_{s\in S} u_i(x,s)P^S(\{s\}|t)$. It is obvious that for any fixed $x \in \mathbb{R}^m_+$, $U(i,x,t)$ is $\mathcal{I}\otimes\mathcal{T}$-measurable in $(i,t) \in I \times T$ and continuous in $x \in \mathbb{R}^m_+$. The collection $\mathcal{E}^p = \{(I \times \Omega, \mathcal{I}\boxtimes\mathcal{F}, \lambda\boxtimes P), u, e, (\tilde{t}_i, i \in I), \tilde{s}\}$ is called a *Private Information Economy* (PIE). For each fixed $t \in T$, $\mathcal{E}^p_t = \{(I, \mathcal{I}, \lambda), U(\cdot,\cdot,t), e\}$ is a large deterministic economy. In contrast to allocations in the CIE, where agents can trade contingently on the true states $s \in S$, allocations in the PIE depend on the agents' shared information, i.e., on the agents' announced signal profile $t \in T$. The following is an analog of Definition 1 in the setting of a PIE.

Definition 2. 1. An *allocation* for the PIE is an integrable function x^p from $(I \times T, \mathcal{I}\boxtimes\mathcal{T}, \lambda\boxtimes P^T)$ to $\mathbb{R}^m_+$. Let $\mathcal{A}^p$ be the collection of all the allocations for the PIE.

2. A PIE allocation x^p is said to be *ex post individually rational* if for P^T-almost all $t \in T$, x^p_t is individually rational in $\mathcal{E}^p_t$.

3. A PIE allocation x^p is *ex post feasible* if for P^T-almost all $t \in T$, x^p_t is feasible in $\mathcal{E}^p_t$.

4. A feasible PIE allocation x^p is said to be *ex post efficient* if for P^T-almost all $t \in T$, x^p_t is efficient in $\mathcal{E}^p_t$.

5. A feasible PIE allocation x^p is said to be an *ex post Walrasian allocation* (ex post competitive equilibrium allocation) if there is a measurable price function p from $(T, \mathcal{T})$ to Δ_m such that for P^T-almost all $t \in T$, (x^p_t, p_t) is a competitive equilibrium in $\mathcal{E}^p_t$.

6. A feasible PIE allocation x^p is said to be in the *ex post core* of the PIE if for P^T-almost all $t \in T$, x^p_t is in the core of $\mathcal{E}^p_t$.

In the PIE, each agent i is privately informed with her signal t_i. A major issue is whether the agent will have any incentive to mis-report that signal. The following definition of incentive compatibility is standard.

Definition 3. For a PIE allocation x^p, an agent $i \in I$, private signals $t_i, t'_i \in T^0$ for agent i, let

$$U_i(x^p_i, t'_i|t_i) = \int_{S\times T_{-i}} u_i(x^p_i(t_{-i}, t'_i), s)\, dP^{S\times T_{-i}}(\cdot|t_i),$$

which is the expected utility of agent i when she receives private signal t_i but mis-reports as t'_i. The PIE allocation x^p is said to be *incentive compatible* if λ-almost all $i \in I$,

$$U_i(x^p_i, t_i|t_i) \geqslant U_i(x^p_i, t'_i|t_i)$$

holds for τ_i-almost all $t_i, t'_i \in T^0$.

3.2. Perfect competition in a large economy with asymmetric information

The fundamental idea of perfect competition is that there are many economic agents, and that each individual agent has negligible influence in the market. Though each individual agent has non-negligible consumption in general, her share of consumption in the aggregate in terms of per capita consumption is negligible; that property can be guaranteed by using an atomless measure space as the space of agents.

When individuals have asymmetric information, a heuristic way to capture the idea of perfect competition is that the private signal of an individual agent can only influence a negligible set of agents, and moreover those signals associated with the individual agents that play a particular role in the model (for example, used in the utility functions or in calculating the aggregate signal distribution in some sense) are essentially independent of each other. The following definition formalizes this intuitive idea.

Definition 4. Let G^0 be a finite set $\{g_1, g_2, \ldots, g_M\}$, (with power set $\mathcal{G}^0$), and F be a measurable process from $(I \times T, \mathcal{I} \boxtimes \mathcal{T})$ to G^0. For agent $i \in I$, $F(i,t)$ is the derived signal of agent i from the signal profile t. The process F is called an *idiosyncratic signal process* if it has the following two properties.

(1) The process F is a signal process with negligible influence from private signals. That is, for λ-almost all $i \in I$, there is a set $A_i \in \mathcal{I}$ with $\lambda(A_i) = 1$ such that for any $t \in T$ and $t'_i \in T^0$, $F(j, (t_{-i}, t_i)) = F(j, (t_{-i}, t'_i))$ holds for each $j \in A_i$.

(2) The process F is essentially pairwise independent conditioned on $\tilde{s}$.

Condition (1) means that agent i's private signal t_i can only possibly influence the value of $F(j,t)$ for a null set of agents $j \in I - A_i$. Thus, whenever agent i mis-reports her private signal t_i has no effect on $F(j,t)$ for almost all agents $j \in I$. Condition (2) says that when a true state s is realized, agent i's derived signal $F(i,\cdot)$ is independent of agent j's derived signal $F(j,\cdot)$ for almost all agents $i, j \in I$. A formal definition of essential pairwise independence is given in Definition 7 of Appendix.

Notice that t_i is the private signal of agent i, and we simply call $F(i,t)$ her signal.

Note that the property in Definition 4(1) can also be defined for the case when I has finitely many agents and λ is the counting probability measure. Since any single agent is not negligible, the validity of (1) implies that for any $i, j \in I, t \in T$ and $t'_i \in T^0$, $F(j, (t_{-i}, t_i)) = F(j, (t_{-i}, t'_i))$, which implies that for any $i \in I$, $F(i,\cdot)$ is constant. Thus, in order for the property in (1) to be meaningful, one has to work in a model with an atomless measure space of agents.

Our idiosyncratic signal process is a general function of the agents' announcements satisfying the above two conditions. [16] We shall consider two special cases in the following two remarks; one involves only agents' private signals and the other replication of signals.

Remark 1. When $F(i,t) = t_i$ for all $(i,t) \in I \times T$, i.e. $F(i,\cdot)$ only takes agent i's private signal as its value, one can simply take $A_i = I \setminus \{i\}$ for any $i \in I$. Since λ is assumed to be atomless, any single agent is negligible, and hence $\lambda(A_i) = 1$. It is obvious that $F(j, (t_{-i}, t_i)) = t_j = F(j, (t_{-i}, t'_i))$ for $j \in A_i$. Therefore, F is a signal process with negligible influence from private signals. If, in addition, F is essentially pairwise independent conditioned on $\tilde{s}$, then F is an idiosyncratic signal process. Note that the property in Definition 4(1) can only guarantee that the private signal of an agent has negligible influence in the functional form. Some underlying correlations conditioned on $\tilde{s}$ may still exist in a non-trivial way. For example, one can construct P so that for a non-negligible set A of agents $i \in I$, $\tilde{t}_i, i \in A$ are correlated conditioned on $\tilde{s}$; then an individual agent may still have non-negligible influence. Thus, Condition (2) is needed.

[16] This kind of general function allows great flexibility and generality for interpreting our model; for example, one can base on the agents' announcements in a coalition. This point alone has already been considered to be a worthy topic for further research in the second paragraph in [9, p. 2440]; see also footnote 26 below.

Remark 2. (a) When a differential information economy with k agents is replicated as in [4], [9] and [10], the agents are divided into many cohorts of k agents and the signals within each cohort may be used in the utility functions or used for calculating the joint distributions within the cohort. As an analog in the continuum setting, we assume that the space of agents is in the form $(I, \mathcal{I}, \lambda) = (I^k \times I', \mathcal{I}^k \otimes \mathcal{I}', \lambda_k \otimes \lambda')$, where $I^k = \{1, 2, \ldots, k\}$, $\mathcal{I}^k$ the power set on I^k, λ_k the counting probability measure, and $(I', \mathcal{I}', \lambda')$ is an atomless probability space. For $i' \in I'$, the agents $(1, i'), (2, i'), \ldots, (k, i')$ are said to be in the same cohort. For an agent $i = (l, i') \in I^k \times I'$, $t \in T$, let $F((l, i'), t) = (t_{(1,i')}, t_{(2,i')}, \ldots, t_{(k,i')})$. Then, F is a process from $(I \times T, \mathcal{I} \boxtimes \mathcal{T})$ to $G^0 = (T^0)^{I^k}$ that takes the signals of the agents in the same cohort. For $i = (l, i') \in I^k \times I'$, let $A_i = I \setminus \{(1, i'), (2, i'), \ldots, (k, i')\}$; then, $j = (q, j') \in A_i$ implies that $j' \neq i'$, $F(j, (t_{-i}, t_i)) = (t_{(1,j')}, \ldots, t_{(k,j')}) = F(j, (t_{-i}, t_i'))$ for any $t \in T$ and $t_i' \in T^0$. Since finitely many agents are still negligible, $\lambda(A_i) = 1$, and F is a signal process with negligible influence from private signals.

(b) We can define another process F' from $(I' \times T, \mathcal{I}' \boxtimes \mathcal{T})$ to G^0 by letting $F(i', t) = (t_{(1,i')}, t_{(2,i')}, \ldots, t_{(k,i')})$ for $(i', t) \in I' \times T$. An analogous property to that of independent replicas is that for all $i', j' \in I'$ with $i' \neq j'$, $F'(i', \cdot)$ and $F'(j', \cdot)$ are independent with identical distributions conditioned on $\tilde{s}$. We do not need this strong condition in order for F to be an idiosyncratic signal process. It is easy to see that F is essentially pairwise independent conditioned on $\tilde{s}$ if and only if so is F'; thus, if F or F' has this property, then F is an idiosyncratic signal process.

When the true state is s, the signal distribution of agent i conditioned on the true state is $P_s^T F_i^{-1}$, i.e., the probability for agent i to have g_l as her signal is $P_s^T(F_i^{-1}(\{g_l\}))$ for each $1 \leqslant l \leqslant M$, where $F_i = F(i, \cdot)$. Let μ_s be the agents' *average signal distribution* conditioned on the true state s, i.e.,

$$\mu_s(\{g_l\}) = \int_I P_s^T(F_i^{-1}(\{g_l\}))\, d\lambda = \int_I \int_T 1_{\{g_l\}}(F(i, t))\, dP_s^T\, d\lambda,$$

where $1_{\{g_l\}}$ is the indicator function of the singleton set $\{g_l\}$. By the Fubini property for $(I \times T, \mathcal{I} \boxtimes \mathcal{T}, \lambda \boxtimes P_s^T)$, [17] μ_s is actually the distribution $(\lambda \boxtimes P_s^T) F^{-1}$ of F, viewed as a random variable on the product space $I \times T$. [18]

From now on, we shall impose the following *non-triviality assumption* on the process F:

$$\forall s, \quad s' \in S, \quad s \neq s' \Rightarrow \mu_s \neq \mu_{s'}. \tag{1}$$

This says that different true states of nature correspond to different average conditional distributions of agents' signals.

Next, we define the following sets

$$\forall s \in S, \quad L_s = \{t \in T : \lambda F_t^{-1} = \mu_s\}; \quad L_0 = T - \bigcup_{s \in S} L_s. \tag{2}$$

The non-triviality assumption implies that for any $s, s' \in S$ with $s \neq s'$, $L_s \cap L_{s'} = \emptyset$. The measurability of the sets $L_s, s \in S$ and L_0 follows from the measurability of F. Thus, the

[17] For a formal definition of the Fubini property, see Definition 6.

[18] As we will see in Appendix, under the assumption of essential pairwise conditional independence, the exact law of large numbers in [12,14] (see Lemma 2 in the Appendix) implies that $(\lambda \boxtimes P_s^T) F^{-1} = \lambda F_t^{-1}$ for P_s^T-almost all $t \in T$.

collection $\{L_0\} \cup \{L_s,\ s \in S\}$ forms a measurable partition of T. That partition will play a central role in later sections.[19]

3.3. Incentive compatibility and ex post efficient, Walrasian and core allocations

Define a mapping Φ from the set $\mathcal{A}^c$ of CIE allocations to the set $\mathcal{A}^p$ of PIE allocations as follows. For any CIE allocation $x^c \in \mathcal{A}^c$, define

$$\Phi(x^c)(i,t) = \begin{cases} e(i) & \text{if } t \in L_0, \\ x^c(i,s) & \text{if } t \in L_s,\ s \in S \end{cases} \tag{3}$$

for $(i,t) \in I \times T$. It is obvious that $\Phi(x^c)$ is integrable on $(I \times T, \mathcal{I} \otimes \mathcal{T}, \lambda \otimes P^T)$, (and thus integrable on the extension $(I \times T, \mathcal{I} \boxtimes \mathcal{T}, \lambda \boxtimes P^T)$), and consequently is a PIE allocation. This means that Φ is indeed a mapping from $\mathcal{A}^c$ to $\mathcal{A}^p$.

Theorem 1 below shows that Φ plays a central role between the two economies $\mathcal{E}^c$ and $\mathcal{E}^p$. In particular, under assumption that F is an idiosyncratic signal process, any CIE allocation x^c can be transformed to an incentive compatible PIE allocation $\Phi(x^c)$, and the ex post efficiency of $\Phi(x^c)$ is equivalent to the efficiency of x^c. The same type of equivalence also holds for core and Walrasian allocations. Thus, the existence of incentive compatible, ex post Walrasian, (and thus ex post individually rational and ex post efficient) allocations, follows from the usual existence result on Walrasian allocations, as in [6]. The theorem below is proved in Section A.3 of the Appendix.

Theorem 1. (1) *If F is a signal process with negligible influence from private signals, then the PIE allocation $\Phi(x^c)$ is always incentive compatible for any CIE allocation x^c.*

(2) *Assume that the process F is essentially pairwise independent conditioned on $\tilde{s}$. Let x^c be any CIE allocation. Then x^c is individually rational, or feasible, or efficient, or a Walrasian allocation, or a core allocation in the CIE if and only if $\Phi(x^c)$ has the corresponding ex post version of the property in the PIE.*[20] *In addition, we have*

$$\int_T u_i(\Phi(x^c)(i,t), s)\, dP_s^T(t) = u_i(x^c(i,s), s), \tag{4}$$

which means that the expected utility of $\Phi(x^c)(i,\cdot)$ conditioned on the true state s is always the utility of $x^c(i,s)$.

(3) *If F is an idiosyncratic signal process, then there exists an incentive compatible PIE allocation x^p that is an ex post Walrasian allocation (and thus ex post individually rational and ex post efficient).*

4. Economies with type dependent utility functions

Section 3 focuses on an atomless economy with asymmetric information, where no agents' types enter utility functions. In this section, we shall consider the more general case that allows agents' types to appear in the utility functions.

[19] As noted in footnote 18, under the condition of essential pairwise independence, the exact law of large numbers in [12,14] implies that $P_s^T(L_s) = 1$ for each $s \in S$.

[20] By the usual core equivalence theorem in [2,6], a PIE allocation is in the ex post core if and only if it is an ex post Walrasian allocation. Thus, core equivalence is still valid in this perfectly competitive framework.

We shall follow the definition and notation in Sections 2 and 3.2. The common consumption set is $\mathbb{R}^m_+$. Let v be a function from $I \times \mathbb{R}^m_+ \times S \times G^0$ to $\mathbb{R}_+$ such that for any given $i \in I$, $v(i, x, s, g)$ is the utility of agent i at consumption bundle x, true state s, and the agent's signal g. For any given $s \in S$, and $g \in G^0$, assume that $v(i, x, s, g)$ is $\mathcal{I}$-measurable in $i \in I$, continuous and monotonic in $x \in \mathbb{R}^m_+$. For given (s, t), let $u(i, x, s, t) = v(i, x, s, F(i, t))$. It can be easily checked that for any fixed $x \in \mathbb{R}^m_+$, $s \in S$, $u(i, x, s, t)$ is $\mathcal{I} \boxtimes \mathcal{T}$-measurable.[21] Let e be an integrable function from I to $\mathbb{R}^m_+$ with $\int_I e(i)\, d\lambda(i) \in \mathbb{R}^m_{++}$, where $e(i)$ is the initial endowment of agent i.

We now define a PIE, where the agents are informed with their signals but not the true state. The ex post utility $U_i(x|t)$ of agent i, (also denoted by $U(i, t, x)$), for the agent's consumption $x \in \mathbb{R}^m_+$ with the given signal profile t is $\sum_{s \in S} u(i, x, s, t) P^S(\{s\}|t)$. It is obvious that for any fixed $x \in \mathbb{R}^m_+$, $U(i, t, x)$ is $\mathcal{I} \boxtimes \mathcal{T}$-measurable. For each fixed $t \in T$, $\mathcal{E}^p_t = \{(I, \mathcal{I}, \lambda), U(\cdot, \cdot, t), e\}$ is a large deterministic economy. The collection $\mathcal{E}^p = \{(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P), u, e, F, (\tilde{t}_i, i \in I), \tilde{s}\}$ is called a PIE. Definition 2 is still applicable for the PIE in this section.

The following definition of incentive compatibility is the same as Definition 3 except that the utility functions u_i are now signal dependent.

Definition 5. For a PIE allocation x^p, an agent $i \in I$, a signal profile $t \in T$, and a signal $t_i' \in T^0$, let

$$U_i(x_i^p, t_i'|t_i) = \int_{S \times T_{-i}} u_i(x_i^p(t_{-i}, t_i'), s, (t_{-i}, t_i))\, dP^{S \times T_{-i}}(\cdot|t_i),$$

which is the expected utility of agent i when she receives private signal t_i but mis-reports as t_i'. The PIE allocation x^p is said to be *incentive compatible* if λ-almost $i \in I$,

$$U_i(x_i^p, t_i|t_i) \geqslant U_i(x_i^p, t_i'|t_i)$$

holds for τ_i-almost all $t_i, t_i' \in T^0$.

Now we present the following theorem that corresponds to the result in Part (3) of Theorem 1 (the proof is given in Section A.4 of the Appendix).

Theorem 2. *If F is an idiosyncratic signal process, then there exists an incentive compatible allocation x^p in the PIE such that x^p is an ex post Walrasian allocation* (*and thus ex post individually rational*, *and ex post efficient*).

5. Asymptotic interpretation

In this section, we translate Theorem 2 to an asymptotic setting. Fix $n \geqslant 1$. We shall first define the nth PIE $\mathcal{E}^p_n$. Let I^n be $\{1, 2, \ldots, n\}$ with the counting probability measure λ_n on its power set $\mathcal{I}^n$; $(I^n, \mathcal{I}^n, \lambda_n)$ represents the space of agents for the nth economy.[22] The sets S and T_0 have the same meanings as in Section 2, i.e., $S = \{s_1, s_2, \ldots, s_K\}$, (with power set $\mathcal{S}$), is the space of true

[21] For any fixed $x \in \mathbb{R}^m_+$, $s \in S$, and $r \in \mathbb{R}$, one can simply observe that

$$\{(i, t) \in I \times T : u(i, x, s, t) < r\} = \bigcup_{g \in G^0} \left[F^{-1}(\{g\}) \cap \{(i, t) \in I \times T : v(i, x, s, g) < r\} \right].$$

[22] We shall use both superscript and subscript n to index objects in the nth economy.

states that are not known to the agents; and $T^0 = \{q_1, q_2, \ldots, q_L\}$ is the space of all the possible private signals for individual agents.

Let $T^n = (T^0)^{I^n}$ be the space of all the functions from I^n to T^0 with its power set $\mathcal{T}^n$, and $(\Omega^n, \mathcal{F}^n)$ the product of $(S, \mathcal{S})$ and $(T^n, \mathcal{T}^n)$. Let $(\Omega^n, \mathcal{F}^n, P_n)$ be a probability space representing all the uncertainty in the nth economy, P_n^S and $P_n^{T^n}$ the marginal probability measures of P_n, respectively, on $(S, \mathcal{S})$ and on $(T^n, \mathcal{T}^n)$. Let $\tilde{s}^n$ be the projection mapping from Ω^n to S.

For $t^n \in T^n$, t^n is a function from I^n to T_0 representing a signal profile for all the agents in I^n, and $t^n(i^n)$, also denoted by $t^n_{i^n}$, the private signal received by agent $i^n \in I^n$. Let $\tilde{t}^n_{i^n}$ be the projection mapping from Ω^n to T^0 with $\tilde{t}^n_{i^n}(s, t^n) = t^n_{i^n}$. For $t^n \in T^n$ and $i^n \in I^n$, let $t^n_{-i^n}$ be the restriction of the signal profile t^n to the set $I^n \setminus \{i^n\}$; $T^n_{-i^n}$ denotes the set of all such $t^n_{-i^n}$.

Let $P_{ns}^{T^n}$ denote the conditional probability measure $P_n^{T^n}(\cdot|s)$ on $(T^n, \mathcal{T}^n)$ when the random variable $\tilde{s}^n$ takes value s. Let $P_n^S(\cdot|t^n)$ be the conditional probability measure on S given the signal profile $t^n \in T^n$. For $i^n \in I^n$, let $\tau^n_{i^n}$ be the signal distribution $P_n\left(\tilde{t}^n_{i^n}\right)^{-1}$ of agent i^n on the space T^0, and $P_n^{S\times T^n_{-i^n}}(\cdot|t^n_{i^n})$ the conditional probability measure on $S \times T^n_{-i^n}$ when the signal for agent i^n is $t^n_{i^n} \in T^0$.

Let F^n be a signal process from $(I^n \times T^n, \mathcal{I}^n \otimes \mathcal{T}^n)$ to a finite space $G^0 = \{g_1, g_2, \ldots, g_M\}$ of derived signals for the agents; $F^n(i^n, t^n)$ will enter the utility function of agent i^n. For each $i^n \in I^n$, let $F^n_{i^n}$ denote the function $F^n(i^n, \cdot)$ on T^n.

For each $s \in S$, let

$$\mu_s^n(\cdot) = \left(\lambda_n \otimes P_{ns}^{T^n}\right)\left(F^n\right)^{-1}(\cdot) = \frac{1}{n}\sum_{i^n=1}^{n} P_{ns}^{T^n}\left(\left(F^n_{i^n}\right)^{-1}(\cdot)\right),$$

which is the average signal distribution conditioned on the true state s. Assume that there exists a positive number δ_0 such that for each $n \geqslant 1$,

$$\forall s \in S, \quad \pi_s^n = P_n^S(\{s\}) \geqslant \delta_0; \quad \forall s, s' \in S, \quad s \neq s' \Rightarrow \|\mu_s^n - \mu_{s'}^n\| \geqslant \delta_0, \tag{5}$$

where μ_s^n and $\mu_{s'}^n$ are viewed as points in the unit simplex Δ_K in $\mathbb{R}^K$, and $\|\mu_s^n - \mu_{s'}^n\|$ is their Euclidean distance. This condition corresponds to the non-triviality assumption on the measures P^S and μ_s, $s \in S$ in the limit model in Section 3.

We shall now define the utilities and endowments for the nth PIE. For simplicity, we take a compact subset $\mathcal{U}_0$ of the space $\mathcal{U}(\mathbb{R}^m_+)$ of non-negative continuous and monotonic functions on $\mathbb{R}^m_+$ satisfying the condition of linear growth [23] that is endowed with the supnorm topology, and a compact subset E_0 of $\mathbb{R}^m_{++}$. Let v^n and e^n be mappings, [24] respectively, from $I^n \times S \times G^0$ to $\mathcal{U}_0$ and from I^n to E_0, where $v^n(i^n, s, g)(\cdot)$ is the utility function of agent i^n at true state $s \in S$ and her signal g, and $e^n(i^n)$ the initial endowment of agent i^n. For $t^n \in T^n$, the ex post utility $U^n_{i^n}(x|t^n)$ of agent i^n, (also denoted by $U^n(i^n, x, t^n)$), for her consumption bundle $x \in \mathbb{R}^m_+$ with the given signal profile t^n is $\sum_{s\in S} u^n_{i^n}(x, s, t^n)P_n^S(\{s\}|t^n)$, where $u^n_{i^n}(x, s, t^n) = v^n(i^n, s, F^n(i^n, t^n))(x)$.

The nth *PIE* is simply the collection $\mathcal{E}_n^p = \{(I^n \times \Omega^n, \mathcal{I}^n \otimes \mathcal{F}^n, \lambda_n \otimes P_n), u^n, e^n, (\tilde{t}^n_{i^n}, i^n \in I^n), \tilde{s}^n\}$. An allocation for $\mathcal{E}_n^p$ is a function from $(I^n \times T^n, \mathcal{I}^n \otimes \mathcal{T}^n, \lambda_n \otimes P_n^{T^n})$ to $\mathbb{R}^m_+$.

[23] The purpose of including the condition of linear growth as defined in footnote 11 is to guarantee the relevant expected utilities in the limiting case to have finite values. Note that the utility functions in [4] are assumed to be bounded (p. 1277).

[24] The compactness assumption on both $\mathcal{U}_0$ and E_0 can be relaxed, respectively, to a tightness condition on the induced distribution of u^n on $\mathcal{U}(\mathbb{R}^m_+)$ and to a uniform integrability condition on e^n.

For an allocation x_n^p of the PIE $\mathcal{E}_n^p$, an agent $i^n \in I^n$, private signals $t_{i^n}^n, (t_{i^n}^n)' \in T^0$ for agent i^n, let

$$U_{i^n}^n\left((x_n^p)_{i^n}, (t_{i^n}^n)'|t_{i^n}^n\right) = \int_{S\times T_{-i^n}^n} u_{i^n}^n\left(x_n^p(i^n, (t_{-i^n}^n, (t_{i^n}^n)'), s, t^n)\right) dP_n^{S\times T_{-i^n}^n}(\cdot|t_{i^n}^n),$$

which is the expected utility of agent i^n when she receives private signal $t_{i^n}^n$ but mis-reports as $(t_{i^n}^n)'$.

The following theorem is an asymptotic analog of Theorem 2. The proof can be found in [15].

Theorem 3. *For the sequence $\mathcal{E}_n^p$, $n\geqslant 1$ of PIEs, assume that the signal processes F^n, $n\geqslant 1$ are asymptotically idiosyncratic in the sense that for any $\delta > 0$, and $s \in S$, both of the following sequences converge to one as n goes to infinity*[25]:

$$\begin{aligned}&\lambda_n\otimes\lambda_n\left(\left\{(i^n, j^n)\in I^n\times I^n : \|P_{ns}^{T^n}\left(F_{i^n}^n, F_{j^n}^n\right)^{-1} - P_{ns}^{T^n}\left(F_{i^n}^n\right)^{-1}\right.\right.\\&\qquad\left.\left.\otimes P_{ns}^{T^n}\left(F_{j^n}^n\right)^{-1}\|\leqslant\delta\right\}\right),\end{aligned}\tag{6}$$

$$\begin{aligned}&\lambda_n\otimes\lambda_n\left(\left\{(i^n, j^n)\in I^n\times I^n : \forall t^n\in T^n,\ (t_{i^n}^n)'\in T^0,\ F^n(j^n, t^n)\right.\right.\\&\qquad\left.\left.= F^n(j^n, (t_{-i^n}^n, (t_{i^n}^n)'))\right\}\right).\end{aligned}\tag{7}$$

Then for any given $\varepsilon > 0$, there is a positive integer N such that for any $n > N$, there exists an allocation x_n^p for the PIE $\mathcal{E}_n^p$, a price function p^n from T^n to the price simplex Δ_m, and sets $B_n \subseteq I^n$ and $C^n \subseteq T^n$ with $\lambda_n(B^n) > 1-\varepsilon$ and $P_n^{T^n}(C^n) > 1-\varepsilon$ satisfying the following properties.

(a) *For each $i^n \in B_n$,*

$$U_{i^n}^n\left((x_n^p)_{i^n}, t_{i^n}^n|t_{i^n}^n\right) + \varepsilon \geqslant U_{i^n}^n\left((x_n^p)_{i^n}, (t_{i^n}^n)'|t_{i^n}^n\right)$$

holds for all $t_{i^n}^n, (t_{i^n}^n)' \in T^0$ with $\tau_{i^n}^n(\{t_{i^n}^n\})\geqslant\varepsilon$ and $\tau_{i^n}^n(\{(t_{i^n}^n)'\})\geqslant\varepsilon$.

(b) *For any $(i^n, t^n) \in I^n \times T^n$, $p_n(t^n)x_n^p(i^n, t^n) = p_n(t^n)e^n(i^n)$.*

(c) *For all $t^n \in C^n$,*

  (i) $\left\|\int_{I^n} x_n^p(i^n, t^n)\, d\lambda_n - \int_{I^n} e^n(i^n)\, d\lambda_n(i^n)\right\| \leqslant\varepsilon$;

  (ii) $\lambda_n\left(\left\{i^n \in I^n : \forall y \in \mathbb{R}_+^m,\ p_n(t^n)y \leqslant p_n(t^n)e^n(i^n) \Rightarrow U_{i^n}^n(x_n^p(i^n, t^n)|t^n) + \varepsilon \geqslant U_{i^n}^n(y|t^n)\ \right\}\right) \geqslant 1-\varepsilon$;

  (iii) $\lambda_n(\{i^n \in I^n : U_{i^n}^n(x_n^p(i^n, t^n)|t^n) + \varepsilon \geqslant U_{i^n}^n(e^n(i^n)|t^n)\}) \geqslant 1-\varepsilon$;

  (iv) *there does not exist an allocation y_n from I^n to $\mathbb{R}_+^m$ such that $\int_{I^n} y_n(i^n)\, d\lambda_n(i^n) = \int_{I^n} e^n(i^n)\, d\lambda_n(i^n)$, and for all $i^n \in I^n$, $U_{i^n}^n(y_n(i^n)|t^n) > U_{i^n}^n(x_n^p(i^n, t^n)|t^n) + \varepsilon$.*

Theorem 3(a) says that the PIE allocation x_n^p for $\mathcal{E}_n^p$ is approximately incentive compatible; (b)(i) and (ii) of (c) mean that x_n^p is an approximate ex post Walrasian allocation; (iii) and (iv) of (c) show that x_n^p is both ex post individually rational and ex post efficient in an approximate sense. Since the common value model can be regarded as a special case of the model with type

[25] In this paper, $\|\cdot\|$ denotes the Euclidean norm.

dependent utilities, exactly the same result in Theorem 3 also provides an asymptotic version of Part (3) of Theorem 1.

6. Relationship to Gul–Postlewaite [4] and McLean–Postlewaite [9]

One needs to choose a mathematically meaningful model for an atomless measure space of agents who act independently *conditioned* on true states of nature. This has not appeared anywhere before. The well-known measurability problem associated with a continuum of independent random variables are automatically resolved in our mathematical model. This is important since we use the distribution of the realized signals to determine the true states; if the relevant sample functions are not known to be measurable in a general analytic framework, this would become meaningless.

Our notion of negligible private information is new. In particular, as it is explained in Section 3.2, the private signal of an individual can only influence a negligible set of agents (through their utilities and/or through computing the realized signal distributions) and furthermore those signals associated with the individual agents are essentially pairwise independent. In contrast to the replica models in [4,9], where all the private announcements in a cohort with a fixed number n of agents influence everyone in the cohort, our formulation is based on a general function of the private announcements for all the agents, which allows great flexibility and generality for interpreting our model. While the formulation of a replica model simply relies on a fixed finite-agent economy, our notion of negligible influence of private signals is reduced to complete triviality for a fixed finite economy, as noted in Section 3.2. Thus, our new formulation has no meaningful exact analog for a fixed finite economy. In addition, an individual agent can have correlation with a negligible set (which can be infinite) of other agents conditioned on the true states while the replica models only do so for agents in the same cohort with a fixed number of n agents.

Our asymptotic results in Theorem 3 are related to the approximate results in Gul–Postlewaite [4], Krasa–Shafer [7], McLean–Postlewaite [9] and Palfrey–Srivastava [10]. In particular, Palfrey–Srivastava [10] discussed the idea of "information smallness", and showed that when the economy is replicated independently, the incentive for an agent to manipulate her private information goes to zero.

Since our asymptotic results in Theorem 3 are closely related to those of Gul–Postlewaite [4], and McLean–Postlewaite [9], we make detailed comparisons below. While we consider here a very general sequence of large, but finite economies with possibly non-concave, type-dependent utilities, a special sequence of replica economies with strictly concave, type-dependent utilities was considered in [4]. Note that the model in [4] relies on a regularity condition that requires the demands of two different types for an agent to be never identical for all prices in some open ball for every realization of the relevant uncertainty. It is not clear what type of utility functions will produce this regularity condition. Since the regularity condition in [4] requires that an individual agent's utility cannot be independent of her type, the independent replica model with type independent utilities (common values) as considered in [9] is thus ruled out in the model of [4]. We do not need the type of regularity condition in this paper, which means that the basic common value model can indeed be regarded as a special case for the more general models with type dependent utilities.

Next, we compare the restrictions on the information structures. In the independent replica model in [4], the private signal of an individual agent has influence over a fixed number of k agents in the same cohort, and the discrete parameter process that takes the signals of all the k

agents in the relevant cohort as its values are mutually independent and identically distributed (iid) conditioned on the true states. In comparison, our asymptotically idiosyncratic signal processes F^n are general functions of the agents' announcements that satisfy Eqs. (6) and (7).[26] It is more general than the replica case in two aspects. First, it allows the private signal of an individual agent to influence a small corner of the market without a fixed bound k (in particular, any fixed finitely many agents in the large finite markets). Second, the signal process is not assumed to be iid but with the much more general condition of approximate pairwise independence conditioned on the true states. In addition, we formulate the non-triviality condition in Eq. (5) in terms of non-negligible distances between the average signal distributions while [4, p. 1277, 9, p. 2434] assumed that the conditional probabilities on the signal space were different with different given true states.

Since our asymptotic model is much more general than the independent replica models considered in [4,9], our conclusion in Theorem 3 is less sharp than the corresponding conclusions in [4,9] in the sense that approximate incentive compatibility is used here.[27] Furthermore, [4,9] have no limit economy for the independent replica models. Therefore, our limit results in Theorems 1 and 2 for the atomless economy are not directly comparable with the replica results in [4,9].[28] We simply say that our Theorems 1 and 2 present both exact incentive compatibility and exact ex post efficiency in the limit model with an information structure based on a general function of the agents' private announcements and with possibly non-concave utility functions. In contrast, the countable replica model in [4,9] is based on the announcements from the same cohort of n agents and obtains exact incentive compatibility but approximate ex post efficiency for strictly concave (or concave) utility functions with a special regularity condition in [4].

Finally, we compare our proofs with those in [4,9]. The proofs for the approximate results in [4,9] require intricate and ingenious computations. In comparison, the proofs of our exact results are simple and transparent in measure-theoretic terms. In order to obtain exactly incentive compatible and approximately ex post efficient allocations, [9] requires the notion of "variability of beliefs". We do not need this condition for proving our limit results on the existence of incentive compatible and ex post efficient allocations in the exact sense. The main technical tool in [9] is the classical law of large numbers for a sequence of iid random variables, which allows the agents to use their announcements to estimate the true states approximately. As an analog, we use the exact law of large numbers for a continuum of (conditionally) independent random variables for the agents to identify the true states with probability one. Also, the proof of our asymptotic results in Theorem 3 is based on our exact results (see [15]). Given the fact that the special replica case as considered in [4,9] is already rather difficult to prove, we believe that it would be very difficult to prove our asymptotic results using finite approximations directly, without using the corresponding

[26] Our idiosyncratic signal process in the limit case is a general function satisfying the two conditions in Definition 4. In particular, it can use just the announcements from a coalition A of agents by letting $F_i(t) = t_i$ for $i \in A$ and $F_i(t) = g_0$ for $i \notin A$, where g_0 is a point in G^0 (for this case, we take $G^0 = \{g_0\} \cup T^0$); F is an idiosyncratic signal process when the private signals for agents in coalition A are essentially independent conditioned on the true states (see the suggestion of [9, p. 2440]). See also footnote 16.

[27] Note that approximate incentive compatibility is also used in [10].

[28] As in the classical literature, the Debreu–Scarf replica economy with convex preferences, the Aumann economy with a measure space of agents and non-convex preferences, and the Kannai and Hildenbrand approximation of the Aumann limit economy for a general sequence of large but finite economies belong to three different classes of models; see [6] for details.

exact results. Thus, our Theorem 2 not only provides the exact results but also plays a key role in proving the asymptotic results in our Theorem 3.

7. Concluding remarks

This paper shows that the introduction of a suitable mathematical model to capture the meaning of perfect competition in a differential information economy has useful consequences. In particular, not only for the first time we model the idea of information negligibility in a differential information economy, and therefore generalize the Aumann model, but also we resolve exactly the incompatibility of incentive compatibility and Pareto efficiency. Furthermore, our results for the limit economies guarantee the corresponding asymptotic results for large but finite economies, and also a number of assumptions needed in [4,9] can be dispensed with in our general setting.

Appendix A.

A.1. The exact law of large numbers

In order to work with independent processes constructed from signal profiles, we need to work with an extension of the usual measure-theoretic product having the Fubini property. Below is a formal definition of the Fubini extension in Definition 2.2 of [14].

Definition 6. Let $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ be probability spaces. A probability space $(I \times \Omega, \mathcal{W}, Q)$ extending the usual product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ is said to be a *Fubini extension* of $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ if for any real-valued Q-integrable function f on $(I \times \Omega, \mathcal{W})$,

(1) The two functions f_i and f_ω are integrable respectively on $(\Omega, \mathcal{F}, P)$ for λ-almost all $i \in I$, and on $(I, \mathcal{I}, \lambda)$ for P-almost all $\omega \in \Omega$;

(2) $\int_\Omega f_i \, dP$ and $\int_I f_\omega \, dP$ are integrable respectively on $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$, with $\int_{I\times\Omega} f \, dQ = \int_I \left(\int_\Omega f_i \, dP \right) d\lambda = \int_\Omega \left(\int_I f_\omega \, d\lambda \right) dP$.[29]

To reflect the fact that the probability space $(I \times \Omega, \mathcal{W}, Q)$ has $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ as its marginal spaces, as required by the Fubini property, it will be denoted by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

The following is an exact law of large numbers for a continuum of independent random variables shown in [12,14].[30]

Lemma 1. *Let g be a measurable process from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to a complete separable metric space X. If the process g is essentially pairwise independent in the sense that for λ-almost all $i \in I$, the random variables g_i and g_j are independent for λ-almost all $j \in I$, then for P-almost all $\omega \in \Omega$, the cross-sectional distribution λg_ω^{-1} of the sample function g_ω*

[29] The classical Fubini Theorem is only stated for the usual product measure spaces. It does not apply to integrable functions on $(I \times \Omega, \mathcal{W}, Q)$ since these functions may not be $\mathcal{I} \otimes \mathcal{F}$-measurable. However, the conclusions of that theorem do hold for processes on the enriched product space $(I \times \Omega, \mathcal{W}, Q)$ that extends the usual product.

[30] This result was originally stated on the Loeb measure spaces in [12] (Theorem 5.2). However, it is noted in [14] that the result can be proved for an extension of the usual product with the Fubini property (Corollary 2.9).

is the same as the distribution $(\lambda \boxtimes P)g^{-1}$ *of the process* g *viewed as a random variable on* $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

We shall now follow the notation of Section 2. When the probability space $(I \times T, \mathcal{I} \boxtimes \mathcal{T}, \lambda \boxtimes P_s^T)$ is a Fubini extension of the usual product space $(I \times T, \mathcal{I} \otimes \mathcal{T}, \lambda \otimes P_s^T)$, for each $s \in S$, it can be checked that $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, defined in the last paragraph of Section 2, is a Fubini extension of the usual product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$. Below is a definition of conditional independence.

Definition 7. Let G be a $\mathcal{I} \boxtimes \mathcal{T}$-measurable process from $I \times T$ to a complete separable metric space X. It is said to be essentially pairwise independent conditioned on the true state random variable $\tilde{s}$ if for each $s \in S$, the process G from $(I \times T, \mathcal{I} \boxtimes \mathcal{T}, \lambda \boxtimes P_s^T)$ to X is essentially pairwise independent.

For the convenience of the reader, we restate Lemma 1 to the setting of conditional independence using our notation.

Lemma 2. *If a process* G *from* $I \times T$ *to a complete separable metric space* X *is essentially pairwise independent conditioned on* $\tilde{s}$, *then for each* $s \in S$, *the cross-sectional distribution* λG_t^{-1} *of the sample function* $G_t(\cdot) = G(t, \cdot)$ *is the same as the distribution* $(\lambda \boxtimes P_s^T)G^{-1}$ *of the process* G *viewed as a random variable on* $(I \times T, \mathcal{I} \boxtimes \mathcal{T}, \lambda \boxtimes P_s^T)$ *for* P_s^T*-almost all* $t \in T$.

A.2. The conditional probability $P^S(\cdot|t), t \in T$

Let δ_s be the Dirac measure on S that gives probability one to the point s and zero to other points. Define a function H from T to the space of probability measures on S by letting

$$H(t) = \begin{cases} \delta_s & \text{for } t \in L_s,\ s \in S, \\ \delta_{s_1} & \text{for } t \in L_0. \end{cases}$$

Lemma 3 below shows that the conditional independence of the signal process F together with the associated exact law of large numbers as shown in [12,14] implies that $P^T(\bigcup_{s' \in S} L_{s'}) = 1$, and $H(t), t \in T$ is a version of the conditional probability $P^S(\cdot|t), t \in T$. Note that using H as the conditional probability $P^S(\cdot|t), t \in T$ means the following. When all the signals are reported by the agents to form a signal profile t, the agents will be able to determine the true state to be s if the cross-sectional signal distribution λF_t^{-1} is observed to be μ_s.

Lemma 3. *If* F *is essentially pairwise independent conditioned on* $\tilde{s}$, *then* $P^T(\bigcup_{s' \in S} L_{s'}) = 1$, *and* $H(t), t \in T$ *is a version of* $P^S(\cdot|t), t \in T$.

Proof. The exact law of large numbers as stated in Lemma 2 says that the set $L_s = \{t \in T : \lambda F_t^{-1} = \mu_s\}$ has P_s^T-probability one. Thus, $P_s^T(L_{s'}) = 0$ for $s \neq s' \in S$, $P_s^T(\bigcup_{s' \in S} L_{s'}) = 1$, and $P^T(\bigcup_{s' \in S} L_{s'}) = \sum_{s \in S} \pi_s P_s^T(\bigcup_{s' \in S} L_{s'}) = 1$. Hence, P^T is a convex combination of mutually singular probability measures $P_s^T, s \in S$.

Fix any $s' \in S$, $B \in \mathcal{T}$. Since $B \setminus L_{s'}$ is a subset of $T \setminus L_{s'}$, which is a $P_{s'}^T$-null set, we have $P_{s'}^T(B \setminus L_{s'}) = 0$. Thus, $P_{s'}^T(B \cap L_{s'}) = P_{s'}^T(B) - P_{s'}^T(B \setminus L_{s'}) = P_{s'}^T(B)$. Hence, for any $s \in S$, we have the following identities

$$\begin{aligned}\int_B H(t)(\{s\})\,dP^T(t) &= \sum_{s'\in S} \pi_{s'} \int_B H(t)(\{s\})\,dP^T_{s'}(t)\\ &= \sum_{s'\in S} \pi_{s'} \int_{B\cap L_{s'}} H(t)(\{s\})\,dP^T_{s'}(t)\\ &= \sum_{s'\in S} \pi_{s'} \int_{B\cap L_{s'}} \delta_{s'}(\{s\})\,dP^T_{s'}(t) = \pi_s P^T_s(B\cap L_s) = \pi_s P^T_s(B)\\ &= P(\{s\}\times B) = \int_B P^S(\{s\}|t)\,dP^T_s(t) \qquad (8)\end{aligned}$$

which implies that $H(t)$, $t \in T$ is indeed a version of $P^S(\cdot|t)$, $t \in T$, by the arbitrary choices of $s \in S$ and $B \in \mathcal{T}$.

A.3. Proof of Theorem 1

(1) By Definition 4(1), we know that there is a set $A^* \in \mathcal{I}$ with $\lambda(A^*) = 1$ such that for any $i \in A^*$, there is a set $A_i \in \mathcal{I}$ with $\lambda(A_i) = 1$ such that for any $t \in T$ and $t'_i \in T^0$, the sample functions $F_{(t_{-i},t_i)}(\cdot)$ and $F_{(t_{-i},t'_i)}(\cdot)$ agree on A_i. Since the society's signal distribution cannot be influenced by a negligible set of agents outside the set A_i, we have $\lambda F^{-1}_{(t_{-i},t_i)} = \lambda F^{-1}_{(t_{-i},t'_i)}$. This means that for any $i \in A^*$, $t \in T$, $t'_i \in T^0$, and $s \in S$,

$$t \in L_s \Leftrightarrow \lambda F_t^{-1} = \mu_s \Leftrightarrow \lambda F^{-1}_{(t_{-i},t'_i)} = \mu_s \Leftrightarrow (t_{-i}, t'_i) \in L_s. \qquad (9)$$

Since L_0 is $T \setminus \bigcup_{s\in S} L_s$, we also know that $t \in L_0 \Leftrightarrow (t_{-i}, t'_i) \in L_0$.

Denote $\Phi(x^c)$ by x^p. We have for any $i \in A^*$, $x^p(i,t) = x^p(i,(t_{-i},t'_i))$ for any $t \in T$ and $t'_i \in T^0$. Therefore, the condition of incentive compatibility in Definition 3 is satisfied by x^p. Thus, part (1) is shown.

(2) Assume that F is essentially pairwise independent conditioned on $\tilde{s}$. The exact law of large numbers as stated in Lemma 2 says that the set $L_s = \{t \in T : \lambda F_t^{-1} = \mu_s\}$ has P^T_s-probability one. Since, for any $s \in S$, $i \in I$, one always has $x^p(i,t) = \Phi(x^c)(i,t) = x^c(i,s)$ for $t \in L_s$, and then by integration, we obtain that

$$\int_T u_i(\Phi(x^c)(i,t), s)\,dP^T_s(t) = \int_{L_s} u_i(\Phi(x^c)(i,t), s)\,dP^T_s(t) = u_i(x^c(i,s), s),$$

which is Eq. (4).

Lemma 3 says that H is a version of the conditional probability $P^S(\cdot|t)$. Thus, for any version of the conditional probability $P^S(\cdot|t)$, we always have $P^S(\cdot|t) = \delta_s$ for P^T-almost all $t \in L_s$. Hence, for P^T-almost all $t \in L_s$,

$$U(i,\cdot,t) = \sum_{s'\in S} u(i,\cdot,s')P^S(\{s'\}|t) = \sum_{s'\in S} u(i,\cdot,s')\delta_s(\{s'\}) = u(i,\cdot,s)$$

for all $i \in I$,[31] and $\mathcal{E}^p_t = \mathcal{E}^c_s$. Let $B_s = \{t \in L_s : \mathcal{E}^p_t = \mathcal{E}^c_s\}$; then $P^T(L_s \setminus B_s) = 0$. Hence, for any $t \in B_s$, the ex post economy-allocation pair $(\mathcal{E}^p_t, x^p(\cdot,t))$ is exactly the same as the complete

[31] Here we note that the Dirac measure δ_s has probability one at the point s and zero at those points $s' \in S$ with $s' \neq s$.

information economy-allocation pair $(\mathcal{E}_s^c, x^c(\cdot, s))$, which means that they must have the same properties.

Thus, if x^c is efficient, then for each fixed $s \in S$, $x^c(\cdot, s)$ is efficient for $\mathcal{E}_s^c$. This means that $x^p(\cdot, t)$ is efficient for $\mathcal{E}_t^p$ for any $t \in B_s$. Since $P^T(L_s \setminus B_s) = 0$ for each $s \in S$, $P^T(\bigcup_{s\in S} B_s) = P^T(\bigcup_{s\in S} L_s) = 1$. Hence $x^p(\cdot, t)$ is efficient for $\mathcal{E}_t^p$ for P^T-almost all $t \in T$. Hence x^p is ex post efficient.

For the other direction, fix any $s \in S$. If x^p is ex post efficient, then $x^p(\cdot, t)$ is efficient for $\mathcal{E}_t^p$ for P^T-almost all $t \in T$, and in particular for P^T-almost all $t \in L_s$. Since $P^T(B_s) = P^T(L_s) = \sum_{s'\in S} \pi_{s'} P_{s'}^T(L_s) = \pi_s P_s^T(L_s) = \pi_s > 0$, we can certainly find a $t \in B_s$ such that $x^p(\cdot, t)$ is efficient for $\mathcal{E}_t^p$. For such a $t \in B_s$, since the economy-allocation pairs $(\mathcal{E}_t^p, x^p(\cdot, t))$ and $(\mathcal{E}_s^c, x^c(\cdot, s))$ are the same, we obtain that $x^c(\cdot, s)$ is efficient for $\mathcal{E}_s^c$. Since s is arbitrarily chosen in S, we know that x^c is efficient.

The rest of the proof for part (2) follows clearly from the definition of each of the properties in Definition 1 and their ex post versions in Definition 2 by using the argument adopted for the proof of efficiency.

(3) By the usual existence result on Walrasian allocations in [6], there exists an allocation x^c that is a Walrasian allocation for the CIE. By parts (1) and (2) of this theorem, the PIE allocation $x^p = \Phi(x^c)$ is an incentive compatible, ex post Walrasian allocation, which is obviously also individually rational and ex post efficient. Hence part (3) follows.

A.4. Proof of Theorem 2

To make those variables with given parameters in the particular context clear, we use subscripts extensively in this section. Fix $s \in S$. Define a mapping Γ from $I \times T$ to $I \times G^0$ by letting $\Gamma(i, t) = (i, F(i, t))$ for all $(i, t) \in I \times T$. Then, Γ is a measurable mapping in the sense that for any measurable set $D \in \mathcal{I} \otimes \mathcal{G}^0$, $\Gamma^{-1}(D)$ is measurable in $\mathcal{I} \boxtimes \mathcal{T}$. Let v_s be the induced measure on $I \times G^0$ of the measure $\lambda \boxtimes P_s^T$ under Γ, i.e., for any $D \in \mathcal{I} \otimes \mathcal{G}^0$, $v_s(D) = \left(\lambda \boxtimes P_s^T\right)\left(\Gamma^{-1}(D)\right)$.

Define a large deterministic economy $\bar{\mathcal{E}}_s = \{(I \times G^0, \mathcal{I} \otimes \mathcal{G}^0, v_s), V_s, e\}$, where the utility function for agent $(i, g) \in I \times G^0$ is $V_s(i, g, \cdot) = v(i, \cdot, s, g)$ and the initial endowment for agent (i, g) is $e(i)$. By the usual existence result on Walrasian allocations in [6], there is a Walrasian allocation y_s with a strictly positive price system p_s for the economy $\bar{\mathcal{E}}_s$. By modifying the values of y_s on a null set (if necessary), we can assume that for every agent $(i, g) \in I \times G^0$, $y_s(i, g)$ is a maximal element in her budget set.

Consider another large deterministic economy $\mathcal{E}_s = \{(I \times T, \mathcal{I} \boxtimes \mathcal{T}, \lambda \boxtimes P_s^T), u_s, e\}$, where the utility function for agent $(i, t) \in I \times T$ is $u_s(i, t, \cdot) = u(i, \cdot, s, t)$ and the initial endowment for agent (i, t) is $e(i)$. Define a mapping x_s from $I \times T$ to $\mathbb{R}_+^m$ by letting $x_s(i, t) = y_s(\Gamma(i, t)) = y_s(i, F(i, t))$. Since y_s is a feasible allocation for $\bar{\mathcal{E}}_s$, it is obvious that x_s is a feasible allocation for $\mathcal{E}_s$. It is also clear that for each agent $(i, t) \in I \times T$, $x_s(i, t)$ is a maximal element in her budget set under the price system p_s. Therefore, x_s is a Walrasian allocation with a strictly positive price system p_s for the economy $\mathcal{E}_s$.

For each $i \in I$, $t, t' \in T$, agents (i, t) and (i, t') have the same endowment, and consequently the same budget set. Hence the utility of agent (i, t) at $x_s(i, t)$ is greater than or equal to her utility at $x_s(i, t')$ since $x_s(i, t')$ belongs to the budget set of agent (i, t). This means that

$$\forall i \in I, \quad s \in S, \quad t, t' \in T, \quad u(i, x_s(i, t), s, t) \geqslant u(i, x_s(i, t'), s, t). \tag{10}$$

As in Eq. (3), define a mapping x^p from $I \times T$ to $\mathbb{R}_+^m$ by letting

$$x^p(i,t) = \begin{cases} e(i) & \text{if } t \in L_0, \\ x_s(i,t) & \text{if } t \in L_s,\ s \in S \end{cases}$$

for $(i,t) \in I \times T$. It is obvious that x^p is integrable on $(I \times T, \mathcal{I} \boxtimes \mathcal{T}, \lambda \boxtimes P^T)$, and hence a PIE allocation.

By the same argument in the first paragraph of the proof of Theorem 1 in Section A.3, we obtain that for λ-almost all $i \in I$, for any $t \in T$, $t_i' \in T^0$ and $s \in S$, $t \in L_s$ if and only if $(t_{-i}, t_i') \in L_s$. Thus, for λ-almost all $i \in I$, for any $t \in L_s$ and $t_i' \in T^0$, we have

$$\begin{aligned} u_i(x_i^p(t_{-i}, t_i), s, (t_{-i}, t_i)) &= u_i(x_s(i,t), s, t) \\ &\geqslant u_i(x_s(i, (t_{-i}, t_i')), s, t) = u_i(x_i^p(t_{-i}, t_i'), s, (t_{-i}, t_i)), \end{aligned}$$

when $t \in L_0$, we also have $(t_{-i}, t_i') \in L_0$; hence

$$u_i(x_i^p(t_{-i}, t_i), s, (t_{-i}, t_i)) = u_i(e(i), s, t) = u_i(x_i^p(t_{-i}, t_i'), s, (t_{-i}, t_i)).$$

Thus, for λ-almost all $i \in I$, we have $U_i(x_i^p, t_i | t_i) \geqslant U_i(x_i^p, t_i' | t_i)$ for τ_i-almost all $t_i, t_i' \in T^0$. Therefore, x^p is an incentive compatible PIE allocation.

Fix any $s \in S$. Lemma 3 says that $P^S(\cdot|t) = \delta_s$ for P^T-almost all $t \in L_s$. Since $P^T = \sum_{s' \in S} \pi_{s'} P_{s'}^T$ with $\pi_{s'} > 0$, a P^T-null set is a P_s^T-null set; hence $P^S(\cdot|t) = \delta_s$ for P_s^T-almost all $t \in L_s$.

Since F is essentially pairwise independent conditioned on $\tilde{s}$, we know that for each fixed $s \in S$, the random variables $x_s(i, \cdot)$, $i \in I$ are essentially pairwise independent. By the exact law of large numbers in Lemma 1,[32] we know that for P_s^T-almost all $t \in T$, $x_{(s,t)}$ has the same distribution (hence the same mean) as x_s, which implies that $x_{(s,t)}$ is a feasible allocation for the large deterministic economy $\mathcal{E}_{(s,t)} = \{(I, \mathcal{I}, \lambda), u_{(s,t)}, e\}$. As noted earlier, for any given $t \in T$, $x_{(s,t)}(i)$ is also a maximal element in the budget set of agent i under the price system p_s. Therefore, for P_s^T-almost all $t \in T$, $(p_s, x_{(s,t)})$ is a Walrasian equilibrium for $\mathcal{E}_{(s,t)}$.

Let E_s be the set of all $t \in L_s$ such that $P^S(\cdot|t) = \delta_s$, and $(p_s, x_{(s,t)})$ is a Walrasian equilibrium for $\mathcal{E}_{(s,t)}$. Then, the above two paragraphs imply that $P_s^T(E_s) = 1$, and for any $t \in E_s$, we have $\int_I x_{(s,t)}(i)\, d\lambda(i) = \int_I e(i)\, d\lambda(i)$.

Let $E = \bigcup_{s \in S} E_s$. Then

$$P^T(E) = \sum_{s' \in S} \pi_{s'} P_{s'}^T \left(\bigcup_{s \in S} E_s \right) = \sum_{s' \in S} \pi_{s'} P_{s'}^T(E_{s'}) = \sum_{s' \in S} \pi_{s'} = 1.$$

Since the sets L_s, $s \in S$ are disjoint, so are E_s, $s \in S$. For any $t \in E$, there is a unique $s \in S$ such that $t \in E_s$ and

$$\int_I x^p(i,t)\, d\lambda(i) = \int_I x_{(s,t)}(i)\, d\lambda(i) = \int_I e(i)\, d\lambda(i),$$

which implies that x^p is feasible for the PIE.

[32] Note that Lemma 2 is not directly applicable to the process x_s since x_s may depend on $s \in S$ while the process G in Lemma 2 is independent of $s \in S$.

Define a measurable function p^* from $(T, \mathcal{T})$ to $\mathbb{R}^m_{++}$ by letting

$$p^*(t) = \begin{cases} e^* & \text{if } t \in L_0, \\ p_s & \text{if } t \in L_s,\ s \in S \end{cases}$$

where e^* is the vector whose components are $1/m$.

Fix any $s \in S$ and $t \in E_s$. Then, $U_t = u_{(s,t)}$, $p^*(t) = p_s$, $\mathcal{E}^p_t = \mathcal{E}_{(s,t)}$ and $x^p_t = x_{(s,t)}$. Since $(p_s, x_{(s,t)})$ is a Walrasian equilibrium for $\mathcal{E}_{(s,t)}$, $(p^*(t), x^p_t)$ is a Walrasian equilibrium for $\mathcal{E}^p_t$.

Hence, for any $t \in E$, $(p^*(t), x^p_t)$ is a Walrasian equilibrium for $\mathcal{E}^p_t$. Since $P^T(E) = 1$, x^p is thus an incentive compatible and ex post Walrasian allocation in the PIE, and therefore ex post individually rational, and ex post efficient.

References

[1] B. Allen, N.C. Yannelis, Differential information economies: introduction, Econ. Theory 18 (2001) 263–273.
[2] R.J. Aumann, Markets with a continuum of traders, Econometrica 32 (1964) 39–50.
[3] D. Glycopantis, N.C. Yannelis, Differential Information Economies, Studies in Economic Theory, vol. 19, Springer, Berlin, 2005.
[4] F. Gul, A. Postlewaite, Asymptotic efficiency in large exchange economies with asymmetric information, Econometrica 60 (1992) 1273–1292.
[5] G. Hahn, N. Yannelis, Efficiency and incentive compatibility in differential information economies, Econ. Theory 10 (1997) 383–411.
[6] W. Hildenbrand, Core and Equilibria of a Large Economy, Princeton University Press, Princeton, NJ, 1974.
[7] S. Krasa, W. Shafer, Informational robustness of competitive equilibria, J. Econ. Theory 101 (2001) 494–518.
[8] P.A. Loeb, Conversion from nonstandard to standard measure spaces and applications in probability theory, Trans. Amer. Math. Soc. 211 (1975) 113–122.
[9] R. McLean, A. Postlewaite, Informational size and incentive compatibility, Econometrica 70 (2002) 2421–2453.
[10] T. Palfrey, S. Srivastava, Private information in large economies, J. Econ. Theory 39 (1986) 34–58.
[11] E. Prescott, R. Townsend, Pareto-optima and competitive equilibria with adverse selection and moral hazard, Econometrica 52 (1984) 21–45.
[12] Y.N. Sun, A theory of hyperfinite processes: the complete removal of individual uncertainty via exact LLN, J. Math. Econ. 29 (1998) 419–503.
[13] Y.N. Sun, The almost equivalence of pairwise and mutual independence and the duality with exchangeability, Probability Theory and Related Fields 112 (1998) 425–456.
[14] Y.N. Sun, The exact law of large numbers via Fubini extension and characterization of insurable risks, J. Econ. Theory 12 (2006) 31–69.
[15] Y.N. Sun, N.C. Yannelis, Perfect competition in asymmetric information economies: compatibility of efficiency and incentives, Working Paper, October 2005; Available at ⟨http://netfiles.uiuc.edu/nyanneli/www/SYexpost.pdf⟩.
[16] N.C. Yannelis, The core of an economy with differential information, Econ. Theory 1 (1991) 183–198.