RESEARCH ARTICLE

Existence of Walrasian equilibria with discontinuous, non-ordered, interdependent and price-dependent preferences, without free disposal, and without compact consumption sets

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Abstract

We extend a result on existence of Walrasian equilibria in He and Yannelis (Econ Theory 61:497–513, 2016) by replacing the compactness assumption on consumption sets made there by the standard assumption that these sets are closed and bounded from below. This provides a positive answer to a question explicitly raised in He and Yannelis (Econ Theory 61:497–513, 2016). Our new equilibrium existence theorem generalizes many results in the literature as we do not require any transitivity or completeness or continuity assumption on preferences, initial endowments need not be in the interior of the consumption sets, preferences may be interdependent and price-dependent, and no monotonicity or local non satiation is needed for any of the agents.

Keywords Continuous inclusion property · Exchange economy · Existence of Walrasian equilibrium

JEL Classification C02 · C62 · D51 · D62

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1 Introduction

In seminal contributions to general equilibrium theory, Arrow and Debreu (1954) and McKenzie (1954) proved the existence of a Walrasian equilibrium. McKenzie's proof contains the first direct application of the Kakutani fixed point theorem. The existence proof of Arrow–Debreu is based on the social equilibrium existence theorem of Debreu (1952), (a generalization of Nash's noncooperative equilibrium theorem), which in turned is proved via the Eilenberg–Montgomery fixed point theorem.

The idea of the proof of Arrow–Debreu was to convert the economy into a social system (or abstract economy). By applying the Debreu social equilibrium result, they established that the abstract economy has a social equilibrium (or abstract equilibrium). Having done this groundwork, it can be easily shown that the existence of an abstract equilibrium in the abstract economy implies the existence of a Walrasian equilibrium in the original economy. Here we should stress that the proof of Arrow–Debreu was an application of the Debreu social equilibrium existence theorem. The proof of McKenzie was based on a clever composition of mappings to which the Kakutani fixed point theorem can be applied to yield a Walrasian equilibrium. Gale (1955), Nikaido (1956) and Debreu (1956) presented alternative proofs using the excess demand approach which is elaborated on the masterful treatise of Debreu (1959), "Theory of Value."

The Arrow–Debreu–McKenzie proofs, as well as the excess demand approach, were based on the assumption that preferences are continuous, complete, transitive, and reflexive; in particular preferences are representable by utility functions. Early behavioral work in economics indicated that the transitivity assumption is typically violated and experimental work clearly demonstrated this fact. Motivated by this, in the mid-seventies, Sonnenschein (1971), Shafer (1974), Mas-Colell (1974), Gale and Mas-Colell (1975), Shafer and Sonnenschein (1975), Borglin and Keiding (1976) and Shafer (1976) showed that the transitivity assumption is not needed to prove either the existence of a Walrasian equilibrium or to maintain a maximizing consumer behavior. That was a major breakthrough, as the proofs of Arrow and Debreu (1954), McKenzie (1954), Gale (1955), Nikaido (1956) or Debreu (1956, 1959) did not work with non-transitive and incomplete preferences and new arguments were required.

Concerning the continuity assumption on preferences, a breakthrough towards relaxing this assumption was first made by Dasgupta and Maskin (1986) in the context of game theory. Motivated by tie breaks in auctions, they presented Cournot–Nash equilibrium existence theorems for games with discontinuous utility functions. This work was further improved and extended in a pioneering paper by Reny (1999). Since then, several author have presented extensions; see, e.g., Carmona and Podczeck (2016), He and Yannelis (2016, 2017), Reny (2016) or Cornet (2020), among others.

In the context of general equilibrium theory, existence results in these regards were presented by He and Yannelis (2016). These results are the state of the art on the existence of a Walrasian equilibrium as they do not require preferences to be transitive, complete, or continuous, and preferences may be interdependent and price-dependent.

However, the main existence result on Walrasian equilibrium in He and Yannelis (2016) was obtained by assuming that consumption sets are compact. It was left as an open question whether the compactness assumption can be relaxed, as the standard

truncation arguments that have been employed in the literature do not seem to work when preferences are discontinuous.

The purpose of this paper is to provide a solution to this open question. We show that, indeed, the compactness assumption on consumption sets can be replaced by the standard assumption made in general equilibrium theory that these sets are closed and bounded from below. We emphasize that this is done without any costs, i.e., no extra assumptions, compared with those made in He and Yannelis (2016), are needed. Thus the result of this paper generalizes the corresponding results in He and Yannelis (2016) properly.

2 Terminology

Definition 1 Let X be a topological space and Y a linear topological space. A correspondence ψ from X to Y is said to have the *continuous inclusion property at x* if there exists an open neighborhood O_x of x and a correspondence $F_x : O_x \to 2^Y$ such that $F_x(z) \subseteq \psi(z)$ for any $z \in O_x$ and $co F_x$ is upper hemicontinuous with nonempty compact values.

Remark 1 In He and Yannelis (2016), the continuous inclusion property was stated in terms of closed correspondences, rather than in term of upper hemicontinuous correspondences with compact values. However, in He and Yannelis (2016) the continuous inclusion property was used in the context of compact consumption (or action) sets, in which case both statements are equivalent. The statement in terms of upper hemicontinuous correspondences with compact values seems to be the key in getting a generalization of the Walrasian equilibrium existence result of He and Yannelis (2016) to cover non-compact consumption sets.

3 The model and the result

In the economies we consider, the commodity space is $\mathbb{R}^l$ and the price space is $\Delta = \{p \in \mathbb{R}^l : \|p\| \leq 1\}$.

Definition 2 An economy $\mathcal{E}$ is a set of triples $\{(X_i, P_i, e_i) : i \in I\}$ where

- I is a nonempty finite set of agents;
- X_i is the consumption set of agent i, and $X = \prod_{i \in I} X_i$;
- $P_i : \Delta \times X \to 2^{X_i}$ is the preference correspondence of agent i;
- $e_i \in X_i$ is the endowment of agent i, and $\sum_{i \in I} e_i \neq 0$.

Definition 3 Let $\mathcal{E}$ be an economy.

(a) Given $p \in \Delta$, the budget set of agent i is $B_i(p) = \{x_i \in X_i : px_i \leq pe_i\}$.
(b) A (non-free disposal) Walrasian equilibrium is a pair (p, x), where $p \in \Delta$ and $x \in X$, such that

    (i) $p \neq 0$;
    (ii) $x_i \in B_i(p)$ for each $i \in I$;

(iii) $B_i(p) \cap P_i(x, p) = \emptyset$ for each $i \in I$;

(iv) $\sum_{i \in I} x_i = \sum_{i \in I} e_i$.

Let $\mathcal{E}$ be an economy. For each $i \in I$, let $\psi_i : \Delta \times X \to 2^{X_i}$ be the correspondence defined by setting $\psi_i(p, x) = B_i(p) \cap P_i(p, x)$ for $(p, x) \in \Delta \times X$. Write $\mathcal{A}$ for the set of feasible allocation of $\mathcal{E}$; thus $\mathcal{A} = \{x \in X : \sum_{i \in I} x_i = \sum_{i \in I} e_i\}$. We consider the following assumptions.

(A1) For each $i \in I$, X_i is a closed and convex subset of $\mathbb{R}_+^l$.

(A2) For each $i \in I$, each $p \in \Delta$, and each $x \in X$, $x_i \notin \text{co} \psi_i(p, x)$.

(A3) If $x \in \mathcal{A}$ and $p \in \Delta$ is a price system such that $x_i \in B_i(p)$ for all $i \in I$ but the set $I' \subseteq I$ of all i with $\psi_i(p, x) \neq \emptyset$ is non-empty, there is an $i \in I'$ such that ψ_i has the continuous inclusion property at (p, x).

(A4) If $x \in \mathcal{A}$ and $p \in \Delta$ is a price system such that $x_i \in B_i(p)$ for all $i \in I$, then there is an $i \in I$ such that $P_i(p, x) \neq \emptyset$.

(Concerning assumption (A3), see also Remarks 3 and 5 below.)

Theorem *If the economy $\mathcal{E} = \{(X_i, P_i, e_i) : i \in I\}$ satisfies (A1)–(A4), then it has a Walrasian equilibrium.*

Proof Let $\langle C_n \rangle_{n \in \mathbb{N}}$ be an increasing sequence of closed balls in $\mathbb{R}^l$, with $\bigcup_{n \in \mathbb{N}} C_n = \mathbb{R}^l$, and set $K_{i,n} = C_n \cap X_i$ for each i and each n. For each $n \in \mathbb{N}$, write $K_n = \prod_{i \in I} K_{i,n}$. Noting that $\mathcal{A}$ is compact, we can assume that $\mathcal{A}$ is included in K_n for each n. Write $y = (p, x)$ for elements of $\Delta \times X$. Note also that $K_{i,n}$ is non-empty, compact, and convex for each i and each n, and hence so is K_n for each n.

Let $\mathcal{H}$ be a collection of correspondences F_y^i witnessing that (A3) is satisfied. Since in a Euclidean space any neighborhood of any point includes a compact neighborhood of this point, and X is closed in its ambient Euclidean space, we can assume by Definition 1 that for each correspondence in $\mathcal{H}$ the domain O_y^i is bounded in $\Delta \times X$ and the image $F_y^i(O_y^i)$ is bounded in X_i (shrinking the domains of the members of $\mathcal{H}$ appropriately). Thus, each F_y^i in $\mathcal{H}$ is such that there is an $n_{i,y} \in \mathbb{N}$ such that for $n \geq n_{i,y}$, O_y^i is included in $\Delta \times K_n$, and $F_y^i(O_y^i)$ is included in $K_{i,n}$. For each $i \in I$ and $n \in \mathbb{N}$, write $\mathcal{C}_n^i$ for the collection of those O_y^i which are included in $\Delta \times K_n$ and are such that $F_y^i(O_y^i)$ is included in $K_{i,n}$.

Fix any $i \in I$ and any $n \in \mathbb{N}$. Let $V_n^i = \bigcup \mathcal{C}_n^i$. Note that $\mathcal{C}_n^i$ is an open cover of V_n^i. Being included in a Euclidean space, V_n^i is metrizable, therefore paracompact; see, e.g., Engelking (1989, p. 300, Theorem 5.1.3). Thus $\mathcal{C}_n^i$ has a closed locally finite refinement $\mathcal{F}_n^i = \{E_{j,n}^i : j \in J_n^i\}$, where J_n^i is an index set (and $E_{j,n}^i$ is closed in V_n^i); see Engelking (1989, p. 302, Theorem 5.1.11).[1]

For each $j \in J_n^i$ choose a $y_j \in \Delta \times K_n$ such that $E_{j,n}^i \subseteq O_{y_j}^i$, where $O_{y_j}^i$ belongs to $\mathcal{C}_n^i$. For each $y \in V_n^i$ let $I_n^i(y) = \{j \in J_n^i : y \in E_{j,n}^i\}$. Then $I_n^i(y)$ is finite for each $y \in V_n^i$. Let $\phi_n^i(y) = \text{co}\left(\bigcup_{j \in I_n^i(y)} F_{y_j}^i(y)\right)$ for $y \in V_n^i$.

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[1] Recall that a topological space W is called *paracompact* if it is Hausdorff and every open cover of W has an open locally finite refinement. Recall also that a *refinement* of a cover $\mathcal{A}$ of a set W is a cover $\mathcal{B}$ of W such that every member of $\mathcal{B}$ is included in some member of $\mathcal{A}$. Finally, recall that a family $\mathcal{B}$ of subsets of a topological space W is called *locally finite* if every point of W has an open neighborhood which meets only finitely many members of $\mathcal{B}$.

Define $H_n^i : \Delta \times K_n \to 2^{K_{i,n}}$ by setting, for each $y = (p, x) \in \Delta \times K_n$,

$$H_n^i(y) = \begin{cases} \phi_n^i(y) & \text{if } y \in V_n^i \\ B_i(p) \cap K_{i,n} & \text{otherwise.} \end{cases}$$

Evidently H_n^i has non-empty compact convex values and is upper hemicontinuous (note that V_n^i is open in $\Delta \times K_n$ and that $\phi_n^i(y) \subseteq B_i(p) \cap K_{i,n}$ for all $y = (p, x) \in V_n^i$; the fact that $y \mapsto \bigcup_{j \in I_n^i(y)} \mathrm{co} F_{y_j}^i(y) : V_n^i \to 2^{\mathbb{R}^l}$ is upper hemicontinuous may be checked elementary; because $\mathrm{co}\big(\bigcup_{j \in I_n^i(y)} F_{y_j}^i(y)\big) = \mathrm{co}\big(\bigcup_{j \in I_n^i(y)} \mathrm{co} F_{y_j}^i(y)\big)$, it follows from Hildenbrand (1974, p. 26, Proposition 6) that $y \mapsto \mathrm{co}\big(\bigcup_{j \in I_n^i(y)} F_{y_j}^i(y)\big) : V_n^i \to 2^{\mathbb{R}^l}$ is upper hemicontinuous as well.)

Do this construction for each $i \in I$ and each $n \in \mathbb{N}$. Moreover, for each $n \in \mathbb{N}$, define a correspondence $G_n : \Delta \times K_n \to 2^{\Delta}$ by setting

$$G_n(y) = \mathrm{argmax}_{q \in \Delta} \, q \sum_{i \in I} (x_i - e_i)$$

for each $y = (p, x) \in \Delta \times K_n$.

By Kakutani's fixed point theorem, for each n the correspondence $G_n \times \prod_{i \in I} H_n^i$ has a fixed point, (p_n^*, x_n^*) say. As in the proof of Theorem 4 in He and Yannelis (2016) it follows that $x_n^* \in \mathcal{A}$ for each n. To see this, fix any n. Write $z_n^* = \sum_{i \in I} (x_{n,i}^* - e_i)$. By the definition of G_n, we must have $p_n^* z_n^* \geq q z_n^*$ for each $q \in \Delta$, and by the definition of the correspondences H_n^i, we have $0 \geq p_n^* z_n^*$. Thus $0 \geq q z_n^*$ for each $q \in \Delta$. Because the definition of Δ implies that each non-zero excess demand vector can be given a positive value by an appropriate $q \in \Delta$, it follows that $z_n^* = 0$, i.e., $x_n^* \in \mathcal{A}$.

Because $\mathcal{A}$ and Δ are compact, we can assume, therefore, that the sequence $\langle (p_n^*, x_n^*) \rangle$ is convergent, say to (p^*, x^*). Thus $x^* \in \mathcal{A}$. By construction, $x_{n,i}^* \in B_i(p_n^*)$ for each $i \in I$ and each n, which implies that $x_i^* \in B_i(p^*)$ for each $i \in I$.

Suppose there is an $i \in I$ such that $\psi_i(p^*, x^*) \neq \emptyset$. By (A3) we can assume that i is such that the continuous inclusion property holds for ψ_i at (p^*, x^*). Let $O_{(p^*, x^*)}^i$ and $F_{(p^*, x^*)}^i$ be chosen according to the second paragraph of this proof. We can pick an $n_0 \in \mathbb{N}$ so large that we have both $O_{(p^*, x^*)}^i \subseteq \Delta \times K_n$ and $F_{(p^*, x^*)}^i(O_{(p^*, x^*)}^i) \subseteq K_{i,n}$ for $n \geq n_0$. Thus $O_{(p^*, x^*)}^i \subseteq V_n^i$ for $n \geq n_0$. Also, since $\langle (p_n^*, x_n^*) \rangle$ converges to (p^*, x^*) and $O_{(p^*, x^*)}^i$ is open in $\Delta \times X$, we can pick an $n_1 \in \mathbb{N}$ such that $(p_n^*, x_n^*) \in O_{(p^*, x^*)}^i$ whenever $n \geq n_1$. Set $\bar{n} = \max\{n_0, n_1\}$. Then $(p_{\bar{n}}^*, x_{\bar{n}}^*) \in V_{\bar{n}}^i$. Thus $H_{\bar{n}}^i(p_{\bar{n}}^*, x_{\bar{n}}^*) = \phi_{\bar{n}}^i(p_{\bar{n}}^*, x_{\bar{n}}^*)$. It follows that $x_{\bar{n},i}^* \in \phi_{\bar{n}}^i(p_{\bar{n}}^*, x_{\bar{n}}^*) \subseteq \mathrm{co}\psi_i(p_{\bar{n}}^*, x_{\bar{n}}^*)$, and we get a contradiction to (A2).

Thus $\psi_i(p^*, x^*) = \emptyset$ for each $i \in I$. Finally, again as in the proof of Theorem 4 in He and Yannelis (2016), it follows from (A4) that $p \neq 0$. To see this, suppose, if possible, otherwise. Then $B_i(p^*) = X_i$ and $\psi_i(p^*, x^*) = \emptyset$ for each i, which is impossible in view of (A4). Thus (p^*, x^*) is a Walrasian equilibrium. □

4 Remarks

Remark 2 Specializing the above theorem to compact consumption sets, we are exactly in the situation of Theorem 4 in He and Yannelis (2016). This is so because, as already noted in Remark 1, in that case the notion of the continuous inclusion property as defined in He and Yannelis (2016) coincides with that employed in the present paper.

Remark 3 The truncation technique used in the proof of our result differs from standard approaches (see, e.g., Debreu 1959, pp. 85–87 or Hildenbrand 1974, p. 50). The reason is that the commodity space need not have any compact subsets relative to which (A3) holds. In other words, there need not be any compact truncation of the commodity space relative to which (A3) holds. In particular, we cannot appeal to the equilibrium existence result for abstract economies in He and Yannelis (2016), as this result requires compact action sets. Instead, we use a sequence of compact truncations of the commodity space. To get such a sequence so that it is suitable for the needs of our result, we have formulated the continuous inclusion property in terms of upper hemicontinuous correspondences, rather that in terms of closed correspondences (cf. Remark 1).

Remark 4 The method of the above proof also yields a free disposal Walrasian equilibrium. A free disposal Walrasian equilibrium is defined by adding "$p \geq 0$" to (i)–(iv) above and by changing (iv) into "$\sum_{i \in I} x_i \leq \sum_{i \in I} e_i$." Now in the list of assumptions and in the proof replace $\mathcal{A}$ by $\mathcal{A}' = \{x \in X : \sum_{i \in I} x_i \leq \sum_{i \in I} e_i\}$ and Δ by $\Delta' = \{p \in \mathbb{R}_+^l : \sum_{j=1}^l p_j = 1\}$. Instead of appealing to the proof of Theorem 4 in He and Yannelis (2016), appeal to the proof of Theorem 2 there, and note that the second sentence in the last paragraph of the above proof becomes superfluous. Along these lines, we also get Theorem 2 of He and Yannelis (2016) as a special case.

Remark 5 Making a requirement stronger than (A3), it was assumed in He and Yannelis (2016) that, for every consumer i, whenever (p, x) is such that $\psi_i(p, x) \neq \emptyset$ then ψ_i has the continuous inclusion property at (p, x). However, if the endowments of the consumers may be on the boundaries of their consumption sets, this latter assumption has problems, even when consumption sets are compact. In fact, as may easily be seen by examples, if the endowment e_i of a consumer i is on the boundary of his consumption set, and the price vector p is such that there are no cheaper points in his budget set, there are cases in which $\psi_i(p, x) \neq \emptyset$ but ψ_i cannot have the continuous inclusion property at (p, x). Such cases can occur even in contexts in which preferences and the aggregate endowment are such that existence of a Walrasian equilibrium follows from the standard theory. On the other hand, it is not hard to see that if only (A3) is required, i.e., if under the conditions of this assumption there be at least one consumer i for whom ψ_i has the continuous inclusion property, then the standard results on existence of Walrasian equilibrium for exchange economies are covered, including the case in which individual endowments may be on the boundaries of the consumption sets.

Remark 6 Independently of this paper, Anderson et al. (2021) also addressed the question whether it is possible to relax the compactness assumption on consumption sets

made in He and Yannelis (2016). They did so in a theorem which, however, is stated just in terms of a free disposal Walrasian equilibrium (see Remark 4 for definition). On the other hand, this theorem allow for an infinite set of agents. Apart from that, the conditions in this theorem are stronger than those imposed in the result of the present paper. E.g., it is required by Anderson et al. (2021) in the mentioned theorem that, for each $(p, x) \in \Delta \times X$ and each consumer i, whenever $y_i \in P_i(p, x)$ and $\lambda \in (0, 1)$, then $\delta y_i + (1 - \delta)x_i \in P_i(p, x)$ for some $\delta \in (0, \lambda)$. This also shows that, when specialized to the case of compact consumption sets, this theorem does not reduce to the corresponding result in He and Yannelis (2016).

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