JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 135, 664–670 (1988)

On a Carathéodory-Type Selection Theorem

TAESUNG KIM*

Department of Economics, University of Minnesota, Minneapolis, Minnesota 55455

KAREL PRIKRY†

Department of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

AND

NICHOLAS C. YANNELIS‡

Department of Economics, University of Minnesota, Minneapolis, Minnesota 55455

Submitted by Michael Magill

Received June 15, 1986

We offer a new Carathéodory-type selection theorem. This result arose naturally from the authors' study of equilibria in abstract economies (generalized games) with a measure space of agents.

1. INTRODUCTION

In the present paper we obtain a Carathéodory-type selection theorem. This result is used in [9] to extend the theory of Nash equilibria, developed in [12, 4, 2, 7, 8, 10, and 13–17], to the setting of an arbitrary measure space of agents and an infinite dimensional strategy space.

More specifically, we consider the following problem. Let T be a measure space, Z be a complete separable metric space, and Y be a separable Banach space. Let $\phi: T \times Z \to 2^Y$ be a convex valued (*possibly empty*) correspondence. Let $U = \{(t, x) \in T \times Z: \phi(t, x) \neq \varnothing\}$. Under appropriate assumptions, does there exist a *Carathéodory-type selection* for ϕ, i.e., a

* Present address: California Institute of Technology, Pasadena, CA 91125.

† Research partially supported by N.S.F. grant.

‡ Present address: Department of Economics, University of Illinois, Champaign, IL 61820.

0022-247X/88 $3.00

function $f: U \to Y$ such that $f(t, x) \in \phi(t, x)$ for all $(t, x) \in U$ and $f(\cdot, x)$ is measurable for each x and $f(t, \cdot)$ is continuous for each t. We provide a positive answer in our main theorem, stated in Section 2.

Although Fryszkowski [5] proved a Carathéodory-type selection theorem, he considers only the case when $U = T \times Z$ and his techniques do not readily extend to the general case.

The paper is organized in the following way. Section 2 contains definitions and notation. The main result is stated in Section 3. Section 4 contains several lemmata needed for the proof of the main result. Finally, the proof of the main theorem is given in Section 5.

2. Notation and Definitions

2.1. *Notation*

2^A denotes the set of all subsets of the set A.

$B(x, \lambda)$ denotes the open ball centered at x of radius λ.

$\setminus$ denotes the set theoretic subtraction.

If $\phi: X \to 2^Y$ is a correspondence then $\phi|_U: U \to 2^Y$ denotes the restriction of ϕ to U,

diam denotes diameter

dist denotes distance

proj denotes projection.

2.2. *Definitions*

Let X and Y be sets. The *graph* G_ϕ of a correspondence $\phi: X \to 2^Y$ is the set $G_\phi = \{(x, y) \in X \times Y: y \in \phi(x)\}$. If X is a topological space, ϕ is said to have *open* lower sections if for each $y \in Y$ the set $\phi^{-1}(y) = \{x \in X: y \in \phi(x)\}$ is open in X. If $(X, \mathcal{A})$ and $(Y, \mathcal{S})$ are measurable spaces, $\phi: X \to 2^Y$ is said to have a *measurable graph* if G_ϕ belongs to the product σ-algebra $\mathcal{A} \otimes \mathcal{S}$.

A *selection for a correspondence* ϕ as above is a mapping f from $S = \{x \in X: \phi(x) \neq \varnothing\}$ into Y such that $f(x) \in \phi(x)$ for every $x \in S$. In the topological setting, a *continuous selection* is a selection f which is continuous on S. If ϕ is a correspondence between measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{S})$, and $S \in \mathcal{A}$, then a *measurable selection* is a selection f which is measurable on S, i.e., with respect to the σ-algebra $\mathcal{A} \cap \mathcal{P}(S)$. It is easy to extend this last concept to the general S, i.e., not in $\mathcal{A}$, but we shall not do so at this point.

Recall that an open cover U of a topological space X is *locally finite* if every $x \in X$ has a neighborhood intersecting only finitely many sets in U.

3. The Main Theorem

Measurable and continuous selections have been extensively used and studied—see, for instance, Castaing and Valadier [3] or Michael [11]. There is also a growing literature on Carathéodory-type selections; see Fryszkowski [5]. Below we state our Carathéodory-type selection theorem.

Main Theorem. *Let (T, τ, μ) be a complete finite measure space, Y be a separable Banach space and Z be a complete separable metric space. Let $X: T \to 2^Y$ be a correspondence with a measurable graph, i.e., $G_X \in \tau \otimes \mathscr{B}(Y)$, and $\phi: T \times Z \to 2^Y$ be a convex valued correspondence (possibly empty) with a measurable graph, i.e., $G_\phi \in \tau \otimes \mathscr{B}(Z) \otimes \mathscr{B}(Y)$, where $\mathscr{B}(Y)$ and $\mathscr{B}(Z)$ are the Borel σ-algebras of Y and Z, respectively, and furthermore suppose that the conditions* (i), (ii), *and* (iii) *below hold*:

(i) *for each $t \in T$, $\phi(t, x) \subset X(t)$ for all $x \in Z$.*

(ii) *for each t, $\phi(t, \cdot)$ has open lower sections in Z, i.e., for each $t \in T$, and each $y \in Y$, $\phi^{-1}(t, y) = \{x \in Z: y \in \phi(t, x)\}$ is open in Z.*

(iii) *for each $(t, x) \in T \times Z$, if $\phi(t, x) \neq \varnothing$, then $\phi(t, x)$ has a nonempty interior in $X(t)$.*

Let $U = \{(t, x) \in T \times Z: \phi(t, x) \neq \varnothing\}$ and for each $x \in Z$, $U_x = \{t \in T: (t, x) \in U\}$ and for each $t \in T$, $U^t = \{x \in Z: (t, x) \in U\}$. Then for each $x \in Z$, U_x is a measurable set in T and there exists a Carathéodory-type selection from $\phi|_U$, i.e., there exists a function $f: U \to Y$ such that $f(t, x) \in \phi(t, x)$ for all $(t, x) \in U$ and for each $x \in Z$, $f(\cdot, x)$ is measurable on U_x and for each $t \in T$, $f(t, \cdot)$ is continuous on U^t. Moreover, $f(\cdot, \cdot)$ is jointly measurable.

4. Lemmata

Lemma 4.1 (Aumann). *If (T, τ, μ) is a complete finite measure space, Y is a complete, separable metric space, and $F: T \to 2^Y$ is a correspondence with a measurable graph, i.e., $G_F \in \tau \otimes \mathscr{B}(Y)$, then there is a measurable function $f: \operatorname{proj}_T(G_F) \to Y$ such that $f(t) \in F(t)$ for almost all $t \in \operatorname{proj}_T(G_F)$.*

Proof. See Aumann [1] or Castaing and Valadier [3].

Lemma 4.2 (Projection theorem). *Let (T, τ, μ) be a complete finite measure space and Y be a complete, separable metric space. If G belongs to $\tau \otimes \mathscr{B}(Y)$, its projection $\operatorname{proj}_T(G)$ belongs to τ.*

Proof. See Theorem III.23 in Castaing and Valadier [3].

LEMMA 4.3. *Let (T, τ, μ) be a complete finite measure space, and Y be a complete separable metric space. Let $X: T \to 2^Y$ be a correspondence with a measurable graph. Then there exist $\{f_k: k = 1, 2, ...\}$ such that*:

(i) *for all k, f_k is a measurable function from $\mathrm{proj}_T(G_X)$ into Y and*

(ii) *for each $t \in \mathrm{proj}_T(G_X)$, $\{f_k(t): k = 1, 2, ...\}$ is a dense subset of $X(t)$.*

Proof. For each $n = 1, 2, ...$, let $\{E_l^n: l = 1, 2, ...\}$ be an open cover of Y such that $\mathrm{diam}(E_l^n) < 1/2^n$. For each n, $l = 1, 2, ...$, define $T_l^n = \{t \in T: X(t) \cap E_l^n \neq \emptyset\}$. Since $T_l^n = \mathrm{proj}_T\{(t, y) \in T \times Y: y \in X(t) \cap E_l^n\}$ and $X(\cdot) \cap E_l^n$ has a measurable graph in $T \times Y$, $T_l^n \in \tau$ by Lemma 4.2. It can be easily checked that $\bigcup_{l=1}^{\infty} T_l^n = \mathrm{proj}_T(G_X) = S$.

For each n, $l = 1, 2, ...$, define the correspondence $X_l^n: T \to 2^Y$ by

$$X_l^n(t) = \begin{cases} X(t) \cap E_l^n & \text{if } \ t \in T_l^n \\ X(t) & \text{if } \ t \notin T_l^n. \end{cases}$$

Since the graph of X_l^n is $\{(t, y) \in T_l^n \times Y: \ y \in X(t) \cap E_l^n\} \cup \{(t, y) \in T \setminus T_l^n \times Y: y \in X(t)\}$, the correspondence X_l^n has a measurable graph. Also, $X_l^n(t) \neq \emptyset$, iff $X(t) \neq \emptyset$, hence the graphs of X_l^n and X have the same projection onto T. By Lemma 4.1, for each n, $l = 1, 2, ...$, there exists a measurable function $f_l^n: S \to Y$ such that $f_l^n(t) \in X_l^n(t)$ for all $t \in T$. Fix t in T. Let $y \in X(t)$. Since for each n, $\{E_l^n: l = 1, 2, ...\}$ is an open cover of Y, for each n, there is some l such that $y \in X(t) \cap E_l^n$. Therefore $\{f_l^n(t): n, l = 1, 2, ...\}$ is dense in $X(t)$. Thus the sequence f_l^n, after a suitable reindexing, gives the desired sequence f_k. This completes the proof of the lemma.

LEMMA 4.4. *Let $(S_i, \mathcal{A}_i)$ for $i = 1, 2$ be measurable spaces, $h: S_1 \to S_2$ be a measurable function and $A \in \mathcal{A}_1 \otimes \mathcal{A}_2$. Then*

$$\mathrm{Proj}_{S_1}(G_h \cap A) \in \mathcal{A}_1.$$

Proof. (a) If $A = A_1 \times A_2$, where $A_i \in \mathcal{A}_i$, $i = 1, 2$, then $\mathrm{Proj}_{S_1}(G_h \cap A) = A_1 \cap h^{-1}(A_2) \in \mathcal{A}_1$.

(b) If $\mathrm{Proj}_{S_1}(G_h \cap A) \in \mathcal{A}_1$, then $\mathrm{Proj}_{S_1}(G_h \cap A^c) \in \mathcal{A}_1$, where $A^c = S_1 \times S_2 \setminus A$. For, $\mathrm{Proj}_{S_1}(G_h \cap A^c) = S_1 \setminus \mathrm{Proj}_{S_1}(G_h \cap A)$.

(c) If $\mathrm{Proj}_{S_1}(G_h \cap A_n) \in \mathcal{A}_1$ for all $n = 1, 2, ...$, then $\mathrm{Proj}_{S_1}(G_h \cap (\bigcup_{n=1}^{\infty} A_n)) \in \mathcal{A}_1$. For, $\mathrm{Proj}_{S_1}(G_h \cap (\bigcup_{n=1}^{\infty} A_n)) = \bigcup_{n=1}^{\infty} \mathrm{Proj}_{S_1}(G_h \cap A_n)$.

Therefore, $\mathrm{Proj}_{S_1}(G_h \cap A) \in \mathcal{A}_1$ for all $A \in \mathcal{A}_1 \otimes \mathcal{A}_2$.

LEMMA 4.5. *Let (T_i, τ_i) for $i = 1, 2, 3$ be measurable spaces, $y: T_1 \to T_3$*

be a measurable function and ϕ: $T_1 \times T_2 \to 2^{T_3}$ *be a correspondence with a measurable graph, i.e.,* $G_\phi \in \tau_1 \otimes \tau_2 \otimes \tau_3$. *Let* W: $T_1 \to 2^{T_2}$ *be defined by*

$$W(t) = \{x \in T_2 \colon y(t) \in \phi(t, x)\}.$$

Then W *has a measurable graph, i.e.,* $G_W \in \tau_1 \otimes \tau_2$.

Proof. Define h: $T_1 \times T_2 \to T_3$ by $h(t, x) = y(t)$ for all $t \in T_1$ and $x \in T_2$. Let $S_1 = T_1 \times T_2$, $\mathcal{A}_1 = \tau_1 \otimes \tau_2$, $S_2 = T_3$, $\mathcal{A}_2 = \tau_3$, and $A = G_\phi$. Then h: $S_1 \to S_2$ is a measurable function and $A \in \mathcal{A}_1 \otimes \mathcal{A}_2$. So, by Lemma 4.4,

$$G_W = \{(t, x) \colon (t, x, h(t, x)) \in A\} \in \mathcal{A}_1 = \tau_1 \otimes \tau_2.$$

LEMMA 4.6. *Let* (T, τ) *be a measurable space,* Z *be an arbitrary topological space and* W_n, $n = 1, 2, \ldots$ *be correspondences from* T *into* Z *with measurable graphs. Then the correspondences* $\bigcup_n W_n(\cdot)$, $\bigcap_n W_n(\cdot)$, *and* $Z \setminus W_n(\cdot)$ *have measurable graphs.*

The proof is obvious.

LEMMA 4.7. *Let* (T, τ, μ) *be a complete finite measure space,* Z *be a complete separable metric space, and* W: $T \to 2^Z$ *be a correspondence with measurable graph. Then for every* $x \in Z$, $\operatorname{dist}(x, W(\cdot))$ *is a measurable function, where* $\operatorname{dist}(x, \varnothing) = +\infty$.

Proof. We first observe that $S = \{t \in T \colon W(t) \neq \varnothing\}$ belongs to τ by Lemma 4.2. Now $\{s \in S \colon \operatorname{dist}(x, W(s)) < \lambda\} = \{s \in S \colon W(s) \cap B(x, \lambda) \neq \varnothing\} = \operatorname{proj}_T[G_W \cap (T \times B(x, \lambda))]$. Another application of Lemma 4.2 concludes the proof.

LEMMA 4.8. *Let* (T, τ, μ) *be a complete finite measure space,* Z *be a complete separable metric space, and* W: $T \to 2^Z$ *be a correspondence with measurable graph. Then the correspondence* V: $T \to 2^Z$ *defined by*

$$V(t) = \{x \in Z \colon \operatorname{dist}(x, W(t)) > \lambda\} \qquad (\textit{where } \lambda \textit{ is any real number})$$

has a measurable graph.

Proof. Consider the function g: $T \times Z \to [0, +\infty]$ given by $g(t, x) = \operatorname{dist}(x, W(t))$. By Lemma 4.7, $g(\cdot, x)$ is measurable for each x, and obviously $g(t, \cdot)$ is continuous for each t. It is well known that g is therefore jointly measurable, i.e., measurable with respect to the product σ-algebra $\tau \otimes \mathcal{B}(Z)$. For this result see, e.g., Lemma III.14, [3]. Finally, $G_V = g^{-1}([\lambda, +\infty])$, hence $G_V \in \tau \otimes \mathcal{B}(Z)$.

5. Proof of the Main Theorem

To begin with, we can assume without loss of generality that $X(t) \neq \emptyset$ for all $t \in T$. For otherwise we can replace the original measure space by the space $(S, \tau \cap \mathscr{P}(S), \mu \mid \tau \cap \mathscr{P}(S))$, where $S = \text{proj}_T(G_X)$. Note that $S \in \tau$ by Lemma 4.2, since $G_X \in \tau \otimes \mathscr{B}(Y)$.

Let $\phi_x(t) \equiv \phi(t, x)$ for all $x \in Z$. Notice that for each $x \in Z$, $\phi_x(\cdot)$ has a measurable graph in $T \times Y$. Observe that

$$U_x = \{t \in T: \phi_x(t) \neq \emptyset\} = \text{proj}_T(G_\phi).$$

By Lemma 4.2, $U_x \in \tau$. By Lemma 4.3 there exist measurable functions $\{y_n(\cdot): n = 1, 2, ...\}$ such that for each t, $\{y_n(t)\}$ is a countable dense subset of $X(t)$. For each $t \in T$, let $W_n(t) = \{x \in Z: y_n(t) \in \phi(t, x)\}$. By assumption (ii) $W_n(t)$ is open in Z. Since by (iii) for each $(t, x) \in U$, $\phi(t, x)$ has nonempty interior in $X(t)$ and $\{y_n(t): n = 1, 2, ...\}$ is dense in $X(t)$, it follows that $\{W_n(t): n = 1, 2, ...\}$ is a cover of the set U^t. By Lemma 4.5, $W_n(\cdot)$ has a measurable graph. For each $m = 1, 2, ...$ define the operator $(\ )_m$ by

$$(W)_m = \{w \in W: \text{dist}(w, Z \setminus W) \geqslant 1/2^m\}.$$

For $n = 1, 2, ...$ and t in T let $V_n(t) = W_n(t) \setminus \bigcup_{k=1}^{n-1} (W_k(t))_n$. Obviously, $V_n(t)$ is open in Z. It can be easily checked that $\{V_n(t): n = 1, 2, ...\}$ is a locally finite open cover of the set U^t. Since $W_n(\cdot)$ has a measurable graph, $V_n(\cdot)$ has a measurable graph by Lemmas 4.6 and 4.8. Let $\{g_n(t, \cdot): n = 1, 2, ...\}$ be a partition of unity subordinated to the open cover $\{V_n(t): n = 1, 2, ...\}$; for instance, for each $n = 1, 2, ...$, let

$$g_n(t, x) = \frac{\text{dist}(x, Z \setminus V_n(t))}{\sum_{k=1}^{\infty} \text{dist}(x, Z \setminus V_k(t))}.$$

Then $\{g_n(t, \cdot): n = 1, 2, ...\}$ is a family of continuous functions $g_n(t, \cdot): U^t \to [0, 1]$ such that $g_n(t, x) = 0$ for $x \notin V_n(t)$ and $\sum_{n=1}^{\infty} g_n(t, x) = 1$ for all $(t, x) \in U$. Define $f: U \to Y$ by $f(t, x) = \sum_{n=1}^{\infty} g_n(t, x)\, y_n(t)$. Since $\{V_n(t): n = 1, 2, ...\}$ is locally finite, each x has a neighborhood N_x which intersects only finitely many $V_n(t)$. Hence, $f(t, \cdot)$ is a finite sum of continuous functions on N_x and it is therefore continuous on N_x. Consequently, $f(t, \cdot)$ is continuous. Furthermore, for any n such that $g_n(t, x) > 0$, $x \in V_n(t) \subset W_n(t) = \{z \in Z: y_n(t) \in \phi(t, z)\}$, i.e., $y_n(t) \in \phi(t, x)$. So $f(t, x)$ is a convex combination of elements $y_n(t)$ from the convex set $\phi(t, x)$. Consequently, $f(t, x) \in \phi(t, x)$ for all $(t, x) \in U$. Since $V_n(\cdot)$ has a measurable graph, $\text{dist}(x, Z \setminus V_n(\cdot))$ is a measurable function by Lemmas 4.6 and 4.7. Therefore for each n and x, $g_n(\cdot, x)$ is a measurable function. Since for each n, $y_n(\cdot)$ is a measurable function, it follows that $f(\cdot, x)$ is measurable for each x, i.e., $f(t, x)$ is a Carathéodory-type selection from $\phi|_U$.

The joint measurability of f follows essentially the same way as the joint measurability of g in Lemma 4.8. However, the situation is somewhat more delicate since the domain of f is not the entire product space $T \times Z$. We shall take this issue up in more detail in a subsequent paper.

Acknowledgments

This paper was written during the stay of the first and third authors at the University of Minnesota. We wish to thank M. Ali Khan for several useful discussions and suggestions.

References

1. R. J. Aumann, Measurable utility and the measurable choice theorem, *in* "La Decision," pp. 15–26, C.N.R.S., Aix-en-Provence, 1967.
2. P. A. Borglin and H. Keiding, Existence of equilibrium actions and equilibrium: A note on the "new" existence theorems, *J. Math. Econom.* **3** (1976), 313–316.
3. C. Castaing and M. Valadier, "Convex Analysis and Measurable Multifunctions," Lecture Notes in Mathematics Vol. 580, Springer-Verlag, New York, 1977.
4. G. Debreu, A social equilibrium existence theorem, *Proc. Nat. Acad. Sci. U.S.A.* **38** (1952), 886–893.
5. A. Fryszkowski, Carathéodory type selectors of set-valued maps of two variables, *Bull. Acad. Polon. Sci.* **25** (1977), 41–46.
6. J. C. Himmelberg, "Measurable Relations," *Fundamenta Mathematicae*, LXXXVII, (1975), 53–72.
7. M. A. Khan, Equilibrium points of nonatomic games over a non-relexive Banach space, *J. Approx. Theory*, in press.
8. M. A. Khan and N. S. Papageorgiou, "On Cournot-Nash Equilibria in Generalized Quantitative Games with a Continuum of Players," University of Illinois, 1985.
9. T. Kim, K. Prikry, and N. C. Yannelis, Equilibria in abstract economies with a measure space of agents and with an infinite dimensional strategy space, *J. Approx. Theory*, in press.
10. A. Mas-Colell, On a theorem of Schmeidler, *J. Math. Econom.* **13** (1984), 201–206.
11. E. Michael, Continuous selections I, *Ann. of Math.* **63** (1956), 363–382.
12. J. F. Nash, Non-cooperative games, *Ann. of Math.* **54** (1951), 286–295.
13. D. Schmeidler, Equilibrium points of nonatomic games, *J. Statist. Phys.* **7** (1973), 295–300.
14. W. Shafer and H. Sonnenschein, Equilibrium in abstract economies without ordered preferences, *J. Math. Econom.* **2** (1975), 345–348.
15. S. Toussaint, On the existence of equilibrium with infinitely many commodities and without ordered preferences, *J. Econom. Theory* **33** (1984), 98–115.
16. N. C. Yannelis and N. D. Prabhakar, Existence of maximal elements and equilibria in linear topological spaces, *J. Math. Econom.* **12** (1983), 233–245.
17. N. C. Yannelis and N. D. Prabhakar, "Equilibrium in Abstract Economies with an Infinite Number of Agents, an Infinite Number of Commodities and without Ordered Preferences," University of Minnesota, 1983.