Econ Theory Bull (2018) 6:183–199
https://doi.org/10.1007/s40505-018-0139-6

RESEARCH ARTICLE

The maximin equilibrium and the PBE under ambiguity

Dionysius Glycopantis[1] · **Nicholas C. Yannelis**[2]

Received: 11 January 2018 / Accepted: 14 March 2018 / Published online: 23 March 2018

Abstract This note refers to the recent work on ambiguous implementation by de Castro–Liu–Yannelis (Econ Theory 63:233–261, 2017). The authors discuss, under condition of ambiguity, the implementation as maximin equilibria of maximin individually rational and ex ante maximin efficient allocations. An explicit example is used to support their analysis. We analyse further the example used by de Castro–Liu–Yannelis (2017). We show that in the formulated game tree the proposed allocation is implementable through a backward induction argument. Also it is shown that a perfect Bayesian equilibrium (PBE) exists, leading to different allocations. Comparisons are drawn between the maximin and the PBE implementations. We consider also briefly the meaning of the incentive compatibility (IC) of proposed allocations.

Keywords Ambiguity · Maximin preferences · Maximin efficient allocations · Maximin equilibrium · Implementation · Mechanism design · Perfect Bayesian equilibrium · Nash equilibrium

JEL Classification D51 · D61 · D81 · D82

1 Introduction

In their paper de Castro et al. (2017) examine the possible implementation of allocations as maximin equilibria, in a partition model of an asymmetric information

✉ Dionysius Glycopantis
dglycopantis@yahoo.co.uk; dionysius.glycopantis@city.ac.uk

1 Department of Economics, City, University of London, Northampton Square, London EC1V 0HB, UK

2 Department of Economics, Tippie College of Business, University of Iowa, Iowa City, USA

exchange economy. The research is in the area of implementation of allocations under condition of ambiguity and maximin preferences (Wald 1950). For each agent, probabilities are attached only to the information sets of his partition of the states of nature. This work generalizes the results of de Castro et al. (2015).

A main result of the paper is that each maximin individually rational and ex ante maximin efficient allocation of a single good economy is implementable as a maximin equilibrium. The area of investigation undertaken is of wide significance.

This new framework allows the authors to consider further differential information economies studied in the literature now under ambiguity. See, for example, de Castro et al. (2011) and Yannelis (1991).

Maximin core, value and Walrasian expectations equilibrium allocations are individually rational and ex ante maximin efficient (see de Castro and Yannelis 2009; Angelopoulos and Koutsougeras 2015; He and Yannelis 2015) and hence implementable.

Any efficient allocation is incentive compatible (IC) with respect to the maximin preferences. Relevant issues are discussed in de Castro and Yannelis (2009, 2013) and Liu (2014). On the other hand as shown by Holmström and Myerson (1983), an efficient allocation may not be IC with respect to the Bayesian preferences. Furthermore, Palfrey and Srivastava (1987) showed that under the Bayesian preferences, neither efficient allocations nor core allocations define an implementable social choice correspondence, when agents are incompletely informed. Thus under ambiguity results which are impossible with Bayesian preferences become possible.

In this note we analyse further the example used by de Castro et al. (2017). Examples are employed to consolidate the understanding of the theory.

Here backward induction is used to obtain the maximin efficient allocation. It is also shown that a PBE, i.e. a Nash equilibrium (NE), exists leading to a different redistribution of the endowments. Comparisons between the actual solutions are not meaningful.

This note is organized as follows. Section 2 defines ambiguous asymmetric information economies. Section 3 describes the example, explains the criterion for maximin preferences and gives the definitions of the maximin individually rational and maximin efficient allocation. Sections 4 and 5 explain the calculation of the payoffs at the end of the tree and then the maximin equilibrium through the revelation mechanism. Section 6 applies to the same example with ambiguity the alternative idea of the perfect Bayesian equilibrium (PBE) and Sect. 7 compares the two equilibria concepts. Section 8 concludes. The Appendix shows that the proposed allocation is individually rational, ex ante maximin efficient but not unique.

2 Ambiguous asymmetric information economies

The economy consists of a finite set of states of nature and a finite set of agents with differential information. Each agent is characterized by an information partition, a multi-prior set, random initial endowments and an ex post utility function. Ambiguity is expressed by the fact that each agent attaches probabilities to his information sets rather than to individual states, unless of course the latter are isolated. Furthermore, the agents have maximin type preferences (Wald 1950) rather than the more often assumed expected utility formulation. Each player takes into account his information, the actions of the other players, and then maximizes the minimum of his payoff.

An ambiguous asymmetric information exchange economy is a set $\mathcal{E} = \{\Omega; (\mathcal{F}_i, P_i, e_i, u_i) : i \in I\}$. The interpretation of the notation is as follows.

Ω is a *finite set of states* of nature and $\omega \in \Omega$ a particular state. $\mathcal{F}_i$ is Agent i's partition of Ω and $E^{\mathcal{F}_i} \in \mathcal{F}_i$ denotes an event and $\omega \in E^{\mathcal{F}_i}$ a state in the event. An event is also denoted by $\hat{E}^{\mathcal{F}_i}$. A partition means that, as the game unfolds and state ω occurs, only Agent i knows that the event $E^{\mathcal{F}_i}$ has occurred.

$\mathbf{R}^l_+$ will denote the non-negative orthant of $\mathbf{R}^l$ where l is the finite number of goods per state. Agent i's *random initial endowment* is $e_i : \Omega \longrightarrow \mathbf{R}^l_+$. $I = \{1, \ldots, N\}$ is a set of N players. Each agent receives his endowment in the interim. That is, e_i is $\mathcal{F}_i$-measurable, meaning that $e_i(.)$ is constant on each element of $\mathcal{F}_i$. In general, measurability means that we are dealing with nice functions.

A *strategy* of Player i is a function $s_i : \mathcal{F}_i \longrightarrow \mathcal{F}_i$. One strategy for each player is summarized as a strategy profile.

μ_i is the *probability measure* on $\sigma(\mathcal{F}_i)$, the algebra generated by the partition of Agent i, that is $\mu_i : \sigma(\mathcal{F}_i) \longrightarrow [0,\ 1]$. It is assumed that for each agent all information sets have positive probabilities. That is for each i and for each event $E^{\mathcal{F}_i} \in \mathcal{F}_i$, $\mu_i(E^{\mathcal{F}_i}) > 0$.

Each μ_i is a well-defined probability, but it is not defined on every state of nature. Indeed, if $E^{\mathcal{F}_i} = \{\omega, \omega'\}$ with $\omega \neq \omega'$, then the probability of the event $E^{\mathcal{F}_i}$ is well defined, but not the probability of the states of nature ω or ω'.

The set of all probability measures over 2^{Ω} (the power set $P(\Omega)$) that agree with μ_i is given by

$$\Delta_i = \{\text{probability measure } \pi_i : 2^{\Omega} \longrightarrow [0,1] \mid \pi_i(A) = \mu_i(A), \forall A \in \sigma(\mathcal{F}_i)\}.$$

Let P_i, a non-empty, closed and convex subset of Δ_i, be the multi-prior set of Agent i.

Let L denote the set of all functions from Ω to $\mathbf{R}^l_+$. Agent i's *allocation* specifies his consumption bundle at each state of nature, i.e. $x_i \in L$. Let $x = (x_1, \ldots, x_N)$ denote an allocation of the above economy $\mathcal{E}$. An allocation x is said to be *feasible*, if for each $\omega \in \Omega$, $\Sigma_{i \in I} x_i(\omega) = \Sigma_{i \in I} e_i(\omega)$. We are seeking such an allocation of the total endowments.

Agent i's *ex post utility function* is $u_i : \mathbf{R}^l_+ \times \Omega \to \mathbf{R} = u_i(c_i; \omega)$. It is assumed to be strictly monotone in c_i, the agent's consumption vector, and $\mathcal{F}_i$-measurable for fixed c_i.

Given any $c_i \in \mathbf{R}^l_+$, and any two states $\omega, \omega' \in \Omega$, with $\omega \neq \omega'$ we have $u_i(c_i; \omega) = u_i(c_i; \omega')$, whenever $\omega \in E^{\mathcal{F}_i(\omega')}$. In effect the $\mathcal{F}_i$ -measurability of the ex post utility functions characterizes the agent's "type".

Let $D_i(x - e, (E^{\mathcal{F}_1}, \ldots, E^{\mathcal{F}_N}))$ denote the *actual redistribution* allocated to Player i. It depends on the planned redistribution $x - e$ and the players' reports.

We denote by g_i the outcome function of Player i. It depends on the planned redistribution, the reports of the players and the realized state of nature. Explicitly,

$$g_i = g_i\left(x - e, \left(E^{\mathcal{F}_1}, \ldots, E^{\mathcal{F}_N}\right), \omega\right) = e_i(\omega) + D_i\left(x - e, \left(E^{\mathcal{F}_1}, \ldots, E^{\mathcal{F}_N}\right)\right),$$

where $e_i(\omega) + D_i(x - e, (E^{\mathcal{F}_1}, \ldots, E^{\mathcal{F}_N}))$ is the bundle that Player i ends up consuming.

Finally, we define for each Player i the *final utility payoff function* $v_i = v_i(x - e, (E^{\mathcal{F}_1}, \ldots, E^{\mathcal{F}_N}); \omega) = u_i(e_i(\omega) + D_i(x - e, (E^{\mathcal{F}_1}, \ldots, E^{\mathcal{F}_N})); \omega)$.

Instead of $v_i(x - e, (E^{\mathcal{F}_1}, \ldots, E^{\mathcal{F}_N}); \omega)$ we can write, for convenience, $v_i((E^{\mathcal{F}_1}, \ldots, E^{\mathcal{F}_N}); \omega)$ or $v_i\left(E^{\mathcal{F}_i}, E^{\mathcal{F}_{-i}}; \omega\right)$, where $E^{\mathcal{F}_{-i}}$ sums up the reports of all other players, different from i.

All assumptions in Sect. 2 of de Castro et al. are satisfied in the example we discuss. For example, it is evident that if all agents truthfully report their information, i.e. nobody lies, the actually realized state will occur. Further concepts and variables, such as the ideas of a maximin preference and of a direct revelation mechanism, etc., are introduced below.

3 The example with ambiguity

We look at Example 1, Section 5.2 of the de Castro et al. (2017) paper. We show that the allocation discussed is indeed individually rational and maximin efficient. We also show that there are other such allocations as well. The numbering of the example and definitions are the same as in their paper.

Example 1 There are two agents, $I = \{1, 2\}$, one commodity, and three possible states of nature $\Omega = \{a, b, c\}$. The ex post utility function of each Agent i is $u_i(c_i; \omega) = \sqrt{c_i}$. The agents' random initial endowments, information partitions and multi-prior sets are

$$(e_1(a), e_1(b), e_1(c)) = (5, 5, 1);\ \mathcal{F}_1 = \{\{a, b\}, c\}$$
$$(e_2(a), e_2(b), e_2(c)) = (5, 1, 5);\ \mathcal{F}_2 = \{\{a, c\}, b\}$$
$$P_1 = \{\text{probability measure } \pi_1 : 2^{\Omega} \longrightarrow [0, 1]\ \ |\pi_1(\{a,\ b\}) = 2/3 \text{ and } \pi_1(c) = 1/3\}$$
$$P_2 = \{\text{probability measure } \pi_2 : 2^{\Omega} \longrightarrow [0, 1]\ \ |\pi_2(\{a,\ c\}) = 2/3 \text{ and } \pi_2(b) = 1/3\}.$$

The priors are completed by attaching to P_1 probabilities, p_1 and p_2 with $p_1 + p_2 = 2/3$, and to P_2 probabilities, q_1 and q_2 with $q_1 + q_3 = 2/3$, where subscripts 1, 2, 3, refer to states $\{a, b, c\}$, respectively.

Here, we want to give an explanation that the *proposed allocation*

$$x = \begin{pmatrix} x_1(a) & x_1(b) & x_1(c) \\ x_2(a) & x_2(b) & x_2(c) \end{pmatrix} = \begin{pmatrix} 5 & 4.8 & 1.2 \\ 5 & 1.2 & 4.8 \end{pmatrix}$$

is maximin individually rational and ex ante maximin efficient and whether it is unique. We also look into the reason why this allocation was proposed in the first instance.

Definition 1 Take two allocations f_i and h_i. Agent i prefers f_i to h_i under the maximin preferences (written as $f_i \succeq_i^{\text{MP}} h_i$) if

$$\min_{\pi \in P_i} \sum_{\omega \in \Omega} u_i(f_i(\omega); \omega)\pi_i(\omega) \geq \min_{\pi \in P_i} \sum_{\omega \in \Omega} u_i(h_i(\omega); \omega)\pi_i(\omega). \tag{1}$$

Furthermore Agent i strictly prefers f_i to h_i, that is $f_i \succ_i^{\text{MP}} h_i$, if he prefers f_i to h_i but not the reverse, i.e. $f_i \succeq_i^{\text{MP}} h_i$ but $h_i \not\succeq_i^{\text{MP}} f_i$.

With respect to the notation, $f_i(\omega)$ signifies that an allocation specifies the consumption bundle at each state of nature ω. The separate reference to ω determines a specific state.

Definition 1 requires knowledge of the probabilities of all states of nature. Given allocation f_i one has to look at all the priors on the left-hand side and choose the one that minimizes the expected utility. Then one does the same thing on the right-hand side for allocation h_i. The prior that minimizes the right-hand side is not necessarily the same as the one for the left-hand side of the weak inequality. In effect the definition refers to the maximum among minima (maximin).

Definition 2 A feasible allocation $x = (x_i)_{i \in I}$ is said to be (maximin) individually rational, if for each i, $x_i \succeq_i^{\text{MP}} e_i$.

Definition 3 A feasible allocation $x = (x_i)_{i \in I}$ is said to be ex ante maximin efficient, if there does not exist another feasible allocation $y = (y_i)_{i \in I}$, such that $y_i \succeq_i^{\text{MP}} x_i$ and $y_i \succ_i^{\text{MP}} x_i$ for at least one i.

We are given μ_i the probability measure on the $\sigma(\mathcal{F}_i)$. Since this is not the complete multi-prior set, P_1 above must include $p_1 + p_2 = 2/3$ and P_2 must include $q_1 + q_3 = 2/3$. This is because Definition 1 requires the probability of all states ω.

In the Appendix we show that the proposed allocation is individually rational, ex ante maximin efficient. We also show that it is not unique.

4 The tree and the payoffs

First we look at the construction of the game tree shown here in Fig. 1(i), which is basically Fig. 2 in de Castro et al. (2017). It shows the players' information sets $A_1 = \{a, b\}, c_1 = \{c\}, A_2 = \{a, c\}$, and $b_2 = \{b\}$ with their probabilities and the payoffs in quantities which makes clear the implied redistribution of the initial endowments. The translation into utilities is straightforward. The heavy black lines will be explained below.

The payoffs are calculated as follows (de Castro et al. 2015). We are given the initial endowments and we use a particular specific allocation. Together they imply a planned redistribution of the endowments. Then employing certain rules referring to strategies used by the players, i.e. reports made, we calculate the payoffs. Finally *a choice mechanism* is invoked to select the equilibrium strategies and, hence, the payoffs.

From the initial endowments, e, and the proposed, above, reallocation among the players, x, it follows that the planned redistribution is

$$x - e = \begin{pmatrix} x_1(a) - e_1(a) & x_1(b) - e_1(b) & x_1(c) - e_1(c) \\ x_2(a) - e_2(a) & x_2(b) - e_2(b) & x_2(c) - e_2(c) \end{pmatrix} = \begin{pmatrix} 0 & -0.2 & 0.2 \\ 0 & 0.2 & -0.2 \end{pmatrix}.$$

We show below that the strategy profile of telling the truth, $s_1(A_1) = A_1, s_1(c_1) = c_1; s_2(A_2) = A_2, s_2(b_2) = b_2$, is an equilibrium of the game tree under the maximin choice rule. It implements the allocation x.

On the tree we distinguish between A and A', b and b', c and c', simply to indicate that they are played from different information sets. Of course as strategies, they are in each particular case the same.

First we look at the rules for calculating payoffs. When the reports $E^{\mathcal{F}_1}, \ldots, E^{\mathcal{F}_N}$ of the players are compatible, they end up with $e_i(\omega) + x_i(\tilde{\omega}) - e_i(\tilde{\omega})$, where $\tilde{\omega}$ is the agreed state, and $x_i(\tilde{\omega}) - e_i(\tilde{\omega})$ is the planned redistribution specified for the state $\tilde{\omega}$. Clearly, if all the players tell the truth, then $\tilde{\omega} = \omega$ and the players get what they planned to get, $e_i(\omega) + x_i(\omega) - e_i(\omega) = x_i(\omega)$. However, since a player may lie, the players may not end up with the planned allocation.

When the reports are not compatible at the realized state ω, in the end lies are detected. There are many ways to resolve the players' payoffs.

Furthermore, another approach would be that each player keeps his initial endowments as was done in Glycopantis et al. (2001, 2003), in another context. In Fig. 1 we indicate with an arrow the corresponding change in payoffs when reports are incompatible. We do not pursue this point.

5 The direct revelation mechanism and the maximin equilibrium

After the formulation of the game, there are rules that the players will follow in choosing their strategies. These are based on the consistent applications of their utility functions. The person who imposes the game to be played is called *the mechanism designer* and he applies a direct revelation mechanism to implement the allocation x.

More formally, a direct revelation mechanism concerning a planned allocation x for the underlying ambiguous asymmetric information economy $\mathcal{E} = \{\Omega; (\mathcal{F}_i, P_i, e_i, u_i) : i \in I\}$ is a set $\Gamma = \langle I, S, x - e, \{g_i\}_{i \in I}, \{v_i\}_{i \in I} \rangle$ which defines a non-cooperative game.

The players have Wald (1950) type preferences and adopt the maximin equilibrium of de Castro et al. (2015). Each player maximizes the minimum of his payoff taking into account the information he has received and the worst actions of all other players.

Let $\mathbb{ME}(\Gamma)$ denote the set of maximin equilibria of the mechanism. We say an allocation x is implementable, if it can be realized as a maximin equilibrium of the direct revelation mechanism.

Definition 10 In a direct revelation mechanism $\Gamma = \langle I, S, x - e, \{g_i\}_{i \in I}, \{v_i\}_{i \in I} \rangle$, a strategy profile $s^* = (s_1^*, \ldots, s_N^*)$ constitutes a *maximin equilibrium* ($\mathbb{ME}$), if for each Player i, his strategy s_i maximizes his interim payoff lower bound, that is, the function $s_i : \mathcal{F}_i \longrightarrow \mathcal{F}_i$ satisfies, for each $E^{\mathcal{F}_i} \in \mathcal{F}_i$,

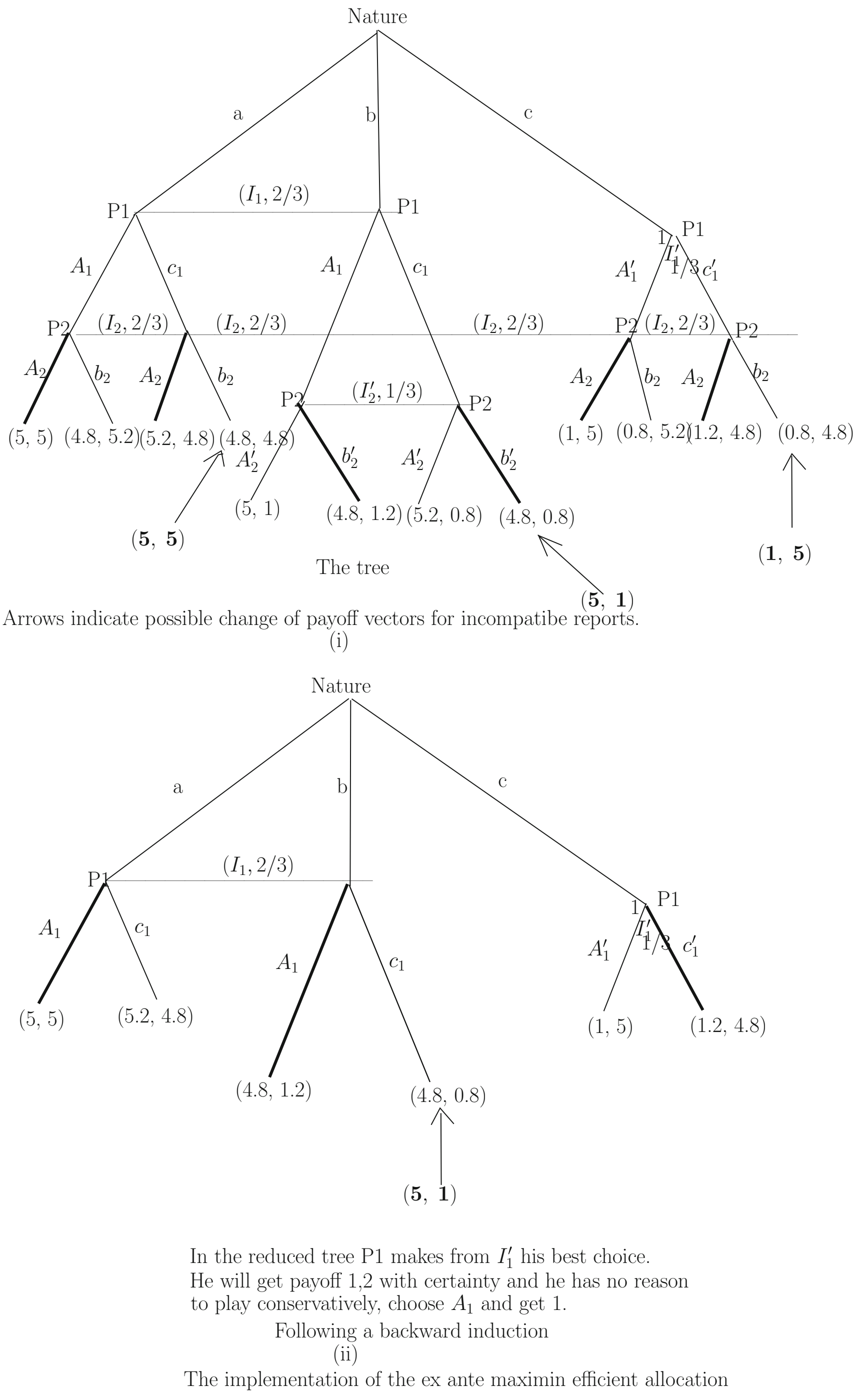

Fig. 1 The maximin equilibrium

$$\min_{\substack{E^{\mathcal{F}_{-i}} \in \mathcal{F}_{-i};\\ \omega' \in E^{\mathcal{F}_i}}} v_i\left(s^*(E^{\mathcal{F}_i}), E^{\mathcal{F}_{-i}}; \omega'\right) \geq \min_{\substack{E^{\mathcal{F}_{-i}} \in \mathcal{F}_{-i};\\ \omega' \in E_i^{\mathcal{F}_i}}} v_i\left(\hat{E}^{\mathcal{F}_i}, E^{\mathcal{F}_{-i}}; \omega'\right)$$

for all $\hat{E}^{\mathcal{F}i} \in \mathcal{F}_i$, where $E^{\mathcal{F}_{-i}}$ denotes the reports from all the other players, so $E^{\mathcal{F}_{-i}} \in \mathcal{F}_{-i} = \times_{j \neq i}\mathcal{F}_j$.

This has the flavour of a NE but the interdependence of the agents is different. The strategies of the players, given the reports of everybody else, maximize their lowest payoff.

Definition 11 Let x be a maximin IR and PO, (individually rational, Pareto optimal) allocation of an ambiguous asymmetric information economy $\mathcal{E}$, and $\mathbb{ME}(\Gamma)$ the set of maximin equilibria of the mechanism $\Gamma = \langle I, S, x - e, \{g_i\}_{i \in I}, \{v_i\}_{i \in I}\rangle$. The allocation x is *implementable as a maximin equilibrium* of the mechanism Γ if, $\exists s^* \in \mathbb{ME}(\Gamma)$, such that $g_i(x - e, s^*(\omega), \omega) = x_i(\omega)$, for all ω and i.

Implementation of the proposed IR and PO allocation

$$x = \begin{pmatrix} x_1(a) & x_1(b) & x_1(c) \\ x_2(a) & x_2(b) & x_2(c) \end{pmatrix} = \begin{pmatrix} 5 & 4.8 & 1.2 \\ 5 & 1.2 & 4.8 \end{pmatrix}$$

is achieved by the agents doing their calculations independently and simultaneously. Their reports determine the net transfers. Although the authors cast the game in tree form, the idea is more like a normal form game. The detailed calculations of the agents are explained.

The development of the figures in de Castro et al. is easy to follow. For the calculation of the payoffs at the terminal nodes, the rules explained above for compatible and incompatible reports of the agents are followed. The authors discuss how one arrives at the strategy profile which implements the ex ante maximin efficient allocation.

For each agent and for every information set, a separate sub-graph is extracted which extends to terminal nodes. In the spirit of the maximin preferences, precise calculations are made, to find the lower bound of his payoffs and this leads to the decision to tell the truth as to what information he has received. That is, when an agent observes an event, his chosen strategy will be to declare the event he has seen; that is he has no incentive to lie. As an example, when he observes the event A_2, Agent 2 will play the strategy $s_2(A_2) = A_2$ and this is part of the equilibrium. Through a truthful strategy profile, we obtain the maximin equilibrium, and hence the required net transfers. The calculations imply that the identified by the authors ex ante maximin efficient allocation is implemented through the constructed game with its rules, i.e. the particular direct revelation mechanism.

We vary the approach but of course in the end the same maximin idea is applied. We shall do the calculations through the application of the familiar, straightforward idea of backward induction. We start with our Fig. 1(i). We assume with the authors (page 247, footnote 12) that "a player lies only if he can benefit from doing so". As said above, we ignore the indicated replacements that will be made if the punishment is removed.

Information sets I_2 and I_2' belong to Agent 2, denoted by P2. The revelation mechanism employed, i.e. the structure of the constructed game, implies that from both information sets Agent 2 will choose the strategies to tell the truth. They will secure the lower bound of his payoffs. Hence he has no incentive to lie with respect to the information he has observed. This is indicated by the heavy black lines.

We go to the folded up tree which is in Fig. 1(ii). With respect to I_1 there is no problem. Now with respect to the singleton I_1' we note that the maximin and the maximum utility approaches come together. The player chooses with certainty the strategy which brings the best outcome.

Putting all the information together, we have obtained the result that the truth-telling, maximin solution allocation is $x_1(a) = 5, x_1(b) = 4.8, x_1(c) = 1.2$ and $x_2(a) = 5, x_2(b) = 1.2, x_2(c) = 4.8$.

Indeed this allocation is IR and PO and it has been shown that it is implementable as a unique maximin equilibrium.[1]

6 The perfect Bayesian equilibrium and its calculation

The construction of the game tree is based on the initial endowments, a particular allocation on which we anchor and certain rules for calculating payoffs. Therefore we could look for the PBE solution and then compare it with the maximin solution.

We shall now turn our attention to the application of the game-theoretic equilibrium concept of PBE. It is appropriate for analyzing the equilibrium and its implementation through extensive form game trees. All PBE are of course NE. The idea is significant because it employs the Bayesian updating of beliefs of the agents. It utilizes extensive form, dynamic game trees.

More formally, a PBE consists of a set of players' optimal behavioural strategies, and consistent with these, a set of beliefs which attach a probability distribution to the nodes of each information set. Consistency requires that the decision from an information set is optimal given the particular player's beliefs about the nodes of this set and the strategies from all other sets, and that beliefs are formed from updating, using the available information. If the optimal play of the game enters an information set, then updating of beliefs must be Bayesian. Otherwise appropriate beliefs are assigned arbitrarily to its nodes.

In the context of differential information economies, we are interested in the implementation or non-implementation properties, in terms of PBE, of various equilibrium notions.

The static concept of the *coalitional Bayesian incentive compatibility* (CBIC) implies that no agent has an incentive to lie with respect to the state(s) he has observed and the PBE satisfies basic rationality criteria in a game where the agents are asymmetrically informed.

The issue is whether cooperative and non-cooperative static solutions can be supported through a non-cooperative solution concept. It has been shown that CBIC

[1] See also de Castro et al. (2017) for more details.

allocations can be supported by a PBE. Such considerations can help us to decide how to choose from the available equilibrium concepts the most appropriate one.

The ambiguity model is different in structure from the ones in Glycopantis et al. (2001, 2003). In de Castro et al. (2017) decisions are taken simultaneously, and when player P2 is to act he does not know what P1 has chosen. Furthermore it is characteristic that the agents attach probabilities to the information sets.

For the PBE we assume that Nature chooses states a, b, c, each with probability 1/3 and the agents no longer attach probabilities to their information sets. On the other hand, we retain the assumption that when P2 is to act he does not know what P1 has chosen. The PBE, i.e. a NE with consistent beliefs, will be calculated on this basis. The relevant graph is Fig. 2. The payoffs at the terminal nodes are as in Fig. 1, so that comparisons with the maximin solution can be made.

This section answers the following question:

Can the maximin Efficient and IR (individually rational) allocation of the previous example be implemented as a PBE? The previous section proved that it can be implemented as a unique maximin equilibrium. It would be of interest to know if it can be implemented as a PBE.

This section will provide a negative answer. We will show that the maximin IR and maximin efficient allocation $x = (x_1, x_2) = ((5, 4.8, 1.2), (5, 1.2, 4.8))$ *cannot* be implemented as a PBE contrary to Sect. 5, which showed that it can be implemented as a maximin equilibrium.

The probabilities 1/2 in the nodes of information set I_1 reflect the fact that states a and b are equally likely. That is when player P1 finds himself in information set I_1 he believes rationally that he is at each of its nodes with probability 1/2.

Next, we call the nodes of information set I_2, from left to right, $\eta_1, \eta_2, \eta_3, \eta_4$ and the optimal choices, calculated through backward induction are shown through heavy lines, which go through η_2 and η_3. Bayesian updating is now different from the one in Glycopantis et al. (2001, 2003). Consider node η_2. In the present context we have

$$\begin{aligned}\text{Prob}(\eta_2/I_2) &= \text{Prob}(I_2/\eta_2)\text{Prob}(\eta_2)/(\text{Prob}(I_2/\eta_2)\text{Prob}(\eta_2)\\ &\quad+\text{Prob}(I_2/\eta_3)\text{Prob}(\eta_3))\\ &= 1 \times 1/3/(1 \times 1/3 + 1 \times 1/3) = 1/2.\end{aligned}$$

This is the belief of P2 that he is at node η_2. An explanation of the terms in the formula is as follows: $\text{Prob}(I_2/\eta_i) = 1$, $i = 1, 2, 3, 4$, because if you find yourself in any of these η_i's you know that you are in I_2. On the other hand, $\text{Prob}(\eta_i)$ is the probability of reaching node η_i through the optimal path. This is the product of the probability of a state and the probability of choice by the player. We have $\text{Prob}(\eta_2), \text{Prob}(\eta_3) = 1/3$ and $\text{Prob}(\eta_1), \text{Prob}(\eta_4) = 0$.

Similarly we obtain the belief 1/2 for node η_3, and 0 beliefs for nodes η_1, η_4.

Next, we call the nodes of information set I_2', from left to right, η_5, η_6 and the optimal paths go through η_6. I_2' belongs to P2.

Consider now node η_6 of information set I_2'. We have

$$\text{Prob}(\eta_6/I_3) = \text{Prob}(I_3/\eta_6)\text{Prob}(\eta_6)/(\text{Prob}(I_3/\eta_5)\text{Prob}(\eta_5)$$

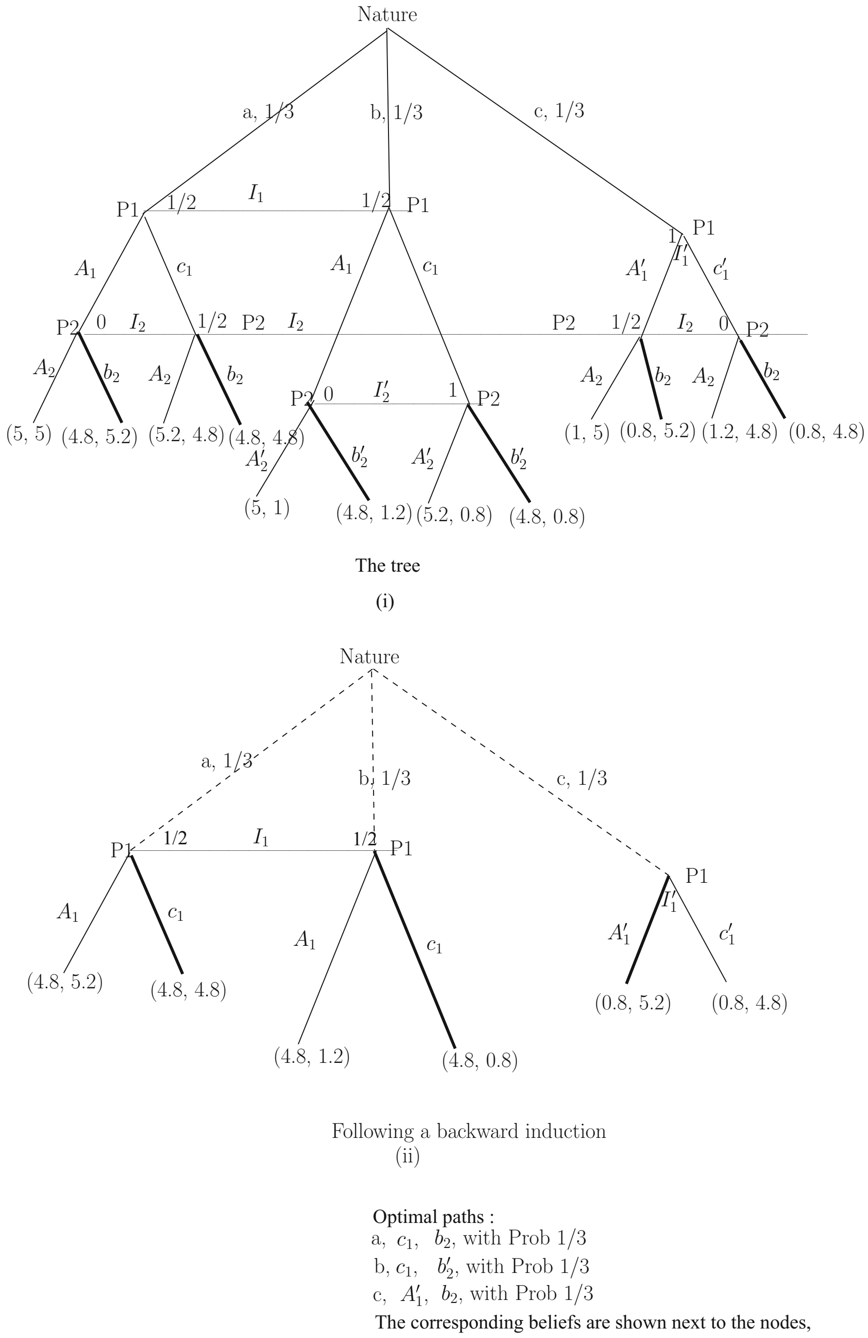

Fig. 2 The perfect Bayesian equilibrium

$$+\,\text{Prob}(I_3/\eta_6)\text{Prob}(\eta_6))$$
$$= 1 \times (1/3)/(1 \times 0 + 1 \times 1/3) = 1.$$

That is P2 believes that he is definitely at node η_6.

So we have obtained a PBE, i.e. optimal strategies and consistent with these, beliefs obtained through Bayesian updating. The optimal expected utility payoffs of the players are $E_1 = \frac{1}{3}(u_1(4.8)+u_1(4.8)+u_1(0.8))$ and $E_2 = \frac{1}{3}(u_2(4.8)+u_2(0.8)+u_2(5.2))$.

The PBE shown above is not unique.[2] For example in Fig. 2(ii) player P1 can decide to play A_1 instead of c_1. This would imply a corresponding change in beliefs, and the alternative PBE allocation

$$x = \begin{pmatrix} x_1(a) & x_1(b) & x_1(c) \\ x_2(a) & x_2(b) & x_2(c) \end{pmatrix} = \begin{pmatrix} 4.8 & 4.8 & 0.8 \\ 5.2 & 1.2 & 5.2 \end{pmatrix}$$

which is discussed below.

7 Comparison of the maximin equilibrium and the PBE

We pursue the de Castro et al. (2017) example further. It sets probabilities on the states of nature, and applies the idea of a PBE in implementing an allocation. The PBE implementation is not unique while the maximin equilibrium is.

The maximin approach implements the IR and PO allocation x which is used itself in the first instance to calculate the payoffs of the tree. On the other hand, one can take the constructed game tree as given, add probabilities on the states of nature and apply the alternative, expected utility idea of a PBE for implementation of allocations. We note that the structure of the information sets is now different from earlier work on asymmetric differential information economies.

In comparing the two ideas, we make the following observations: (i) First the note shows that a PBE is again available and the calculations of beliefs can be done through Bayesian updating. (ii) Second the PBE is not unique. Hence comparisons of the two solutions do not really help. For example, consider Fig. 2(ii) here. If from I_1 player P1 decides to play A_1 rather than c_1, then under P2 he will get $y_1(a) = 5.2$, $y_1(b) = 1.2$, $y_1(c) = 5.2$, more than under the maximin solution. The issue is simply that the PBE approach is valid. (iii) The maximin solution is not a NE. We can see this as follows. Consider Fig. 1(i). Suppose that P2 finds himself in information set I_2. Then he would be better off to play b_2 rather than A_2.

An explanation on (ii) follows: The maximin solution allocation is $x_1(a) = 5, x_1(b) = 4.8, x_1(c) = 1.2$ and $x_2(a) = 5, x_2(b) = 1.2, x_2(c) = 4.8$. We have also found that a PBE allocation is $y_1(a) = 4.8, y_1(b) = 4.8, y_1(c) = 0.8$ and $y_2(a) = 4.8, y_2(b) = 0.8, y_2(c) = 5.2$. Observation (ii) points out that we cannot conclude now that the ex post maximin solution performs better.

On the one hand, direct comparison between the quantities of the maximin allocation x and this particular PBE allocation y, above, shows that the maximin allocation

[2] This is contrary to maximin equilibrium which is unique.

performs better. However, as pointed out, if from I_1 player P1 chose A_1 rather than c_1, then we obtain another PBE allocation in which P2 gets quantities {5.2, 1.2, 5.2}, more than he gets under the maximin solution. Therefore comparisons between maximin and PBE solutions are not meaningful.

Now, the allocation y can be implemented as a PBE if the agents' beliefs on the nodes of the tree are q*. The question is whether the mechanism designer can change the agents' original ambiguous beliefs to q*. This might require a mechanism in stages, up to the point when the game tree has been allocated beliefs q* to its nodes.

On a different issue, one could consider Nature as a third, fictitious player or an intermediary. It facilitates trade and drops out at the end, like the Walrasian auctioneer. Its endowments, utility functions and payoffs will be zero and the probability per state 1/3. Although the zero utility function is not strictly concave, Definitions 1, 2 and 3 are satisfied. We could talk now about expected payoffs and the tree would be slightly easier to comprehend. The two players act independently of each other but they get some information from Nature.

Next, we want to discuss the IC property of the allocations in the example. First we make general remarks.

Taking as the basis the example of an economy with asymmetric information discussed here, we consider in some detail the ex ante idea of IC allocations. These exhibit stability in that no agent has an incentive to lie about the information he has received. Of course when the state of nature has been revealed ex post, then everybody knows everything and there is no scope in lying.

The situation that the idea of IC attempts to capture is when one is neither in an ex post revelation of the state nor in the limited knowledge of the initial structure of the model.

Now, before the actual state is revealed, Agent 1 'strongly believes' that Agent 2 will accept the statement that it is state c and act accordingly. Therefore Agent 1 can lie and say that it is state c. This will imply that state b can be excluded as a possibility because if the state was b then Agent 2 would see it and he would not accept the lie. Therefore Agent 1 can work on the basis that it is state a and he lies, saying that it is state c.

Analogously, Agent 2 'strongly believes' that Agent 1 will accept the statement that it is state b and act accordingly.

In order to declare that an allocation is not incentive compatible it must be possible that an agent can lie about a state of nature and end up with higher utility than what he has been allocated.

We discuss two cases.

Case 1. Consider the PBE allocation

$$x = \begin{pmatrix} x_1(a) & x_1(b) & x_1(c) \\ x_2(a) & x_2(b) & x_2(c) \end{pmatrix} = \begin{pmatrix} 4.8 & 4.8 & 0.8 \\ 5.2 & 1.2 & 5.2 \end{pmatrix}.$$

We wish to check its incentive compatibility. We ask whether it is possible that an agent can lie about a state and end up with higher utility than what he has been allocated under the PBE.

If A1 lies that he has seen c, he will end up with $u_1(5 + 0.8 - 1) = u_1(4.8)$. The same as the utility $u_1(x_1(a))$ of the PBE. If A2 lies that he has seen b, he will end up with $u_2(5 + 1.2 - 1) = u_2(5.2)$. The same as the utility $u_2(x_2(a))$ of the PBE. Therefore nobody can lie and benefit.

The incentive compatibility of the allocation matches up with the fact that it can be supported as a PBE of a game tree analysis.

Case 2. Next we consider the maximin allocation.

Agent 1 is in the set $\{a, b\}$ and expects utilities $u_1(5.2)$, from lying that he has seen state c, and $u_1(4.8)$ from state b. Applying the maximin utility criterion, he will expect the utility $u_1(4.8)$ which is the same as the maximin utility of the proposed allocation for the set $\{a, b\}$.

Similarly, Agent 2 is in the set $\{a, c\}$ and expects utilities $u_2(5.2)$, from lying that he has seen state b, and $u_2(4.8)$ from state c. Applying the maximin utility criterion, he will expect the utility $u_2(4.8)$ which is the same as the maximin utility of the proposed allocation for the set $\{a, c\}$.

Therefore when Agent i lies he takes the $\min(u_i(5.2), u_i(4.8)) = u_i(4.8)$. Telling the truth yields $\min(u_i(5), u_i(4.8)) = u_i(4.8)$. Hence there is no incentive to lie. The above implies that *the maximin proposed allocation is incentive compatible*.

8 Concluding remarks

In the context of non-cooperative behaviour under ambiguity, de Castro et al. (2017) introduce conditions under which a maximin individually rational and ex ante maximin efficient allocation, x, is implementable. It is implemented through the mechanism $\Gamma = \langle I, S, x - e, \{g_i\}_{i \in I}, \{v_i\}_{i \in I} \rangle$ as its unique maximin equilibrium outcome.

In general, theory and examples explain and support each other. The theory explains how a result can be justified and an example can open new theoretical possibilities. First we confirm that the proposed allocation by the authors in their example is indeed individually rational and ex ante maximin efficient. We explain that it is not unique.

We also show how implementation of the proposed allocation, x, can be achieved through backward induction, although the agents cannot finally calculate the expected utilities.

The payoffs in the de Castro et al. game tree are based on rules which anchor on x, and the implementation through maximin preferences chooses x again. On the other hand, one can set probabilities on the states of nature and implement an outcome, y, as a PBE. In relation to the game trees in Glycopantis et al. (2001, 2003), the messages that the agents receive now have a different structure. Even so it is straightforward to calculate the conditional expectations which express the beliefs used to obtain the agents' optimal strategies. This settles the issue of the existence of a PBE.

It is pointed out that comparisons of the maximin and the expected utility PBE solutions are not meaningful. Nevertheless the maximin solution provides higher expected utility than one of the PBE allocations. Finally it is shown that the maximin allocation is IC while a PBE is not. This is due to their diametrically different utility choice formulations.

Acknowledgements Very helpful comments by Zhiwei Liu on earlier drafts were greatly appreciated. We are also very grateful to a referee for very helpful comments. Of course all inadequacies are our responsibility.

Appendix

In order to check the *individual rationality* of the proposed allocation, we consider Agent 1. We want to solve the following:

Problem 1

$$\text{Minimize } p_1\sqrt{5} + p_2\sqrt{4.8} \quad \text{subject to } p_1 + p_2 = 2/3.$$

It is straightforward that we obtain the corner solution $p_1 = 0$ and $p_2 = 2/3$. This justifies the formula used by de Castro et al.:

$$\frac{2}{3}\min\{\sqrt{5}, \sqrt{5}\} + \frac{1}{3}\sqrt{1} = 1.824 < \frac{2}{3}\min\{\sqrt{5}, \sqrt{4.8}\} + \frac{1}{3}\sqrt{1.2} = 1.826.$$

Similarly we obtain the identical relation for Agent 2. Hence the proposed feasible allocation is individually rational and Definition 2 is satisfied. Each agent loses 0.2 and gains 0.2 units of different goods but from distinct starting quantities and the agent becomes strictly better off.

Next we want to show that the proposed allocation above is *ex ante maximin efficient*. First, we want to give a justification of the choice of the proposed allocation. One can trace the steps through a social welfare function. Alternatively we can start the analysis from Problem 3.

However it is of some interest to see that the proposed allocation emerges also as the one that maximizes a social welfare function, W, in which the two agents have equal weights. We are looking for the solution of:

Problem 2

$$\begin{array}{ll} \text{Maximize} & W = 2\sqrt{x_1(a)} + 2\sqrt{x_1(b)} + \sqrt{x_1(c)} + 2\sqrt{x_2(a)} + \sqrt{x_2(b)} + 2\sqrt{x_2(c)} \\ \text{Subject to} & x_1(a) + x_2(a) = 10, \quad x_1(b) + x_2(b) = 6, \quad x_1(c) + x_2(c) = 6, \end{array}$$

where the coefficients 2 and 1 are suggested by $\frac{2}{3}$ and $\frac{1}{3}$ of the de Castro et al. formula above.

The problem is separable into the sub-problems per individual constraint, as there is no transferability of endowments between periods. We have the unique solutions $x_1(a) = x_2(a) = 5$, $x_1(b) = 4.8$, $x_2(b) = 1.2$ and $x_1(c) = 1.2$, $x_2(c) = 4.8$. There does not exist another feasible allocation which both agents prefer and at least one of them prefers strictly. It follows that we also have the unique solution of:

Problem 3

$$\begin{array}{ll} \text{Maximize} & W'' = 2\sqrt{x_1(b)} + \sqrt{x_1(c)} + \sqrt{x_2(b)} + 2\sqrt{x_2(c)} \\ \text{Subject to} & x_1(b) + x_2(b) = 6, \quad x_1(c) + x_2(c) = 6. \end{array}$$

The solution is $X_1 = (x_1(b), x_1(c)) = (4.8, 1.2)$ and $X_2 = (x_2(b), x_2(c)) = (1.2, 4.8)$; and $u_1 = 1.826$ and $u_1 = 1.826$.

But of course for Definition 3 we have to go to the maximin criterion of Definition 1.

We must take into account that we have priors implying, as shown in Problem 1, multiplication of a minimum by 2/3. We know that for Problem W'' we have obtained unique solutions X_1 and X_2. Suppose we attach to these the quantities $y_1 = y_2 = 5$. Then we get the solution of Problem 2 which is also what was called above the proposed allocation.

We shall show that this resulting allocation is ex ante maximin efficient. We argue as follows. Suppose one increases y_1 and reduces y_2, only. The utility of Agent 1 cannot increase because $\min(y_1, 4.8)$ stays at 4.8 and the utility of Agent 2 can only be reduced if for example $\min(y_2, 4.8) = y_2$ which will now itself carry the coefficient 2/3.

Analogously, looking at it from the point of view of Agent 2, if we only decrease y_1 and increase y_2, we do not obtain a superior allocation according to the maximin criterion.

Next, suppose that we only change the vectors X_1 and X_2. The uniqueness of the solution of Problem 3 implies that the utility value of at least one of the X_is has been reduced. Without loss of generality, let this be that of X_1.

In all circumstances, there will be, according to the maximin criterion, a reduction in the utility for Agent 1. Therefore changes in X_1 and X_2 alone cannot lead to a strictly preferred allocation.

Suppose now we consider a combination of all changes at the same time. Suppose, without loss of generality, that the utility value of X_1 has been reduced. No matter what the change in y_1 is, the utility of Agent 1 according to the maximin criterion will be reduced. Hence the proposed allocation is ex ante maximin efficient.

Now we turn our attention to the question of whether the proposed allocation is the unique one which has this property. Consider the allocation

$$x = \begin{pmatrix} x_1(a) & x_1(b) & x_1(c) \\ x_2(a) & x_2(b) & x_2(c) \end{pmatrix} = \begin{pmatrix} 5-\epsilon & 4.8 & 1.2 \\ 5+\epsilon & 1.2 & 4.8 \end{pmatrix},$$

where ϵ is very small. Notice that ϵ and $-\epsilon$ are chosen to preserve feasibility because we do not assume free disposal. Also ϵ is chosen small enough so that we still have $\min(5-\epsilon, 4.8) = 4.8$, i.e. $-0.2 < \epsilon < 0.2$.

The new allocation is individually rational and this is easy to see. Analogously to the previous proof above we obtain, for Agent 1 and Agent 2, respectively,

$$\frac{2}{3}\min\{\sqrt{5}, \sqrt{5}\} + \frac{1}{3}\sqrt{1} < \frac{2}{3}\min\{\sqrt{5-\epsilon}, \sqrt{4.8}\} + \frac{1}{3}\sqrt{1.2}, \quad \text{and}$$
$$\frac{2}{3}\min\{\sqrt{5}, \sqrt{5}\} + \frac{1}{3}\sqrt{1} < \frac{2}{3}\min\{\sqrt{5+\epsilon}, \sqrt{4.8}\} + \frac{1}{3}\sqrt{1.2}.$$

Next, we have to show that, in spite of the new $(y_1', y_2') = (x_1(a), x_2(a))$, any such allocation is ex ante maximin efficient. We repeat the previous argument.

Suppose now considering simultaneous changes of the allocation vectors, that X_1 has been reduced. No matter what the change in y_1' is, the utility of Agent 1 according to the maximin criterion will be reduced. Hence the proposed allocation is ex ante maximin efficient.

References

Angelopoulos, A., Koutsougeras, L.C.: Value allocation under ambiguity. Econ. Theory **59**(1), 147–167 (2015)

de Castro, L.I., Yannelis, N.C.: Ambiguity aversion solves the conflict between efficiency and incentive compatibility. Working Paper (2009)

de Castro, L.I., Yannelis, N.C.: An interpretation of Ellsberg's Paradox based on information and incompleteness. Econ. Theory Bull. **1**(2), 139–144 (2013)

de Castro, L.I., Pesce, M., Yannelis, N.C.: Core and equilibria under ambiguity. Econ. Theory **48**, 519–548 (2011)

de Castro, L.I., Liu, Z., Yannelis, N.C.: Implementation under ambiguity. Games Econ. Behav. (2015). https://doi.org/10.1016/j.geb.2015.10.010

de Castro, L.I., Liu, Z., Yannelis, N.C.: Ambiguous implementation: the partition model. Econ. Theory **63**, 233–261 (2017). https://doi.org/10.1007/s00199-016-1023-y

Glycopantis, D., Muir, A., Yannelis, N.C.: An extensive form interpretation of the private core. Econ. Theory **18**, 293–319 (2001)

Glycopantis, D., Muir, A., Yannelis, N.C.: On extensive form implementation of contracts in differential information economies. Econ. Theory **21**, 495–526 (2003)

He, W., Yannelis, N.C.: Equilibrium theory under ambiguity. J. Math. Econ. **61**, 86–95 (2015)

Holmström, B., Myerson, R.B.: Efficient and durable decision rules with incomplete information. Econometrica **51**(6), 1799–1819 (1983)

Liu, Z.: A note on the welfare of the maximin rational expectations. Econ. Theory Bull. **2**, 213–218 (2014)

Palfrey, T.R., Srivastava, S.: On Bayesian implementable allocations. Rev. Econ. Stud. **54**(2), 193–208 (1987)

Wald, A.: Statistical Decision Functions. Wiley, New York (1950)

Yannelis, N.C.: The Core of an economy with differential information. Econ. Theory **1**, 183–198 (1991)