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Markets with Many More Agents than Commodities: Aumann's "Hidden" Assumption[1]

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We address a question posed by J.-F. Mertens and show that, indeed, R. J. Aumann's classical existence and equivalence theorems depend on there being "many more agents than commodities." We show that for an *arbitrary* atomless measure space of agents there is a fixed non-separable infinite dimensional commodity space in which one can construct an economy that satisfies all the standard assumptions but which has no equilibrium, a core allocation that is not Walrasian, and a Pareto efficient allocation that is not a valuation equilibrium. We identify the source of the failure as the requirement that allocations be strongly measurable. Our main example is set in a commodity–measure space pair that displays an "acute scarcity" of strongly measurable allocations—where strong measurability necessitates that consumer choices be closely correlated no matter the prevailing prices. This makes the core large since there may not be any strongly measurable improvements even though there are many weakly measurable strict improvements. Moreover, at some prices the aggregate demand correspondence is empty since disaggregated demand has no strongly measurable selections, though it does have weakly measurable selections. We note that our example can be constructed in any vector space whose dimension is greater than the cardinality of the continuum—that is, *whenever there are at least as many commodities as agents*. We also prove a positive core equivalence result for economies in non-separable commodity spaces. *Journal of Economic Literature* Classification Numbers: C62, C71, D41, D50. © 2001 Elsevier Science

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1. INTRODUCTION

The Arrow–Debreu–McKenzie model of exchange under perfect competition is formulated in terms of a finite set of agents taking prices as given and engaging in the sale and purchase of a finite number of commodities (Arrow and Debreu [3], McKenzie [30], Debreu [11]). This formulation, though at times technically convenient, raises conceptual difficulties. For example, a finite number of agents should mean that individuals are able to exercise some influence and that the assumption of price taking behavior is nonsensical. Furthermore, a finite number of commodities postulates a predetermined termination date beyond which all economic activity ceases and a predetermined finite set of uncertain states of the world. It also excludes the many important models of general equilibrium that are set in infinite dimensional commodity spaces.

In his classical papers, Aumann [4, 5] suggested that the appropriate model for perfectly competitive markets is one with a continuum of traders. The insignificance of individual traders is thus captured by the idea of a set with zero measure. Furthermore, summation or aggregation is generalized by the notion of the Lebesgue integral. Aumann also showed that the set of core allocations in his model coincide with the set of Walrasian allocations, which in turn is not empty (see also Hildenbrand's classical book [19]).

Since it is impossible to integrate functions that are not measurable the class of "admissible" allocations in Aumann's model is restricted to that of measurable functions. Thus, Walrasian allocations, Pareto improvements, and the improving allocations that can be considered by blocking coalitions must all be measurable. This restriction, however, seems innocuous when there is a finite number of commodities *and* Aumann's measurability assumption on preferences is satisfied. In the main this paper is concerned with Aumann's model when there are infinitely many commodities. It shows that when the commodity space is very large this restriction may be so severe that it renders Aumann's [4, 5] theorems false. The paper solves an open question which appears to have first been posed by Mertens who writes:

> "*If I remember correctly that conversation with Aumann, he was stressing the importance of going beyond the separable case* (*strong measurability*), *to check really whether equivalence did depend on there being many more agents than commodities.*" [31, Footnote 2, p. 189]

Though the Lebesgue integral is a finite dimensional integral it has a straightforward infinite dimensional abstraction termed the Bochner integral (see Dunford and Schwartz [14], Diestel and Uhl [12], Yannelis [45]). Indeed, the Bochner integral has been used to generalize Aumann's core equivalence theorem and to prove the existence of equilibrium in various infinite dimensional gener equilibrium and game theoretic models. Moreover, in sharp contrast to other more general integrals,[3] the Bochner integral retains a most elementary property of economic feasibility: net trades associated with feasible allocations cannot be positive and non-zero for every consumer (see Section 10).

This paper, however, provides a class of counter examples which show that if aggregation is generalized by the Bochner integral then the core equivalence result of Aumann [4] and the existence of equilibrium result of Aumann [5] do not hold in non-separable infinite dimensional commodity spaces.[4] We show that for an *arbitrary* atomless measure space of agents there is a fixed non-separable infinite dimensional commodity space in which one can construct an economy that satisfies all the desirable assumptions but which has no equilibrium, a core allocation that is not Walrasian, and a Pareto efficient allocation that is not a valuation equilibrium. By desirable assumptions we mean those assumptions which guarantee that these results hold in the finite dimensional setting as well as in the separable infinite dimensional setting:

1. The positive cone of the commodity space has a non-empty interior.
2. Initial endowments are strictly positive.
3. Consumption sets are the positive cone of the commodity space.[5]
4. Preferences are induced by continuous convex strictly monotone utility functions.
5. Preferences satisfy Aumann's measurability assumption: If x and y are *strongly measurable* allocations then the set of agents that prefer x over y is measurable.

It is well understood that what drives the results of Aumann is the requirement that the space of agents be atomless. When the number of commodities is finite this requirement underscores the "hidden" assumption that the economy has "many more" agents than commodities. This paper shows that Aumann's classical existence and equivalence theorems depend on there being many more agents than commodities.

When the number of commodities is finite Aumann's "hidden" assumption has two important manifestations. First, Lyapunov's convexity theorem, which guarantees that the integral of set-valued functions is closed and convex.

[3] E.g., the Pettis or Gelfand or Dunford integrals (see Diestel and Uhl [12], Talagrand [43]).

[4] The paper also shows that the theorems in Khan and Yannelis [24] and Rustichini and Yannelis [38, 40], which are set in infinite dimensional separable Banach spaces, do not extend to non-separable spaces.

[5] For non-existence we truncate the economy so that consumption sets coincide with a weakly compact convex subset of the positive cone. Also, this set contains initial endowments in its norm interior.

Second, the various measurable selection results that are due to the "abundance" of strongly measurable functions from the space of agents into the commodity space—which ensure that strongly measurable functions are rich enough abstractions of perfectly competitive allocations. Both of these manifestations can fail in infinite dimensional settings, albeit for two very different reasons. Lyapunov's convexity theorem depends on the topolinear relationship between the commodity space and the space of essentially bounded measurable functionals.[6] It may fail even when the commodity space is separable, see Diestel and Uhl [12, p. 265] or Yannelis [45, p. 24]. However, the abundance of strongly measurable functions is a topological property that is inherited by separable metrizable commodity spaces.[7] The main example in this paper is set in commodity-measure space pairs that display an "acute scarcity" of strongly measurable functions.

It is impossible to Bochner integrate a function unless that function is strongly measurable and can be essentially approximated by a sequence of step functions. Thus, a purely technical consequence of adopting the Bochner integral is that feasible allocations must be strongly measurable and that coalitions can only consider those improvements that are strongly measurable. We identify the source of the failure of existence of equilibrium and the failure of core equivalence as this technical requirement of strong measurability.[8] In large spaces, this restriction makes the core large since there may not be any strongly measurable improvements even though there are many weakly measurable strict improvements. It also makes the set of Walrasian allocations small. In our example, there exists a Pettis integrable only weakly measurable Walrasian allocation, even though there are no strongly measurable equilibria.

Technically, the argument goes as follows. Suppose that the set of consumers is the continuum $[0, 1]$. We consider the non-separable Hilbert space $l_2([0, 1])$ (see Section 2 for definition). Here $[0, 1]$ may be interpreted as the set of commodities. We define an ordering on $l_2([0, 1])$ and an economy that satisfies the desired assumptions. In this economy each consumer $t \in [0, 1]$ extremely desires the commodity t. We note, however, that measurable allocations have the property that almost every consumer is allocated zero of every commodity in a fixed set of full measure, see Proposition 8.2. This property, which has no analogue in the separable

[6] Lyapunov's convexity theorem holds if for each coalition the cardinality of the family of distinct sub-coalitions is larger than the dimension of the commodity space, see Sections 7.

[7] We see from the proofs in Rustichini and Yannelis [38] that the Aumann core equivalence result can be proven without regard to whether or not the exact version of Lyapunov's convexity theorem holds. Thus, Rustichini and Yannelis' generalization of the Aumann core equivalence theorem to separable infinite dimensional spaces seems to inherit the "hidden" assumption of Aumann.

[8] See also the discussion in Mertens [31] on the importance of measurability.

spaces l_2^n and l_2, is the source of the failure of both existence of equilibrium and core equivalence. The property could be interpreted as saying that, in our non-separable setting, strong measurability makes it seem as though there are many more commodities than agents—hence violating Aumann's "hidden" assumption—and that strong measurability necessitates that consumer choices be closely correlated no matter the prevailing prices.

To establish that our economy does not have a price equilibrium we show that if $f: [0,1] \to l_2([0,1])$ is a strongly measurable Walrasian allocation, with an equilibrium price p, then there must be some $t \in [0,1]$ with the following contradictory properties:

1. The value of commodity t under p is zero.
2. Consumer t consumes zero of commodity t.
3. $f(t)$ maximizes consumer t's utility subject to the budget constraint.

To show core-Walras non-equivalence we identify an allocation and show that any improvement upon this allocation has the following property:

There is some non-negligible coalition $S' \in [0,1]$ such that each consumer $t \in S'$ must be allocated a strictly positive amount of the commodity t.

Such improvements cannot be essentially separably valued and thus cannot be strongly measurable. Therefore, this allocation cannot be improved upon by any coalition and is in the core. This is the case even though the allocation is not a Walrasian equilibrium and there would have been many improvements had we considered the more general Pettis integral.

The negative results in this paper points to the insight that Aumann's measurability assumption is far less restrictive in the non-separable setting than it is in the separable setting. Indeed, because of the "scarcity" of strongly measurable allocations it is almost superfluously satisfied in our example. We emphasize this insight by first showing that one cannot hope to prove core equivalence without some measurability assumption on preferences, even in the finite dimensional setting. We then strengthen Aumann's measurability assumption and prove a positive equivalence result for *Bochner* economies in non-separable commodity spaces.

The lack of atomic coalitions is crucial for Aumann's proof of core equivalence. However, every non-atomic measure space admits a non-measurable atom: A non-measurable set that does not contain non-negligible measurable subsets (see Section 4). In separable commodity spaces, the role of Aumann's measurability assumption in the proof of core equivalence is to negate the effects of such non-measurable atoms. We strengthen that role and assume the following technical assumption:

Suppose that the commodity space has a base with cardinality $\mathbf{m}$. *There is a family of coalitions* $\mathfrak{I}$ *with the property that the union of* $\mathbf{m}$ *null sets in* $\mathfrak{I}$

is a null set. Moreover, if $\tilde{x}$ is a core allocation, $t \mapsto e_t$ is the endowments mapping, and z is an arbitrary commodity bundle then the set of agents that prefer $t \mapsto e_t + z$ over $\tilde{x}$ is not only measurable but is also in $\mathfrak{I}$.

This strengthened measurability assumption is adequately onerous in large spaces;[9] it is also implied by Aumann's measurability assumption when the commodity space is separable. Consequently our positive theorem implies Aumann's classical equivalence result as well as the related infinite dimensional theorem of Rustichini and Yannelis [38].

It should be noted that existence of an equilibrium and core equivalence results with a continuum of agents in non-separable Banach commodity spaces are already available in the literature. For example, for the commodity space L_∞ [10] (see Bewley [8, 9], Mertens [31], Noguchi [32], Podczeck [35]) and the commodity space $M(\Omega)$[11] (see for example Mas-Colell [29]). However, the space L_∞ in these results is endowed with the Mackey-(L_∞, L_1) topology which is metrizable and separable. Moreover, the weak-$(M(\Omega), C(\Omega))$ topology of the the closed unit ball of $M(\Omega)$ is metrizable and separable. Therefore these positive results preserve Aumann's "hidden" assumption. In particular, the usual measurable selection theorems hold in such settings.

Our counter examples indicate that once this "hidden" assumption is violated existence of an equilibrium and core-equivalence fail. Indeed, a corollary of our main result states that for any choice of atomless measure space of agents and any non-separable Hilbert commodity space there is an economy that satisfies all the desirable assumptions but which has no equilibrium, a core allocation that is not Walrasian, and a Pareto efficient allocation that is not a valuation equilibrium. Non-separable Hilbert commodity spaces arise in models of asset trade under uncertainty (e.g., Khan and Sun [22]). When the sample space of uncertain states is not separable, the space of all asset returns with finite variance is a non-separable Hilbert space. These spaces are covered by our negative results.

Moreover, since the mid-1980's many of the important theorems from the finite dimensional literature have been extended to exceptionally general infinite dimensional spaces, but for economies with a finite number of traders. This paper shows that it may not be possible to emulate that literature without adopting a different notion of aggregation than the Bochner integral and possibly sacrificing economic content.

[9] In one sense it requires that consumer characteristics be less heterogeneous that in our counter example, see Corollary 10.2.1 in Section 10.

[10] The space of essentially bounded measurable functionals on a measure space.

[11] The space of finite measures on the compact metric space Ω.

The next section contains the notation, basic definitions, and some introductory results. Section 3 defines the notion of a Bochner economy and the notion of a Pettis economy.

Our counter examples are set-out as theorems in Sections 4 and 5, where we establish a more general result than the one we have already advertised. We show that existence of equilibrium, Core-equivalence, and the second theorem of welfare economics fail whenever the following holds.

There is a non-negligible coalition that is the union of a family of null sets smaller than dimension of the commodity space.

One implication of this result is that if the space of agents is atomless, then our counter example can be constructed in any vector space whose dimension is greater than or equal to $\mathbf{c}$—we define a vector ordering on the commodity space whose order topology is non-separable and Banachable; we then define an economy in this Banach space. That is, our example can be constructed *whenever there are at least as many commodities as agents.*

We note that our counter example is constructed using an arbitrary atomless measure space of agents, and thus includes Loeb measure spaces. It therefore may be possible to use "lifting" arguments to infer from our negative theorems a negative result for large but finite economies (see Loeb [28], Anderson [1]).[12]

In Section 6 we explore ways of getting around the negative results of the previous section. We establish a close converse of the main result of Section 5. We list a theorem on the equivalence of the core and Walrasian allocations for Bochner as well as Pettis economies. We do this, however, by non-trivially strengthening Aumann's measurability assumption. The proof is in Section 9.

In Section 7 we discuss the relationship between Lyapunov's convexity theorem and the statements of the main theorems in Sections 5 and 6. Sections 10 and 11 contain concluding remarks and open questions.

2. PRELIMINARIES

We begin with some notation and definitions. We also list some known results, which we shall refer to later in the paper.

The set of all subsets of a set X, including the empty set, will be denoted 2^X. The cardinal number of a set X will be denoted $\operatorname{card}(X)$. The cardinal number assigned to the set of natural numbers is $\aleph_0$. The cardinal number

[12] See also the work of Lewis [26] who uses non-standard analysis and the non-standard analog of the Lyapunov theorem of Loeb [27] to model the idea of "many more" agents than commodities. See also the work of Gretsky–Ostroy [17] who introduced the idea of thick and thin markets.

assigned to the set of all real numbers is denoted by $\mathbf{c}$. The *product* $\mathbf{mn}$ of cardinal numbers $\mathbf{m}$ and $\mathbf{n}$ is the cardinality of the set $X \times Y$, where $\mathrm{card}(X) = \mathbf{m}$ and $\mathrm{card}(Y) = \mathbf{n}$. For every cardinal number $\mathbf{m}$, the number $2^{\mathbf{m}}$ is defined as the cardinality of the family of all subsets of a set X satisfying $\mathrm{card}(X) = \mathbf{m}$. We define $\mathbf{n}^{\mathbf{m}}$ as the cardinality of the set of all functions from X to Y, where $\mathrm{card}(X) = \mathbf{m}$ and $\mathrm{card}(Y) = \mathbf{n}$. We know that $(\mathbf{n}^{\mathbf{m}_1})^{\mathbf{m}_2} = \mathbf{n}^{\mathbf{m}_1 \mathbf{m}_2}$. We assume the Generalized Continuum Hypothesis and use the fact that if $\aleph_\alpha$ is a regular cardinal, then

$$\aleph_\alpha^{\aleph_\beta} = \begin{cases} \aleph_\alpha & \text{if} \quad \aleph_\beta < \aleph_\alpha, \\ \aleph_{\beta+1} & \text{if} \quad \aleph_\alpha \leqslant \aleph_\beta; \end{cases}$$

see Takeuti and Zaring [42, Theorem 11.28, p. 100].

We shall restrict our attention to vector spaces whose dimension is a regular cardinal number. Note that $\aleph_n$, $n = 0, 1, 2, \ldots$, are regular cardinals. Moreover, every successor cardinal $\aleph_{\alpha+1}$ is a regular cardinal, see Takeuti and Zaring [42, Theorem 11.13, p. 92].

Let X be a topological space. The set of all cardinal numbers of the form $\mathrm{card}(\mathscr{B})$, where $\mathscr{B}$ is some base for the topology of X, is well ordered. This set has a smallest element, which is called the *weight* of the topological space, see Engelking [15, p. 12]. A *second-countable* space has weight $\leqslant \aleph_0$.

PROPOSITION 2.1. *If X is a topological space with weight $\leqslant \mathbf{m}$, then X has a dense subset of cardinality $\leqslant \mathbf{m}$.*

Proof. Let $\mathscr{B}$ be a base for the topology of X with $\mathrm{card}(\mathscr{B}) \leqslant \mathbf{m}$. Choose one point from each $B \in \mathscr{B}$. If Y is the collection of these points, then $X = \bar{Y}$. ∎

Let E be a topological vector space. Denote by E' the topological dual of E and by E'' the strong dual of E'. We use the notation $\langle p, x \rangle$ to denote the value of $p \in E'$ at $x \in E$. Denote by $\dim(E)$ the dimension of the vector space E.

An *ordered vector space* is a vector space E that is ordered by an ordering $\leqslant$ which in addition to being reflexive, transitive, and anti-symmetric satisfies the following condition: for arbitrary $z \in E$ and real $r > 0$,

$$x \leqslant y \Rightarrow rx + rz \leqslant ry + rz.$$

The set $\{x \in E : 0 \leqslant x\}$ is a convex cone with vertex 0, which is called the *positive cone* of E; and is denoted E_+. A proper cone is a convex cone C with vertex 0 having the property $C \cap -C = \{0\}$. Each proper cone $C \subseteq E$ defines, by virtue of "$x \leqslant y$" if and only if "$y - x \in C$", an ordering $\leqslant$ on E under which E is an ordered vector space with positive cone $E_+ = C$. The space E is an *ordered Banach space* if it is a Banach space, an ordered vector space, and E_+ is norm closed. E is an *ordered Hilbert space* if it is a Hilbert space, an ordered vector space, and E_+ is norm closed.

Let E be an ordered vector space, we define the *order topology* $\mathscr{T}_0$ of E to be the finest locally convex topology on E for which every order interval is bounded, (see Schaefer [41, p. 230]). We shall use the notation $[\cdot, \cdot]$ to denote order intervals.

PROPOSITION 2.2. *Let $(E, \mathscr{T})$ be an ordered Banach space. If the positive cone E_+ has a $\mathscr{T}$-interior point e and $[-e, e]$ is $\mathscr{T}$-bounded, then $\mathscr{T}_0 = \mathscr{T}$.*

Proof. The set $[-e, e]$ is radial in E and absorbs order intervals. Thus, order intervals are $\mathscr{T}$-bounded and $\mathscr{T}_0$ is finer than $\mathscr{T}$; since $[-e, e]$ is a $\mathscr{T}$-neighborhood of 0 and absorbs $\mathscr{T}$-bounded sets, it follows that $\mathscr{T}_0 = \mathscr{T}$. ∎

We shall only consider measure spaces that are complete finite positive and non-trivial (*i.e., if (T, τ, μ) is a measure space, then $\mu(T) > 0$.*) For the measure space (T, τ, μ), let S be an arbitrary subset of T and define the real number $\mu^*(S)$, called the *outer measure* of S, as follows

$$\mu^*(S) = \inf_{F \in \tau,\, F \supset S} \mu(F).$$

Let E be a Banach space. Following Diestel and Uhl [12], a function $f\colon T \to E$ is *simple* if there are $x_1, x_2, ..., x_n$ in E and $S_1, S_2, ..., S_n$ in τ such that

$$f = \sum_{i=1}^{m} x_i \chi_{s_i},$$

where $\chi_{s_i}(t) = 1$ if $t \in S_i$ and $\chi_{s_i}(t) = 0$ if $t \notin S_i$.

A function $f\colon T \to E$ is *strongly μ-measurable* if there is a sequence of simple functions $f_n\colon T \to E$ such that

$$\lim_n \|f_n(t) - f(t)\| = 0,$$

for μ-almost all $t \in T$. A function $f\colon T \to E$ is *weakly or scalar μ-measurable* if the functional $t \mapsto \langle p, f(t) \rangle$ is measurable for every $p \in E'$.

We know from Pettis' measurability theorem, see Diestel and Uhl [12, Theorem II.1.2, p. 42], that $f\colon T \to E$ is strongly μ-measurable if and only if it is weakly μ-measurable and it is μ-essentially separably valued, i.e., there is $M \in \tau$ with $\mu(M) = 0$ such that $\{f(t)\colon t \in T \backslash M\}$ is a norm separable subset of E. We shall also use the fact if $f\colon T \to E$ is strongly μ-measurable then $f^{-1}(G) \in \tau$ for each open set $G \subseteq E$, see Dunford and Schwartz [14, Theorem III.6.10, p. 148].

A strongly μ-measurable function $f\colon T \to E$ is *Bochner integrable* if there is a sequence of simple functions $f_n\colon T \to E$ such that

$$\lim_n \int_{t \in T} \|f_n(t) - f(t)\| = 0.$$

In this case, for each $S \in \tau$ we denote by $\int_S f(t)\, d\mu(t)$ the limit

$$\lim_n \int_S f_n(t)\, d\mu(t),$$

where $\int_S f_n(t)\, d\mu(t)$ is defined in the obvious way.

The relationship between two functions f and g on T into E, which is expressed by the statement $f - g = 0$ μ-a.e., is an equivalence relation. We let $[f]$ be the class of functions equivalent to f and let $L_1(\mu, E)$ (or Bochner-$L_1(\mu, E)$) denote the space of all equivalence classes of Bochner integrable functions on T into E. We shall occasionally speak of elements of $L_1(\mu, E)$ as if they were functions.

We use the fact that if $p \in E'$ and f is integrable, then $\langle p, f(t) \rangle$ is integrable and

$$\left\langle p, \int_S f(t)\, d\mu(t) \right\rangle = \int_S \langle p, f(t) \rangle\, d\mu(t),$$

for any $S \in \tau$, see Dunford and Schwartz [14, Theorem III.2.19, p. 113].

Bochner integration requires that integrable functions be strongly measurable. We turn to the more general Pettis integral, which allows us to integrate functions that may be only weakly measurable.

If $f\colon T \to E$ is weakly μ-measurable and $t \mapsto \langle p, f(t) \rangle$ is in $L_1(\mu, \mathbb{R})$ for all $p \in E'$, then f is called *Dunford integrable*. The *Dunford integral* of f over $S \in \tau$ is that element $x_S \in E''$, which exists by Diestel and Uhl [12, Lemma II.3.1, p. 52], such that

$$\langle p, x_S \rangle = \int_S \langle p, f(t) \rangle\, d\mu(t),$$

for each $p \in E'$. In the case that $x_S \in E$ for all $S \in \tau$, the function f is called *Pettis integrable* and we write

$$\text{Pettis-}\int_S f(t)\, d\mu(t) = x_S$$

to denote the Pettis integral of f over $S \in \tau$.

Similarly, if $f: T \to E'$ is a function such that $t \mapsto \langle f(t), p \rangle$ is in $L_1(\mu, \mathbb{R})$ for all $p \in E$, then for each set $S \in \tau$ there is $x_S \in E'$ such that

$$\langle x_S, p \rangle = \int_S \langle f(t), p \rangle \, d\mu(t),$$

for each $p \in E$. The element x_S is called the *Gelfand integral* of f over S. Note that in reflexive spaces the Gelfand and Pettis integrals coincide.

Finally, we introduce the idea of the "weight" of a measure space.

DEFINITION 2.3. Let (T, τ, μ) be a measure space, $S \subseteq T$, and $\mathfrak{F} \subseteq \tau$. Let $\mathcal{O}$ be the set of all cardinal numbers of the form $\operatorname{card}(\mathcal{K})$, where $\mathcal{K}$ is some collection $\{S_k: k \in K\}$ of sets such that

1. $\forall k \in K, S_k \in \mathfrak{F}$;
2. $\forall k \in K, \mu^*(S_k) = 0$;
3. $S \subseteq \bigcup_{k \in K} S_k$.

If the set $\mathcal{O}$ is empty, then let $w(S, \mathfrak{F}, p) = 0$. If the set $\mathcal{O}$ is not empty, then it is well ordered—for the usual partial ordering of cardinal numbers—and has a smallest element, which we denote $w(S, \mathfrak{F}, \mu)$.

The cardinal number $w(S, \mathfrak{F}, \mu)$ depends on the collection of μ-null sets in $\mathfrak{F}$. If $w(S, \mathfrak{F}, \mu) = 0$, then S cannot be covered by any family of μ-null sets in $\mathfrak{F}$. If $w(S, \mathfrak{F}, \mu) = \mathbf{m} > 0$, then S can be covered by a family of (no less than) $\mathbf{m}$ μ-null sets in $\mathfrak{F}$. Also, if $w(S, \mathfrak{F}, \mu) > 0$ and $\mu(S) > 0$, then $w(S, \mathfrak{F}, \mu) > \aleph_0$.

PROPOSITION 2.3. *Let (T, τ, μ) be a measure space. If $S \subseteq T$, $\mathfrak{F} \subseteq \tau$, and $w(S, \mathfrak{F}, \mu) = \mathbf{m} > 0$, then S is the union of $\mathbf{m}$ disjoint non-empty μ-null sets.*

Proof. There is a collection $\{S_k: k \in K\}$ of μ-null sets in $\mathfrak{F}$ such that $\operatorname{card}(K) = \mathbf{m}$ and $S \subseteq \bigcup_{k \in K} S_k$. Let $<$ well order K, and denote by 0 the smallest element of K. Define a collection of sets inductively:

$$V_0 = S_0, \qquad V_k = S_k \Big\backslash \bigcup_{k' < k} V_{k'}.$$

Evidently, $\{V_k: k \in K\}$ is a disjointed collection of μ-null sets. Also, $S \subseteq \bigcup_{k \in K} V_k$.

Notice that

$$S = \bigcup_{k \in K:\, V_k \cap S \neq \varnothing} (V_k \cap S) \subseteq \bigcup_{k \in K:\, V_k \cap S \neq \varnothing} S_k.$$

Therefore,

$$\mathbf{m} = w(S, \mathfrak{F}, \mu) \leqslant \operatorname{card}(k \in K : V_k \subseteq S \neq 0) \leqslant \operatorname{card}(K) = \mathbf{m},$$

and $\{V_k \cap S : k \in K, V_k \cap S \neq 0\}$ is the required collection of sets. ∎

3. THE BOCHNER AND THE PETTIS ECONOMIES

An *economy* (or a *Bochner economy*) $\mathscr{E}$ is a quintuple $[E, (T, \tau, \mu), X, \succ, e]$, where E is the *commodity space*, which is an ordered Banach space, (T, τ, μ) is the *measure space of agents*, $X\colon T \to 2^E$ is the *consumption correspondence*, $t \mapsto \succ(t) \subseteq X(t) \times X(t)$ is the *preference correspondence*, and $e\colon T \to E$ is the *initial endowments function*, which is a Bochner integrable function satisfying $e(t) \in X(t)$ for all $t \in T$.

For notational convenience, we shall denote any function $x\colon T \to E$ by $t \mapsto x_t$. In particular e_t will denote $e(t)$. Also, X_t and $\succ_t$ will denote $X(t)$ and $\succ(t)$, respectively.

An *allocation* is a strongly μ-measurable function $t \mapsto x_t \in X_t$. An allocation $t \mapsto x_t$ is *feasible* if $\int_T x_t \, d\mu(t) = \int_T e_t \, d\mu(t)$.

A *coalition* S is a measurable set S such that $\mu(S) > 0$. A coalition S can *improve* upon an allocation $t \mapsto x_t$ if there is an allocation $t \mapsto y_t$ such that $y_t \succ_t x_t$ for every $t \in S$, and $\int_S y_t \, d\mu(t) = \int_S e_t \, d\mu(t)$. The set of all feasible allocations for the economy $\mathscr{E}$ that no coalition can improve upon is called the *core* of the economy. The set of all equivalence classes of functions in the core of $\mathscr{E}$ is denoted Boc-$\mathbf{C}(\mathscr{E})$.

A pair $(t \mapsto x_t, \pi)$, where $t \mapsto x_t$ is a feasible allocation and $\pi \in E'_+$, is an *equilibrium* if for each $t \in T$, $\langle \pi, x_t \rangle \leqslant \langle \pi, e_t \rangle$, and $\langle \pi, y \rangle > \langle \pi, e_t \rangle$ if $y \succ_t x_t$; μ-a.e. The set of all equivalence classes of allocations $t \mapsto x_t$ such that $(t \mapsto x_t, \pi)$ is an equilibrium for some π is denoted Boc-$\mathbf{W}(\mathscr{E})$.

A *Pettis economy* $\mathscr{E}$ is a quintuple $[E, (T, \tau, \mu), X, \succ, e]$, defined in exactly the same way as a Bochner economy except that the initial endowments function $e\colon T \to E$ is a Pettis integrable function.

A *Pettis allocation* is a weakly μ-measurable function $t \mapsto x_t \in X_t$. A Pettis allocation $t \mapsto x_t$ is a *Pettis feasible allocation* if

$$\text{Pettis-}\int_T x_t \, d\mu(t) = \text{Pettis-}\int_T e_t \, d\mu(t).$$

A coalition S can *Pettis improve* upon a Pettis allocation $t \mapsto x_t$ if there is a Pettis allocation $t \mapsto y_t$ such that $y_t \succ_t x_t$ for every $t \in S$, and

$$\text{Pettis-}\int_S y_t \, d\mu(t) = \text{Pettis-}\int_S e_t \, d\mu(t).$$

The set of all Pettis feasible allocations for the economy $\mathscr{E}$ that no coalition can Pettis improve upon is called the *Pettis core* of the economy. The set of all equivalence classes of functions in the Pettis core of $\mathscr{E}$ is denoted Pet-$\mathbf{C}(\mathscr{E})$.

A pair $(t \mapsto x_t, \pi)$, where $t \mapsto x_t$ is a Pettis feasible allocation and $\pi \in E'_+$, is a *Pettis equilibrium* if for each $t \in T$, $\langle \pi, x_t \rangle \leqslant \langle \pi, e_t \rangle$, and $\langle \pi, y \rangle > \langle \pi, e_t \rangle$ if $y \succ_t x_t$; μ-a.e. The set of all equivalence classes of Pettis allocations $t \mapsto x_t$ such that $(t \mapsto x_t, \pi)$ is a Pettis equilibrium for some π is denoted Pet-$\mathbf{W}(\mathscr{E})$.

Theorem 5.3, below, shows that Boc-$\mathbf{C}(\mathscr{E})$ is not always a subset of Pet-$\mathbf{C}(\mathscr{E})$, even when $\mathscr{E}$ is a Bochner economy. However, if $\mathscr{E}$ is a Bochner economy then the following statements hold:

1. $\mathscr{E}$ is a Pettis economy.
2. [Bochner-$L_1(\mu, E) \cap$ Pet-$\mathbf{C}(\mathscr{E})$] $\subseteq$ Boc-$\mathbf{C}(\mathscr{E})$.
3. [Bochner-$L_1(\mu, E) \cap$ Pet-$\mathbf{W}(\mathscr{E})$] $=$ Boc-$\mathbf{W}(\mathscr{E})$.
4. If Boc-$\mathbf{C}(\mathscr{E}) \subseteq$ Boc-$\mathbf{W}(\mathscr{E})$, then Boc-$\mathbf{C}(\mathscr{E}) \subseteq$ Pet-$\mathbf{C}(\mathscr{E})$.[13]

We list the assumptions that are used in the statements of our negative results. The binary relation $\succ_t$ is called *strictly monotone* if $x, y \in X_t$, $y > x \Rightarrow y \succ_t x$. It is continuous if $\{x \in X_t : x \succ_t y\}$ and $\{x \in X_t : y \succ_t x\}$ are open in X_t. The collection $\{\succ_t : t \in T\}$ is *Aumann measurable* if the set $\{t \in T : x_t \succ_t y_t\}$ is measurable for any pair of allocations $t \mapsto x_t$ and $t \mapsto y_t$.

A1.

1. $\forall t \in T$, the initial endowment e_t is an interior point of E_+;
2. $\forall t \in T$, the consumption set $X_t = E_+$;
3. $\forall t \in T$, $\succ_t$ is induced from a norm-continuous, convex, strictly monotone, utility function.

A2. $\{\succ_t : t \in T\}$ is Aumann measurable.

4. NECESSITY OF AUMANN MEASURABILITY

We show that it is not possible to drop Aumann's measurability assumption and still hope to obtain Aumann's core–Walras equivalence result (even in finite dimensional separable commodity spaces).

THEOREM 4.1. *Let E be an ordered Banach space that has at least two linearly independent strictly positive linear functionals, and whose*

[13] Boc-$\mathbf{W}(\mathscr{E}) \subseteq$ Pet-$\mathbf{W}(\mathscr{E}) \subseteq$ Pet-$\mathbf{C}(\mathscr{E})$.

positive cone has a non-empty interior. Each of statements 1 *and* 2 *implies statement* 3.

1. *If* $\mathscr{E} = [E, (T, \tau, \mu), \cdot, \cdot, \cdot]$, *is a Bochner economy satisfying* A1, *then* $Boc\text{-}\mathbf{C}(\mathscr{E}) \subseteq Boc\text{-}\mathbf{W}(\mathscr{E})$.

2. *If* $\mathscr{E} = [E, (T, \tau, \mu), \cdot, \cdot, \cdot]$ *is a Pettis economy satisfying* A1, *then* $Pet\text{-}\mathbf{C}(\mathscr{E}) \subseteq Pet\text{-}\mathbf{W}(\mathscr{E})$.

3. *Every subset of* T *is* μ*-measurable.*

The *non-existence* of an atomless measure space (T, τ, μ) such that every subset of T is measurable is consistent with the usual assumptions of set theory—including the axiom of choice—see Federer [16, p. 59] and Oxtoby [34, p. 26]. Indeed, we know that [0, 1] contains a set that is not Lebesgue measurable, e.g., Oxtoby [34, Chapter 5]. Endowing $\mathbb{R}^n$ with its canonical ordering, we obtain the following corollary.

COROLLARY 4.1.1. *For every* $n \geqslant 2$, *there is an economy* $\mathscr{E} = [\mathbb{R}^n, [0, 1], X, \succ, e]$ *that satisfies* A1 *but* $Boc\text{-}\mathbf{C}(\mathscr{E}) \nsubseteq Boc\text{-}\mathbf{W}(\mathscr{E})$.

Before proving Theorem 4.1 we need the following well known proposition, which states that the existence of a non-measurable set implies the existence of a non-measurable atom, cf. Bartle [7, p. 169].

PROPOSITION 4.2. *If* $V \subseteq T$ *is not measurable, then there exists a non-measurable set* $V^* \subseteq V$ *that does not contain a non-null measurable subset.*

Proof. Let

$$\alpha = \sup_{S \in \tau,\, S \subseteq V} \mu(S).$$

If $\alpha = 0$ then let $V^* = V$. If $\alpha > 0$ then choose a sequence $\alpha_n \uparrow \alpha$ such that $\alpha_n < \alpha$ for all n. For each n there is $S_n \in \tau$, $S_n \subseteq V$ such that $\mu(S_n) > \alpha_n$. The set $V' = \bigcup_n S_n \subseteq V$ is measurable and for every n, $\mu(V') > \alpha_n$. Thus, $\mu(V') \geqslant \alpha$, which implies $\mu(V') = \alpha$.

The set $V^* = V \setminus V'$ is not measurable and is the required set. Else, if there is $S \in \tau$, $S \subseteq V^*$ with $\mu(S) > 0$ then $V' \cup S \in \tau$, $V' \cup S \subseteq V$, and $\mu(V' \cup S) > \alpha$. ∎

Proof (Proof of Theorem 4.1). Suppose by way of contradiction that $1 \nRightarrow 3$ or that $2 \nRightarrow 3$. Then there is by Proposition 4.2 a non-measurable set V that does not contain any non-null measurable subsets.

Let g be an interior point of E_+ and let p and q be linearly independent strictly positive linear functionals on E such that $\langle p, g \rangle = \langle q, g \rangle = 1$.

We define a Bochner economy $\mathscr{E} = [E, (T, \tau, \mu), E_+, \succ, e]$ that satisfies A1. Initial endowments are $t \mapsto e_t = g$, and utility functions are given as follows:

$$U_t(x) = \begin{cases} \langle p, x \rangle & \text{if} \quad t \in T \setminus V; \\ \langle q, x \rangle & \text{if} \quad t \in V. \end{cases}$$

Since $p \neq q$ and g is an interior point of E_+, then $t \mapsto e_t \notin$ Boc-$\mathbf{W}(\mathscr{E})$ and $t \mapsto g \notin$ Pet-$\mathbf{W}(\mathscr{E})$.

We show that $t \mapsto e_t \in$ Pet-$\mathbf{C}(\mathscr{E})$ and thus $t \mapsto e_t \in$ Boc-$\mathbf{C}(\mathscr{E})$. Suppose the contrary and that some coalition $S \in \tau$ (with $\mu(S) > 0$) can Pettis improve upon $t \mapsto e_t$ with a Pettis allocation $f \colon T \to E_+$ such that

$$\text{Pettis-}\int_S f_t \, d\mu(t) = \int_S e_t \, d\mu(t). \tag{1}$$

The set $\{t \in S : \langle p, f_t \rangle \leqslant 1\} \subseteq V$ is a null set, because it is a measurable subset of V. Therefore,

$$\begin{aligned} \left\langle p, \text{Pettis-}\int_S f_t \, d\mu(t) \right\rangle &= \int_S \langle p, f_t \rangle \, d\mu(t) \\ &> \int_S \langle p, e_t \rangle \, d\mu(t) = \left\langle p, \int_S e_t \, d\mu(t) \right\rangle, \end{aligned}$$

which contradicts Eq. (1). ∎

5. FAILURE IN LARGE COMMODITY SPACES

In this section, E will denote an arbitrary infinite dimensional vector space whose dimension is a regular cardinal number. We shall list a main proposition and several important consequences, the most important of which is Theorem 5.3. Proofs are given in Section 8.

The next proposition says that the existence of a coalition that is covered by $\leqslant \dim(E)$ negligible sets implies the existence of an unpleasant Bochner economy that satisfies the very pleasant assumptions in A1 as well as Aumann's measurability assumption A2.

Proposition 5.1. *If (T, τ, μ) satisfies*

$$\exists S \in \tau, \qquad \mu(S) > 0, \qquad 0 < w(S, \tau, \mu) \leqslant \dim(E), \tag{2}$$

then there is an ordering on E, whose order topology $\mathcal{T}_0$ *is normable (with norm* $\|\cdot\|$*) and complete, and an economy* $\mathscr{E} = [(E, \|\cdot\|), (T, \tau, \mu), X, \succ, e]$ *satisfying* A1 *and* A2 *but*

1. $Boc\text{-}\mathbf{C}(\mathscr{E}) \nsubseteq Boc\text{-}\mathbf{W}(\mathscr{E})$.

2. $\exists f \in Boc\text{-}\mathbf{C}(\mathscr{E})$ *that can be Pettis improved upon by the coalition S.*

3. $\exists f \in Boc\text{-}\mathbf{C}(\mathscr{E})$ *that is not an equilibrium allocation for* $\mathscr{E}^* = [(E, \|\cdot\|), (T, \tau, \mu), X, \succ, f]$.

4. $Boc\text{-}\mathbf{W}(\mathscr{E}) = \varnothing$ *and there is a truncation* $\hat{\mathscr{E}} = [(E, \|\cdot\|), (T, \tau, \mu), \hat{X}, \succ, e]$ *of* $\mathscr{E}$ *satisfying*[14]

   (i) $\forall t, X_t = \hat{X}$, *which is a weakly compact and convex subset of* E_+;

   (ii) $\forall t, e_t$ *is a* $\mathcal{T}_0$*-interior point of* $\hat{X}$;

but $Boc\text{-}\mathbf{W}(\hat{\mathscr{E}}) = \varnothing$.

5. $Pet\text{-}\mathbf{W}(\mathscr{E}) \neq \varnothing$ *and* $Pet\text{-}\mathbf{W}(\hat{\mathscr{E}}) \neq \varnothing$.

The next proposition tells us that every atomless measure space can be covered by $\mathbf{c}$ non-empty null sets.

PROPOSITION 5.2. *If* (T, τ, μ) *is an atomless measure space, then*

$$w(T, \tau, \mu) = \mathbf{c}.$$

That is, T is the union of a disjointed family $\{S_k : k \in K\}$ *of non-empty* μ*-null sets with* $\operatorname{card}(K) = \mathbf{c}$.

Proof. Let $\{S_0, S_1\}$ be a partition of T into two disjoint sets of equal measure. Inductively, if r is a finite string of 0s and 1s let $\{S_{r0}, S_{r1}\}$, be a partition of S_r into two sets of equal measure.

Let K be the family of all countable sequences of 0's and 1s. For any natural number $n > 0$ let k_n be the first n elements of $k \in K$. Let $S_k = \bigcap_{n=1}^{\infty} S_{k_n}$.

For any $k \in K$, S_k is a μ-null set. Also, $\{S_k : k \in K\}$ is a disjointed family of sets whose union is T; thus $w(T, \tau, \mu) > 0$. The union of the family $\{S_k : k \in K, S_k \neq \varnothing\}$ is still T. Since $\mu(T) > 0$,

$$\aleph_0 < w(T, \tau, \mu) \leqslant \operatorname{card}(\{k \in K : S_k \neq \varnothing\}) \leqslant \operatorname{card}(K) \leqslant \mathbf{c},$$

which implies—by the continuum hypothesis—that $w(T, \tau, \mu) = \mathbf{c}$. ∎

Proposition 5.1 and Proposition 5.2 easily yield the following devastating theorem.

[14] By trunction we mean $\hat{X} \subseteq X$ and $\succ$ is restricted to $\hat{X}$.

Theorem 5.3 (Main Theorem). *Assume that* $\dim(E) \geqslant \mathbf{c}$. *If* (T, τ, μ) *satisfies*

$$\exists S \in \tau, \qquad \mu(S) > 0, \qquad (S, \tau, \mu) \text{ is atomless,}$$

then there is an ordering on E, *whose order topology* $\mathscr{T}_0$ *is normable (with norm* $\|\cdot\|$*) and complete, and an economy* $\mathscr{E} = [(E, \|\cdot\|), (T, \tau, \mu), X, \succ, e]$ *satisfying* A1 *and* A2 *but*

1. *Boc*-$\mathbf{C}(\mathscr{E}) \nsubseteq$ *Boc*-$\mathbf{W}(\mathscr{E})$.
2. $\exists f \in$ *Boc*-$\mathbf{C}(\mathscr{E})$ *that can be Pettis improved upon by the coalition* S.
3. $\exists f \in$ *Boc*-$\mathbf{C}(\mathscr{E})$ *that is not an equilibrium allocation for* $\mathscr{E}^* = [(E, \|\cdot\|), (T, \tau, \mu), X, \succ, f]$.
4. *Boc*-$\mathbf{W}(\mathscr{E}) = \varnothing$ *and there is a truncation* $\hat{\mathscr{E}} = [(E, \|\cdot\|), (T, \tau, \mu), \hat{X}, \succ, e]$ *of* $\mathscr{E}$ *satisfying*[15]

    (i) $\forall t$, $X_t = \hat{X}$, *which is a weakly compact and convex subset of* E_+;
    (ii) $\forall t$, e_t *is a* $\mathscr{T}_0$*-interior point of* $\hat{X}$;

but Boc-$\mathbf{W}(\hat{\mathscr{E}}) = \varnothing$.

5. *Pet*-$\mathbf{W}(\mathscr{E}) \neq \varnothing$ *and Pet*-$\mathbf{W}(\hat{\mathscr{E}}) \neq \varnothing$.

In Aumann [4, 5] the space of agents is the continuum $[0, 1]$ with its Lebesgue measure, the proof of Proposition 5.1 betrays a stronger result for Hilbert commodity spaces.

Corollary 5.3.1. *Let* (T, τ, μ) *be atomless. If* H *is a non-separable Hilbert space, then there is an ordering on* H, *which makes* H *an ordered Hilbert space, and an economy* $\mathscr{E} = [H, (T, \tau, \mu), X, \succ, e]$ *satisfying* A1 *and* A2 *but items* 1, 3–5 *of Theorem* 5.3 *hold.*

6. CORE-WALRAS EQUIVALENCE

In this section we present some interesting positive results. We begin with a list of the assumptions that are used in the statements of these results.

B1.

1. (T, τ, μ) is an atomless measure space.
2. Pettis-$\int_T e_t \, d\mu(t)$ is an interior point of E_+;

[15] By truncation we mean $\hat{X} \subseteq X$ and $\succ$ is restricted to $\hat{X}$.

3. $\forall t \in T$, the consumption set $X_t = E_+$;

4. $\forall t \in T$, $\succ_t$ is norm continuous, irreflexive, transitive, and strictly monotone.

Let f be a function on T with values $f_t \in X_t$. For each $z \in E$ define the set

$$A_z^f = \{t \in T : z + e_t \succ_t f_t\}.$$

We turn to the measurability assumption used in Hildenbrand [20].

B2. For any allocation $t \mapsto f_t$ and any $z \in E$, the set A_z^f is measurable.

We list a theorem and a corollary. The proof is given in Section 9.

THEOREM 6.1. *Let E be an ordered Banach space with weight* $\mathbf{m}$. *Assume that* $\mathscr{E} = [E, (T, \tau, \mu), X, \succ, e]$ *satisfies* B1. *If* $[f] \in \mathit{Boc}\text{-}\mathbf{C}(\mathscr{E})$ *and there exits* $\mathfrak{F} \subseteq \tau$ *such that*

$$\nexists S \subseteq T, \qquad \mu^*(S) > 0, \qquad 0 < w(S, \mathfrak{F}, \mu) \leqslant \mathbf{m}; \qquad \textit{and} \tag{3}$$

$$\forall z \in E, \qquad A_z^f \in \mathfrak{F}, \tag{4}$$

then $[f] \in \mathit{Boc}\text{-}\mathbf{W}(\mathscr{E})$. *If* $\mathscr{E}$ *is a Pettis economy and* $[f] \in \mathit{Pet}\text{-}\mathbf{C}(\mathscr{E})$, *then* $[f] \in \mathit{Pet}\text{-}\mathbf{W}(\mathscr{E})$.

Equation (3) says that there is no non-negligible set of agents that can be covered by $\leqslant \mathbf{m}$ null sets in $\mathfrak{F}$.

Every separable Banach space has weight $\aleph_0$. Also, every atomless measure space (T, τ, μ) satisfies Eq. (3) for $\mathbf{m} = \aleph_0$ and $\mathfrak{F} = \tau$. Thus, Theorem 6.1 implies Aumann's [4] result for the commodity space $\mathbb{R}^n$ and the Rustichini and Yannelis [38, 40] results for separable ordered Banach spaces whose positive cones have interior points.

COROLLARY 6.1.1. *Let E be a separable ordered Banach space. If* $\mathscr{E} = [E, (T, \tau, \mu), X, \succ, e]$ *satisfies* B1 *and* B2, *then* $\mathit{Boc}\text{-}\mathbf{C}(\mathscr{E}) = \mathit{Boc}\text{-}\mathbf{W}(\mathscr{E})$.

7. LYAPUNOV'S CONVEXITY THEOREM

In this section we prove results on the relationship between Eqs. (2) and (3), of Proposition 5.1 and Theorem 6.1, respectively, and Lyapunov's convexity theorem in the weak topology.

DEFINITION 7.1. Let E be a Banach space and let (T, τ, μ) be a measure space. Let $G: \tau \to E$ be a countably additive vector measure satisfying $G(S \cap F) = 0$ for all $F \in \tau$ if and only if $\mu(S) = 0$. *Lyapunov's convexity* *theorem holds* for the triplet $((T, \tau, \mu), E, G)$ whenever $\{G(F \cap S) : F \in \tau\}$ is a weakly compact and convex set in E, for each $S \in \tau$.

Let E, (T, τ, μ), and G be as in Definition 7.1. We know from Diestel and Uhl [12, Theorem IX.1.4, p. 263] that Lyapunov's convexity theorem holds for $((T, \tau, \mu), E, G)$ if and only if the following is satisfied:

If $S \in \tau$ and $\mu(S) > 0$, then the operator $f \mapsto \int_S f(x)\, dG(x)$ on the subspace of functions in $L_\infty(\mu)$ vanishing off S is **not** *one to one.*

Denote by $A \varDelta B$ the symmetric difference $(A \backslash B) \cup (B \backslash A)$. Let (T, τ, μ) be a measure space. Let $\mathcal{N}$ be the equivalence relation on τ defined by $S \mathcal{N} S'$ if $\mu(S \varDelta S') = 0$. For $S \in \tau$ let $S(\mu) = \{S' \in \tau : S' \subseteq S\}$ and denote by $S(\mu) \mid \mathcal{N}$ the corresponding quotient space and by $[S]$ the equivalence class of S.

The next theorem says that Lyapunov's convexity theorem holds whenever each coalition contains more distinguished sub-coalitions than commodities.

THEOREM 7.1 (Rustichini and Yannelis [39]). *Let E, (T, τ, μ), and G be as in Definition* 7.1. *Assume that E is infinite dimensional. If (T, τ, μ) satisfies*

$$\nexists S \in \tau, \qquad \mu(S) > 0, \qquad \operatorname{card}(S(\mu) \mid \mathcal{N}) \leqslant \dim(E), \tag{5}$$

then Lyapunov's convexity theorem holds for the triplet $((T, \tau, \mu), E, G)$.

Proof. For any $S \in \tau$ let $L^S_\infty(\mu)$ be the subspace of functions in $L_\infty(\mu)$ vanishing off S. Let $[S], [S'] \in S(\mu) \mid \mathcal{N}$ and $[S] \neq [S']$. The functions χ_S and $\chi_{S'}$, correspond to two distinct elements of $L^S_\infty(\mu)$. This and Eq. (5) tell us that if $S \in \tau$ with $\mu(S) > 0$ then

$$\operatorname{card}(L^S_\infty(\mu)) \geqslant \operatorname{card}(S(\mu) \mid \mathcal{N}) > \dim(E). \tag{6}$$

Suppose that Lyapunov's convexity theorem does not hold for the triplet $((T, \tau, \mu), E, G)$. There is some $S \in \tau$ with $\mu(S) > 0$ and a one to one linear function from $L^S_\infty(\mu)$ into E. So $\dim(E) \geqslant \dim(L^S_\infty(\mu))$. However, $\dim(L^S_\infty(\mu)) = \operatorname{card}(L^S_\infty(\mu))$, since $\operatorname{card}(L^S_\infty(\mu)) > \dim(E) \geqslant \mathbf{c}$. This contradicts Eq. (6). ∎

If E is an infinite dimensional separable Banach space, then $\dim(E) = \mathbf{c}$. We obtain the following corollary.

COROLLARY 7.1.1. *Let E, (T, τ, μ), and G be as in Definition* 7.1. *Assume that E is separable and infinite dimensional. If (T, τ, μ) satisfies*

$$\nexists S \in \tau, \qquad \mu(S) > 0, \qquad \operatorname{card}(S(\mu) \mid \mathcal{N}) \leqslant \mathbf{c},$$

then Lyapunov's convexity theorem holds for the triplet $((T, \tau, \mu), E, G)$.

The next result is an easy but interesting consequence of Proposition 2.3 and Theorem 7.1.

COROLLARY 7.1.2. *Let E, (T, τ, μ), and G be as in Definition* 7.1. *Assume that E is infinite dimensional. If $\mathfrak{F} \subseteq \tau$ and (T, τ, μ) satisfies*

$$\begin{aligned} &\nexists S \in \tau, \qquad \mu(S) > 0, \qquad \operatorname{card}(S(\mu) \mid \mathcal{N}) < \operatorname{card}(S); \qquad \textit{and} \\ &\nexists S \in \tau, \qquad \mu(S) > 0, \qquad 0 < w(S, \mathfrak{F}, \mu) \leqslant \dim(E), \end{aligned} \tag{7}$$

then Lyapunov's convexity theorem holds for $((T, \tau, \mu), E, G)$.

Proof. Fix some $S \in \tau$ with $\mu(S) > 0$. From Proposition 2.3 and Eq. (7), $\operatorname{card}(S) > \dim(E)$. Thus,

$$\operatorname{card}(S(\mu) \mid \mathcal{N}) \geqslant \operatorname{card}(S) > \dim(E),$$

which implies Eq. (5) of Theorem 7.1. ▮

8. PROOFS OF PROPOSITION 5.1

Let A be a non-empty set. Consider the set of all real-valued functions x defined on A such that $\operatorname{card}(\{a \in A : x(a) \neq 0\}) \leqslant \aleph_0$ and $\sum_{a \in A} x(a)^2 < \infty$. These functions form a real vector space under pointwise addition and scalar multiplication. If we define an inner product $\langle x, y \rangle = \sum_{a \in A} x(a)\, y(a)$ this space becomes the Hilbert space $l_2(A)$. If two vectors x, y are orthogonal then we write $x \perp y$. We write $X^{\perp}$ to denote the set of all orthogonal vectors to X.

PROPOSITION 8.1. *If* $\operatorname{card}(A) = \mathbf{m} > \aleph_0$ *is a regular cardinal number, then* $\dim(l_2(A)) = \mathbf{m}$.

Proof. We know that $l_2(A)$ has weight $\mathbf{m}$, see Engelking [15, 4.4.K, p. 288]. Hence, there is a dense subset $\mathcal{D}$ of $l_2(A)$ such that $\operatorname{card}(\mathcal{D}) = \mathbf{m}$. Since $l_2(A)$ is metrizable, each $x \in l_2(A)$ is the limit of some sequence in $\mathcal{D}$. So $\mathbf{m} \leqslant \dim(l_2(A)) \leqslant \operatorname{card}(l_2(A)) \leqslant \mathbf{m}^{\aleph_0}$. Since $\mathbf{m}$ is a regular cardinal number, $\mathbf{m} = \mathbf{m}^{\aleph_0}$; hence $\dim(l_2(A)) = \mathbf{m}$. ▮

Now we highlight the role of strong measurability in the proof of our main result.

PROPOSITION 8.2. *If $t \mapsto f_t$ is a strongly μ-measurable function with values in $l_2(A)$, then there is $M \in \tau$, $\mu(M) = 0$, such that*

$$\operatorname{card}(\{a \in A : \exists t \in T \backslash M,\ f_t(a) \neq 0\}) \leqslant \aleph_0.$$

Proof. Since $t \mapsto f_t$ is a strongly μ-measurable function, it is essentially separably valued. Therefore, there is $M \in \tau$, $\mu(M) = 0$, such that $\{f_t : t \in T \setminus M\}$ is a norm separable subset of $l_2(A)$. Let $\mathscr{H}$ be a countable dense subset of this set.

From the definition of $l_2(A)$ we see that if $h \in \mathscr{H}$, then $\operatorname{card}(\{a \in A : h(a) \neq 0\}) \leqslant \aleph_0$. Therefore,

$$\operatorname{card}(\{a \in A : \exists h \in \mathscr{H}, h(a) \neq 0\}) \leqslant \aleph_0, \tag{8}$$

since the union of countably many sets of cardinality $\leqslant \aleph_0$ is a countable set.

Let $a' \in \{a \in A : \exists t \in T \setminus M, f_t(a) \neq 0\}$ be arbitrarily chosen. Consider the point $x \in l_2(A)$ such that $x(a') = 1$ and $x(a) = 0$ for $a \neq a'$. The point $f_{t'}$ is contained in the open set $\{y \in l_2(A) : \langle x, y \rangle \neq 0\}$. Hence, there is $h \in \mathscr{H}$ that is also contained in this open set and $h(a') \neq 0$. That is

$$\{a \in A : \exists t \in T \setminus M, f_t(a) \neq 0\} \subseteq \{a \in A : \exists h \in \mathscr{H}, h(a) \neq 0\},$$

which in view of Eq. (8) implies

$$\operatorname{card}(\{a \in A : \exists t \in T \setminus M, f_t(a) \neq 0\}) \leqslant \aleph_0.$$

This proves the proposition. ▮

8.1. *The Commodity Space*

Let S^* be the set from Eq. (2) of Proposition 5.1. That is,

$$\mu(S^*) > 0, \qquad 0 < w(S^*, \tau, \mu) \leqslant \dim(E).$$

Since $\mu(S^*) > 0$ it must be the case that $w(S^*, \tau, \mu) > \aleph_0$—only null sets are covered by countably many null sets. Since we assume the continuum hypothesis, it must be the case that $w(S^*, \tau, \mu) \geqslant \mathbf{c}$. Therefore, letting $\mathbf{m} = \dim(E)$ we see that $\mathbf{m} \geqslant \mathbf{c}$.

Let A be a set such that $\operatorname{card}(A) = \mathbf{m}$. We see from Proposition 8.1 that the space $l_2(A)$ has dimension $\mathbf{m} = \dim(E)$. Thus, E is identical (up to algebraic isomorphism) to $l_2(A)$. Without loss of generality, identify the space E with $l_2(A)$.

We wish to construct an ordering on E whose order topology is the same as the canonical norm topology of $l_2(A)$.

By Proposition 2.3 the set S^* is the union of a disjointed collection $\{S_k^* : k \in K\}$ of non-empty μ-null sets with $0 < \operatorname{card}(K) < \mathbf{m}$. The index set K can be taken as though it were a proper subset of A—since A is not finite. Fix $\theta \in A \setminus K$.

Let $g_\theta \in l_2(A)$ satisfy $g_\theta(\theta)=1$ and $g_\theta(a)=0$ for $a \in A \backslash \{\theta\}$. Let B denote the closed unit ball in $l_2(A)$. Let E_+ be the convex cone with vertex 0 generated by $3g_\theta + B$. That is

$$E_+ = \bigcup_{\alpha \geqslant 0} \alpha(3g_\theta + B).$$

It is easy to verify that E_+ is a norm closed convex proper cone. Also, g_θ is a norm interior point of E_+ and $[-g_\theta, g_\theta]$ is norm bounded. Thus, E_+ induces a vector ordering on $l_2(A)$ making it an ordered Banach space. By Proposition 2.2 the order topology $\mathscr{T}_0$ of this ordering is the same as norm topology of $l_2(A)$. Endow E with the canonical norm of $l_2(A)$.

8.2. *The Economy*

For each $t \in T$ let the consumption set be E_+; i.e., $X_t = E_+$. For each $t \in T$ let the individual endowment be g_θ; i.e., $e_t = g_\theta$. A1.1.-2. *are satisfied.*

Consider the set $\{g_k \in l_2(A) : k \in K\}$ where

$$g_k(a) = \begin{cases} 1 & \text{if} \quad a = k; \\ 0 & \text{otherwise.} \end{cases}$$

For $t \in S_k^*$ let the utility function $U_t \colon E_+ \to \mathbb{R}$ be defined as $U_t(f) = \langle g_\theta + g_k, f \rangle = f(\theta) + f(k)$. Intuitively, each agent in S_k^* extremely desires commodity θ and the k-th commodity. For $t \in T \backslash S^*$ let $U_t(f) = \langle g_\theta, f \rangle = f(\theta)$.

Evidently, U_t is norm-continuous and convex for all $t \in T$. We show that U_t is strictly monotone with respect to the ordering induced by E_+. For some $k \in K$, take a fixed $t \in S_k^*$ and let $f = (3g_\theta + h)$ with $\|h\| \leqslant 1$. We get $U_t(f) = 3 + h(\theta) + h(k) > 0$, since $h(\theta), h(k) \in [-1, 1]$. Also, for $t \in T \backslash S^*$, $U_t(f) = 3 + h(\theta) > 0$. Since the functionals U_t are linear then $U_t(E_+ \backslash \{0\}) > 0$ and U_t are strictly monotone with respect to the ordering induced by E_+. A1.3. *is satisfied.*

Finally, we show that preferences satisfy Aumann's measurability assumption: $S = \{t \in T : U_t(x_t) > U_t(y_t)\}$ is measurable for any pair of allocations $t \mapsto x_t$, $t \mapsto y_t$. By Proposition 8.2 there is a μ-null set M such that the set

$$K' = \{k \in K : \exists t \in T \backslash M, x_t(k) \neq y_t(k)\}$$

is countable. Consider the measurable sets $F = \{t \colon x_t(\theta) > y_t(\theta)\}$ and $G = \{t \colon x_t(\theta) \leqslant y_t(\theta)\}$. Evidently, $F \backslash S \subseteq (\bigcup_{k \in K'} S_k^* \cup M)$ and $S \cap G \subseteq (\bigcup_{k \in K'} S_k^* \cup M)$. That is, $S = (F \backslash N) \cup O$ for some μ-null sets N and O, which means that S is measurable. A2 *is satisfied.*

8.3. *Items* (1)–(3)

We show that the feasible allocation $t \mapsto g_\theta$ is in the core. Suppose otherwise. Then there is some allocation $t \mapsto f_t$, which is strongly measurable, and a measurable set S such that $\mu(S) > 0$,

$$\int_S f_t \, d\mu(t) = \int_S e_t \, d\mu(t) = \mu(S) \, g_\theta, \tag{9}$$

and $U_t(f_t) > U_t(g_\theta) = 1$ for all $t \in S$. Evidently, $\mu(S \cap S^*) > 0$ since the converse implies

$$\left\langle g_\theta, \int_S f_t \, d\mu(t) \right\rangle = \int_S \langle g_\theta, f_t \rangle \, d\mu(t)$$
$$= \int_S U_t(f_t) \, d\mu(t) > \mu(S) = \langle g_\theta, \mu(S) \, g_\theta \rangle,$$

which contradicts Eq. (9). Clearly, if $t \in S \backslash S^*$ then $f_t(\theta) > 1$. Thus the set $S' = \{ t \in (S \cap S^*) : f_t(\theta) \leqslant 1 \}$ is measurable and $\mu(S') > 0$. Otherwise,

$$\left\langle g_\theta, \int_S f_t \, d\mu(t) \right\rangle = \int_S \langle g_\theta, f_t \rangle \, d\mu(t) = \int_S f_t(\theta) \, d\mu(t) > \mu(S) = \langle g_\theta, \mu(S) \, g_\theta \rangle,$$

which contradicts Eq. (9).

Since $U_t(f_t) > 1$ and $f_t(\theta) \leqslant 1$ for all $t \in S' \subseteq S^*$ then $f_t(k) > 0$ if $t \in S_k^* \cap S'$. Recall that the collection $\{ S_k^* : k \in K \}$ is disjointed, $\mu(S') > 0$, $\mu(S_k^*) = 0$ for all $k \in K$, and $S' \subseteq \bigcup_{k \in K} S_k^*$. Therefore for any μ-null set M

$$\operatorname{card}(\{ a \in A : \exists t \in T \backslash M, \, f_t(a) \neq 0 \})$$
$$\geqslant \operatorname{card}(\{ k \in K : S_k^* \cap (S' \backslash M) \neq \varnothing \}) > \aleph_0,$$

which contradicts Proposition 8.2.

Now that we have shown that the allocation $t \mapsto g_\theta$ is in Boc-$\mathbf{C}(\mathscr{E})$, we show that it is not in Boc-$\mathbf{W}(\mathscr{E})$ and that it can be Pettis improved upon by the coalition S^*. Consider the function $t \mapsto f_t$,

$$f_t = \begin{cases} \alpha g_\theta + (1 - \alpha)(g_k + 1/2 g_\theta) & \text{if } \; t \in S_k^*; \\ 0 & \text{if } \; t \notin S^*. \end{cases}$$

This function has values in E_+ for all $1 > \alpha \geqslant 1 - 2/(3\sqrt{5})$. Fix such an α and note

$$\forall t \in S^*, \qquad U_t(f_t) = f_t(\theta) + f_t(k) = 3/2 - \alpha/2 > 1 = U_t(g_\theta). \tag{10}$$

Take an arbitrary $p \in l_2(A)$. Then $\langle p, g_k \rangle = 0$ for all except countably many $k \in K$. The first consequence of this is that if $\langle p, g_\theta \rangle > 0$ then

$$\langle p, f_t \rangle = \langle p, (1/2 + \alpha/2)\, g_\theta) \rangle < \langle p, g_\theta \rangle = \langle p, e_t \rangle, \qquad \mu\text{-a.e.} \quad t \in S^*. \tag{11}$$

From Eqs. (10) and (11) and monotonicity, we see that the allocation $t \mapsto g_\theta$ is not in Boc-$\mathbf{W}(\mathscr{E})$. The second consequence is that $t \mapsto f_t$ is weakly μ-measurable and

$$\text{Pettis-}\int_{S^*} f_t \, d\mu(t) = \mu(S^*)(1/2 + \alpha/2)\, g_\theta < \mu(S^*)\, g_\theta = \int_{S^*} e_t \, d\mu(t). \tag{12}$$

From Eqs. (10) and (12) and monotonicity, the coalition S^* can Pettis improve upon $t \mapsto g_\theta$.

8.4. *Item* (4)

Truncate the consumption sets E_+ to $\hat{X} = \text{co}(\{3g_\theta + B\} \cup \{0\})$. This set is closed and bounded. Thus, it is weakly compact. We also see that $e_t \in \text{int}\, \hat{X}$ for each $t \in T$. Thus, (i)–(ii) of item (4) are satisfied.

Suppose that $(t \mapsto f_t, p)$ is an equilibrium for this truncated economy. Let N be the μ-null such that f_t is a utility maximizer for each $t \in T \backslash N$.

For $t \in T \backslash (S^* \cup N)$ we have $f_t(\theta) \geqslant 1$, for otherwise $U_t(e_t) \geqslant U_t(f_t)$.

Let $S' = \{t \in S^* : f_t(\theta) \leqslant 1\}$. Then as before $\mu(S') > 0$, since otherwise

$$\begin{aligned}\left\langle g_\theta, \int_T f_t \, d\mu(t) \right\rangle &= \int_T \langle g_\theta, f_t \rangle \, d\mu(t) \\ &= \int_T f_t(\theta) \, d\mu(t) > \mu(T) = \left\langle g_\theta, \int_T e_t \, d\mu(t) \right\rangle,\end{aligned}$$

which contradicts feasibility.

The allocation $t \mapsto f_t$ is strongly μ-measurable. Thus, by Proposition 8.2 there is a μ-null set M such that

$$\begin{aligned}&\text{card}(\{k \in K : \exists t \in S_k^* \backslash M, f_t(k) \neq 0\}) \\ &\qquad \leqslant \text{card}(\{a \in A : \exists t \in T \backslash M, f_t(a) \neq 0\}) \leqslant \aleph_0.\end{aligned} \tag{13}$$

Also, $\langle p, g_k \rangle = 0$ for all but countably many $k \in K$, and $\mu(S') > 0$. Thus, we can summarize

$$\text{card}(\{k \in K : \langle p, g_k \rangle \neq 0\}) \leqslant \aleph_0; \tag{14}$$

$$\text{card}(\{k \in K : (S' \backslash (M \cup N)) \cap S_k^* \neq \varnothing\}) > \aleph_0. \tag{15}$$

Thus, there is $k \in K$ that is an element of the set in Eq. (15) but that is not an element of the sets in Eqs. (13) and (14). Pick $t \in (S' \backslash (M \cup N)) \cap S_k^*$. Then, $f_t(\theta) \leqslant 1$, f_t maximizes utility on agent t's budget set, $f_t(k) = 0$, and $\langle p, g_k \rangle = 0$. It is easy to see that $f_t(\theta) = 1$, since the contrary implies $U_t(e_t) > U_t(f_t)$.

Consider the point $z = 1/2 f_t + 1/2 e_t$, which is in int $\hat{X}$. Then $\langle p, z \rangle \leqslant \langle p, e_t \rangle$ and for some $1 > \alpha > 0$, $\alpha z + (1-\alpha) 2g_k$ is also in int $\hat{X}$. Since $\langle p, g_k \rangle = 0$ and $p > 0$ we get

$$\langle p, \alpha z + (1-\alpha) 2g_k \rangle = \langle p, \alpha z \rangle \leqslant \langle p, z \rangle \leqslant \langle p, e_t \rangle. \tag{16}$$

Also, $U_t(z) = U_t(f_t)$, since $f_t(k) = 0$ and $f_t(\theta) = 1$. Thus,

$$U_t(\alpha z + (1-\alpha) 2g_k) = U_t(\alpha f_t + (1-\alpha) 2g_k) = 2 + \alpha(f_t(\theta) - 2) > 1 = U_t(f_t). \tag{17}$$

Equations (16) and (17) contradict the fact that f_t maximizes utility on the budget set. So Boc-$\mathbf{W}(\hat{\mathscr{E}}) = \varnothing$—it is also easy to see that Boc-$\mathbf{W}(\mathscr{E}) = \varnothing$.

8.5. *Item* (5)

Consider the allocation $t \mapsto f_t$, which is defined as

$$f_t = \begin{cases} e_t & \text{if} \quad t \in T \backslash S^*; \\ e_t + 1/(2\sqrt{2})\, g_k & \text{if} \quad t \in S_k^*. \end{cases}$$

It is easy to check that $f_t \in \hat{X}$ for all t. Since for each $p \in E'$ we have $\langle p, g_k \rangle = 0$ for all but countably many k then $t \mapsto f_t$ is a Pettis feasible allocation.

It is also easy to check that $U_t(x) > U_t(f_t)$ for some $x \in E_+$ implies $x(\theta) > f_t(\theta)$. This is obviously the case for $t \in T \backslash S^*$. For $t \in S_k^*$ (for some k) note that $e_t + \alpha g_k \notin E_+$ if $\alpha > 1/(2\sqrt{2})$. Thus, $(t \mapsto f_t, g_\theta)$ is a Pettis equilibrium for $\mathscr{E}$ and $\hat{\mathscr{E}}$.

9. PROOF OF THEOREM 6.1

We adapt the classical argument used by Aumann [4], see also Hildenbrand [20]. The proof for the Pettis economy follows with obvious changes that are identified in the footnotes.

Suppose that $t \mapsto f_t$ is a feasible allocation in Boc-$\mathbf{C}(\mathscr{E})$.[16]

For each $z \in E$ let $A_z^f = \{t \in T : z + e_t \succ_t f_t\}$, which is in $\mathfrak{F}$ and thus measurable.

[16] Suppose that $t \mapsto f_t$ is a Pettis feasible allocation in Pet-$\mathbf{C}(\mathscr{E})$.

Let

$$C = \mathrm{co}\{z \in E : \mu(A^f_z) > 0\}.$$

We claim that $C \neq \varnothing$ and prove the claim later.

We show that $C \cap -E_+ = \varnothing$. Assume the contrary. Then we can write some $x \in -E_+$ in the form $\sum_{i=1}^r \alpha_i z_i$, where $\alpha_i > 0$, $\sum_{i=1}^r \alpha_i = 1$ and $\mu(A^f_{z_i}) > 0$ for each i. Since the measure space (T, τ, μ) is atomless, there is a small enough real $\lambda > 0$ and disjoint measurable sets $S_i \subseteq A^f_{z_i}$ such that $\mu(S_i) = \lambda\alpha_i$, for every i. But $\sum_{i=1}^r \lambda\alpha_i z_i = \lambda x \in -E_+$ and from the monotonicity and transitivity of preferences the coalition $S = \bigcup_{i=1}^r S_i$ can improve upon $t \mapsto f_t$ by using the redistribution $g_t = z_i - (1/(\alpha_i r))\, x + e_t$ for $t \in S_i$. This is a contradiction, which implies that $C \cap -E_+ = \varnothing$.

Since $-E_+$ has an interior point, there is a non-zero positive continuous linear functional π such that $(\pi, z) \geqslant 0$ for every $z \in C$.

Suppose that for some $S \subseteq T$ that is not μ-null set there is a function $t \mapsto g_t$ such that $g_t + e_t \in \operatorname{int} E_+$, $g_t + e_t \succ_t f_t$, and $\langle \pi, g_t \rangle < 0$, for all $t \in S$. By Proposition 2.1 the set $\{g_t : t \in S\}$ has a dense subset H with cardinal number $\leqslant \mathbf{m}$. For each $h \in H$, $\mu(A^f_h) = 0$ since $\langle \pi, h \rangle < 0$. From the continuity of preferences and since $g_t + e_t \in \operatorname{int} E_+$, for each $t \in S$ there is some $h \in H$ such that $h + e_t \succ_t f_t$. Thus, $S \subseteq \bigcup_{h \in H} A^f_h$. Since $\mu(A^f_h) = 0$ and $A^f_h \in \mathfrak{F}$ for all h, then by Eq. (3) $\mu^*(S) = 0$, which contradicts the supposition that S is not a μ-null set.

Let $\Gamma_t = \{z \in E : z + e_t \succ_t f_t\}$. We see from the previous argument that $\langle \pi, \operatorname{int} \Gamma_t \rangle \geqslant 0$, μ-a.e. We show that $\Gamma_t \subseteq \overline{\operatorname{int} \Gamma_t}$. Let $z + e_t \succ_t f_t$ and let $v + e_t \in \operatorname{int} E_+$. The open line segment (z, v) is in the interior of $E_+ - e_t$. From the continuity of preferences there is $x \in (z, v)$ such that the open line segment (z, x) is in $\operatorname{int} \Gamma_t$. Thus, $\Gamma_t \subseteq \overline{\operatorname{int} \Gamma_t}$ and $\langle \pi, \Gamma_t \rangle \geqslant 0$, μ-a.e.

We outline standard arguments that show that $t \mapsto f_t$ is Walrasian. Since preferences are strictly monotonic then $\pi > 0$ and $\langle \pi, f_t \rangle > \langle \pi, e_t \rangle$, μ-a.e. Since $t \mapsto f_t$ is feasible then $\langle \pi, f_t \rangle = \langle \pi, e_t \rangle$, μ-a.e.[17]

Now $\int e_t \, d\mu(t)$ is an interior point of E_+. So for some $t \in T$, $\langle \pi, f_t \rangle = \langle \pi, e_t \rangle > 0$ and $\langle \pi, \Gamma_t \rangle \geqslant 0$. This together with the continuity and strict monotonicity of preferences implies that $\langle \pi, E_+ \setminus \{0\} \rangle > 0$. Hence, for μ-almost every $t \in T$, $\langle \pi, \Gamma_t \rangle > 0$, which implies that $t \mapsto f_t$ is in Boc-$\mathbf{W}(\mathscr{E})$.[18]

Finally, we establish the claim that $C \neq \varnothing$. Take the integrable function $t \mapsto g_t = f_t + v - e_t$, where $v \in \operatorname{int} E_+$.[19] Note that $g_t + e_t \in \operatorname{int} E_+$ and $g_t + e_t \succ_t f_t$. Let H be a dense subset of $\{g_t : t \in T\}$ with cardinal number $\leqslant \mathbf{m}$. Once again $\bigcup_{h \in H} (A^f_h) = T$. Thus, for some $h \in H$, $\mu(A^f_h) > 0$ and $h \in C$.

[17] Since $t \mapsto f_t$ is Pettis feasible then $\langle \pi, f_t \rangle = \langle \pi, e_t \rangle$, μ-a.e.

[18] Which implies that $t \mapsto f_t$ is in Pet-$\mathbf{W}(\mathscr{E})$.

[19] Take the Pettis integrable function $t \mapsto g_t = f_t + v - e_t$, where $v \in \operatorname{int} E_+$.

10. REMARKS

Remark 10.1. One can see from the proof of Proposition 5.1 that two important results on the Bochner integral of set valued functions fail in non-separable spaces: Aumann's measurable selection theorem and its consequence due to Hiai and Umegaki [18, Theorem 2.2]. A related counter example of the Hiai and Umegaki theorem can be found in Khan and Rustichini [21, Proposition 5.5, p. 184]. These results are crucial to the proof of equivalence in Rustichini and Yannelis [38]. Similarly, it can be seen that the demand correspondence for the truncated economy $\hat{\mathscr{E}}$ has no strongly μ-measurable selections at some prices. For example, at price go the demand correspondence has no strongly measurable selections but has a weakly measurable selection, which is the Pettis equilibrium allocation.

Remark 10.2. The Bochner integral has been used to generalize Aumann's core equivalence theorem (e.g., Cheng [10], Glazyrina [17], Rustichini and Yannelis [38, 40]) and to prove the existence of equilibrium in various infinite dimensional models (e.g., Yannelis [44], Balder and Yannelis [6], Rustichini and Yannelis [39], Khan and Yannelis [24], Noguchi [33], Kim and Yannelis [25]). If f is Bochner integrable then $\int_S f(t)\, d\mu(t) = 0$ for all $S \in \tau$ if and only if f vanishes μ-a.e. So net trades associated with feasible allocations cannot be positive and non-zero for every consumer.

PROPOSITION 10.1. *Let (T, τ, μ) be a positive non-trivial finite measure space and let E be an ordered Banach space. If $\int_T f(t)\, d\mu(t) = 0$, then $\mu(\{t \in T : f(t) > 0\}) < \mu(T)$.*

Proof. Suppose that $\mu(\{t \in T : f(t) > 0\}) < \mu(T)$. Since E is locally convex and E_+ is closed and convex, then $\int_S f(t)\, d\mu(t) \geqslant 0$ for all $S \in \tau$. Since $\int_T f(t)\, d\mu(t) = 0$, $E_+ = E_+ + E_+$, and $E_+ \cap (-E_+) = \{0\}$ then $\int_S f(t)\, d\mu(t) = 0$ for all $S \in \tau$. Thus, f vanishes μ-a.e. and $\mu(\{t \in T : f(t) > 0\}) = 0$. ∎

This is not the case for the more general notions of integration. Hence, there could be an interpretive problem with using the Pettis or Gelfand or Dunford integrals. Indeed, there is a Pettis integrable function f which is everywhere non-zero but for which Pettis-$\int_S f(s)\, d\mu(t) = 0$ for every $S \in \tau$ (cf. Khan and Sun [23]). Consider an orthonormal basis $\{e_t : t \in [0, 1]\}$ for the non-separable Hilbert space $l_2([0, 1])$. Define $f: [0, 1] \to l_2([0, 1])$ by $f(t) = e_t$. For any $p \in l_2([0, 1])$ we have $\langle p, f(t) \rangle = 0$ for all but countably many t. Thus, for any measurable set $S \subseteq [0, 1]$ we have

$$\forall P \in l_2([0, 1]), \quad \int_S \langle p, f(t) \rangle\, d\mu(t) = 0 \quad \text{and} \quad \text{Pettis-}\int_S f(t)\, d\mu(t) = 0.$$

A similar interpretive problem arises with the allocation identified as the Pettis equilibrium in Proposition 5.1, where each individual in a non-negligible coalition consumes a positive amount of some commodity not initially available in the economy.[20]

Note, however, that the problem elucidated above disappears when the commodity space admits strictly positive prices.

PROPOSITION 10.2. *Let (T, τ, μ) be a measure space. Let E be an ordered Banach space such that there exists $p \in E'$, $p \gg 0$. If Pettis-$\int_T f(t)\, d\mu(t) = 0$ then $\mu(\{t \in T : f(t) > 0\}) < \mu(T)$.*

Remark 10.3. Several corollaries of Theorem 6.1 can be proved.

We define the notion of the equal treatment core. For each $t' \in T$ let

$$D(t') = \{t \in T : \succ_t = \succ_{t'}, X_t = X_{t'}, e_t = e_{t'}\},$$

which is the class of all agents identical to agent t'. Let Boc-$\hat{\mathbf{C}}(\mathscr{E}) \subseteq$ Boc-$\mathbf{C}(\mathscr{E})$ have the property $[f] \in$ Boc-$\hat{\mathbf{C}}(\mathscr{E})$ if $f(t) = f(t')$ for any $t \in T$, $t' \in D(t)$, μ-a.e. Define in the obvious analogous way the set Pet-$\hat{\mathbf{C}}(\mathscr{E}) \subseteq$ Pet-$\mathbf{C}(\mathscr{E})$.

Suppose that $\mu^*(D(t)) > 0$, μ-a.e., and that $[f] \in$ Boc-$\hat{\mathbf{C}}(\mathscr{E})$ or $[f] \in$ Pet-$\hat{\mathbf{C}}(\mathscr{E})$. Then there is a μ-null set M such that $\mu(A^f_h) = 0$ implies $A^f_z = \varnothing$ or $A^f_z \subseteq M$. Letting

$$\mathfrak{F} = \{S \in \tau : \mu(S) \neq 0 \text{ or } S \subseteq M\},$$

we see that Eq. (3) is satisfied:

$$\nexists S \subseteq T, \qquad \mu^*(S) > 0, \qquad w(S, \mathfrak{F}, \mu) > 0.$$

COROLLARY 10.2.1. *Let E be an ordered Banach space. If $\mathscr{E} = [E, (T, \tau, \mu), X, \succ, e]$ is an economy that satisfies* B1, B2, *and $\mu^*(D(t)) > 0$ μ-a.e., then Boc-$\hat{\mathbf{C}}(\mathscr{E}) \subseteq$ Boc-$\mathbf{W}(\mathscr{E})$.*

If $\mathscr{E}$ is a Pettis economy, B1 *is satisfied, and for any Pettis allocation $t \mapsto f_t$ and any $z \in E$, the set A^f_z is measurable, then Pet-$\hat{\mathbf{C}}(\mathscr{E}) \subseteq$ Pet-$\mathbf{W}(\mathscr{E})$.*

By an *Ulam number*, or a cardinal of *measure zero*, we mean a cardinal $\mathbf{m}$ with the following property (see Federer [16], Oxtoby [34]):

If μ is a finite measure defined on 2^X, $\operatorname{card}(X) \leqslant \mathbf{m}$, *and $\mu(\{x\}) = 0$ for all $x \in X$, then $\mu(X) = 0$.*

[20] See Dilworth and Girardi [13] for an exposition on differences between Pettis and Bochner integrable functions in infinite-dimensional Banach spaces.

A measure μ on the class of Borel sets of a topological space X is called a *Borel measure*. We shall use the following result, see Oxtoby [34, Theorems 16.3-4, p. 63].

PROPOSITION 10.3. *Let μ be a Borel measure on a metric space X.*

1. *If S is the union of a family of* $\mathbf{m}$ *open sets of measure zero and if* $\mathbf{m}$ *is an Ulam number, then $\mu(s)=0$.*

2. *If X has weight* $\mathbf{m}$ *which is an Ulam number, then the union of any family of open sets of measure zero has measure zero.*

Suppose that T is a topological space with the property that the union of $\mathbf{n} \leqslant \mathbf{m}$ open μ-null sets is a μ-null set. Let $\mathscr{M}$ be the class of all open μ-null subsets of T. Letting

$$\mathfrak{F} = \{ S \in \tau : \mu(S) \neq 0 \text{ or } \exists M \in \mathscr{M},\ S \subseteq M \},$$

we see that Eq. (3) is satisfied. The following two corollaries are consequences of 1 and 2 of Proposition 10.3, respectively.

COROLLARY 10.3.1. *Let E be an ordered Banach space with weight* $\mathbf{m}$, *which is an Ulam number, and let T be a metric space. Let μ be a Borel measure on T and let $\mathscr{E} = [E, (T, \tau, \mu), X, \succ, e]$ be an economy that satisfies* B1 *and* B2. *If $[f] \in Boc\text{-}\mathbf{C}(\mathscr{E})$ satisfies*

$$\forall z \in E; \qquad \mu(A_z^f) = 0 \Rightarrow \exists M \in \mathscr{M}, \qquad A_z^f \subseteq M, \tag{18}$$

then $[f] \in Boc\text{-}\mathbf{W}(\mathscr{E})$.

If $\mathscr{E}$ is a Pettis economy, $[f] \in Pet\text{-}\mathbf{C}(\mathscr{E})$, and for any Pettis allocation $t \mapsto f_t$ and any $z \in E$, the set A_z^f is measurable, then $[f] \in Pet\text{-}\mathbf{W}(\mathscr{E})$.

COROLLARY 10.3.2. *Let E be an ordered Banach space and let T be a metric space with weight* $\mathbf{m}$, *which is an Ulam number. Let μ be a Borel measure on T and let $\mathscr{E} = [E, (T, \tau, \mu), X, \succ, e]$ be an economy that satisfies* B1 *and* B2. *If $[f] \in Boc\text{-}\mathbf{C}(\mathscr{E})$ satisfy*

$$\forall z \in E; \qquad \text{if} \quad \mu(A_z^f) = 0 \Rightarrow \exists M \in \mathscr{M}, \quad A_z^f \subseteq M, \tag{19}$$

then $[f] \in Boc\text{-}\mathbf{W}(\mathscr{E})$.

If $\mathscr{E}$ is a Pettis economy, $[f] \in Pet\text{-}\mathbf{C}(\mathscr{E})$, and for any Pettis allocation $t \mapsto f_t$ and any $z \in E$, the set A_z^f is measurable, then $[f] \in Pet\text{-}\mathbf{W}(\mathscr{E})$.

Several variants of the ideas in Corollaries 10.3.1 and 10.3.2 can be proved. All that is needed is to identify a family of μ-null sets with the property that the union of every sub-family of μ-null sets with cardinality $\leqslant \mathbf{m}$ is a μ-null set. For example, the conditions in Corollaries 10.3.1 and 10.3.2 requiring that E or T have weight that is an Ulam number can be weakened if we restrict our attention to Radon or Loeb measures, see Ross [37].

Remark 10.4. We can see from the proof of Proposition 5.1 that there is no free-disposal equilibrium for the truncated economy $\hat{\mathscr{E}}$, with or without positive prices. Furthermore, the positive results in Section 6 can be proved in the useful setting of a Banach lattice without order unit but with uniformly proper preferences, cf. Rustichini and Yannelis [38].

Remark 10.5. In addition to the notion of a Pettis economy we can define a Gelfand-economy and a Dunford-economy by appropriately changing the method of integration. For the Dunford-economy we say a weakly measurable allocation f is feasible if $f - e$ is Pettis integrable and

$$\text{Pettis-}\int_T f(t) - e(t)\, d\mu(t) = 0.$$

Note that f and e need not be Pettis integrable and hence their Dunford-integral, if it exists, need not be in E.

Remark 10.6. We do not investigate whether core allocations can be approximately decentralized by prices. That is, whether core allocations of economies with sufficiently large (but finite) numbers of agents are approximately competitive. It appears that the finite dimensional results on approximate decentralizability of core allocations fail even in *separable* commodity spaces (see for instance [2]). This indicates an entirely different syndrome to the failure of Core-Walras equivalence in non-separable spaces highlighted in the present paper. As noted in the introduction, however, our counter example is constructed using an arbitrary atomless measure space of agents, and thus includes Loeb measure spaces. It therefore may be possible to use "lifting" arguments to infer from our negative theorems a negative result for large but finite economies.

11. OPEN QUESTIONS

Remark 11.1. The counter examples in Proposition 5.1 are constructed using a non-separable ordered Hilbert commodity space with order unit. Can we view the construction in Proposition 5.1 as a concrete version of the proof of a more general result. One that characterizes a class of Banach spaces as those spaces in which Bochner existence and Bochner Core-Walras equivalence hold.[21] This important question remains open. Curiously, Proposition 5.1, and the examples in Khan and Rustichini [21], appear to crucially require the existence of a Pettis integrable function f which is everywhere non-zero but for which Pettis-$\int_S f(s)\,d\mu(t) = 0$ for every $S \in \tau$, see also Remark 10.

Remark 11.2. Since the commodity space in the proof of Proposition 5.1 is reflexive, the Pettis integral in the statement of Proposition 5.1 can be changed to the Gelfand or Dunford integral. An important question left open by Proposition 5.1 and Theorem 6.1 is whether we can obtain results in non-separable spaces for the Pettis or Gelfand or Dunford integrals that are as general as the known results for the Bochner economies in separable spaces.

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[21] This question was recently taken up by Podczeck [36], who provides some results in this direction.

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