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ABSTRACT

We prove the existence of a competitive equilibrium in an exchange economy with incomplete markets, where preferences may be discontinuous, non-ordered, price-dependent, and interdependent (externalities allowed), and no survival or interiority of the initial endowments assumption is imposed. Thus, we bring the existence of an equilibrium with incomplete asset markets to the “state of the art” level.

1. Introduction

Since the pioneering work of Dasgupta and Maskin (1986), several advances were made on existence of an equilibrium in games with discontinuous payoffs/preferences (see for example Reny (1999, 2016a, 2020)).

Recently, similar advances were made on the existence of a competitive (Walrasian) equilibrium for exchange economies (see for example the recent works of He and Yannelis (2016), Podczeck and Yannelis (2022), Anderson et al. (2022), Bhowmik and Yannelis (2024), and Khan et al. (2025a,b), among others).

A short historical introduction to the existence of a Walrasian equilibrium follows below. Walras (1874) was the one to introduce the concept of an equilibrium and it is interesting to remark that he did not call it competitive. Rigorous proofs were provided first by McKenzie (1954) and a few months later by Arrow and Debreu (1954).¹ Both papers required preferences to be continuous, convex, transitive and complete. Major advances were made in 1970's in demand theory and equilibrium theory indicating that preferences need not be transitive or complete or convex (see Mas-Colell (1974), Shafer (1974), Shafer and Sonnenschein (1975), Borglin and Keiding (1976), and Shafer (1976)). It is only recently that the above works were

generalized to non-continuous preferences and fixed points for non-continuous correspondences were established (see He and Yannelis (2017)). It should be remarked that the key condition introduced was the continuous inclusion property (see He and Yannelis (2016, 2017) and Reny (2016b)). This condition is different than the payoff security used in game theory (see Reny (1999)).

The purpose of this paper is to generalize the existence of equilibrium to an economy with incomplete (asset) markets by allowing for discontinuous preferences. By doing so, we dispense with several standard assumptions, i.e., interiority of the initial endowments or irreducibility, and monotonicity and convexity of preferences. Also, preferences may be price-dependent and interdependent (externalities allowed). Thus, we bring the existence of an equilibrium with incomplete markets to the “state of the art” level.

A short historic review of general equilibrium with incomplete markets is as follows: Radner (1972) introduced a general equilibrium model with incomplete markets and he alluded to the problems arising in proving the existence of an equilibrium. Hart (1975) provided a nonexistence example of a competitive equilibrium based on a discontinuity in the span of the assets when the prices change. However, Duffie and Shafer (1985) showed that generically a competitive equilibrium exists in the Hart economy. Cass (1984) introduced the idea of nominal

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¹ It should be noted that the main result of Arrow and Debreu (1954) was an application of the earlier social equilibrium theorem of Debreu (1952). The contribution of Arrow and Debreu (1954) was to convert the economy to a social system/abstract economy and apply the Debreu's (1952) existence theorem.

financial assets and in his modeling a competitive equilibrium exists. The work of Cass generated literature on incomplete markets and this is the modeling we follow in this paper (see Geanakoplos (1990) for an excellent survey).

In Section 5, we indicate how our main theorem includes as a special case the incomplete markets equilibrium existence results of Cass (1984), Geanakoplos and Polemarchakis (1986), Werner (1985, 1989), and Gottardi and Hens (1996).

Furthermore, when our incomplete market economy is specialized to an exchange economy, our main theorem gives as a corollary the recent Podczeck and Yannelis (2022) existence result of a competitive equilibrium. Obviously in an exchange economy framework we generalize and extend substantially the existence theorems of Arrow and Debreu (1954), McKenzie (1954, 1959, 1981), Mas-Colell (1974), and Shafer (1976) among others.

The new proof we offer relies on the new assumption called continuous inclusion property (see He and Yannelis (2016, 2017), Reny (2016b), and Podczeck and Yannelis (2022)). Although, our proof follows a novel compactification introduced in Podczeck and Yannelis (2022), the introduction of incomplete markets necessitates new arguments that are far from trivial.

The paper proceeds as follows. Section 2 provides a model of exchange economy with incomplete markets. In Section 3, it is shown that a competitive equilibrium exists under the condition of continuous inclusion property. In Section 4, we present an example where preferences are not continuous but a competitive equilibrium exists if the continuous inclusion property is satisfied. No other existence theorem in the literature can cover this case. We also provide an example which shows that a competitive equilibrium does not exist when the continuous inclusion property fails. In Section 5, we discuss how our model encompasses Gottardi and Hens (1996), Geanakoplos and Polemarchakis (1986), and Werner (1985, 1989) in the incomplete market literature, and extends Podczeck and Yannelis (2022) to an exchange economy with incomplete markets.

2. The model

2.1. Economy and competitive equilibrium

A two-period exchange economy is considered where asset markets are open in the first period (denoted by 0) and markets for consumption goods are open in both the first period and the second period (denoted by 1). The uncertainty is described by the set of states of nature in period 1, denoted by $S = \{1, \dots, S\}$. In period 0 (considered state 0), agents are unaware of which state will occur in period 1. Assets deliver monetary payoffs contingent on the states of period 1, that is, assets are nominal. In each state $s \in S_0 \equiv S \cup \{0\}$, there exist L consumption goods, the set of which is denoted by $\mathcal{L} = \{1, 2, \dots, L\}$. Therefore the number of commodities equals $\ell := L(S + 1)$. The Euclidean space $\mathbb{R}^\ell$ is taken as the commodity space of the economy.

Let $I = \{1, \dots, m\}$ denote the finite set of agents and $J = \{1, 2, \dots, J\}$ the set of financial assets. Each agent $i \in I$ chooses his consumption bundle x_i from the consumption set $X_i \subset \mathbb{R}^\ell$, which contains his endowment e_i . We assume that there is no friction in asset markets so that every agent i can choose his portfolio θ_i in the portfolio choice set $\Theta_i \equiv \mathbb{R}^J$. The sets $X := \prod_{i \in I} X_i$ and $\Theta := \prod_{i \in I} \Theta_i$ indicate the sets of consumption and portfolio allocations, respectively. Let $p \in \mathbb{R}^\ell$ denote the commodity price (vector) and $q \in \mathbb{R}^J$ the asset price (vector). Agent i 's preferences are represented by the preference correspondence $P_i : (\mathbb{R}^\ell \times \mathbb{R}^J) \times (X \times \Theta) \rightarrow 2^{X_i}$ where $P_i(p, q, x, \theta)$ indicates the set of consumption bundles that agent i prefers to x_i given price-allocation profile $(p, q, x, \theta) \in (\mathbb{R}^\ell \times \mathbb{R}^J) \times (X \times \Theta)$. Note that P_i incorporates the dependence on prices and the externalities of consumptions and portfolios.

Let A_X and A_Θ denote the set of attainable consumption and portfolio allocations, respectively, that is, $A_X = \{x \in X : \sum_{i \in I} x_i = \sum_{i \in I} e_i\}$ and

$A_\Theta = \{\theta \in \Theta : \sum_{i \in I} \theta_i = 0\}$. The set of attainable allocations is defined by $A = A_X \times A_\Theta$.

Each asset $j \in J$ pays $r_j(s)$ in state $s \in S$. The payoffs of J assets in state $s \in S$ are given by the J -dimensional row vector $r(s) = (r_j(s))_{j \in J}$, whereas those of asset j in S states are given by a S -dimensional column vector $r_j = (r_j(s))_{s \in S}$. The asset (payoff) structure is described by the $S \times J$ matrix $R = [(r_j)_{j \in J}]$, with $S \geq J$. For a price pair $(p, q) \in \mathbb{R}^\ell \times \mathbb{R}^J$, we employ the following notations: $p \square y$ denotes the column vector of $(p(s) \cdot y(s))_{s \in S_0}$, $p \square_1 y$ denotes the column vector of $(p(s) \cdot y(s))_{s \in S}$, and

$$W(q) := \begin{bmatrix} -q \\ R \end{bmatrix}.$$

The following definition provides a notion of arbitrage for incomplete markets:

Definition 2.1. A portfolio $\hat{\theta} \in \mathbb{R}^J$ is defined to be an arbitrage with respect to asset price $q \in \mathbb{R}^J$ if it satisfies $W(q) \cdot \hat{\theta} > 0$.² An asset price $q \in \mathbb{R}^J$ is said to admit no arbitrage if there is no arbitrage, and then it is called no-arbitrage asset price.

Let Q denote the set of no-arbitrage asset prices. It is well known that $Q = \{q \in \mathbb{R}^J : q = \lambda \cdot R \text{ for some } \lambda \in \mathbb{R}_{++}^S\}$, which is a convex cone with $0 \notin Q$ but may not be closed. Define the set of normalized prices as³

$$\Delta = \{(p, q) \in \mathbb{R}^\ell \times \mathcal{C}^\ell(Q) : p = (p(s))_{s \in S_0}, \\ \|(p(0), q)\| \leq 1, \|p(s)\| \leq 1, \forall s \in S\}.$$

The budget correspondence $B_i : \Delta \rightarrow 2^{X_i \times \Theta_i}$ and the open budget correspondence $B_i^\circ : \Delta \rightarrow 2^{X_i \times \Theta_i}$ of agent i are defined by

$$B_i(p, q) = \{(x_i, \theta_i) \in X_i \times \Theta_i : p \square (x_i - e_i) \leq W(q) \cdot \theta_i\}, \\ B_i^\circ(p, q) = \{(x_i, \theta_i) \in X_i \times \Theta_i : p \square (x_i - e_i) \ll W(q) \cdot \theta_i\}.$$

For a given price-allocation profile $(p, q, x, \theta) \in \Delta \times (X \times \Theta)$, agent $i \in I$ chooses a P_i -maximal consumption–portfolio pair (x_i, θ_i) in $B_i(p, q)$, i.e., $B_i(p, q) \cap (P_i(p, q, x, \theta) \times \Theta_i) = \emptyset$. Let $\mathcal{E} = \langle (X_i, P_i, e_i, \Theta_i)_{i \in I}, R \rangle$ denote the economy described above.

Definition 2.2. A competitive (Walrasian) equilibrium of economy $\mathcal{E}$ is a price-allocation profile $(p^*, q^*, x^*, \theta^*) \in \Delta \times (X \times \Theta)$ such that

- (i) For each $s \in S_0$, $p^*(s) \neq 0$, and $q^* \in Q \setminus \{0\}$;
- (ii) For each $i \in I$, $(x_i^*, \theta_i^*) \in B_i(p^*, q^*)$;
- (iii) For each $i \in I$, $B_i(p^*, q^*) \cap (P_i(p^*, q^*, x^*, \theta^*) \times \Theta_i) = \emptyset$;
- (iv) $(x^*, \theta^*) \in A$.

2.2. Assumptions and statement of theorem

In He and Yannelis (2016), the continuous inclusion property was stated in terms of closed correspondences, rather than in terms of upper hemicontinuous correspondences with compact values. However, He and Yannelis (2016) used the continuous inclusion property in the context of compact consumption sets.

Recently, by adopting the continuous inclusion property in terms of upper hemicontinuous correspondences with nonempty compact values, Podczeck and Yannelis (2022) generalized the results of He and Yannelis (2016) in an exchange economy with non-compact consumption sets. Following Podczeck and Yannelis (2022), we define the continuous inclusion property.

² For two vectors v and v' in an Euclidean space $\mathbb{R}^n$, $v \geq v'$ implies that $v - v' \in \mathbb{R}_+^n$; $v > v'$ implies that $v \geq v'$ and $v \neq v'$; $v \gg v'$ implies that $v - v' \in \mathbb{R}_{++}^n := \{x \in \mathbb{R}^n : x_k > 0, \forall k = 1, \dots, n\}$.

³ We denote by $\|\cdot\|$ the Euclidean norm and by $\|\cdot\|_1$ the ℓ_1 -norm. Given a set A in a Euclidean space, let $\mathcal{C}^\ell(A)$, $\text{int}(A)$, and $\text{co}(A)$ denote its closure, interior, and convex hull of A , respectively.

Definition 2.3. Let X be a topological space and Y a linear topological space. A correspondence $\psi : X \rightarrow 2^Y$ is said to have the *continuous inclusion property* at x if there exists an open neighborhood O_x of x and a correspondence $F_x : O_x \rightarrow 2^Y$ such that $F_x(z) \subset \psi(z)$ for any $z \in O_x$ and $\text{co}F_x$ is upper hemicontinuous with nonempty, compact, and convex values.⁴

For each $i \in I$, we now define the correspondence $\psi_i : \Delta \times (X \times \Theta) \rightarrow 2^{X_i \times \Theta_i}$ by

$$\psi_i(p, q, x, \theta) = B_i(p, q) \cap (P_i(p, q, x, \theta) \times \Theta_i).$$

For the proof of equilibrium existence, we need the following assumptions.

- (A1) For each $i \in I$, X_i is a closed convex subset of $\mathbb{R}_+^{\ell}$ containing e_i , and $\sum_{i \in I} e_i(s) \neq 0$ for each $s \in S_0$.
- (A2) For each $i \in I$, $\Theta_i = \mathbb{R}^J$.⁵
- (A3) For each $i \in I$, $(x_i, \theta_i) \notin \text{co}\psi_i(p, q, x, \theta)$ for every $(p, q, x, \theta) \in \Delta \times (X \times \Theta)$.
- (A4) If $(p, q, x, \theta) \in \Delta \times A$ is a price-allocation profile such that $(x_i, \theta_i) \in B_i(p, q)$ for each $i \in I$ and $I(p, q, x, \theta) \equiv \{i \in I : \psi_i(p, q, x, \theta) \neq \emptyset\} \neq \emptyset$, then there is an agent $i \in I(p, q, x, \theta)$ such that ψ_i has the continuous inclusion property at (p, q, x, θ) .
- (A5) If $(p, q, x, \theta) \in \Delta \times A$ is a price-allocation profile such that $(x_i, \theta_i) \in B_i(p, q)$ for each $i \in I$, then for each $s \in S_0$, there is an agent $i \in I$ with $\omega_i(s) \in \mathbb{R}_+^L$ such that $x_i + \omega_i(s)\mathbf{1}_s \in P_i(p, q, x, \theta)$.⁶
- (A6) There is no redundant asset.
- (A7) There exists a portfolio $\hat{\theta} \in \mathbb{R}^J$ such that $R \cdot \hat{\theta} > 0$.

Assumptions (A1)–(A3) and (A6) are self-explanatory. [Shafer and Sonnenschein \(1975\)](#) assume that $x_i \notin \text{co}P_i(x)$ for every $x \in X$. This assumption can be translated to $x_i \notin \text{co}P_i(p, q, x, \theta)$ for every $(p, q, x, \theta) \in \Delta \times (X \times \Theta)$ in our model, which implies our assumption (A3). Obviously, preferences need not be convex, and they are irreflexive. Assumption (A4) corresponds to the continuous inclusion property of [Podczeck and Yannelis \(2022\)](#). This assumption is very weak and automatically holds if the preference correspondences have an open graph and the irreducibility or strict positivity of the initial endowments assumptions are satisfied. [Lemma 5.1](#) and [Remark 5.2](#) in Section 5 clarify this point. This assumption is not related to the payoff security assumption introduced in game theory by [Reny \(1999\)](#). Assumption (A5) stipulates that, in each state s , some agent i is (statewise) non-satiated with consumption bundle $\omega_i(s) \in \mathbb{R}_+^L$ such that $x_i + \omega_i(s)\mathbf{1}_s \in P_i(p, q, x, \theta)$. This assumption corresponds to the global non-satiation assumption of [Podczeck and Yannelis \(2022\)](#). By virtue of this assumption, the equilibrium commodity price cannot be zero and the equilibrium asset price admits no-arbitrage. If there is a risk-free asset, then Assumption (A7) trivially holds. Both Assumptions (A5) and (A7) imply that the equilibrium asset price cannot be zero. Observe that A_X is compact due to Assumption (A1) and A_Θ is closed but unbounded. Thus A is closed but is not guaranteed to be compact.

Theorem 2.1. *If the economy $\mathcal{E}$ satisfies Assumptions (A1)–(A7), then it has a competitive equilibrium.*

3. Proof of the theorem

The idea of the proof is to construct an abstract economy and apply a fixed point theorem. However the lack of continuity makes the

argument a bit more complicated than usual. One has to be careful in constructing mappings that satisfy the continuous inclusion property. To this end, we compactify the economy in a way that the continuous inclusion property is satisfied and construct an equilibrium mapping that satisfies the assumptions of the Kakutani's fixed point theorem. Thus, we obtain an equilibrium for the compactified economy. Then a limiting argument is operated to show that a competitive equilibrium exists in the limit. The continuous inclusion property plays an important role throughout the arguments.

Let $\{C^n\}$ be an increasing sequence of closed balls in $\mathbb{R}^\ell \times \mathbb{R}^J$ such that $\bigcup_{n \in \mathbb{N}} C^n = \mathbb{R}^\ell \times \mathbb{R}^J$. Set $K_i^n = C^n \cap (X_i \times \Theta_i)$ for each $i \in I$ and each $n \in \mathbb{N}$. Write $K^n = \prod_{i \in I} K_i^n$ and $y = (p, q, x, \theta) \in \Delta \times (X \times \Theta)$. Note that K_i^n is nonempty, compact, and convex for each $i \in I$ and each $n \in \mathbb{N}$, and thus so is K^n for each $n \in \mathbb{N}$. Since A_X is compact, we can assume that A_X is included in the projection of K^n onto X for each $n \in \mathbb{N}$.

Let $\mathcal{H}$ be a collection of correspondences F_y^i 's from $\Delta \times (X \times \Theta)$ to $2^{X_i \times \Theta_i}$ such that Assumption (A4) is satisfied. Since, in a Euclidean space, any neighborhood of any point includes a compact neighborhood of this point and $X \times \Theta$ is closed in its ambient Euclidean space, we can assume by [Definition 2.3](#) that, for each correspondence in $\mathcal{H}$, the domain O_y^i is bounded in $\Delta \times (X \times \Theta)$ and its image $F_y^i(O_y^i)$ is bounded in $X_i \times \Theta_i$. Thus, for each F_y^i in $\mathcal{H}$, there is an $n_y^i \in \mathbb{N}$ such that, for each $n \geq n_y^i$, O_y^i is included in $\Delta \times K^n$ and $F_y^i(O_y^i)$ is included in K_i^n . For each $i \in I$ and each $n \in \mathbb{N}$, denote by $\mathcal{C}_i^n$ the collection of those O_y^i 's which are included in $\Delta \times K^n$ and are such that $F_y^i(O_y^i)$ is included in K_i^n . That is, we write $\mathcal{C}_i^n = \{O_y^i : O_y^i \subset \Delta \times K^n, F_y^i(O_y^i) \subset K_i^n\}$.

For each $i \in I$ and each $n \in \mathbb{N}$, let $V_i^n = \bigcup_{O_y^i \in \mathcal{C}_i^n} O_y^i$. Note that V_i^n is open in $\Delta \times K^n$ and $\mathcal{C}_i^n$ is an open cover of V_i^n . Being included in a Euclidean space, V_i^n is metrizable and therefore paracompact.⁷ Thus $\mathcal{C}_i^n$ has a closed locally finite refinement $\mathcal{F}_i^n = \{E_{i,k}^n : k \in \mathcal{J}_i^n\}$, where $\mathcal{J}_i^n$ is an index set and $E_{i,k}^n$ is closed in V_i^n .⁸ For each $k \in \mathcal{J}_i^n$, choose a point $y_k \in \Delta \times K^n$ such that $E_{i,k}^n \subset O_{y_k}^i \in \mathcal{C}_i^n$. For each $y \in V_i^n$, let $\zeta_i^n(y) = \{k \in \mathcal{J}_i^n : y \in E_{i,k}^n\}$. Then $\zeta_i^n(y)$ is finite for each $y \in V_i^n$.

Define the correspondence $\varphi_i^n : V_i^n \rightarrow 2^{X_i \times \Theta_i}$ by

$$\varphi_i^n(y) = \text{co} \left(\bigcup_{k \in \zeta_i^n(y)} F_{y_k}^i(y) \right), \quad \forall y \in V_i^n.$$

It is obvious that φ_i^n has nonempty, compact, and convex values. Define the correspondence $\xi_i^n : V_i^n \rightarrow 2^{X_i \times \Theta_i}$ by $\xi_i^n(y) = \bigcup_{k \in \zeta_i^n(y)} \text{co}F_{y_k}^i(y)$. It is clear that ξ_i^n is upper hemicontinuous. Since $\text{co}(\bigcup_{k \in \zeta_i^n(y)} F_{y_k}^i(y)) = \text{co}(\bigcup_{k \in \zeta_i^n(y)} \text{co}F_{y_k}^i(y))$, we see that $\varphi_i^n(y) = \text{co}(\xi_i^n(y))$ for each $y \in V_i^n$ and therefore φ_i^n is upper hemicontinuous as well. Moreover, noting that, for each $i \in I$ and each y_k , $F_{y_k}^i(y) \subset \psi_i(y)$ for every $y \in O_{y_k}^i$, we obtain

$$\varphi_i^n(y) \subset \text{co}\psi_i(y), \quad \forall y \in V_i^n. \quad (3.1)$$

For each $i \in I$ and each $n \in \mathbb{N}$, define the correspondence $H_i^n : \Delta \times K^n \rightarrow 2^{K_i^n}$ by setting, for each $y = (p, q, x, \theta) \in \Delta \times K^n$,

$$H_i^n(y) = \begin{cases} \varphi_i^n(y), & \text{if } y \in V_i^n, \\ B_i(p, q) \cap K_i^n, & \text{if } y \notin V_i^n. \end{cases}$$

Observing that V_i^n is open in $\Delta \times K^n$ and $\varphi_i^n(y) \subset B_i(p, q) \cap K_i^n$ for each $y = (p, q, x, \theta) \in V_i^n$, one can easily show that H_i^n is upper hemicontinuous with nonempty, compact, and convex values.

For each $n \in \mathbb{N}$, define the correspondence $G^n : \Delta \times K^n \rightarrow 2^A$ by setting, for each $y = (p, q, x, \theta) \in \Delta \times K^n$,

$$\begin{aligned} G^n(y) = & \left\{ (\hat{p}, \hat{q}) \in \Delta : \sum_{s \in S_0} \hat{p}(s) \cdot z(s) + \hat{q} \cdot \sum_{i \in I} \theta_i \right. \\ & \left. \geq \sum_{s \in S_0} p'(s) \cdot z(s) + q' \cdot \sum_{i \in I} \theta_i, \quad \forall (p', q') \in \Delta \right\}, \end{aligned}$$

⁴ Given a correspondence $\varphi : X \rightarrow 2^Y$, the correspondence $\text{co}\varphi : X \rightarrow 2^Y$ is defined by $\text{co}\varphi(x) = \text{co}(\varphi(x))$.

⁵ If not confusing, for notational convenience, we usually use Θ_i and Θ instead of $\mathbb{R}^J$ and $(\mathbb{R}^J)^m$, respectively.

⁶ We denote by $\mathbf{1}_s$ the indicator function on S_0 such that $\mathbf{1}_s(s') = 1$ if $s' = s$ and $\mathbf{1}_s(s') = 0$ otherwise.

⁷ See Theorem 5.1.3 of [Engelking \(1989\)](#).

⁸ See Theorem 5.1.11 of [Engelking \(1989\)](#).

where $z(s) := \sum_{i \in I} (x_i(s) - e_i(s))$ for each $s \in S_0$. It can be easily checked that G^n is upper hemicontinuous with nonempty, compact, and convex values.

Clearly the correspondence $G^n \times \prod_{i \in I} H_i^n : \Delta \times K^n \rightarrow 2^{\Delta \times K^n}$ is upper hemicontinuous with nonempty, compact, and convex values. By Kakutani's fixed point theorem, for each $n \in \mathbb{N}$, $G^n \times \prod_{i \in I} H_i^n$ has a fixed point $(p^n, q^n, x^n, \theta^n) \in \Delta \times K^n$.

By construction, it is obvious that

$$(x_i^n, \theta_i^n) \in B_i(p^n, q^n), \forall i \in I, \forall n \in \mathbb{N}. \quad (3.2)$$

Write $z^n = \sum_{i \in I} (x_i^n - e_i)$ and $\theta_0^n = \sum_{i \in I} \theta_i^n$. We claim that $z^n = 0$ and $\theta_0^n = 0$, i.e., $(x^n, \theta^n) \in A$ for each $n \in \mathbb{N}$. To see this, suppose to the contrary that $z^n(0) \neq 0$ or $\theta_0^n \neq 0$ for some $n \in \mathbb{N}$. Then, by the definition of G^n , we have $p^n(0) \cdot z^n(0) + q^n \cdot \theta_0^n > 0$. However, we know from (3.2) that $p^n(0) \cdot z^n(0) + q^n \cdot \theta_0^n \leq 0$, a contradiction. This implies that $z^n(0) = 0$ and $\theta_0^n = 0$ for each $n \in \mathbb{N}$. If $z^n(s) \neq 0$ for some $s \in S$ and some $n \in \mathbb{N}$, then we have $p^n(s) \cdot z^n(s) > 0$. However, it follows from (3.2) that $p^n(s) \cdot z^n(s) \leq r(s) \cdot \theta_0^n = 0$, which is a contradiction, implying that $z^n(s) = 0$ for each $s \in S$ and each $n \in \mathbb{N}$. Hence we have $(x^n, \theta^n) \in A$ for each $n \in \mathbb{N}$.

Now we claim that the sequence $\{(p^n, q^n, x^n, \theta^n)\}$ is bounded in $\Delta \times A$. First observe that the sequence $\{(p^n, q^n, x^n)\}$ is bounded since $(p^n, q^n, x^n) \in \Delta \times A_X$ for each $n \in \mathbb{N}$. Thus we need to show that the sequence $\{\theta^n\}$ is bounded. Suppose to the contrary that it is unbounded. Define $a_n^{-1} := \sum_{i \in I} \|\theta_i^n\|$. Then a_n^{-1} goes to infinity as $n \rightarrow \infty$. Since the sequences $\{a_n \theta^n\}$ and $\{(p^n, q^n, x^n)\}$ are bounded, we can assume that the sequence $\{(p^n, q^n, x^n, a_n \theta^n)\}$ converges to (p^*, q^*, x^*, v^*) . Noting $a_n[p^n \square (x_i^n - e_i)] \leq W(q^n) \cdot (a_n \theta_i^n)$, in the limit, we obtain $W(q^*) \cdot v_i^* \geq 0$ for each $i \in I$. Observe that $\sum_{i \in I} \|v_i^*\| = 1$ and $\sum_{i \in I} v_i^* = 0$. Therefore $W(q^*) \cdot v_i^* = 0$ for each $i \in I$ but it is impossible that $v_i^* = 0$ for each $i \in I$. However, since R has column full rank, the fact that $W(q^*) \cdot v_i^* = 0$ implies $v_i^* = 0$ for each $i \in I$, a contradiction. Consequently, we conclude that the sequence $\{(p^n, q^n, x^n, \theta^n)\}$ is bounded. Hence we can assume that the sequence $\{(p^n, q^n, x^n, \theta^n)\}$ converges to $(p^*, q^*, x^*, \theta^*) \in \Delta \times A$.

Since B_i has closed graph and $(x_i^n, \theta_i^n) \in B_i(p^n, q^n)$ for each $i \in I$ and each $n \in \mathbb{N}$, we have $(x_i^*, \theta_i^*) \in B_i(p^*, q^*)$ for each $i \in I$. Let $y^* = (p^*, q^*, x^*, \theta^*)$. We claim that $\psi_i(y^*) = \emptyset$ for each $i \in I$. Suppose to the contrary that $I(y^*) = \{i \in I : \psi_i(y^*) \neq \emptyset\} \neq \emptyset$. By Assumption (A4), there is an agent $i \in I(y^*)$ such that ψ_i has the continuous inclusion property at y^* . For that agent i , we consider $O_{y^*}^i$ and $F_{y^*}^i$. Now we can choose $n_0 \in \mathbb{N}$ so large that we have both $O_{y^*}^i \subset \Delta \times K^n$ and $F_{y^*}^i(O_{y^*}^i) \subset K_i^n$ for all $n \geq n_0$. Thus $O_{y^*}^i \subset V_i^n$ for all $n \geq n_0$. Since $O_{y^*}^i$ is an open neighborhood of y^* in $\Delta \times (X \times \Theta)$, we can pick an $n_1 \in \mathbb{N}$ such that $y^n = (p^n, q^n, x^n, \theta^n) \in O_{y^*}^i$ for all $n \geq n_1$. Taking $\hat{n} = \max\{n_0, n_1\}$, we have $y^{\hat{n}} \in V_i^{\hat{n}}$ and thus $(x_i^{\hat{n}}, \theta_i^{\hat{n}}) \in H_i^{\hat{n}}(y^{\hat{n}}) = \varphi_i^{\hat{n}}(y^{\hat{n}})$. It follows from (3.1) that $(x_i^{\hat{n}}, \theta_i^{\hat{n}}) \in \varphi_i^{\hat{n}}(y^{\hat{n}}) \subset \text{co}\psi_i(y^{\hat{n}})$, which is a contradiction to Assumption (A3). Therefore, we obtain

$$B_i(p^*, q^*) \cap (P_i(p^*, q^*, x^*, \theta^*) \times \Theta_i) \equiv \psi_i(y^*) = \emptyset, \forall i \in I. \quad (3.3)$$

Now we claim that $p^*(s) \neq 0$ for each $s \in S_0$. To see this, suppose to the contrary that $p^*(\hat{s}) = 0$ for some state $\hat{s} \in S_0$. In view of Assumption (A5), for this $\hat{s} \in S_0$, there is an agent i with $\omega_i(\hat{s}) \in \mathbb{R}_+^L$ such that $x_i^* + \omega_i(\hat{s})\mathbf{1}_{\hat{s}} \in P_i(p^*, q^*, x^*, \theta^*)$. Since $p^*(\hat{s}) = 0$, it is obvious that $(x_i^* + \omega_i(\hat{s})\mathbf{1}_{\hat{s}}, \theta_i^*) \in B_i(p^*, q^*)$. As a consequence, it follows that

$$(x_i^* + \omega_i(\hat{s})\mathbf{1}_{\hat{s}}, \theta_i^*) \in B_i(p^*, q^*) \cap (P_i(p^*, q^*, x^*, \theta^*) \times \Theta_i) \equiv \psi_i(y^*) \neq \emptyset,$$

which contradicts (3.3), implying that $p^*(s) \neq 0$ for each $s \in S_0$.

To show that $q^* \in Q$, suppose to the contrary that $q^* \in c\ell(Q) \setminus Q$. Then there exists an arbitrage $\eta \in \mathbb{R}^J$ such that $W(q^*) \cdot \eta > 0$ with strict inequality holds for some state $\hat{s} \in S_0$. Due to Assumption (A5), for the state $\hat{s} \in S_0$, there is an agent i with $\omega_i(\hat{s}) \in \mathbb{R}_+^L$ such that $x_i^* + \omega_i(\hat{s})\mathbf{1}_{\hat{s}} \in P_i(p^*, q^*, x^*, \theta^*)$. On the other hand, for the state $\hat{s} \in S_0$, we see that $p^* \square (x_i^* + \omega_i(\hat{s})\mathbf{1}_{\hat{s}} - e_i) < W(q^*) \cdot (\theta_i^* + \lambda\eta)$ for sufficiently large $\lambda > 0$, that is, $(x_i^* + \omega_i(\hat{s})\mathbf{1}_{\hat{s}}, \theta_i^* + \lambda\eta) \in B_i(p^*, q^*)$. Therefore, we have

$$(x_i^* + \omega_i(\hat{s})\mathbf{1}_{\hat{s}}, \theta_i^* + \lambda\eta) \in B_i(p^*, q^*) \cap (P_i(p^*, q^*, x^*, \theta^*) \times \Theta_i) \equiv \psi_i(y^*) \neq \emptyset,$$

which contradicts (3.3), implying that $q^* \in Q$.

Furthermore, it is true that $q^* \neq 0$. Suppose otherwise. By Assumption (A7), there is a portfolio $\hat{\theta} \in \mathbb{R}^J$ such that $R \cdot \hat{\theta} > 0$, which implies that there is some state $\hat{s} \in S$ in which $r(\hat{s}) \cdot \hat{\theta} > 0$. By virtue of Assumption (A5), for the state $\hat{s} \in S$, there is an agent i with $\omega_i(\hat{s}) \in \mathbb{R}_+^L$ such that $x_i^* + \omega_i(\hat{s})\mathbf{1}_{\hat{s}} \in P_i(p^*, q^*, x^*, \theta^*)$. On the other hand, for that $\hat{s} \in S$, we have $p^* \square (x_i^* + \omega_i(\hat{s})\mathbf{1}_{\hat{s}} - e_i) < W(q^*) \cdot (\theta_i^* + \lambda\hat{\theta})$ for sufficiently large $\lambda > 0$, that is, $(x_i^* + \omega_i(\hat{s})\mathbf{1}_{\hat{s}}, \theta_i^* + \lambda\hat{\theta}) \in B_i(p^*, q^*)$. Therefore we obtain

$$(x_i^* + \omega_i(\hat{s})\mathbf{1}_{\hat{s}}, \theta_i^* + \lambda\hat{\theta}) \in B_i(p^*, q^*) \cap (P_i(p^*, q^*, x^*, \theta^*) \times \Theta_i) \equiv \psi_i(y^*) \neq \emptyset,$$

which contradicts (3.3), implying that $q^* \neq 0$.

Hence we conclude that $(p^*, q^*, x^*, \theta^*)$ is a competitive equilibrium. ■

4. Examples

The following examples are generalization of the ones in He and Yannelis (2016) to incomplete markets. The first example below covers the case where the continuous inclusion property (i.e., Assumption (A4)) is violated and there is no competitive equilibrium.

Example 4.1. Consider an exchange economy with incomplete markets, where there are two agents ($m = 2$), one good ($L = 1$), two assets ($J = 2$), and three states ($S = \{a, b, c\}$) in period 1. Take the good as a numéraire. Assume that there is no consumption and endowment in period 0, that is, $x_i(0) = e_i(0) = p(0) = 0$ and thus $X_i = \mathbb{R}_+^3$ for each $i \in I$. But agent i chooses his portfolio $\theta_i \in \mathbb{R}^2$ in period 0 subject to $q \cdot \theta_i \leq 0$. Agents' preferences and initial endowments are given as follows:

$$\begin{aligned} P_1(p, q, x, \theta) &= \{x'_1 \in X_1 : x'_1(a) + x'_1(b) \\ &\quad + x'_1(c) > x_1(a) + x_1(b) + x_1(c)\}, \quad e_1 = (\tfrac{1}{2}, 0, \tfrac{1}{2}), \\ P_2(p, q, x, \theta) &= \{x'_2 \in X_2 : x'_2(c) > x_2(c)\}, \quad e_2 = (0, 1, \tfrac{1}{2}). \end{aligned}$$

The asset structure in period 1 is given by

$$R = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It can be easily checked that this economy satisfies Assumptions (A1)–(A3) and (A5)–(A7). Since there is only one good in each state, we can set $p = (1, 1, 1)$. The economy allows us to restrict the asset prices to the set Q of no-arbitrage asset prices, which is an open convex cone with vertex 0 generated by $\{(1, 0), (0, 1)\}$, that is, $Q = \mathbb{R}_{++}^2$. For convenience, normalize the no-arbitrage asset price $q = (q_1, q_2) \in Q$ such that $q_1 + q_2 = 1$.

In this economy, there is no competitive equilibrium. To see this, suppose to the contrary that there is a competitive equilibrium $(p^*, q^*, x^*, \theta^*) \in \Delta^\circ \times A$ where $\Delta^\circ = \{(p, q) \in \mathbb{R}_{++}^3 \times Q : p = (1, 1, 1), q_1 + q_2 = 1\}$.

We know that

$$\begin{aligned} B_1(p^*, q^*) &= \left\{ (x_1, \theta_1) \in \mathbb{R}_+^3 \times \mathbb{R}^2 : q_1^* \theta_{11} + q_2^* \theta_{12} \leq 0, x_1(a) \leq \tfrac{1}{2} + \theta_{11} + \theta_{12}, \right. \\ &\quad \left. x_1(b) \leq \theta_{11}, x_1(c) \leq \tfrac{1}{2} + \theta_{12} \right\}, \\ B_2(p^*, q^*) &= \left\{ (x_2, \theta_2) \in \mathbb{R}_+^3 \times \mathbb{R}^2 : q_1^* \theta_{21} + q_2^* \theta_{22} \leq 0, \right. \\ &\quad \left. x_2(a) \leq \theta_{21} + \theta_{22}, x_2(b) \leq 1 + \theta_{21}, x_2(c) \leq \tfrac{1}{2} + \theta_{22} \right\}. \end{aligned}$$

By the monotonicity of preferences, we have

$$\begin{aligned} q_1^* \theta_{11} + q_2^* \theta_{12} &= 0, & x_1(a) &= \tfrac{1}{2} + \theta_{11} + \theta_{12}, \\ & & x_1(b) &= \theta_{11}, & x_1(c) &= \tfrac{1}{2} + \theta_{12}, \\ q_1^* \theta_{21} + q_2^* \theta_{22} &= 0, & x_2(a) &\leq \theta_{21} + \theta_{22}, \\ & & x_2(b) &\leq 1 + \theta_{21}, & x_2(c) &= \tfrac{1}{2} + \theta_{22}. \end{aligned}$$

Then it holds that

$$\begin{aligned} (\theta_{11}^*, \theta_{12}^*) &\in \arg \max_{(\theta_{11}, \theta_{12})} \frac{1}{2} + 2(\theta_{11} + \theta_{12}), & s.t. & \quad q_1^* \theta_{11} + q_2^* \theta_{12} = 0, \\ (\theta_{21}^*, \theta_{22}^*) &\in \arg \max_{(\theta_{21}, \theta_{22})} \frac{1}{2} + \theta_{22}, & s.t. & \quad q_1^* \theta_{21} + q_2^* \theta_{22} = 0. \end{aligned}$$

with $\theta_{11}^* + \theta_{21}^* = 0$ and $\theta_{12}^* + \theta_{22}^* = 0$. Since $x_i^* \geq 0$ for each $i \in I$, we have $\theta_{11}^* \geq 0$ and $\theta_{21}^* = -1$ if $q_1^* \geq q_2^*$, while $\theta_{12}^* = -\frac{1}{2}$ and $\theta_{22}^* = 0$ otherwise, which obviously violates the market clearing condition. Consequently, there is no competitive equilibrium.

It can be easily checked that the economy does not satisfy the continuous inclusion property. To see this, consider the price-allocation profile $\hat{y} \equiv (\hat{p}, \hat{q}, \hat{x}, \hat{\theta}) \in \Delta' \times A$ where $\Delta' = \{(p, q) \in \mathbb{R}_{++}^3 \times \mathbb{R}_+^2 : p = (1, 1, 1), q \in c\ell(Q), q_1 + q_2 = 1\}$ and

$$\begin{aligned} \hat{p} &= (1, 1, 1), & \hat{x}_1 &= (\frac{1}{2}, 0, \frac{1}{2}), & \hat{x}_2 &= (\frac{1}{2}, 1, \frac{1}{2}), \\ \hat{q} &= (1, 0), & \hat{\theta}_1 &= (0, 0), & \hat{\theta}_2 &= (0, 0). \end{aligned}$$

Note that $(\hat{x}_i, \hat{\theta}_i) \in B_i(\hat{p}, \hat{q})$ for each $i \in I$. Then we can show that $\psi_1(\hat{p}, \hat{q}, \hat{x}, \hat{\theta}) \neq \emptyset$. First observe that

$$\begin{aligned} B_1(\hat{p}, \hat{q}) &= \left\{ (x_1, \theta_1) \in \mathbb{R}_+^3 \times \mathbb{R}^2 : \theta_{11} \leq 0, x_1(a) \leq \frac{1}{2} + \theta_{11} + \theta_{12}, \right. \\ &\quad \left. x_1(b) \leq \theta_{11}, x_1(c) \leq \frac{1}{2} + \theta_{12} \right\}, \\ P_1(\hat{p}, \hat{q}, \hat{x}, \hat{\theta}) &= \left\{ x_1 \in \mathbb{R}_+^3 : x_1(a) + x_2(b) + x_2(c) > 1 \right\}, \quad \Theta_1 = \mathbb{R}^2. \end{aligned}$$

Choosing $(x'_1, \theta'_1) = ((\frac{3}{2}, 0, \frac{3}{2}), (0, 1)) \in B_1(\hat{p}, \hat{q})$, we see that $x'_1(a) + x'_2(b) + x'_2(c) = 3 > 1$ and thus

$$(x'_1, \theta'_1) \in B_1(\hat{p}, \hat{q}) \cap (P_1(\hat{p}, \hat{q}, \hat{x}, \hat{\theta}) \times \Theta_1) \equiv \psi_1(\hat{p}, \hat{q}, \hat{x}, \hat{\theta}) \neq \emptyset.$$

Now we perturb the asset price around $\hat{q} = (1, 0)$ by taking $q' = (1 - \varepsilon, \varepsilon) \in Q$ with $\varepsilon \in (0, 1)$. If $(x_1, \theta_1) \in B_1(\hat{p}, q')$, then it holds that

$$(1 - \varepsilon)\theta_{11} + \varepsilon\theta_{12} \leq 0, \quad 0 \leq \theta_{11}, \quad -\frac{1}{2} \leq \theta_{12}. \quad (4.1)$$

To have $x_1 \notin P_1(\hat{p}, q', \hat{x}, \hat{\theta})$, it is required that $1 + 2\theta_{11} + 2\theta_{12} \leq 1$ and $x_1 \in \mathbb{R}_+^3$, which implies that

$$\theta_{11} + \theta_{12} \leq 0, \quad 0 \leq \theta_{11}, \quad -\frac{1}{2} \leq \theta_{12}. \quad (4.2)$$

Observe that the set of θ_1 's that satisfy (4.2) is nonempty. It can be easily checked that, as long as $\varepsilon \in (0, \frac{1}{2})$, the set of θ_1 's that satisfy (4.1) is nonempty and fulfills (4.2). Then it follows that, for every $q' = (1 - \varepsilon, \varepsilon)$ with $\varepsilon \in (0, \frac{1}{2})$,

$$B_1(\hat{p}, q') \cap (P_1(\hat{p}, q', \hat{x}, \hat{\theta}) \times \Theta_1) \equiv \psi_1(\hat{p}, q', \hat{x}, \hat{\theta}) = \emptyset.$$

Consequently, in any open neighborhood O^1 of $\hat{y} = (\hat{p}, \hat{q}, \hat{x}, \hat{\theta})$, there exists a price-allocation profile $(\hat{p}, q', \hat{x}, \hat{\theta}) \in O^1$ such that $\psi_1(\hat{p}, q', \hat{x}, \hat{\theta}) = \emptyset$. That is, we cannot find an open neighborhood O^1 of $\hat{y} = (\hat{p}, \hat{q}, \hat{x}, \hat{\theta})$ and a correspondence F_y^1 with nonempty values such that $F_y^1(z) \subset \psi_1(z)$ for any $z \in O^1$. This implies that ψ_1 does not have the continuous inclusion property at $\hat{y} = (\hat{p}, \hat{q}, \hat{x}, \hat{\theta})$. Similarly, it can be easily checked that ψ_2 does not have the continuous inclusion property at $\hat{y}$. $\square$

The next example shows that even if preferences are discontinuous, there must exist a competitive equilibrium when the continuous inclusion property (i.e., Assumption (A4)) is satisfied. This demonstrates that our existence theorem is not covered by any other existence result on incomplete markets.

Example 4.2. As in Example 4.1, we consider a one-good, two-agent, three-state economy with two assets where there is no consumption in period 0. But agents' preferences and initial endowments are given as follows. Agents' preferences exhibit discontinuity as follows:

$$\begin{aligned} P_i(p, q, x, \theta) &= \left\{ x'_i \in X_i : x'_i(a)x'_i(b) + x'_i(c) > x_i(a)x_i(b) + x_i(c) \right\} \\ &\quad \setminus \left\{ x'_i \in X_i : x'_i(a) - x_i(a) = x'_i(b) - x_i(b), \right. \\ &\quad \left. x'_i(a) < \frac{3}{2}x_i(a) \right\}, \quad \forall i = 1, 2, \end{aligned}$$

with $e_1 = (1, 3, 1)$ and $e_2 = (3, 1, 1)$. The asset structure is given by

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

As in Example 4.1, we know that the set Q of no-arbitrage asset prices is given by $Q = \mathbb{R}_{++}^2$ and let $\Delta' = \{(p, q) \in \mathbb{R}_{++}^3 \times c\ell(Q) : p = (1, 1, 1), q_1 + q_2 = 1\}$. Observe that this economy satisfies Assumptions (A1)–(A3) and (A5)–(A7).

We shall show that Assumption (A4) is satisfied and therefore there exists a competitive equilibrium due to Theorem 2.1. Take any price-allocation profile $y \equiv (p, q, x, \theta) \in \Delta' \times A$ such that $(x_i, \theta_i) \in B_i(p, q)$ for each $i \in I$ and $I(y) \neq \emptyset$. Without loss of generality, we can assume that agent 1 belongs to $I(y)$, i.e., $\psi_1(y) \neq \emptyset$, so that there exists a consumption–portfolio choice $(\hat{x}_1, \hat{\theta}_1) \in \psi_1(p, q, x, \theta) = B_1(p, q) \cap (P_1(p, q, x, \theta) \times \Theta_1)$. Since $\hat{x}_1 \in P_1(p, q, x, \theta)$, it follows that $\hat{x}_1(a)\hat{x}_1(b) + \hat{x}_1(c) > x_1(a)x_1(b) + x_1(c)$. Then we can choose a consumption bundle $z_1 \in X_1$ such that $z_1 \ll \hat{x}_1$ and $z_1 \in \text{int}(P_1(p, q, x, \theta))$. Since $q \cdot \hat{\theta}_1 \leq 0$ with $q > 0$ and $p \square_1(z_1 - e_1) \ll p \square_1(\hat{x}_1 - e_1) \leq R \cdot \hat{\theta}_1$, we can take a portfolio $\zeta_1 \ll \hat{\theta}_1$ such that $q \cdot \zeta_1 < 0$ and $p \square_1(z_1 - e_1) \ll R \cdot \zeta_1$, implying $p \square_1(z_1 - e_1) \ll W(q) \cdot \zeta_1$, that is, $(z_1, \zeta_1) \in B_1^o(p, q) \equiv \text{int}(B_1(p, q))$. As a consequence, we have $(z_1, \zeta_1) \in B_1^o(p, q) \cap [\text{int}(P_1(p, q, x, \theta)) \times \Theta_1]$, and thus there exists an open neighborhood $\Delta_1 \times O_{(q, x, \theta)}^1$ of $y \equiv (p, q, x, \theta)$ such that⁹

$$\begin{aligned} (z_1, \zeta_1) &\in B_1(p', q') \times (P_1(p', q', x', \theta') \times \Theta_1) \\ &= \psi_1(y'), \quad \forall y' \equiv (p', q', x', \theta') \in \Delta_1 \times O_{(q, x, \theta)}^1. \end{aligned}$$

Set $O_y^1 \equiv \Delta_1 \times O_{(q, x, \theta)}^1$ and define the correspondence $F_y^1 : O_y^1 \rightarrow 2^{X_i \times \Theta_i}$ by $F_y^1(y') \equiv \{(z_1, \zeta_1)\}$ for any $y' \in O_y^1$. Then the following hold:

- $O_y^1 \equiv \Delta_1 \times O_{(q, x, \theta)}^1$ is an open neighborhood of y ;
- $F_y^1(y') \equiv \{(z_1, \zeta_1)\} \subset \psi_1(y')$ for any $y' \in O_y^1$;
- coF_y^1 is a single-valued constant correspondence and therefore is upper hemicontinuous with nonempty, compact, and convex values on O_y^1 .

As a consequence, ψ_1 has the continuous inclusion property at y , that is, Assumption (A4) is satisfied. Hence, by Theorem 2.1, there exists a competitive equilibrium.

Indeed it can be easily checked that there is a competitive equilibrium $(p^*, q^*, x^*, \theta^*) \in \Delta' \times A$, where

$$\begin{aligned} p^* &= (1, 1, 1), & x_1^* &= (2, 2, 1), & x_2^* &= (2, 2, 1), \\ q^* &= (\frac{1}{2}, \frac{1}{2}), & \theta_1^* &= (1, -1), & \theta_2^* &= (-1, 1). \end{aligned}$$

Notice that even if the endowments are on the boundary such that $e_1 = (0, 4, 1)$ and $e_2 = (4, 0, 1)$, there is a competitive equilibrium $(p^*, q^*, x^*, \theta^*) \in \Delta' \times A$, where

$$\begin{aligned} p^* &= (1, 1, 1), & x_1^* &= (2, 2, 1), & x_2^* &= (2, 2, 1), \\ q^* &= (\frac{1}{2}, \frac{1}{2}), & \theta_1^* &= (2, -2), & \theta_2^* &= (-2, 2). \quad \square \end{aligned}$$

5. Relation to the literature

Gottardi and Hens (1996) extend the irreducibility condition of McKenzie (1959, 1981) to the general equilibrium model with incomplete markets (GEI model). If an economy fulfills their irreducibility condition (GEI irreducibility) when a minimum wealth constraint is always satisfied for some agent, every agent possesses the minimum wealth. Using this role of GEI irreducibility condition, they show that a quasi-equilibrium becomes a competitive equilibrium.

In Gottardi and Hens (1996), there is no consumption in period 0 and they consider numéraire assets instead of nominal assets. This

⁹ Recall that there is only one good in each state so that $p'(s) = p(s) = 1$ for each $s \in S$.

setup can be accommodated in our model such that, without loss of generality, we assume that assets are nominal¹⁰ and set $x_i(0) = e_i(0) = p(0) = 0$ for every $i \in I$. Let us define the set of prices by $\Delta = \Delta_1 \times \Delta_0$ where $\Delta_0 = \{q \in \mathcal{CL}(Q) : \|q\| \leq 1\}$ and $\Delta_1 = \{p \in \mathbb{R}_+^{LS} : \|p(s)\|_1 = 1, \forall s \in S\}$. Then the budget correspondence B_i becomes $B_i(p, q) = \{(x_i, \theta_i) \in X_i \times \Theta_i : q \cdot \theta_i \leq 0, p \square_i(x - e_i) \leq R \cdot \theta_i\}$.

Below we specify the assumptions of [Gottardi and Hens \(1996\)](#) using our notations instead of theirs.¹¹

- (G1) For each $i \in I$, utility function $u_i : X_i \rightarrow \mathbb{R}$ is continuous, strictly monotone, and strictly quasi-concave.
- (G2) For each $i \in I$, $e_i \in X_i \setminus \{0\}$ and $\sum_{i \in I} e_i \gg 0$.
- (G3) There exists $\hat{\theta} \in \mathbb{R}^J$ such that $R \cdot \hat{\theta} > 0$.
- (G4) For each $(p, q) \in \text{int}(\Delta_1) \times \Delta_0 \setminus \{0\}$, there is an agent $i \in I$ such that $\{\theta_i \in \mathbb{R}^J : q \cdot \theta_i < 0 \text{ and } p \square_i e_i + R \cdot \theta_i \geq 0\} \neq \emptyset$.
- (G5) For each nontrivial partition $\{I_1, I_2\}$ of I and for each $(p, x) \in \text{int}(\Delta_1) \times A_X$ such that $p \square_{I_1}(x_i - e_i) \in \langle R \rangle$ for each $i \in I_1$, there exists $z = (z_i)_{i \in I} \in (\mathbb{R}^{LS})^m$ with $\sum_{i \in I} z_i = 0$ such that¹²

- (i) $u_i(x_i + z_i) \geq u_i(x_i)$ for each $i \in I_1$ with the inequality being strict for some $i \in I_1$,
- (ii) $p \square_{I_2} z_i \in \langle R \rangle$ and $e_i + z_i \in X_i$ for each $i \in I_2$.

- (G6) There is no redundant asset.¹³

- (G7) For each $i \in I$, $X_i = \mathbb{R}_+^{LS}$ and $\Theta_i = \mathbb{R}^J$.

[Gottardi and Hens \(1996\)](#) combine Assumption (G4) (minimum wealth condition) and Assumption (G5) (GEI irreducibility) to obtain that the interior of “the set of budget feasible asset trades” is nonempty for each $i \in I$ at a quasi-equilibrium.¹⁴ In their model, the preference correspondence P_i is given by $P_i(p, q, x, \theta) = \{x'_i \in X_i : u_i(x'_i) > u_i(x_i)\}$. Notice that Assumption (G1) implies Assumptions (A3) and (A5). Assumption (G2) implies Assumption (A1), while Assumptions (G3) and (G6) are identical to Assumptions (A7) and (A6), respectively. Assumption (G7) implies Assumptions (A1)–(A2). If we replace Assumptions (G4)–(G5) by Assumption (A4), by our Theorem 1, the economy of [Gottardi and Hens \(1996\)](#) has a competitive equilibrium.

Below we provide an example which satisfies all the assumptions of [Gottardi and Hens \(1996\)](#) except Assumption (G5) (GEI irreducibility). This economy satisfies Assumption (A4) (continuous inclusion property assumption) and thus has a competitive equilibrium. In other words, although the economy violates GEI irreducibility, there is a competitive equilibrium since it satisfies Assumption (A4). This shows that Assumption (G5) is a stronger sufficient condition than Assumption (A4) for the existence of a competitive equilibrium.

Example 5.1. As in [Example 4.2](#), we consider a one-good, two-agent, three-state economy with two assets where there is no consumption in period 0. But agents' preferences and initial endowments are given as follows:

$$P_i(p, q, x, \theta) = \{x'_i \in X_i : u_i(x'_i) > u_i(x_i)\} \text{ where } u_i(x_i) = x_i(a) + x_i(b) + x_i(c), \forall i \in I$$

with $e_1 = (2, 2, 1)$ and $e_2 = (0, 0, 2)$. The asset structure is given by

$$R = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

¹⁰ See [Magill and Shafer \(1990\)](#) for this point.

¹¹ Numbering their assumptions, we use (G1)–(G5) for their (A.1)–(A.5).

¹² Denote by $\langle R \rangle$ the marketed subspace (asset span) of asset markets.

¹³ In [Gottardi and Hens \(1996\)](#), Assumptions (G6)–(G7) are not numbered assumptions but they are assumed.

¹⁴ For the notions of “the set of budget feasible asset trades” and “quasi-equilibrium”, see [Gottardi and Hens \(1996\)](#).

As in [Example 4.2](#), we know that the set Q of no-arbitrage asset prices is given by $Q = \mathbb{R}_{++}^2$ and let $\Delta' = \{(p, q) \in \mathbb{R}_{++}^3 \times \mathcal{CL}(Q) : p = (1, 1, 1), q_1 + q_2 = 1\}$. It is obvious that Assumptions (G1)–(G3) and (G6)–(G7) are satisfied, which implies that Assumptions (A1)–(A3) and (A5)–(A7) are also satisfied.

To check Assumption (G4), let $q = (\alpha, 1 - \alpha)$ with $\alpha \in [0, 1]$ and consider agent 1 with $\theta_1 = \left(-\frac{1}{2}, -\frac{1}{2}\right)$. Then $q \cdot \theta_1 = -\frac{1}{2} < 0$ and $e_1 + R \cdot \theta_1 = \left(1, \frac{3}{2}, \frac{1}{2}\right) \gg 0$ and therefore agent 1 satisfies Assumption (G4). But the economy is not GEI irreducible, i.e., Assumption (G5) is violated. To show this, let x be an allocation in A_X such that $x_i - e_i = \langle R \rangle$ for each $i \in I$. Let z_1 be a point in $\mathbb{R}^3$ with $u_1(x_1 + z_1) > u_1(x_1)$ that satisfies $z_1 = R \cdot \eta_1 = (\eta_{11} + \eta_{12}, \eta_{11}, \eta_{12})$ for some $\eta_1 = (\eta_{11}, \eta_{12}) \in \mathbb{R}^2$. Note that the condition $u_1(x_1 + z_1) > u_1(x_1)$ implies that $\eta_{11} + \eta_{12} > 0$. Recalling that $e_2(a) = e_2(b) = 0$, we have $e_2 + z_2 = e_2 - z_1 \notin \mathbb{R}_+^3$. Thus, the economy is not GEI irreducible.

However, it can be easily checked that Assumption (A4) holds. Hence, by [Theorem 2.1](#), there exists a competitive equilibrium, which is given by

$$p^* = (1, 1, 1), \quad x_1^* = (2, 1, 2), \quad x_2^* = (0, 1, 1), \\ q^* = \left(\frac{1}{2}, \frac{1}{2}\right), \quad \theta_1^* = (-1, 1), \quad \theta_2^* = (1, -1). \quad \square$$

Remark 5.1. If $e_i \in \text{int}(X_i)$, it holds that

$$B_i^\circ(p, q) \neq \emptyset, \quad \forall (p, q) \in \hat{\Delta},$$

where $\hat{\Delta} \equiv \{(p, q) \in \Delta : p(s) \neq 0, \forall s \in S_0\}$.

[Remark 5.1](#) is a well-known fact. It characterizes a special case where B_i° has an open graph with nonempty values. In the literature as [Cass \(1984\)](#), [Geanakoplos and Polemarchakis \(1986\)](#), and [Werner \(1989\)](#), it is assumed that $e_i \in \text{int}(X_i)$ for each $i \in I$ (individual interiority on the initial endowments), by [Remark 5.1](#), which implies that B_i° has an open graph with nonempty values for each $(p, q) \in \hat{\Delta}$. Although [Werner \(1985\)](#) adopts a weaker condition than the individual interiority on the initial endowments. It turns out that B_i° has an open graph with nonempty values.

Define the preference correspondence $P_i : \Delta \times (X \times \Theta) \rightarrow 2^{X_i}$ by $P_i(p, q, x, \theta) = \{x'_i \in X_i : x'_i \succ_i x_i\}$. In [Cass \(1984\)](#), [Geanakoplos and Polemarchakis \(1986\)](#), and [Werner \(1985, 1989\)](#), each P_i has an open graph.

Lemma 5.1. Suppose that B_i° has an open graph with nonempty values and P_i has an open graph with convex values for each $i \in I$. Then, the correspondence $\psi_i : \Delta \times (X \times \Theta) \rightarrow 2^{X_i \times \Theta_i}$ defined by $\psi_i(p, q, x, \theta) = B_i(p, q) \cap (P_i(p, q, x, \theta) \times \Theta_i)$ satisfies Assumption (A4).

Proof. Since B_i° has an open graph and thus open lower sections, it is lower hemicontinuous (see [Yannelis and Prabhakar \(1983, Proposition 4.1\)](#)). Noting that it is nonempty-valued, we observe that its closure, i.e., $\mathcal{CL}(B_i^\circ(\cdot)) = B_i(\cdot)$ is lower hemicontinuous and obviously it is convex-valued. Since P_i has an open graph with convex values, by virtue of [Lemma 4.2 of Yannelis \(1987\)](#), the correspondence ψ_i is lower hemicontinuous with convex values.

In order to show that Assumption (A4) is satisfied, take any price-allocation profile $\hat{y} \equiv (\hat{p}, \hat{q}, \hat{x}, \hat{\theta}) \in \Delta \times A$ such that $(\hat{x}_i, \hat{\theta}_i) \in B_i(\hat{p}, \hat{q})$ for each $i \in I$ and $I(\hat{y}) \neq \emptyset$ where $I(\hat{y}) = \{i \in I : \psi_i(\hat{y}) \neq \emptyset\}$. Now select and fix some agent $i \in I(\hat{y})$. Since $i \in I(\hat{y})$ and ψ_i is lower hemicontinuous, the set $U_i \equiv \{y = (p, q, x, \theta) \in \Delta \times A : \psi_i(y) \neq \emptyset\}$ is open and $\hat{y} \in U_i$. Since ψ_i is lower hemicontinuous with nonempty convex values on U_i , we have $\psi_i|_{U_i} : U_i \rightarrow 2^{X_i \times \Theta_i}$ lower hemicontinuous with nonempty convex values.¹⁵ By [Theorem 3.1''' of Michael \(1956\)](#), there exists a continuous function $f_i : U_i \rightarrow X_i \times \Theta_i$ such that $f_i(z) \in \psi_i(z)$ for any $z \equiv (p, q, x, \theta) \in U_i$. Set $O_{\hat{y}}^i = U_i$ and define $F_{\hat{y}}^i : O_{\hat{y}}^i \rightarrow 2^{X_i \times \Theta_i}$ by $F_{\hat{y}}^i(z) = \{f_i(z)\}$ for any $z \in O_{\hat{y}}^i$. Then we see that

¹⁵ We denote by $\psi_i|_{U_i}$ the restriction of correspondence ψ_i on U_i .

- $O_{\hat{y}}^i = U_i$ is an open neighborhood of $\hat{y}$;
- $F_{\hat{y}}^i(z) = \{f_i(z)\} \subset \psi_i(z)$ for any $z \in O_{\hat{y}}^i$;
- $coF_{\hat{y}}^i$ is a single-valued correspondence and thus is upper hemicontinuous with nonempty, compact, and convex values on $O_{\hat{y}}^i$.

It follows that ψ_i has the continuous inclusion property at $\hat{y}$, i.e., Assumption (A4) is satisfied. ■

As mentioned above, in Cass (1984), Geanakoplos and Polemarchakis (1986), and Werner (1985) B_i° has an open graph with nonempty values and P_i has an open graph with convex values. By Lemma 5.1, their models satisfies Assumption (A4) and therefore we obtain their existence theorems as corollaries of our Theorem.¹⁶

Remark 5.2. Werner (1989) assumes that the preference correspondence P_i has an open graph and $e_i \in \text{int}(X_i)$ for each $i \in I$. Here, we set $\psi_i(p, q, x, \theta) = B_i(p, q) \cap (coP_i(p, q, x, \theta) \times \Theta_i)$. If we replace P_i by coP_i in Lemma 5.1 and apply the same arguments in the proof of Lemma 5.1, we can obtain that ψ_i satisfies Assumption (A4). Then our theorem gives as a corollary a generalized version of Werner's theorem which includes price-dependent and interdependent preferences and without assuming the individual/aggregate interiority conditions on the initial endowments.

Remark 5.3. Our model is an extension of Podczeck and Yannelis (2022) to exchange economy with incomplete markets, which is in a two-period setting. If we take $\Theta_i = \{0\}$ for each $i \in I$ and $S = \emptyset$, i.e., there is no asset market and no future, then our model reduces to Podczeck and Yannelis (2022).

Remark 5.4. It would be of interest to extend our analysis to economies with a continuum of agents or with an infinite commodity space. It is an open question to show that there exists a competitive equilibrium when incomplete markets are introduced into the models of Anderson et al. (2022), Bhowmik and Yannelis (2024), Khan et al. (2025a,b), Otsuka (2024), and Podczeck and Yannelis (2024).

Remark 5.5. One can extend our model to incomplete markets with portfolio constraints and redundant assets. This extension could cover the results of Aouani and Cornet (2009) and Hahn and Won (2020).

CRedit authorship contribution statement

Guangsug Hahn: Writing – review & editing, Conceptualization.
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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

¹⁶ Note that Geanakoplos and Polemarchakis (1986) consider numéraire assets as in Gottardi and Hens (1996), whereas assets are nominal in Cass (1984) and Werner (1985).

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