Journal of Mathematical Economics 12 (1983) 233–245. North-Holland

# EXISTENCE OF MAXIMAL ELEMENTS AND EQUILIBRIA IN LINEAR TOPOLOGICAL SPACES

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Received May 1983, final version received October 1983

We present some mathematical theorems which are used to generalize previous results on the existence of maximal elements and of equilibrium. Our main theorem in this paper is a new existence proof for an equilibrium in an abstract economy, which is closely related to a previous result of Borglin–Keiding, and Shafer–Sonneschein, but allows for an infinite number of commodities and a countably infinite number of agents.

## 1. Introduction

The purpose of this paper is two-fold. First, to prove the existence of maximal elements over compact subsets of Hausdorff linear topological spaces generalizing the previous results of Fan (1962), Sonnenschein (1971), Borglin–Keiding (1976) and Aliprantis–Brown (1983). Second, to prove the existence of an equilibrium for an abstract economy as defined in Shafer–Sonnenschein (1975) and Borglin–Keiding (1976). This theorem is closely related to a previous result of Borglin–Keiding (1976, p. 315) but allows for an infinite number of commodities and a countably infinite number of traders.

It should be emphasized that the method of proof given in Borglin–Keiding (1976, p. 315) cannot be carried out to allow for an infinite number of commodities and a countably infinite number of agents. In particular, it fails due to the fact that the countably infinite intersection of open sets in a linear topological space need not be open.[1] Thus, to allow for double

*We wish to thank David Bindschadler, Kim Border, Jim Jordan, M. Ali Khan, Nicholas Papageorgiou and Ket Richter for their comments and suggestions, as well as a careful referee.

[1]We must note that the method of proof given in Shafer–Sonnenschein (1975) cannot be carried out in an infinite dimensional commodity space. In fact, it fails due to the fact that the convex hull of an upper-semicontinuous correspondence need not be upper-semicontinuous [see Larsen (1973, p. 340)].

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infinity, i.e., infinite number of traders and commodities, a new proof of a novel type is required.

In order to prove our results we develop some technical tools. In particular, we offer two new mathematical results, namely, a selection theorem and a fixed point theorem.

The paper proceeds as follows. Section 2 contains notation and definitions. Section 3 presents some mathematical theorems, which constitute the main technical tools used to prove our main results in the next sections. We remark that those technical theorems are quite general and may be useful to a wide field of problems in economics. Section 4 provides a clear understanding of the relationship between preference correspondences which are lower-semicontinuous and preference correspondences which have open sections. Section 5 contains results on the existence of maximal elements, and section 6 presents a proof of existence of equilibrium for an abstract economy. Finally, section 7 contains some technical remarks.

## 2. Notation and definitions

### 2.1. Notation

| | |
|---|---|
| $2^A$ | denotes the set of all subsets of $A$, |
| $\operatorname{con} A$ | denotes the convex hull of the set $A$, |
| $\operatorname{cl} A$ | denotes the closure of the set $A$, |
| $\mathbb{R}$ | denotes the set of real numbers. |
| If | $\varphi$: $X \to 2^Y$ is a correspondence $Q\|_A$ denotes the restriction of $\varphi$ to $A$, i.e., $\varphi\|_A: A \to 2^Y$. |

### 2.2. Definitions

Let $X$, $Y$ be two topological spaces. A correspondence $\varphi: X \to 2^Y$ is said to be *lower-semicontinuous* (l.s.c.) if the set $\{x \in X: \varphi(x) \cap V \neq \emptyset\}$ is open in $X$ for every open subset $V$ of $Y$. A correspondence $\varphi: X \to 2^Y$ is said to be *upper-semicontinuous* (u.s.c.) if the set $\{x \in X: \varphi(x) \subset V\}$ is open in $X$ for every open subset $V$ of $Y$. A correspondence $\varphi: X \to 2^Y$ has an *open graph* if the set $G_\varphi = \{(x, y) \in X \times Y: y \in \varphi(x)\}$ is open in $X \times Y$.

## 3. Preliminary results

### 3.1. A selection theorem

Let $\Delta(Y)$ be the set of all nonempty, convex subsets of $Y$ which are either finite-dimensional or closed or have an interior point. Michael (1956, Theorem 3.1‴, p. 368) showed that if $X$ is a $T_1$-space and $Y$ is a separable

Banach space,[2] then any l.s.c. correspondence $\varphi: X \to \Delta(Y)$ admits a *continuous selection*, i.e., there exists a continuous function $f: X \to Y$ such that $f(x) \in \varphi(x)$ for all $x \in X$.

However, the condition that $Y$ is a separable Banach space cannot be relaxed in Theorem 3.1‴ in Michael (1956). Specifically, the counterexample given in Michael (1956, p. 374) shows that Theorem 3.1‴ fails if the Banach space is not separable. Below we prove a related result to Theorem 3.1‴ which extends $Y$ from a separable Banach space to a linear topological space. This selection theorem is the key mathematical tool to prove our main result in section 6.

*Theorem 3.1. Let $X$ be a paracompact*[3] *Hausdorff space and $Y$ be a linear topological space. Suppose $\varphi: X \to 2^Y$ is a correspondence such that*

*(i) for each $x \in X$, $\varphi(x)$ is nonempty,*
*(ii) for each $x \in X$, $\varphi(x)$ is convex, and*
*(iii) for each $y \in Y$, $\varphi^{-1}(y) = \{x \in X: y \in \varphi(x)\}$ is open in $X$.*

*Then there exists a continuous function $f: X \to Y$ such that $f(x) \in \varphi(x)$ for all $x \in X$.*

*Proof.* For each $y \in Y$ $\phi^{-1}(y)$ is open in $X$, and by (i), for each $x \in X$ there is a $y \in Y$ such that $x \in \varphi^{-1}(y)$. Hence, the collection $\mathscr{C} = \{\varphi^{-1}(y): y \in Y\}$ is an open cover of $X$. Since $X$ is paracompact, there is an open locally finite refinement $\mathscr{F} = \{U_a: a \in A\}$ of $\mathscr{C}$ [Michael (1953, p. 831)]. ($A$ is an index set and $U_a$ is an open set in $X$.) By Proposition 2 in Michael (1953, p. 833) we can find a family of continuous functions $\{g_a: a \in A\}$ such that $g_a: X \to [0, 1]$, $g_a(x) = 0$ for $x \notin U_a$ and $\sum_{a \in A} g_a(x) = 1$ for all $x \in X$. For each $a \in A$ choose $y_a \in Y$ such that $U_a \subset \varphi^{-1}(y_a)$. This can be done since $\mathscr{F}$ is a refinement of $\mathscr{C}$. Define $f: X \to Y$ by $f(x) = \sum_{a \in A} g_a(x) y_a$, for all $x \in X$. By local finiteness of $\mathscr{F}$, each $x \in X$ has a neighborhood $N_x$ which intersects only finitely many $U_a$'s. Hence, $f(x)$ is a finite sum of continuous functions on $N_x$ and is therefore continuous on $N_x$. So $f$ is a continuous function from $X$ to $Y$. Further, for any $a \in A$ such that $g_a(x) \neq 0$, $x \in U_a \subset \varphi^{-1}(y_a)$ and so $y_a \in \varphi(x)$. Thus, $f(x)$ is a convex combination of elements $y_a$ in $\varphi(x)$ and so $f(x) \in \varphi(x)$ for all $x \in X$. Q.E.D.

### *3.2. Fixed point theorems*

Using Theorem 3.1 in conjunction with an extension of Schauder's fixed point theorem [Smart (1974, p. 33)] we can prove the following fixed point result:

[2]See Kelley–Namioka (1963, p. 58) for a definition.
[3]See Michael (1953, p. 831) for a definition.

*Theorem 3.2. Let $X$ be a paracompact, convex, nonempty subset of a Hausdorff locally convex linear topological space; $D$ be a compact subset of $X$; and $P:X \to 2^D$ be a correspondence such that for all $x \in X$ $P(x)$ is convex and nonempty. If for all $y \in D$ $P^{-1}(y)=\{x \in :y \in P(x)\}$ is open in $X$, then there exists $x^* \in D$ such that $x^* \in P(x^*)$.*

*Proof.* For all $x \in X$ $P(x)$ is convex, nonempty and for each $y \in X$ $P^{-1}(y)$ is open in $X$. Hence, by Theorem 3.1 there exists a continuous function $f:X \to X$ such that $f(x) \in P(x)$ for all $x \in X$. By Theorem 4.5.1 in Smart (1974, p. 33) there exists $x^* \in X$ such that $x^* = f(x^*) \in P(x^*) \subset D$. Q.E.D.

We will now prove an analogous result for a Hausdorff linear topological space $E$, which need not be locally convex. The theorem below has been proved independently by Browder (1968).[4] For completeness we give a proof.

*Theorem 3.3. Let $X$ be a compact, convex, nonempty subset of a Hausdorff linear topological space $E$ and $P:X \to 2^X$ be a correspondence such that for all $x \in X$ $P(x)$ is convex and nonempty. If for each $y \in X$ $P^{-1}(y)=\{x \in X: y \in P(x)\}$ is open in $X$, then there exists $x^* \in X$ such that $x^* \in P(x^*)$.*

*Proof.* Since for each $y \in X$ the set $P^{-1}(y)$ is open in $X$ and each $x \in X$ is at least in one of these open sets, the collection $\{P^{-1}(y): y \in X\}$ is an open cover of $X$. Since $X$ is compact, there exists a finite set $\{y_1, \ldots, y_n\}$ such that $X \subseteq \bigcup_{i=1}^n P^{-1}(y_i)$. Let $\{g_1, \ldots, g_n\}$ be a continuous partition of unity subordinated to the above covering [Michael (1953, Proposition 2, p. 833)], i.e., each $g_i:X \to [0,1]$ is continuous and $g_i(x)=0$ for $x \notin P^{-1}(y_i)$ and $\sum_{i=1}^n g_i(x)=1$ for all $x \in X$. Define the continuous mapping $f:X \to X$ by $f(x)=\sum_{i=1}^n g_i(x) y_i$. Note, that for any $i$ such that $g_i(x) \neq 0$, $x \in P^{-1}(y_i)$ or $y_i \in P(x)$. Hence, $f(x)$ is a convex combination of points $y_i$ in the convex set $P(x)$ and so $f(x) \in P(x)$ for all $x \in X$. Let $S$ be the finite dimensional simplex spanned by the finite set $\{y_1, \ldots, y_n\}$. Since the topology induced on any finite dimensional subspace of $E$ by the topology of $E$ coincides with the Euclidean topology [Kelley–Namioka (1963, Theorem 7.3, p. 59)], $f:S \to S$ is a continuous mapping of a finite dimensional simplex $S$ into itself. By Brouwer's fixed point theorem there exists $x^* \in S$ such that $x^* = f(x^*) \in P(x^*)$. Q.E.D.

In a Euclidean space $\mathbb{R}^n$ the assumption in Theorems 3.2 and 3.3 that for each $y \in X$ $P^{-1}(y)$ is open in $X$ can be weakened, to the condition that $P:X \to 2^X$ is l.s.c. The following elementary fixed point theorem is implicitly in Gale–Mas-Colell (1975), but it is not given in the present form.

[4]We thank Kim Border and a referee for pointing this out to us.

*Theorem 3.4.* *Let $X$ be a nonempty, compact, convex subset of $\mathbb{R}^n$, and let $P: X \to 2^X$ be a l.s.c. correspondence such that for all $x \in X$ $P(x)$ is nonempty and convex. Then there exists $x^* \in X$ such that $x^* \in P(x^*)$.*

*Proof.* Since $P: X \to 2^X$ is a l.s.c. correspondence with convex and nonempty values by Theorem 3.1‴ in Michael (1956), there exists a continuous function $f: X \to X$ such that for all $x \in X$ $f(x) \in P(x)$. Since $f$ is a continuous mapping from a nonempty, compact, convex set $X$ into itself by Brouwer's fixed point theorem, there exists $x^* \in X$ such that $x^* = f(x^*) \in P(x^*)$. Q.E.D.

## 4. Lower-semicontinuous and open sectioned preferences

Let $X$ be a topological space. A binary relation $\mathscr{P}$ on $X$ is a subset of $X \times X$. We read $(x, y) \in \mathscr{P}$ as '$x$ is preferred to $y$'. Define the correspondence $P: X \to 2^X$ by $P(x) = \{y \in X: (y, x) \in \mathscr{P}\}$ and the correspondence $P^{-1}: X \to 2^X$ by $P^{-1}(y) = \{x \in X: y \in P(x)\}$. We call $P(x)$ the *upper contour set* or *upper section* of $\mathscr{P}$ and $P^{-1}(y)$ the *lower contour set* or *lower section* of $\mathscr{P}$. We say that $\mathscr{P} \subset X \times X$ has an *open graph* if the set $G = \{(y, x) \in X \times X: (y, x) \in \mathscr{P}\}$ is open in $X \times X$.

Although the relationship of open graph and openness of lower and upper sections is known [see Bergstrom et al. (1976)], the relationship of open sections with lower-semicontinuity is still unknown. Below we examine this relationship.

*Proposition 4.1.* *If for each $y \in X$ $P^{-1}(y)$ is open in $X$, then $P$ is l.s.c.*

*Proof.* We must show that the set $\{x \in X: P(x) \cap V \neq \emptyset\}$ is open in $X$ whenever $V$ is an open subset of $X$. It is easy to check that for any $V \subset X$,

$$\{x \in X: P(x) \cap V \neq \emptyset\} = \bigcup_{y \in V} P^{-1}(y). \tag{1}$$

Since by assumption $P^{-1}(y)$ is open in $X$, $\bigcup_{y \in V} P^{-1}(y)$ has the same property as being the union of open sets in $X$. Hence, by (1) it follows that the set $\{x \in X: P(x) \cap V \neq \emptyset\}$ is open in $X$ whenever $V$ is an open subset of $X$. Consequently, $P$ is l.s.c. Q.E.D.

*Corollary 4.1.* *If $G = \{(y, x) \in X \times X: (y, x) \in \mathscr{P}\}$ is open in $X \times X$, then for all $x \in X$, $P(x)$ is open in $X$ and $P$ is l.s.c.*

*Proof.* Since $G$ is open in $X \times X$, the sets $P(x)$ and $P^{-1}(y)$ are open in $X$. But if $P^{-1}(y)$ is open in $X$, by Proposition 4.1 $P$ is l.s.c. Q.E.D.

*Remark 4.1.* If $P$ is l.s.c. then for each $y \in X$ $P^{-1}(y)$ may not be open in $X$.

*Proof.* Let $P:\mathbb{R}\to 2^{\mathbb{R}}$ be given by $P(x)=[-x,\infty)$. Then

$$P^{-1}(y)=\{x: y\in P(x)\}=\{x: y\in[-x,\infty)\}$$
$$=\{x: -x\leqq y\}=\{x: x\geqq -y\}=[-y,\infty).$$

Notice that $P^{-1}(y)$ is not open in $\mathbb{R}$. However, we will show that $P$ is l.s.c. Since we have proved (Proposition 4.1) that $\bigcup_{y\in V}P^{-1}(y)=\{x: P(x)\cap V\neq\emptyset\}$ whenever $V$ is an open subset of $\mathbb{R}$, then it is sufficient to show that $\bigcup_{y\in V}P^{-1}(y)$ is open. To this end let $V$ be an open subset of $\mathbb{R}$. Define $b=\sup\{v: v\in V\}$ and observe that $b>v$ for all $v\in V$. We will show that $\bigcup_{y\in V}P^{-1}(y)=(-b,\infty)$. Indeed, note that

$$\begin{aligned}
x\in(-b,\infty) &\Leftrightarrow -b<x\Leftrightarrow b>-x \\
&\Leftrightarrow b>y\geqq -x \quad \text{for some} \quad y\in V \\
&\Leftrightarrow -y\leqq x \quad \text{for some} \quad y\in V \\
&\Leftrightarrow x\in[-y,\infty) \quad \text{for some} \quad y\in V \\
&\Leftrightarrow x\in P^{-1}(y) \quad \text{for some} \quad y\in V \\
&\Leftrightarrow x\in \bigcup_{y\in V}P^{-1}(y).
\end{aligned}$$

Thus $\bigcup_{y\in V}P^{-1}(y)=(-b,\infty)$ which is an open subset of $\mathbb{R}$, and consequently, $P$ is l.s.c. Q.E.D.

Hence, we conclude that the assumption that the preference correspondence $P:X\to 2^X$ has an open graph is stronger than the assumption that $P$ is l.s.c. and the upper contour set is open. Furthermore, if the lower contour set is open, the upper contour set is l.s.c. but the reverse may not be true.[5]

## 5. Existence of maximal elements

Let $X$ be a nonempty subset of a topological space and $P:X\to 2^X$ be a preference correspondence defined by $P(x)=\{y\in X:(y,x)\in\mathscr{P}\}$. If there exists $x^*\in X$ such that $P(x^*)=\emptyset$, then $x^*$ is said to be a *maximal element* in $X$.

[5] A simple example of a correspondence which is l.s.c. and does not have open lower sections is the budget set.

Let $X$, $Y$ be topological spaces. A correspondence $\varphi: X \to 2^Y$ is said to have *open lower sections* if the set $\varphi^{-1}(y) = \{x \in X : y \in \varphi(x)\}$ is open in $X$ for every $y$ in $Y$.

*Lemma 5.1. Let* $X, Y$ *be linear topological spaces and* $\varphi: X \to 2^Y$ *be a correspondence with open lower sections. Define the correspondence* $\psi: X \to 2^Y$ *by* $\psi(x) = \operatorname{con} \varphi(x)$ *for all* $x \in X$. *Then* $\psi$ *has open lower sections.*

*Proof.* Let $y_0 \in Y$ and $x_0 \in \psi^{-1}(y_0)$. We shall exhibit an open set $U$ in $X$ such that $x_0 \in U \subset \psi^{-1}(y_0)$. Since $y_0 \in \psi(x_0) = \operatorname{con} \varphi(x_0)$, we can find $y_1, \dots, y_n$ in $\varphi(x_0)$ and reals $a_1, \dots, a_n$ such that $a_i \geqq 0$, $\sum_{i=1}^n a_i = 1$ and $y_0 = \sum_{i=1}^n a_i y_i$. For each $i = 1, \dots, n$, $\varphi^{-1}(y_i)$ is open in $X$ and $x_0 \in \varphi^{-1}(y_i)$. Define $U = \bigcap_{i=1}^n \varphi^{-1}(y_i)$. Then $x_0 \in U$, $U$ is open in $X$. To complete the proof we must show that $U \subset \psi^{-1}(y_0)$. Let $x \in U$, then $x \in \varphi^{-1}(y_i)$ or $y_i \in \varphi(x)$ for all $i = 1, \dots, n$. Hence, $y_0 = \sum_{i=1}^n a_i y_i \in \psi(x)$, i.e., $x \in \psi^{-1}(y_0)$. Consequently, $x_0 \in U \subset \psi^{-1}(y_0)$. Q.E.D.

The following theorem extends the Sonnenschein (1971, Theorem 4, p. 219) result to Hausdorff linear topological spaces. It also generalizes slightly the results of Fan (1962, Lemma 4) [see also Borglin–Keiding (1976, p. 313) and Aliprantis–Brown (1983, Theorem 3.5)].

*Theorem 5.1. Let* $X$ *be a compact, convex subset of a Hausdorff linear topological space and* $P: X \to 2^X$ *be a correspondence such that for all* $x \in X$ $x \notin \operatorname{con} P(x)$. *If for each* $y \in X$ $P^{-1}(y) = \{x \in X : y \in P(x)\}$ *is open in* $X$, *then there exists* $x^* \in X$ *such that* $P(x^*) = \emptyset$.

*Proof.* Suppose otherwise, i.e., for all $x \in X$ $P(x) \neq \emptyset$. Then the correspondence $\varphi: X \to 2^X$ defined by $\varphi(x) = \operatorname{con} P(x)$ for all $x \in X$ is convex and nonempty valued. By Lemma 5.1 for each $y \in X$ $\varphi^{-1}(y) = \{x \in X : y \in \varphi(x)\}$ is open in $X$. Hence, by Theorem 3.3 there exists $x^* \in X$ such that $x^* \in \varphi(x^*) = \operatorname{con} P(x^*)$, a contradiction to the assumption that for all $x \in X$ $x \notin \operatorname{con} P(x)$. Q.E.D.

In a Euclidean space $\mathbb{R}^n$ the assumption that for all $y \in X$ $P^{-1}(y)$ is open in $X$ can be weakened in a simple way.

The following theorem slightly generalizes the Sonnenschein (1971, Theorem 4, p. 219) result:

*Theorem 5.2. Let* $X$ *be a nonempty, compact, convex subset of* $\mathbb{R}^n$, *and* $P: X \to 2^X$ *be a l.s.c. correspondence such that for all* $x \in X$ $x \notin \operatorname{con} P(x)$. *Then there exists* $x^* \in X$ *such that* $P(x^*) = \phi$.

*Proof.* Suppose that for all $x \in X$ $P(x) \neq \phi$. Then the correspondence $\varphi: X \to 2^X$ defined by $\varphi(x) = \operatorname{con} P(x)$ for all $x \in X$ is convex, nonempty valued and by Proposition 2.6 in Michael (1956) l.s.c. By Theorem 3.4 there exists $x^* \in X$ such that $x^* \in \varphi(x^*) = \operatorname{con} P(x^*)$, a contradiction. Q.E.D.

We now extend Theorem 5.1 to a more general class of preference correspondences. We will need the following definition.

*Definition 5.1.* Let $X$ be a subset of a linear topological space. A correspondence $\varphi: X \to 2^X$ is said to be of *class* $\mathscr{L}$, if

(i) $x \notin \operatorname{con} \varphi(x)$ for all $x \in X$,
(ii) $\varphi^{-1}(y) = \{x \in X : y \in \varphi(x)\}$ is open in $X$ for all $y \in X$.

Let $\psi: X \to 2^X$ be a correspondence. The correspondence $\varphi_x: X \to 2^X$ is an $\mathscr{L}$-*majorant* of $\psi$ at $x$ if $\varphi_x$ is of class $\mathscr{L}$ and there is an open neighborhood $N_x$ of $x$ such that for all $z \in N_x$ $\psi(z) \subset \varphi_x(z)$. The correspondence $\psi: X \to 2^X$ is $\mathscr{L}$-*majorized* if for each $x \in X$ such that $\psi(x) \neq \phi$, there is an $\mathscr{L}$-majorant of $\psi$ at $x$.

The following corollary of Theorem 5.1 generalizes Corollary 1 in Borglin–Keiding (1976, p. 314) and it can be used to provide their results.

*Corollary 5.1. Let $X$ be a nonempty, compact, convex subset of a Hausdorff linear topological space and $P: X \to 2^X$ be $\mathscr{L}$-majorized. Then there exists $x^* \in X$ such that $P(x^*) = \phi$.*

*Proof.* It follows from Theorem 5.1 using the same argument adopted in Borglin–Keiding (1976, Corollary 1, p. 314).

The following result is not implied by any of the above theorems since it requires neither the range of the preference correspondence to be convex nor the domain to be compact.

*Theorem 5.3. Let $X$ be a nonempty, paracompact, convex subset of a Hausdorff locally convex linear topological space and $D$ be a compact subset of $X$. Let $P: X \to 2^D$ be a correspondence such that for all $x \in D$ $x \notin \operatorname{con} P(x)$. If for all $y \in D$ $P^{-1}(y) = \{x \in X : y \in P(x)\}$ is open in $X$, then there exists $x^* \in X$ such that $P(x^*) = \phi$.*

*Proof.* Suppose not, i.e., for all $x \in X$ $P(x) \neq \phi$. Then the correspondence $\varphi: X \to 2^D$ defined by $\varphi(x) = \operatorname{con} P(x)$ for all $x \in X$ is convex and nonempty valued. By Lemma 5.1 for each $y \in D$ $\varphi^{-1}(y) = \{x \in X : y \in \varphi(x)\}$ is open in $X$. By Theorem 3.2 there exists $x^* \in D$ such that $x^* \in \varphi(x^*) = \operatorname{con} P(x^*)$, a contradiction. Q.E.D.

## 6. Existence of equilibrium

In this section we prove the existence of equilibrium for an abstract economy with an infinite number of commodities and a countable number of agents.

Before we proceed to our main theorem we will need some facts.

*Fact 6.1.* Let $X, Y$ be linear topological spaces, and $\varphi: X \to 2^Y$, $\psi: X \to 2^Y$ be correspondences having open lower sections. Then the correspondence $\theta: X \to 2^Y$ defined by $\theta(x) = \varphi(x) \cap \psi(x)$ for all $x \in X$ has open lower sections.

*Proof.* Simply, note that $\theta^{-1}(y) = \varphi^{-1}(y) \cap \psi^{-1}(y)$. Since for all $y \in Y$ $\varphi^{-1}(y)$ and $\psi^{-1}(y)$ are open in $X$, $\theta^{-1}(y)$ is open in $X$ for all $y \in Y$, Q.E.D.

*Fact 6.2.* Let $X, Y$ be two topological spaces and $\varphi: X \to 2^Y$ be a correspondence having open lower sections. Then the correspondence $\varphi|_E: E \to 2^Y$ has open lower sections for $E \subset X$.

*Proof.* Since $\varphi^{-1}(y)$ is open in $X$ for every $y \in Y$, then the set $E \cap \varphi^{-1}(y) = \{x \in E: y \in \varphi(x)\}$ is open in $E$. Q.E.D.

*Lemma 6.1. Let $X, Y$ be topological spaces and $E \subset X$ be open in $X$. Let $\varphi: X \to 2^Y$ be an u.s.c. correspondence and $f: E \to Y$ be a continuous selection from $\varphi|_E$. Then the correspondence $\psi: X \to 2^Y$ defined by*

$$\begin{aligned} \psi(x) &= \{f(x)\} \quad \text{if} \quad x \in E, \\ &= \varphi(x) \quad \text{if} \quad x \notin E \end{aligned}$$

*is u.s.c.*

*Proof.* We must show that the set $K = \{x \in X: \psi(x) \subset V\}$ is open in $X$ for every open subset $V$ of $Y$. Let $A = \{x \in X: \varphi(x) \subset V\}$ and $B = \{x \in E: f(x) \in V\}$. It can be easily checked that $K = A \cup B$. It follows from the u.s.c. of $\varphi$ that $A$ is open in $X$. By continuity of $f$, $B$ is open in $E$ and hence is open in $X$ since $E$ is open in $X$. Thus, $K$ is open in $X$. Q.E.D.

Let the set of agents be any countable set denoted by $I$. For each $i \in I$ let $X_i$ be a nonempty set. An *abstract economy* $\Gamma = (X_i, A_i, P_i)_{i \in I}$ is defined as a family of ordered triples $(X_i, A_i, P_i)$, where $A_i: \prod_{j \in I} X_j \to 2^{X_i}$ and $P_i: \prod_{j \in I} X_j \to 2^{X_i}$ are correspondences. An *equilibrium*[6] for $\Gamma$ is an

[6]This definition of an equilibrium is due to Borglin–Keiding (1976, p. 315).

$x^* \in X = \prod_{i \in I} X_i$ satisfying for each $i \in I$:

(i) $x_i^* \in \mathrm{cl}\, A_i(x^*)$,

(ii) $P_i(x^*) \cap A_i(x^*) = \phi$.

We can now state our main result.

*Theorem 6.1. Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ be an abstract economy satisfying for each $i \in I$:*

*(i) $X_i$ is a nonempty, compact, convex, metrizable subset of a locally convex linear topological space,*

*(ii) $A_i(x)$ is convex and nonempty for all $x \in X$,*

*(iii) the correspondence $\bar{A}_i : X \to 2^{X_i}$ defined by $\bar{A}_i(x) = \mathrm{cl}\, A_i(x)$ for all $x \in X$ is u.s.c.,*

*(iv) $A_i$ has open lower sections,*

*(v) $P_i$ has open lower sections,*

*(vi) $x_i \notin \mathrm{con}\, P_i(x)$ for all $x \in X$.*

*Then $\Gamma$ has an equilibrium.*

*Proof.* Define for each $i \in I$ the correspondence $\varphi_i : X \to 2^{X_i}$ by $\varphi_i(x) = A_i(x) \cap \mathrm{con}\, P_i(x)$ for all $x \in X$. By Lemma 5.1 and Fact 6.1 $\varphi_i$ has open lower sections. Hence, by Proposition 4.1 $\varphi_i : X \to 2^{X_i}$ is l.s.c. and so the set $U_i = \{x \in X : \varphi_i(x) \neq \phi\}$ is open[7] in $X$. Since $X$ is a metrizable space [Kelley–Namioka (1963, p. 50)] $U_i$ is paracompact [Michael (1956, p. 831)] Further, the correspondence $\varphi_i | U_i : U_i \to 2^{X_i}$ is nonempty, convex valued and by Fact 6.2 has open lower sections. Hence, by Theorem 3.1 there exists a continuous function $f_i : U_i \to X_i$ such that $f_i(x) \in \varphi_i(x)$ for all $x \in U_i$. Define the correspondence $F_i : X \to 2^{X_i}$ by

$$F_i(x) = \{f_i(x)\} \quad \text{if} \quad x \in U_i, \qquad F_i(x) = \bar{A}_i(x) \quad \text{if} \quad x \notin U_i.$$

By Lemma 6.1 $F_i$ is u.s.c. Define $F : X \to 2^X$ by $F(x) = \prod_{i \in I} F_i(x)$. By Lemma 3 [Fan (1952, p. 124)] $F$ is u.s.c. Since for each $x \in X$ $F(x)$ is convex, closed and nonempty, by Theorem 1 [Fan (1952, p. 122)] there exists $x^* \in X$ such that $x^* \in F(x^*)$. Note that for each $i \in I$, if $x^* \in U_i$, then $x_i^* = f_i(x^*) \in \varphi_i(x^*) \subset \mathrm{con}\, P_i(x^*)$, a contradiction to (vi). Hence, $x^* \notin U_i$ and so for all $i \in I$, $x_i^* \in \bar{A}_i(x^*)$ and $\varphi_i(x^*) = \phi$, i.e., $A_i(x^*) \cap \mathrm{con}\, P_i(x^*) = \phi$ which implies $A_i(x^*) \cap P_i(x^*) = \phi$. Consequently, $\Gamma$ has an equilibrium.[8] Q.E.D.

[7] Since $\varphi_i$ is l.s.c., $U_i = \{x \in X : \varphi_i(x) \neq \emptyset\} = \{x \in X : \varphi_i(x) \cap X_i \neq \emptyset\}$ is open in $X$.

[8] Note that if in Theorem 6.1 one assumes that $P_i(x)$ is open in $X_i$ for all $x \in X$, then condition (ii) of the definition of equilibrium can be strengthened to $P_i(x^*) \cap \mathrm{cl}\, A_i(x^*) = \emptyset$. This is in fact the notion of equilibrium that Shafer–Sonnenschein (1975) prove.

The following useful corollary of Theorem 6.1 may be of independent interest.

*Corollary 6.1.* *Let $\Gamma=(X_i, A_i, P_i)_{i\in I}$ be an abstract economy satisfying for each $i\in I$:*

(*i*) *$X_i$ is a nonempty, weakly compact, convex subset of a separable Banach space,*
(*ii*) *$A_i(x)$ is convex and nonempty for all $x\in X$,*
(*iii*) *the correspondence $\bar{A}_i: X\to 2^{X_i}$ defined by $\bar{A}_i(x)=\operatorname{cl} A_i(x)$ for all $x\in A$ is u.s.c. in the weak topology [see Dunford–Schwartz (1966, p. 419)],*
(*iv*) *$A_i$ has open lower sections with respect to the weak topology,*
(*v*) *$P_i$ has open lower sections in the weak topology,*
(*vi*) *$x_i\notin \operatorname{con} P_i(x)$ for all $x\in X$.*

*Then $\Gamma$ has an equilibrium.*

*Proof.* The proof follows from Theorem 6.1. Indeed, by Theorem 3 [Dunford–Schwartz (1966, p. 434)] the weak topology of a weakly compact subset $X_i$ of a separable Banach space is a metric topology. Q.E.D.

## 7. Remarks

*Remark 7.1.* Theorem 6.1 was proved for metrizable subsets of a locally convex linear topological space. We needed metrizability in order to show that the set $U_i=\{x\in X:\varphi_i(x)\neq\phi\}$ is paracompact. Without the metrizability assumption $U_i$ may not be paracompact [Michael (1956, p. 835)] and, consequently, our selection Theorem 3.1 cannot be applied. Hence, we do not know if Theorem 6.1 can be extended to nonmetrizable subsets without introducing additional assumptions.

*Remark 7.2.* In Theorem 6.1 the assumption of metrizability can be relaxed if one introduces the assumption that the set $E_i=\{x\in X:P_i(x)\neq\phi\}$ is perfectly normal, i.e., every open subset of $E_i$ is an $F_\sigma$, where $F_\sigma$ denotes the countable union of closed sets [see Michael (1956)]. Hence, since in the proof of Theorem 6.1 $U_i$ is an open subset of $E_i$, it is an $F_\sigma$. Consequently, by Proposition 3 in Michael (1956, p. 835) $U_i$ is paracompact and therefore the selection Theorem 3.1 can be applied.

*Remark 7.3.* In Theorem 6.1 the set of agents $I$ was assumed to be a countable set. The reason for this is that, if each $X_i$ is a metrizable subset of a locally convex linear topological space, then $X=\prod_{i\in I}X_i$ is metrizable if $I$ is a countable set [Kelley–Namioka (1963, p. 50)]. However, if the metrizability assumption is relaxed by introducing the additional assumption that the set $E_i = \{x \in X : P_i(x) \neq \phi\}$ is perfectly normal (Remark 7.2) then $I$ can be any countable or uncountable set because $X = \prod_{i \in I} X_i$ is a subset of a locally convex linear topological space [Kelley–Namioka (1963, p. 47)].

*Remark 7.4.* We will now give examples of locally convex linear topological spaces which are metrizable, and have been used in economics.

(i) A normed space[9] is a locally convex metrizable linear topological space [Berge (1963, Example 1, p. 249)].

(ii) A Banach space is a complete normed space[10] and so it is a normed space. Hence, it is a locally convex metrizable linear topological space.

(iii) Let $p$ be a real number $1 \leqq p < \infty$. The space $l_p$ consists of all sequences of scalars $\{a_1, a_2, \ldots\}$ for which $\sum_{i=1}^{\infty} |a_i|^p < \infty$. The norm of an element $x = \{a_i\}$ in $l_p$ is defined by $\|x\| = (\sum_{i=1}^{\infty} |a_i|^p)^{1/p}$. The space $l_\infty$ consists of bounded sequences. The norm of an element $x = \{a_i\}$ in $l_\infty$ is defined by $\|x\|_\infty = \sup_i |a_i|$. The space $l_p$ $1 \leqq p \leqq \infty$ is a Banach space [Luenberger (1969, Example 4, p. 36)] and, consequently, it is a locally convex metrizable linear topological space.

(iv) The space $L_p[0,1]$ consists of those real valued measurable functions $x$ on the interval $[0,1]$ for which $|x(t)|^p$ is Lebesgue integrable. The norm is defined by $\|x\|_p = (\int_0^1 |x(t)|^p \, dt)^{1/p}$. $L_p[0,1]$, $1 \leqq p \leqq \infty$ is a Banach space [Luenberger (1969, Example 5, p. 37)] and, consequently, it is a locally convex metrizable linear topological space.

(v) The space $C[0,1]$ of continuous functions on $[0,1]$ with norm $\|x\| = \sup_{0 \leqq t \leqq 1} |x(t)|$ is a Banach space [Luenberger (1969, Example 1, p. 34)] and so it is a locally convex metrizable linear topological space.

*Remark 7.5.* Theorem 6.1 remains true if assumption (i) is replaced by the condition that $X_i$ is a nonempty, compact, convex subset of any of the five spaces, i.e., (i–v) in Remark 7.4. Hence, the commodity space in Theorem 6.1 is general enough to include the separable Banach commodity spaces used in Khan (1982) and Yannelis–Prabhakar (1983) and the space $L_\infty$ used in Bewley (1972). Note that $l_p$ and $L_p$, $1 \leqq p < \infty$ are separable Banach spaces [Luenberger (1969, pp. 36 and 43)].

[9]See Berge (1963, p. 231) for a definition.
[10]See Berge (1963, p. 252) for a definition.

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