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Existence of Equilibrium in Bayesian Games with Infinitely Many Players*

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We prove the existence of a Bayesian Nash equilibrium for a Bayesian game with infinitely many players. We make three main advances to the existing literature. In particular:

1. We provide existence theorems where the set of agents can be any infinite set (countable or uncountable) as well as a measure space.

2. We use the information partition approach to model the differential information rather than the type set approach and therefore, our equilibrium is in behavioral strategies rather than in distributional strategies. This enables us to dispense with the independent type assumption used by the others in the literature.

3. Our modeling allows the individual's action set to depend on the states of nature and to be an arbitrary subset of an infinite dimensional space.

In order to incorporate all the above generalizations, we offer a new proof of a novel type. *Journal of Economic Literature* Classification Numbers: C72, C11.

* With deep sorrow I am announcing the tragic death of my friend and co-author Taesung Kim. He died in Seoul from a heart attack on March 14, 1997 at the age of 38. I first met Taesung during my stay at the University of Minnesota in 1983–1987. Taesung was very capable, sincere, personable and I was very glad to have him as a friend. He spent January and February of 1996 and 1997 at the University of Illinois. In his second visit he brought his family (his wife Young-Eun and his daughters Na-Yeon and Na-Gyung) and it was a great pleasure to meet them. A gentleman like him will certainly be missed.

† Deceased.

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1. INTRODUCTION

Many aspects of Bayesian games (or games with differential information) have been studied in the literature. The question regarding the existence of equilibria in these games is one of them. Milgrom and Weber [16] noted that the usual fixed point argument of Nash [17] with the standard assumptions is not applicable in proving the existence of Bayesian equilibrium and hence introduced sufficient conditions for the existence. Balder [1, 2] generalized their result and Radner and Rosenthal [21] presented sufficient conditions for the existence of pure strategy Bayesian equilibrium. However, all these existence results are limited to a finite number of agents (or players). Hence, it is of interest to know not only the conditions which guarantee existence of a Bayesian Nash equilibrium (BNE) with infinitely many players, but also how one defines the notion of a BNE in the presence of a continuum of players. To the best of our knowledge this has not been done. The main purpose of this paper is to provide equilibrium existence results for Bayesian games with infinitely many agents.

While equilibrium existence results in Bayesian games have been confined to the finite number of agents, the literature of games or economies *without* differential information has studied models with infinitely many agents. In this literature, two main different approaches have been employed to model infinitely many agents. One is to extend the finite agent's model directly to the infinite case so that the joint strategy profile of all agents is just the product of each agent's strategy (e.g., Yannelis and Prabhakar [26] among others). We call this the *Cartesian Product Approach*. In this approach, even if there are infinitely many agents, a priori each agent's action can unilaterally affect the outcome of the game. The other approach is to impose a measure space structure on the set of agents so that each agent's action is negligible but the joint action by the agents with positive measure can affect the outcome of the game (e.g., Khan [15] an Schmeidler [23] among others). We call this the *Measure Theoretic Approach*. In this paper we will examine both settings in the context of Bayesian games and provide Bayesian equilibrium existence theorems for each one separately.

In modeling differential information in Bayesian games, we use the information partition approach following the models of Postlewaite and Schmeidler [20], Palfrey and Srivastava [18]. In this approach all uncertainty arises from an exogenously given probability measure space denoting the states of the nature of the world and each agent's private information is a partition of the state space. Therefore, under this approach each agent's strategy is a function from the state space to his/her set of available actions, which is measurable with respect to his/her information partition. Thus, one could interpret a strategy as a behavioral strategy. This is in contrast with the Bayesian equilibrium existence results of Milgrom and Weber

[16] and Balder [1, 2] which are based on the Harsanyi type model and proved the existence of an equilibrium in distributional strategies for Bayesian games with a finite number of agents.

There are two other works in the literature which prove Bayesian equilibrium existence with infinitely many agents. First, Balder and Rustichini [3] proved the existence of a Bayesian equilibrium in distributional strategies with infinitely many players. They used the Cartesian product approach and imposed a continuity assumption which amounts to the fact that only a countable number of agents can affect the payoff of each agent even though there are uncountably many. They also assumed the independent types condition which means that each agent's type is drawn independently from the others. Balder and Yannelis [4] showed the existence of a Bayesian equilibrium with a measure space of agents but they considered only the case of symmetric information.

The purpose of this paper is to make three main new advances in the existing literature. First, we provide existence results for Bayesian games with infinitely many agents covering the Cartesian product approach as well as the Measure theoretic approach. Second, we used the information partition approach to model the differential information rather than the type set approach and therefore, our equilibrium is in behavioral strategies rather than in distributional strategies. It should be noted that by using this approach, we are able to eliminate the independent type assumption used in Balder and Rustichini [3]. Third, our results allow the individual's action set to depend on the state of nature and to be an arbitrary subset of a separable Banach space. In order to obtain existence results which allow us to incorporate all the above generalization, a new proof of a novel type is required.

The rest of the paper proceeds as follows: Section 2 contains the notation and the basic definitions. In Section 3, we introduce the game with differential information as well as the notion of a Bayesian Nash equilibrium. Sections 4 and 5 contain our main existence theorems, and Section 6 contains some concluding remarks and open questions. Finally, we have collected the main technical lemmas for our existence theorems in the Appendices.

2. NOTATION AND DEFINITIONS

We begin with some notation and definitions.

Let 2^X denote the set of all nonempty subsets of the set X. If X and Y are sets, the *graph* of the set-valued function (or correspondence) $\phi: X \to 2^Y$ is $G_\phi = \{(x, y) \in X \times Y: y \in \phi(x)\}$.

Let $(\Omega, \mathscr{F}, \mu)$ be a complete, finite measure space and Y be a separable Banach space. The correspondence $\phi: \Omega \to 2^Y$ is said to have a *measurable graph* if

$$G_\phi \in \mathscr{F} \otimes \mathbf{B}(Y),$$

where $\mathbf{B}(Y)$ denotes the Borel σ-algebra on Y and $\otimes$ denotes the product σ-algebra. The measurable function $f: \Omega \to Y$ is called a *measurable selection* of $\phi: \Omega \to 2^Y$ if

$$f(\omega) \in \phi(\omega) \qquad \text{for} \quad \mu\text{-a.e.}$$

Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space and Y be a Banach space. Following Diestel-Uhl [9], the function $f: \Omega \to Y$ is called *simple* if there exist $y_1, y_2, ..., y_n$ in Y and $E_1, E_2, ..., E_n$ in $\mathscr{F}$ such that

$$f = \sum_{i=1}^{n} y_i \chi_{E_i},$$

where $\chi_{E_i}(\omega) = 1$ if $\omega \in E_i$ and $\chi_{E_i}(\omega) = 0$ if $\omega \notin E_i$. A function $f: \Omega \to Y$ is called $\mathscr{F}$*-measurable* if there exists a sequence of simple functions $f_n: \Omega \to Y$ such that

$$\lim_{n \to \infty} \|f_n(\omega) - f(\omega)\| = 0 \qquad \text{for} \quad \mu\text{-a.e.}$$

An $\mathscr{F}$-measurable function $f: \Omega \to Y$ is said to be *Bochner integrable* if there exists a sequence of simple functions $\{f_n: n = 1, 2, ...\}$ such that

$$\lim_{n \to \infty} \int_{\omega \in \Omega} \|f_n(\omega) - f(\omega)\| \, d\mu(\omega) = 0.$$

In this case, for each $E \in \mathscr{F}$ we define the *integral* of f, denoted by $\int_E f(\omega) \, d\mu(\omega)$, as

$$\lim_{n \to \infty} \int_E f_n(\omega) \, d\mu(\omega).$$

It can be shown (see Diestel-Uhl [9, Theorem 2, p. 45]) that if $f: \Omega \to Y$ is an $\mathscr{F}$-measurable function, then f is Bochner integrable if and only if

$$\int_\Omega \|f(\omega)\| \, d\mu(\omega) < \infty.$$

It turns out to be important in our paper that the *Dominated Convergence Theorem* holds for Bochner integrable functions. In particular, if $f_n: \Omega \to Y (n = 1, 2, ...)$ is a sequence of bochner integrable functions such that for μ-a.e.

$$\lim_{n \to \infty} f_n(\omega) = f(\omega) \qquad \text{and} \quad \|f_n(\omega)\| \leqslant g(\omega),$$

where $g: \Omega \to \mathbb{R}$ is an integrable function, then f is Bochner integrable and

$$\lim_{n \to \infty} \int_{\omega \in \Omega} \|f_n(\omega) - f(\omega)\| \, d\mu(\omega) = 0.$$

The space of equivalence classes of Y-valued Bochner integrable functions $y: \Omega \to Y$, normed by

$$\|y\| = \int_{\Omega} \|y(\omega)\| \, d\mu(\omega),$$

is denoted by $L_1(\mu, Y)$. It is a standard result that normed by the functional $\|\cdot\|$ above, $L_1(\mu, Y)$ becomes a Banach space (see Diestel-Uhl [9, p. 50]).

A Banach space Y has the *Radon-Nikodym Property* (RNP) with respect to the measure space (T, τ, v) if for each v-continuous vector measure $G: \tau \to Y$ of bounded variation, there exists some $g \in L_1(v, Y)$ such that for all $E \in \tau$,

$$G(E) = \int_E g(t) \, dv(t).$$

It is a standard result (Diestel and Uhl [9]) that if Y^* (the norm dual for of Y) has the RNP with respect to (T, τ, v), then

$$(L_1(v, Y))^* = L_\infty(v, Y^*).$$

A correspondence $\phi: \Omega \to 2^Y$ is said to be *integrably bounded* if there exists a function $h \in L_1(\mu, \mathbb{R})$ such that

$$\sup \{ \|y\| : y \in \phi(\omega)\} \leqslant h(\omega) \; \mu\text{-a.e.}$$

If $\{F_n: n=1, 2, ...\}$ is a sequence of nonempty subsets of a Banach space Y, we denote by Ls F_n and Li F_n the set of its *limit superior* and *limit inferior* points respectively, i.e.,

$$\text{Ls } F_n = \{x \in Y: x = \lim_{k \to \infty} x_{n_k}, x_{n_k} \in F_{n_k}, k=1, 2, ...\}$$

$$\text{Li } F_n = \{x \in Y: x = \lim_{n \to \infty} x_n, x_n \in F_n, n=1, 2, ...\}.$$

A ω in front of Ls F_n (Li F_n) will mean limit superior (inferior) with respect to the weak topology $\sigma(Y, Y^*)$. If Z is a metric space, Y is a Banach space and $\phi: Z \to 2^Y$ is a correspondence, we say that ϕ is *upper semicontinuous* (u.s.c.) if $\text{Ls } \phi(z_n) \subset \phi(z)$ whenever the sequence $z_n \in Z$ converges to z (written as $z_n \to z$).

3. THE GAME WITH DIFFERENTIAL INFORMATION

Let $(\Omega, \mathscr{F}, \mu)$ be a complete, finite, separable measure space, where Ω denotes the set of states of the world and the σ-algebra $\mathscr{F}$, denotes the set of events. Let Y be a separable Banach space and T be a set of agents (either finite or infinite).

A *Bayesian game* (or a *game with differential information*) is $G = \{(X_t, u_t, \mathscr{F}_t, q_t): t \in T\}$, where

(1) $X_t: \Omega \to 2^Y$ is the *action set-valued function* of agent t, where $X_t(\omega)$ is the set of actions available to t when the state is ω,

(2) for each $\omega \in \Omega$, $u_t(\omega, \cdot): \prod_{s \in T} X_s(\omega) \to \mathbb{R}$ is the *utility function* of agent t, which can depend on the states,

(3) $\mathscr{F}_t$ is a sub σ-algebra of $\mathscr{F}$ which denotes the *private information* of agent t,

(4) $q_t: \Omega \to \mathbb{R}_{++}$ is the *prior* of agent t, (where q_t is a Radon–Nikodym derivative such that $\int q_t(w)\, d\mu(\omega) = 1$).

Let L_{X_t} denote the set of all Bochner integrable and $\mathscr{F}_t$-measurable selections from the action set-valued function $X_t: \Omega \to 2^Y$ of agent t, i.e.,

$$L_{X_t} = \{\tilde{x}_t \in L_1(\mu, Y): \tilde{x}_t \text{ is } \mathscr{F}_t\text{-measurable and } \tilde{x}_t(\omega) \in X_t(\omega)\ \mu\text{-a.e.}\}.$$

The typical element of L_{X_t} is denoted as $\tilde{x}_t$ while that of $X_t(\omega)$, as $x_t(\omega)$ (or x_t). Let $L_X = \prod_{s \in T} L_{X_s}$ and $L_{X_{-t}} = \prod_{s \neq t} L_{X_s}$. Given a Bayesian game G, a *strategy* for agent t is an element $\tilde{x}_t$ in L_{X_t}.

Throughout the paper, we assume that for each $t \in T$, there exists a finite or countable partition Π_t of Ω. Moreover, the σ-algebra $\mathscr{F}_t$ is generated by Π_t. For each $\omega \in \Omega$, let $E_t(\omega)$ ($\varepsilon \Pi_t$) denote the smallest set in $\mathscr{F}_t$ containing ω and we assume that for all t,

$$\int_{\omega' \in E_t(\omega)} q_t(\omega')\, d\mu(\omega') > 0.$$

For each $\omega \in \Omega$, the *conditional* (*interim*) *expected utility function* of agent t, $v_t(\omega, \cdot, \cdot)\colon L_{X_t} \times X_t(\omega) \to \mathbb{R}$ is defined as:

$$v_t(\omega, \tilde{x}_{-t}, x_t) = \int_{\omega' \in E_t(\omega)} u_t(\omega', \tilde{x}_{-t}(\omega'), x_t)\, q_t(\omega' \mid E_t(\omega))\, d\mu(\omega'),$$

where

$$q_t(\omega' \mid E_t(\omega)) = \begin{cases} 0 & \text{if} \quad \omega' \notin E_t(\omega) \\ \dfrac{q_t(\omega')}{\int_{\tilde{\omega} \in E_t(\omega)} q_t(\tilde{\omega})\, d\mu(\tilde{\omega})} & \text{if} \quad \omega' \in E_t(\omega). \end{cases}$$

The function $v_t(\omega, \tilde{x}_{-t}, x_t)$ is interpreted as the conditional expected utility of agent t using the action x_t when the state is ω and the other agents employ the strategy profile $\tilde{x}_{-t}$, where $\tilde{x}_{-t}$ is an element of $L_{X_{-t}}$.

A *Bayesian Nash equilibrium* for G is a strategy profile $\tilde{x}^* \in L_X$ such that for all $t \in T$

$$v_t(\omega, \tilde{x}^*_{-t}, \tilde{x}^*_t(\omega)) = \max_{y_t \in X_t(\omega)} v_t(\omega, \tilde{x}^*_{-t}, y_t)\ \mu\text{-a.e.}$$

We can now state the assumptions needed for our first main theorem.

(A.1) $X_t\colon \Omega \to 2^Y$ *is a nonempty, convex, weakly compact valued and integrably bounded correspondence having a* $\mathscr{F}_t$*-measurable graph, i.e.,* $G_{X_t} \in \mathscr{F}_t \otimes \mathbf{B}(Y)$.

(A.2) (i) *For each* $\omega \in \Omega$, $u_t(\omega, \cdot, \cdot)\colon \prod_{s \neq t} X_s(\omega) \times X_t(\omega) \to \mathbb{R}$ *is continuous where* $X_s(\omega)$ $(s \neq t)$ *is endowed with the weak topology and* $X_t(\omega)$, *with the norm topology.*

(ii) *For each* $x \in \prod_{s \in T} Y_s$ *with* $Y_s = Y$, $u_t(\cdot, x)\colon \Omega \to \mathbb{R}$ *is* $\mathscr{F}$*-measurable.*

(iii) *For each* $\omega \in \Omega$ *and* $x_{-t} \in \prod_{s \neq t} X_s(\omega)$, $u_t(\omega, x_{-t}, \cdot)\colon X_t(\omega) \to \mathbb{R}$ *is concave.*

(iv) u_t *is integrably bounded.*

Remark. Since the norm topology is finer than the weak topology, (A.2) (i) is weaker than the weak continuity of $u_t(\omega, \cdot, \cdot)$ in actions of all agents.

4. COUNTABLY MANY AGENTS

We can now state our first main result.

THEOREM 4.1. *Let T be a countable set. Let $G = \{(X_t, u_t, \mathscr{F}_t, q_t): t \in T\}$ be a Bayesian game satisfying* (A.1)–(A.2). *Then there exists a Bayesian Nash equilibrium for G.*

Proof. It follows from Lemma A.1 in Appendix I that for each $\omega \in \Omega$, $\upsilon_t(\omega, \cdot, \cdot): L_{X_{-t}} \times X_t(\omega) \to \mathbb{R}$ is continuous where $L_{X_s}(s \neq t)$ is endowed with the weak topology and $X_t(\omega)$, with the norm topology. It is easy to see that for each $(\tilde{x}_{-t}, x_t) \in L_{X_{-t}} \times X_t$, $\upsilon_t(\cdot, \tilde{x}_{-t}, x_t): \Omega \to \mathbb{R}$ is $\mathscr{F}_t$-measurable. Since $u_t(\omega, x_{-t}, x_t)$ is concave in x_t, so is $\upsilon_t(\omega, \tilde{x}_{-t}, x_t)$. For each $t \in T$, define $\phi_t: \Omega \times L_{X_{-t}} \to 2^Y$ by

$$\phi_t(\omega, \tilde{x}_{-t}) = \{x_t \in X_t(\omega): \upsilon_t(\omega, \tilde{x}_{-t}, x_t) = \max_{y_t \in X_t(\omega)} \upsilon_t(\omega, \tilde{x}_{-t}, y_t)\}.$$

A direct application of Weierstrass theorem implies that for each $(\omega, \tilde{x}_{-t}) \in \Omega \times L_{X_{-t}}$, $\phi_t(\omega, \tilde{x}_{-t})$ is nonempty. By the Berge maximum theorem for each $\omega \in \Omega$, $\phi_t(\omega, \cdot): L_{X_{-t}} \to 2^{X_t(\omega)}$ is u.s.c. where $X_t(\omega)$ is endowed with the norm topology and $L_{X_{-t}}$, with the weak topology. Moreover, it follows from the concavity of υ_t in x_t that for each $\omega \in \Omega$, $\phi_t(\omega, \cdot)$ is convex-valued.

For each $t \in T$, define $\Phi_t: L_{X_{-t}} \to 2^{L_{X_t}}$ by

$$\Phi_t(\tilde{x}_{-t}) = \{\tilde{x}_t \in L_{X_t}: \tilde{x}_t(\omega) \in \phi_t(\omega, \tilde{x}_{-t})\ \mu\text{-a.e.}\}.$$

By Lemma A.2 in Appendix I, for each $\tilde{x}_{-t} \in L_{X_{-t}}$, $\varphi_t(\cdot, \tilde{x}_{-t})$ has a $\mathscr{F}_t$-measurable graph. Therefore, by the Aumann measurable selection theorem for each fixed $\tilde{x}_{-t} \in L_{X_{-t}}$, there exists an $\mathscr{F}_t$-measurable function $f_t: \Omega \to Y$ satisfying $f_t(\omega) \in \varphi_t(\omega, \tilde{x}_{-t})$ μ-a.e. Since for each $(\omega, \tilde{x}_{-t}) \in \Omega \times L_{X_{-t}}$, $\varphi_t(\omega, \tilde{x}_{-t}) \subset X_t(\omega)$ and $X_t(\cdot)$ is integrably bounded, it follows that $f_t \in L_{X_t}$. Therefore, $f_t \in \Phi_t(\tilde{x}_{-t})$, i.e., Φ_t is nonempty valued. By Lemma A.4 in Appendix I, L_{X_t} is a weakly compact subset of $L_1(\mu, Y)$. Since the weak topology of a weakly compact subset of a separable Banach space is metrizable (Dunford and Schwartz [10, p. 434]), L_{X_t} is metrizable. Since the set of agents T is countable, $L_{X_{-t}}$ is also metrizable. Therefore, by Lemma A.5 in Appendix I, Φ_t is weakly u.s.c. Define $\Phi: L_X \to 2^{L_X}$ by

$$\Phi(\tilde{x}) = \prod_{s \in T} \Phi_s(\tilde{x}_{-s}).$$

Since L_{X_s} is weakly compact for all s, so is L_X. Since $X_t: \Omega \to 2^Y$ has a measurable graph and it is integrably bounded, L_{X_t} is nonempty by the

Aumann measurable selection theorem. It follows from the convex valuedness of X_t that L_{X_t} is convex. Since Φ_t is weakly u.s.c., so is Φ. It is easy to see that Φ is convex, nonempty valued. Therefore, by the Fan–Glicksberg fixed point theorem there exists $\tilde{x}^* \in L_X$ such that $\tilde{x}^* \in \Phi(\tilde{x}^*)$. The reader can easily verify that $\tilde{x}^*$ is a Bayesian Nash equilibrium for G by construction. This completes the proof of the theorem. ∎

5. EXISTENCE RESULT: UNCOUNTABLY MANY AGENTS

In this section, we present Bayesian Nash equilibrium existence theorems with uncountably many agents. There are two different ways of modeling games with uncountably many agents. One is to model games *without* any measure structure on the set of agents T. In this case, the joint strategy of all agents is simply an element in the Cartesian product of each agent's strategy set L_{X_t}. The other is to model games *with* a measure structure on the set of agents T. In this approach, naturally each individual agent's action can be negligible but the aggregate action can matter. Each model has its own merits depending on the situation. In this section, we provide Bayesian equilibrium existence theorems for both settings. We first begin with Bayesian games without a measure structure on T, which we call the *Cartesian Product Approach.*

5.1. *The Cartesian Product Approach*

In this model, all the aspects of the Bayesian game G is the same as the model with the countably many agents except that T is now uncountable. Unfortunately, the proof given in the previous section does not hold since if T is uncountable, $L_{X_{-t}}$ is not metrizable, which means that we can no longer use Lemma A.5 in Appendix I. However, in our proposition below the continuity assumption on the utility function implies that for each agent t, there exists a countable set $J_t(\subset T)$ such that the actions of agents in $T-J_t$ do not affect the utility of agent t. In a different setting, a similar property of the continuous utility function was called *Countable Myopia* by Balder and Rustichini [3].

PROPOSITION 5.1 (Countable Myopia). *If $f \in \mathbf{C}(L_X)$ (where $\mathbf{C}(L_X)$ is the class of continuous functions on L_X), then there exists a countable subset J of T such that for every $\tilde{x}, \tilde{y} \in L_X$, if $\tilde{x}_t = \tilde{y}_t$ for $t \in J$, then $f(\tilde{x}) = f(\tilde{y})$.*

Proof. Note that $L_X(\equiv \prod_{t\in T} L_{X_t})$ is endowed with the product topology of the weak topology in each L_{X_t}. Then L_{X_t} is weakly compact by Lemma A.4 in Appendix I. Since L_{X_t} is a separable Banach space, the weak topology in L_{X_t} is metrizable and hence Hausdorff. Therefore, L_X is compact, Hausdorff. Define the subset $\mathbf{A}$ of $\mathbf{C}(L_X)$ by

$$\mathbf{A} = \left\{ g \in \mathbf{C}(L_X) \colon g(\tilde{x}) = \prod_{s \in J} g_s(\tilde{x}_s) \text{ for some finite index set } J \text{ and } g_s \in \mathbf{C}(L_{X_s}) \right\}.$$

Then $\mathbf{A}$ is an algebra which contains the constant function 1 and separates points of L_X (i.e., for $\tilde{x}, \tilde{y} \in L_X$ with $\tilde{x} \neq \tilde{y}$, there exists $g \in \mathbf{A}$ such that $g(\tilde{x}) \neq g(\tilde{y})$). Therefore, by the Stone-Weierstrass theorem, $\mathbf{A}$ is dense in $\mathbf{C}(L_X)$ by the sup. norm.

Now let $f \in \mathbf{C}(L_X)$. Since $\mathbf{A}$ is dense in $\mathbf{C}(L_X)$, for every n, there exists $f^n \in \mathbf{A}$ such that

$$\sup_{x \in L_X} |f(\tilde{x}) - f^n(\tilde{x})| < \frac{1}{n}.$$

Since $f^n \in \mathbf{A}$, $f^n(\tilde{x}) = \prod_{s \in J_n} f_s(\tilde{x}_s)$ for some finite index set J_n. Moreover, we can have $J_n \subset J_{n+1}$. Let $J = \bigcup_{n=1}^{\infty} J_n$. Then, if $\tilde{x}_t = \tilde{y}_t$ for $t \in J$,

$$f(\tilde{x}) = \prod_{t \in J} f_t(\tilde{x}_t) = \prod_{t \in J} f_t(\tilde{y}_t) = f(\tilde{y}). \quad \blacksquare$$

Now, we are ready to state our second main existence theorem.

THEOREM 5.2. *Let T be an uncountable set. Let $G = \{(X_t, u_t, \mathscr{F}_t, q_t) \colon t \in T\}$ be a Bayesian game satisfying* (A.1)–(A.2). *Then, there exists a Bayesian Nash equilibrium for G.*

Proof. By Lemma A.1 in Appendix I, for each $\omega \in \Omega$, $v_t(\omega, \cdot, \cdot) \colon L_{X_{-t}} \times X_t(\omega) \to \mathbb{R}$ is continuous. By the above Proposition, there exists a countable set $J_t^{\omega} (\subset T)$ such that

$$v_t(\omega, \tilde{x}_{-t}, x) = v_t(\omega, \tilde{y}_{-t}, x_t) \qquad \text{if} \quad \tilde{x}_j = \tilde{y}_j \quad \text{for} \quad j \in J_t^{\omega}. \tag{1}$$

Notice that if ω and ω' are in the same partition,

$$v_t(\omega, \cdot, \cdot) = v_t(\omega', \cdot, \cdot).$$

So, we can assume that $J_t^{\omega} = J_t^{\omega'}$ if ω and ω' are in the same partition. Let $J_t = U_{\omega \in \Omega} J_t^{\omega}$. Since Π_t is a countable partition, J_t is still a countable set as it is a countable union of countable sets. Now by (1) we can define $\tilde{v}_t(\omega, \cdot, \cdot): \prod_{s \in J_t} L_{X_s} \times X_t(\omega) \to \mathbb{R}$ by

$$\tilde{v}_t(\omega, \tilde{x}_{J_t}, x_t) = v_t(\omega, \tilde{x}_{-t}, x_t),$$

where $\tilde{x}_{J_t} = (\tilde{x}_t)_{s \in J_T}$. Define $\phi_t: \Omega \times \prod_{s \in J_t} L_{X_s} \to 2^Y$ by

$$\phi_t(\omega, \tilde{x}_{J_t}) = \{x_t \times X_t(\omega): \tilde{v}_t(\omega, \tilde{x}_{J_t}, x_t) = \max_{y_t \in X_t(\omega)} \tilde{v}_t(\omega, \tilde{x}_{J_t}, y_t)\}.$$

As it was shown in Theorem 4.1, it is easy to show that ϕ_t is nonempty, convex valued and ϕ_t is weakly u.s.c. in $\tilde{x}_{J_t}$. Moreover, $\phi_t(\cdot, \tilde{x}_{J_t})$ has $\mathscr{F}_t$-measurable graph. For each $t \in T$, define $\Phi_t: \prod_{s \in J_t} L_{X_s} \to 2^{L_{X_t}}$ by

$$\Phi_t(\tilde{x}_{J_t}) = \{\tilde{x}_t \in L_{X_t}: \tilde{x}_t(\omega) \in \phi_t(\omega, \tilde{x}_{J_t})\ \mu\text{-a.e.}\}.$$

Since $\prod_{s \in J_t} L_{X_s}$ is metrizable, $\tilde{\Phi}_t$ is weakly u.s.c. (recall Lemma A.5 in Appendix I) and nonempty, convex valued. Define $\Phi: L_X \to 2^{L_X}$ by

$$\Phi(\tilde{x}) = \prod_{t \in T} \Phi_t(\tilde{x}_{J_t})$$

Then again Φ is weakly u.s.c. and nonempty, convex valued. Since L_X is weakly compact, convex set, by the Fan–Glicksberg fixed point theorem there exists $\tilde{x}^* \in L_X$ such that $\tilde{x}^* \in \Phi(\tilde{x}^*)$. It can be easily checked that $\tilde{x}^*$ is a Bayesian equilibrium for G. ∎

5.2. *Measure Space of Agents*

In this section, we study the Bayesian game G with a measure space of agents. A Bayesian game with a measure space of agents $(T, \mathbf{T}, \nu)$ is $G = \{(X, u, \mathscr{F}_t, q_t): t \in T)\}$, where

(1) $X: T \times \Omega \to 2^Y$ is the *action set-valued function*, where $X(t, \omega)$ is interpreted as the set of actions available to agent t when the state is ω,

(2) for each $(t, \omega) \in T \times \Omega$, $u(t, \omega, \cdot, \cdot): L_1(\nu, y) \times X(t, \omega) \to \mathbb{R}$ is the *utility function*, where $u(t, \omega, x, x_t)$ is interpreted as the utility of agent t using action x_t when the state is ω and other players use the joint action x,

(3) $\mathscr{F}_t$ is the sub σ-algebra of $\mathscr{F}$ which denotes the *private information* of agent t,

(4) $q_t: \Omega \to \mathbb{R}_{++}$ is the *prior* of agent t.

As before, let L_{X_t} denote the set of all Bochner integrable, $\mathscr{F}_t$-measurable selections from the action set-valued function $X(t)$ of agent t, i.e.,

$$L_{X_t} = \{\tilde{x}(t) \in L_1(\mu, Y): \tilde{x}(t, \cdot): \Omega \to Y \text{ is } \mathscr{F}_t\text{-measurable and}$$
$$\tilde{x}(t, \omega) \in X(t, \omega)\ \mu\text{-a.e.}\}.$$

Let

$$L_X = \{\tilde{x} \in L_1(\nu, L_1(\mu, Y)): \tilde{x}(t) \in L_{X_t} \text{ for } \nu\text{-a.e.}\}.$$

In a Bayesian game with a measure space of agents, a *strategy* for agent t is an element in L_{X_t} and a *joint strategy profile* is an element in L_X. For each $(t, \omega) \in T \times \Omega$, the conditional expected utility function of agent t, $v(t, \omega, \cdot, \cdot): L_X \times X(t, \omega) \to \mathbb{R}$ is defined as

$$v(t, \omega, \tilde{x}, x_t) = \int_{\omega' \in E_t(\omega)} u(t, \omega', \tilde{x}(\omega'), x_t)\, q_t(\omega' \mid E_t(\omega))\, d\mu(\omega'),$$

where

$$q_t(\omega' \mid E_t(\omega)) = \begin{cases} 0 & \text{if} \quad \omega' \notin E_t(\omega) \\ \dfrac{q_t(\omega')}{\int_{\tilde{\omega} \in E_t(\omega)} q_t(\tilde{\omega})\, d\mu(\tilde{\omega})} & \text{if} \quad \omega' \in E_t(\omega). \end{cases}$$

A *Bayesian Nash equilibrium* for G is a strategy profile $\tilde{x}^* \in L_X$ such that for ν-a.e. and for μ-a.e.,

$$v(t, \omega, \tilde{x}^*, \tilde{x}^*(t, \omega)) = \max_{y \in X(t, \omega)} v(t, \omega, \tilde{x}^*, y).$$

We can now state the assumptions needed for the proof of the next theorem.

(B.1) (i) *$X: T \times \Omega \to 2^Y$ is a nonempty, convex, weakly compact valued and integrably bounded correspondence having a $\mathbf{T} \otimes \mathscr{F}$-measurable graph, i.e., $G_X \in \mathbf{T} \otimes \mathscr{F} \otimes \mathbf{B}(Y)$.*

(ii) *For each $t \in T$, $X(t, \cdot): \Omega \to 2^Y$ has a $\mathscr{F}_t$-measurable graph, i.e., $G_{X(t)} \in \mathscr{F}_t \otimes \mathbf{B}(Y)$.*

(B.2) (i) *For each $(t, \omega) \in T \times \Omega$, $u(t, \omega, \cdot, \cdot): L_1(\nu, Y) \times X(t, \omega) \to \mathbb{R}$ is continuous where $L_1(\nu, Y)$ is endowed with the weak topology and $X(t, \omega)$ with the norm topology.*

(ii) *For each $(x, y) \in L_1(\nu, Y) \times Y$, $u(\cdot, \cdot, x, y): \mathbf{T} \times \Omega \to \mathbb{R}$ is $\mathbf{T} \otimes \mathscr{F}$-measurable.*

(iii) *For each* $(t, \omega, x) \in T \times \Omega \times L_1(v, Y)$, $u(t, \omega, x, \cdot): X(t, \omega) \to \mathbb{R}$ *is concave.*

(iv) u_t *is integrably bounded.*

(B.3) (i) Ω *is a countable set.*

(ii) *The dual* Y^* *of* Y *has the RNP* (*Radon-Nikodym property*) *with respect to* $(T, \mathbf{T}, v)$. (*The definition of RNP is in Section* 2.)

(B.4) *The correspondence* $t \mapsto L_{X_t}$ *has a* $\mathbf{T}$*-measurable graph.*

THEOREM 5.3. *Let* $(T, \mathbf{T}, v)$ *be a finite, complete, separable measure space. Let* $G = \{(X, u, \mathscr{F}_t, q_t): t \in T\}$ *be a Bayesian game satisfying* (B.1)–(B.4). *Then there exists a Bayesian Nash equilibrium for G.*

Proof. It follows from (B.3) and Lemma A.6 in Appendix II that for each $(t, \omega) \in T \times \Omega$, $v(t, \omega, \cdot, \cdot): L_X \times X(t, \omega) \to \mathbb{R}$ is continuous where $L_X (\subset L_1(v, L_1(\mu, Y)))$ and $L_1(\mu, Y)$ are endowed with the weak topology and $X(t, \omega) (\subset Y)$, with the norm topology. It is easy to see that for each $(\tilde{x}, y) \in L_X \times Y$, $v(\cdot, \cdot, \tilde{x}, y): T \times \Omega \to \mathbb{R}$ is $\mathbf{T} \otimes \mathscr{F}$-measurable and for each $(t, \tilde{x}, y) \in T \times L_X \times Y$, $v(t, \cdot, \tilde{x}, y)$ is $\mathscr{F}_t$-measurable. Now define $\phi: T \times \Omega \times L_X \to 2^Y$ by

$$\phi(t, \omega, \tilde{x}) = \{y \in X(t, \omega): v(t, \omega, \tilde{x}, y) = \max_{z \in X(t, \omega)} v(t, \omega, \tilde{x}, z)\}.$$

By the same argument as the proof of Theorem 4.1, ϕ is nonempty, convex valued and weakly u.s.c. in $\tilde{x}$. Moreover, by Lemma A.2 in Appendix I, $\phi(t, \cdot, \tilde{x}): \Omega \to 2^Y$ has a $\mathscr{F}_t$-measurable graph and $\phi(\cdot, \cdot, \tilde{x}): T \times \Omega \to 2^Y$ has a $\mathbf{T} \otimes \mathscr{F}$-measurable graph. Now, define $\Phi: T \times L_X \to 2^{L_1(\mu, Y)}$ by

$$\Phi(t, \tilde{x}) = \{\tilde{y}(t) \in L_1(\mu, Y): \tilde{y}(t, \omega) \in \phi(t, \omega, \tilde{x})\ \mu\text{-a.e.}\} \cap L_{X_t}.$$

It follows from Balder–Yannelis ([4, Proposition 5.3, p. 342]) that the correspondence $t \mapsto \{\tilde{y}(t) \in L_1(\mu, Y): \tilde{y}(t, \omega) \in \phi(t, \omega, \tilde{x})\ \mu\text{-a.e.}\}$ has a $\mathbf{T}$-measurable graph. Moreover, by (B.4) the correspondence $t \mapsto L_{X_t}$ has a $\mathbf{T}$-measurable graph. Therefore, for each fixed $\tilde{x} \in L_X$, $\Phi(\cdot, \tilde{x})$ has a $\mathbf{T}$-measurable graph, too. It follows from the Aumann measurable selection theorem that Φ is nonempty valued. Also, by Lemma A.4 in Appendix I we have that L_{X_t} is a weakly compact subset of $L_1(\mu, Y)$. Therefore, L_X is also a weakly compact subset of $L_1(v, L_1(\mu, Y))$ and we can conclude that L_X is metrizable (Dunford–Schwartz [10, p. 434]). By Lemma A.5 in

Appendix I, the correspondence $\Phi(t, \cdot)\colon L_X \to 2^{L_1(\mu, Y)}$ it weakly u.s.c. Define the correspondence $\Psi\colon L_X \to 2^{L_X}$ by

$$\Psi(\tilde{x}) = \{\tilde{y} \in L_X\colon \tilde{y}(t) \in \Phi(t, \tilde{x})\ \nu\text{-a.e.}\}.$$

Since for each $\tilde{x} \in L_X$, $\Phi(\cdot, \tilde{x})$ has a **T**-measurable graph, $\Psi(\tilde{x})$ is a nonempty set (recall the Aumann measurable selection theorem). Another application of Lemma A.5 in Appendix I enables us to conclude that ψ is weakly u.s.c. It is easy to see that ψ is convex valued and that the set L_X is nonempty, convex and weakly compact. Therefore, by the Fan–Glicksberg fixed point theorem there exists $\tilde{x}^* \in L_X$ such that $\tilde{x}^* \in \Psi(\tilde{x}^*)$. Then by the construction of Ψ, $\tilde{x}^*$ is a Bayesian Nash equilibrium for G. This completes the proof. ▮

6. CONCLUDING REMARKS AND OPEN QUESTIONS

Remark 6.1. In the Cartesian product approach, the continuity assumption dictates that for each agent t, there is only a countable number of agents whose action can affect the agent t's utility. Hence the remaining uncountably many agents' actions are meaningless in this infinite game. It is always a countable few who affect the agent t's utility even though this countable set can change depending on the functional form of the utility. Even a single agent's action can affect the agent t's utility if that agent is in the important countable set. We don't know if one can dispense with this assumption, this is an open question.

Remark 6.2. In the measure theoretic approach, actions of a countable number of agents are meaningless unless they are atoms in the measure space. An individual agent's action is negligible but the joint actions of set of agents with positive measure matter. Hence, the measure theoretic approach is more appropriate if one wants to analyze the models of perfect competition.

Remark 6.3. Note that assumptions (B.1), (B.2) in Section 5.2 are the same as (A.1), (A.2) and that (B.4) is the measurability assumption (needed since we introduce a measure structure on the set of agents, T). The only new assumption is (B.3), which we need to prove the weak continuity of the expected utility function in Lemma 4.6 in Appendix II. If Ω is uncountable and each agent's information partition is uncountable, then to prove the weak continuity of expected utility we need another assumption:

(B.3)′ *For each* $(t, \omega, x_t) \in T \times \Omega \times Y$, $u(t, \omega, \cdot, x_t): L_1(\nu, Y) \to \mathbb{R}$ *is linear.*

Assumption (B.3)′ is rather strong but it is necessary to prove the weak continuity if Ω is uncountable (see, for example, Balder–Yannelis [4]).

Remark 6.4. Assumption (B.3) can also be replaced by the fact that each partition is countable and the proof of Theorem 5.3 will still go through.

Remark 6.5. One may wonder as to whether or not the "information partition" approach adopted in this paper is superior to the "Harsanyi type" approach. It is difficult to answer this question because we are not aware of any existence results for the latter approach with the continuum of players. However, for the finite or countable set of players model, all we can say is that our assumptions seem to be less restrictive. However, this doesn't mean that one cannot eventually obtain more general existence results for the "Harsanyi-type" model. It is important to note that one may be able to show that we can go back and forth from the one approach to the other. However, at the moment in a general setting this seems to be an open question. As far as the applicability of our model is concerned we feel that the "information partition" approach is closer to the one adopted in the implementation literature as well as in the growing literature on economies with differential information.

Remark 6.6. Throughout the paper we employ the concept of Bochner integration. This notion may be a restriction in some cases because it becomes difficult to work with spaces which are not separable. Indeed, one may adopt the notion of Gelfand integral (or Pettis) in order to remedy this difficulty. However, at the moment the corresponding results on Bochner integration, e.g., Fatou's Lemma, integration preserves u.s.c., etc. (see, for example, Yannelis [25]) are not available for the Gelfand or Pettis integrals. Once such results are available, one may be able to obtain equilibrium existence theorems for non-separable spaces. At the moment this is an open question.

Remark 6.7. For the deterministic model with a continuum of players, equilibrium existence results in pure strategies are available and by now we know that the non-atomicity of the measure space of players makes the Lyapunov theorem applicable. However, in the present framework Lyapunov's theorem fails (due to the infinite dimensionality of the strategy space) and approximate or even exact versions under certain conditions may be needed in order to obtain purification results. The work of Podczeck [19] and Rustichini–Yannelis [22] may be useful in order to obtain pure strategy equilibrium results.

APPENDIX I

We begin the Appendix by proving the weak continuity of the conditional expected utility function of each agent.

LEMMA A.1. *Let $u_t\colon \prod_{s\neq t} X_s \times X_t \to \mathbb{R}$ be continuous when $X_s (s \neq t)$ is endowed with the weak topology and X_t, with the norm topology. Then for $E \in \mathscr{F}$, $v_t\colon \prod_{s\neq t} L_{X_s} \times X_t \to \mathbb{R}$ defined by*

$$v_t(\tilde{x}_{-t}, x_t) = \int_{\omega\in E} u_t(\tilde{x}_{-t}(\omega), x_t)\, d\mu(\omega)$$

is continuous where $L_{X_s}(s \neq t)$ is endowed with the weak topology and X_t, with the norm topology.

Proof. Let $\tilde{x}^n_{-t}, \tilde{x}_{-t} \in \prod_{s\neq t} L_{X_s}$ and $x^n_t, x_t \in X_t$ satisfying $\tilde{x}^n_{-t} \to \tilde{x}_{-t}$ and $x^n_t \to x_t$. By the property of the product topology, for each $s(\neq t)$, $\tilde{x}^n_s \to \tilde{x}_s$ weakly (written as $\tilde{x}^n_s \to_\omega \tilde{x}_s$). We need to show that

$$\int_{\omega\in E} u_t(\tilde{x}^n_{-t}(\omega), x^n_t)\, d\mu(\omega) \to \int_{\omega\in E} u_t(\tilde{x}_{-t}(\omega), x_t)\, d\mu(\omega).$$

We prove this via two steps. First, we show:

CLAIM 1. *For each $s(\neq t)$, for each $\omega \in \Omega$, the sequence $\{\tilde{x}^n_s(\omega)\}$ in X_s converges weakly to $\tilde{x}_s(\omega)$.*

Proof of Claim 1. Fix $\omega \in \Omega$. To prove the claim, we need to show that for all $y^* \in Y^*$, $y^*(\tilde{x}^n_s(\omega))$ converges to $y^*(\tilde{x}_s(\omega))$. Since $\prod_s = \{E^1_s, E^2_s, ...\}$ is a countable partition of Ω of agent s, $\tilde{x}^n_s$ and $\tilde{x}_s$ can be written as

$$\tilde{x}^n_s = \sum_{k=1}^{\infty} n^{n,k}_s \chi_{E^k_s} \qquad \text{and} \qquad \tilde{x}_s = \sum_{k=1}^{\infty} x^k_s \chi_{E^k_s},$$

where $x^{n,k}_s, x^k_s \in X_s$. Note that for each $s \in T$, there exists a unique $E^{k(\omega)}_s \in \prod_s$ with $\omega \in E^{k(\omega)}_s$. Then

$$\begin{aligned} y^*(\tilde{x}^n_s(\omega)) &= \int_{\omega'\in\Omega} \tilde{x}^n_s(\omega) \frac{y^*}{\mu(E^{k(\omega)}_s)} \chi_{E^{k(\omega)}_s}(\omega')\, d\omega' \\ &= \int_{\omega'\in\Omega} \tilde{x}^n_s(\omega') \frac{y^*}{\mu(E^{k(\omega)}_s)} \chi_{E^{k(\omega)}_s}(\omega')\, d\omega' \qquad (2) \end{aligned}$$

since $\tilde{x}_s^n(\omega') = \tilde{x}_s^n(\omega)$ if $\omega' \in E_s^{k(\omega)}$. Note that

$$\frac{y^*}{\mu(E_s^{k(\omega)})} \in L_\infty(\mu, Y^*) \qquad \text{and} \quad \tilde{x}_s^n \in L_1(\mu, Y).$$

Since $\tilde{x}_s^n$ converges to $\tilde{x}_s$ weakly in $L_1(\mu, Y)$, (2) converges to

$$\int_{\omega' \in \Omega} \tilde{x}_s(\omega') \frac{y^*}{\mu(E_s^{k(\omega)})} \chi_{E_s^{k(\omega)}}(\omega') \, d\mu(\omega') = y^*(\tilde{x}_s(\omega)).$$

Since the choice of $y^* \in Y^*$ is arbitrary, $\tilde{x}_s^n(\omega)$ converges weakly to $\tilde{x}_s(\omega)$. This proves Claim 1.

Claim 2. *$\int_{\omega \in E} u_t(\tilde{x}^n_{-t}(\omega), x_t^n) \, d\mu(\omega)$ converges to $\int_{\omega \in E} u_t(\tilde{x}_{-t}(\omega), x_t) \, d\mu(\omega)$.*

Proof of Claim 2. By Claim 1, for each $s(\neq t)$, for each $\omega \in \Omega$, $\tilde{x}_s^n(\omega)$ converges weakly to $\tilde{x}_s(\omega)$. By the continuity of u_t with the given topologies, for each $\omega \in \Omega$, $u_t(\tilde{x}^n_{-t}(\omega), x_t^n)$ converges to $u_t(\tilde{x}_{-t}(\omega), x_t)$. Therefore, by the Lebesgue dominated convergence theorem

$$\int_{\omega \in E} u_t(\tilde{x}^n_{-t}(\omega), x_t^n) \, d\mu(\omega) \qquad \text{converges to} \quad \int_{\omega \in E} u_t(\tilde{x}_{-t}(\omega), x_t) \, d\mu(\omega),$$

which completes the proof.

Lemma A.2. *Let $\upsilon: \Omega \times X_t \to \mathbb{R}$ is a measurable function. Then $\phi: \Omega \to 2^{X_t}$ defined by*

$$\phi(\omega) = \{x_t \in X_t: \upsilon(\omega, x_t) = \sup_{y_t \in X_t} \upsilon_t(\omega, y_t)\},$$

has a measurable graph.

Proof. See Castaing–Valadier [7, p. 86] or Debreu [8].

Lemma A.3. *Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space and Y be a separable Banach space. Let $\{\tilde{x}_n: n = 1, 2, ...\}$ be a sequence of functions in $L_1(\mu, Y)$ such that $\tilde{x}_n$ converges weakly to $\tilde{x} \in L_1(\mu, Y)$. Suppose that for all n, $x_n(\omega) \in \mathscr{F}(\omega)$ μ-a.e. ω, where $\mathscr{F}: \Omega \to 2^Y$ is a weakly compact, integrably bounded, nonempty valued correspondence. Then*

$$\tilde{x}(\omega) \in \overline{\text{con}}\, w - Ls\{\tilde{x}_n(\omega)\} \text{ } \mu\text{-a.e.},$$

where $\overline{\text{con}}\, A$ is the closure of the convex hull of A.

Proof. See Yannelis [25, p. 11].

The lemma below is known as Diestel's theorem and several alternative proofs can be found in the literature. For completeness, we provide a proof (see also Yannelis [25, p. 7] and the references therein).

LEMMA A.4. *Let Y be a separable Banach space and $X_t: \Omega \to 2^Y$ be integrably bounded, weakly compact, convex valued correspondence. Then the set*

$$L_{X_t} = \{\tilde{x} \in L_1(\mu, Y): \tilde{x} \text{ is } \mathscr{F}_t - \text{measurable and } \tilde{x}(\omega) \in X_t(\omega)\ \mu\text{-a.e. } \omega\}$$

is weakly compact in $L_1(\mu, Y)$.

Proof. The proof is based on the celebrated theorem of James [14]. Note that the dual of $L_1(\mu, Y)$ is $L_\infty(\mu, Y^*_{w^*})$ where w^* denotes the w^*-topology), i.e., $L_1(\mu, Y)^* = L_\infty(\mu, Y^*_{w^*})$ (see, for instance, Tulcea–Tulcea [24]). Let x be an arbitrary element of $L_\infty(\mu, Y^*_{w^*})$. If we show that x attains its supremum on L_{X_t}, the result will follow from James' theorem (James [14]). Note that

$$\begin{aligned}\sup_{\Psi \in L_{Xt}} \Psi \cdot x &= \sup_{\Psi \in L_{Xt}} \int_{\omega \in \Omega} (\Psi(\omega) \times (\omega))\, d\mu(\omega) \\ &= \int_{\omega \in \Omega} \sup_{\phi \in X_t(\omega)} (\phi \cdot x(\omega))\, d\mu(\omega),\end{aligned}$$

where the second equality follows from Theorem 2.2 of Hiai–Umegaki [13]. Define $g_t: \Omega \to 2^Y$ as

$$g_t(\omega) = \{ y \in X_t(\omega): y \cdot x(\omega) = \sup_{\phi \in X_t(\omega)} \phi \cdot x(\omega)\}.$$

It follows from the weak compactness of $X_t(\omega)$ that for all $\omega \in \Omega$, $g_t(\omega)$ is nonempty. Define $f_t: \Omega \times Y \to \mathbb{R}$ by

$$f_t(\omega, y) = \sup_{\phi \in X_t(\omega)} \phi \cdot x(\omega) - y \cdot x(\omega).$$

It is easy to see that for each ω, $f_t(\omega, \cdot)$ is continuous and for each y, $f_t(\cdot, y)$ is $\mathscr{F}_t$-measurable and hence $f_t(\cdot, \cdot)$ is jointly measurable. Then observe that $G_{g_t} = f_t^{-1}(0) \cap G_{X_t}$ and that since $f_t^{-1}(0)$ and G_{X_t} belong to $\mathscr{F}_t \otimes \mathbf{B}(Y)$, so does G_{g_t}. It follows from the Aumann measurable selection theorem that there exists an $\mathscr{F}_t$-measurable function $z_t: \Omega \to Y$ such that $z_t(\omega) \in g_t(\omega)$ μ-a.e. ω Thus, $z_t \in L_{X_t}$ and we have

$$\sup_{\phi \in L_{Xt}} \phi \cdot x = \int_{\omega \in \Omega} (z_t(\omega)\, x(\omega))\, d\mu(\omega) = z_t \cdot x.$$

Since $x \in L_\infty(\mu, Y^*_{w^*})$ was chosen arbitrarily, we conclude that every element of $(L_1(\mu, Y))^*$ attains its supremum on L_{X_t} and this completes the proof of the fact that L_{X_t} is weakly compact. The result below is taken from Yannelis [25, p. 19] and it is reported for the share of completeness.

LEMMA A.5. *Let $(\Omega, \mathscr{F}, \mu)$ be a complete, finite, separable measure space, Z be a metric space and Y be a separable Banach space. Let $\phi: \Omega \times Z \to 2^Y$ be a nonempty, convex valued correspondence satisfying*

(i) *for each $\omega \in \Omega$, $\phi(\omega, \cdot): Z \to 2^Y$ is weakly u.s.c., i.e., is continuous when Z is endowed with the metric topology and Y, with the weak topology.*

(ii) *for all $(\omega, z) \in \Omega \times Z$, $\phi(\omega, z) \subset X(\omega)$, where $X: \Omega \to 2^Y$ is an integrably bounded, convex, weakly compact and nonempty valued correspondence.*

Then $\Phi: Z \to 2^{L_1(\mu, Y)}$ defined by

$$\Phi(z) = \{\tilde{x} \in L_1(\mu, Y): \tilde{x}(\omega) \in \phi(\omega, z)\ \mu\text{-a.e.}\}$$

is weakly u.s.c.

Proof. First, note that $L_1(\mu, X)$ defined by

$$L_1(\mu, X) = \{\tilde{x} \in L_1(\mu, Y): \tilde{x}(\omega) \times X(\omega)\ \mu\text{-a.e.}\}$$

is weakly compact in $L_1(\mu, Y)$ (recall Lemma A.4). By Lemma A.4, for each $z \in Z$, the set

$$\Phi(z) = \{\tilde{x} \in L_1(\mu, Y): \tilde{x}(\omega) \in \phi(\omega, z)\ \mu\text{-a.e.}\}$$

is weakly compact. Since the measure space $(\Omega, \mathscr{F}, \mu)$ is separable and Y is a separable Banach space, $L_1(\mu, Y)$ is a separable Banach space. Hence, $L_1(\mu, X)$ is metrizable as it is a weakly compact subset of $L_1(\mu, Y)$ (Dunford and Schwartz [10, Theorem V.6.3, p. 434]). Consequently, in order to show that Φ is weakly u.s.c., it suffices to show that Φ has a weakly closed graph. To this end, let $\{z_n\}$ and $\{\tilde{x}_n\}$ be a sequence converging weakly to z and $\tilde{x}$ satisfying $\tilde{x}_n \in \Phi(z_n)$ for all n. We must show that $\tilde{x} \in \Phi(z)$. Since $\tilde{x}_n \in \Phi(z_n)$, $\tilde{x}_n(\omega) \in \phi(\omega, z_n)$ μ-a.e. It follows from Lemma A.3 that

$$\tilde{x}(\omega) \in \overline{\text{con}}\ w - L_s\{\tilde{x}_n(\omega)\} \subset \overline{\text{con}}\ w - L_s\{\phi(\omega, z_n)\}\ \mu\text{-a.e.} \tag{3}$$

Since for each $\omega \in \Omega$, $\phi(\omega, \cdot)$ has a weakly closed graph, we have that

$$w - L_s\{\phi(\omega, z_n)\} \subset \phi(\omega, z). \tag{4}$$

Since $\phi(\omega, z)$ is convex and weakly compact, we can conclude from (3) and (4) that $\tilde{x}(\omega) \in \phi(\omega, z)$ μ-a.e. Since ϕ is integrably bounded, it follows that $\tilde{x} \in \Phi(z)$. This completes the proof of the Lemma. ▮

APPENDIX II

In this Appendix we prove the continuity of the expected utility for the case of a measure space of players.

LEMMA A.6. *Let $(T, \mathbf{T}, \nu)$ and $(\Omega, \mathscr{F}, \mu)$ be finite measure spaces, where Ω is a countable set. Let X be a weakly compact subset of the separable Banach space Y whose dual Y^* has the RNP (Radon–Nikodym property) with respect to $(T, \mathbf{T}, \nu)$. For each $t \in T$, let $u(t, \omega, \cdot, \cdot): L_1(\nu, X) \times X \to \mathbb{R}$ be continuous where $L_1(\nu, X)$ are endowed with the weak topology and X, with the norm topology. Then for each $t \in T$, for $E \in \mathscr{F}$, $v_t: L_1(\nu, L_1(\mu, X)) \times X \to \mathbb{R}$ defined by*

$$v_t(\tilde{x}, x_t) = \int_{\omega \in E} u(t, \tilde{x}(\omega), x_t) \, d\mu(\omega)$$

is continuous where $L_1(\nu, L_1(\mu, X))$ is endowed with the weak topology and X, with the norm topology.

Proof. Let $\tilde{x}^n, \tilde{x} \in L_1(\nu, L_1(\mu, X))$ and $x_t^n, x_t \in Y$ such that $\tilde{x}^n$ converges weakly to $\tilde{x}$ and $\tilde{x}_t^n$ converges (in norm) to x_t. We need to show that

$$\int_{\omega \in E} u(t, \tilde{x}^n(\omega), x_t^n) \, d\mu(\omega) \to \int_{\omega \in E} u(t, \tilde{x}(\omega), x_t) \, d\mu(\omega).$$

We prove this via two steps. First, we show

CLAIM 1. *For each $\omega \in \Omega$, the sequence $\{\tilde{x}^n(\omega)\}$ in $L_1(\nu, X)$ converges weakly to $\tilde{x}(\omega)$.*

Proof of Claim 1. Fix $\omega \in \Omega$. To prove the claim, we need to show that for all $y^* \in [L_1(\nu, Y)]^* = L_\infty(\nu, Y^*)$ [by the RNP of Y^* with respect to $(T, \mathbf{T}, \nu)$],

$$\int_{t \in T} \tilde{x}_t^n(\omega) \, y^*(t) \, d\nu(t) \qquad \text{converges to} \qquad \int_{t \in T} \tilde{x}_t(\omega) \, y^*(t) \, d\nu(t).$$

Since $\Pi_t = \{E_t^1, E_t^2, ...\}$ is a countable partition of Ω of agent t, $\tilde{x}_t^n$ and $\tilde{x}_t$ can be written as

$$\tilde{x}_t^n = \sum_{k=1}^{\infty} x_t^{n,k} \chi_{E_t^k} \qquad \text{and} \qquad \tilde{x}_t = \sum_{k=1}^{\infty} x_t^k \chi_{E_t^k},$$

where $x_t^{n,k}, x_t^k \in X$. Note that for each $t \in T$, there exists a unique $E_t^{k(\omega)} \in \Pi_t$ with $\omega \in E_t^{k(\omega)}$. Moreover, for each $t \in T$,

$$\mu(E_t^{k(\omega)}) > \mu(\{\omega\}) > 0.$$

First, choose $y^* \in L_\infty(v, Y^*)$ such that

$$y^* = a^* \chi_{T_0}, \qquad \text{where} \quad a^* \in Y^* \text{ and } T_0 \in \mathbf{T}.$$

Then

$$\begin{aligned}
\int_{t \in T} \tilde{x}_t^n(\omega)\, y^*(t)\, dv(t) &= \int_{t \in T_0} \tilde{x}_t^n(\omega)\, a^*\, dv(t) \\
&= \int_{t \in T_0} \left[\int_{\omega' \in \Omega} \tilde{x}_t^n(\omega) \frac{a^*}{\mu(E_t^{k(\omega)})} \chi_{E_t^{k(\omega)}}(\omega')\, d\mu(\omega') \right] dv(t) \\
&= \int_{t \in T_0} \left[\int_{\omega' \in \Omega} \tilde{x}_t^n(\omega') \frac{a^*}{\mu(E_t^{k(\omega)})} \chi_{E_t^{k(\omega)}}(\omega')\, d\mu(\omega') \right] dv(t)
\end{aligned} \tag{5}$$

since $\tilde{x}_t^n(\omega') = \tilde{x}_t^n(\omega)$ if $\omega' \in E_t^{k(\omega)}$. Note that for each $t \in T$,

$$\frac{a^*}{\mu(E_t^{k(\omega)})} \in L_\infty(\mu, Y^*) \qquad \text{and} \qquad \tilde{x}_t^n \in L_1(\mu, Y).$$

Since $\mu(E_t^{k(\omega)})$ is uniformly bounded from below by $\mu(\{\omega\})$, the mapping

$$t \mapsto \frac{a^*}{\mu(E_t^{k(\omega)})} \chi_{E_t^{k(\omega)}}$$

is in $L_\infty(v, L_\infty(\mu, Y^*))$. Since $\tilde{x}^n$ converges weakly to $\tilde{x}$ in $L_1(v, L_1(\mu, Y))$, (5) converges to

$$\begin{aligned}
\int_{t \in T_0} \left[\int_{\omega \in \Omega} \tilde{x}_t(\omega') \frac{a^*}{\mu(E_t^{k(\omega)})} \chi_{E_t^{k(\omega)}}(\omega')\, d\mu(\omega') \right] dv(t) &= \int_{t \in T_0} \tilde{x}_t(\omega)\, a^*\, dv(t) \\
&= \int_{t \in T} \tilde{x}_t(\omega)\, y^*(t)\, dv(t),
\end{aligned}$$

where the first equality holds since $\tilde{x}_t^n(\omega') = \tilde{x}_t^n(\omega)$ if $\omega' \in E_t^k(\omega)$. So, for any simple function $y^* \in L_\infty(\nu, Y^*)$,

$$\int_{t \in T} \tilde{x}_t^n(\omega)\, y^*(t)\, d\nu(t) \qquad \text{converges to} \quad \int_{t \in T} \tilde{x}_t(\omega)\, y^*(t)\, d\nu(t).$$

Next, let $y^* \in L_\infty(\nu, Y)$. Since $(T, \mathscr{F}, \nu)$ is a finite measure space, there exists a sequence of simple functions converging to y^* uniformly (recall the Egoroff theorem). Let $\varepsilon > 0$ be given and let $h \in L_1(\nu, Y)$ be a simple function such that

$$\|y^* - h\| < \frac{\varepsilon}{p}$$

$$\text{where} \quad p > \sup\left\{\int_{t \in T} \|\tilde{x}_t^n(\omega)\|\, d\nu(t), \int_{t \in T} \|\tilde{x}_t(\omega)\|\, d\nu(t) \colon n = 1, 2, ...\right\}.$$

Then

$$\begin{aligned}
&\left|\int_{t \in T} \tilde{x}_t^n(\omega)\, y^*(t)\, d\nu(t) - \int_{t \in T} \tilde{x}_t(\omega)\, y^*(t)\, d\nu(t)\right| \\
&\quad \leqslant \left|\int_{t \in T} \tilde{x}_t^n(\omega)(y^*(t) - h(t))\, d\nu(t)\right| + \left|\int_{t \in T} (\tilde{x}_t^n(\omega) - \tilde{x}_t(\omega)) h(t)\, d\nu(t)\right| \\
&\qquad + \left|\int_{t \in T} \tilde{x}_t(\omega)(h(t) - y^*(t))\, d\nu(t)\right| \\
&\quad \leqslant 2\varepsilon + \left|\int_{t \in T} (\tilde{x}_t^n(\omega) - \tilde{x}_t(\omega))\, h(t)\, d\nu(t)\right|.
\end{aligned}$$

Since h is simple, we obtain

$$\lim_{n \to \infty} \left|\int_{t \in T} (\tilde{x}_t^n(\omega) - \tilde{x}_t(\omega))\, h(t)\, d\nu(t)\right| = 0.$$

Thus, the above estimates imply that

$$\int_{t \in T} \tilde{x}_t^n(\omega)\, y^*(t)\, d\nu(t) \qquad \text{converges to} \quad \int_{t \in T} \tilde{x}_t(\omega)\, y^*(t)\, d\nu(t)$$

for all $y^* \in L_\infty(\nu, Y^*)$. This proves Claim 1.

CLAIM 2. *$\int_{\omega \in E} u(t, \tilde{x}^n(\omega), x_t^n)\, d\mu(\omega)$ converges to $\int_{\omega \in E} u(t, \tilde{x}(\omega), x_t)\, d\mu(\omega)$.*

Proof of Claim 2. By Claim 1, for each $\omega \in \Omega$, $\tilde{x}^n(\omega)$ converges weakly to $\tilde{x}(\omega)$. By the continuity property with the given topologies, for each

$\omega \in \Omega$, $u(t, \tilde{x}^n(\omega), x_t^n)$ converges to $u(t, \tilde{x}(\omega), x_t)$. Therefore, by the Lebesgue dominated convergence theorem,

$$\int_{\omega \in E} u(t, \tilde{x}^n(\omega), x_t^n)\, d\mu(\omega) \qquad \text{converges to} \quad \int_{\omega \in E} u(t, \tilde{x}(\omega), x_t)\, d\mu(\omega),$$

which completes the proof.

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