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RESEARCH ARTICLE

Ex ante efficiency implies incentive compatibility

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Abstract We show that when agents become informationally negligible in a large economy with asymmetric information, every ex ante efficient allocation must be incentive compatible. This means that any ex ante core or Walrasian allocation is incentive compatible. The corresponding result is false for fixed finite-agent economies with asymmetric information. An example is also constructed to show that the ex post version of the result does not hold. Furthermore, we show that the result is sharp in the sense that it will fail to hold if one relaxes any of the main assumptions, namely, strong conditional independence on the information structure, strict concavity on the utility functions, type independence on the utility functions and endowments.

Keywords Asymmetric information · Pareto efficiency · Incentive compatibility · Negligible private information · Strong conditional independence · Exact law of large numbers

JEL Classification Numbers D0 · D5 · D6 · D8 · C0

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1 Introduction

It is well known that in a finite-agent economy with asymmetric information, there may not exist any incentive compatible, efficient allocations.[1] However, intuition suggests that a perfectly competitive market should still perform efficiently since no single agent has monopoly power on information. This intuitive idea of perfect competition for an atomless economy with asymmetric information is formalized in Sun and Yannelis (2007). In particular, it is shown in Sun and Yannelis (2007) that there exists an incentive compatible, ex post efficient allocation in a perfectly competitive asymmetric information economy, and thus the well-known conflict between incentive compatibility and Pareto efficiency is resolved exactly.[2]

While the existence of incentive compatible and ex post efficient allocations is shown in Sun and Yannelis (2007). It is easy to construct a large asymmetric information economy satisfying all the relevant conditions, and yet it has an ex post efficient allocation which is not incentive compatible (see Proposition 2 below). The purpose of this paper is to show that if we shift our attention to the ex ante case, then a totally different type of result can be obtained. That is, every ex ante efficient allocation is incentive compatible (see Theorem 1 below). The result is shown under the assumptions of strong conditional independence on the information structure, strict concavity on the utility functions, type independence on the utility functions and endowments. Such a result is obviously false in a fixed finite-agent economy with asymmetric information, and there are no analogous results in the literature.

The proof of Theorem 1 requires four main assumptions, namely, strong conditional independence on the information structure, strict concavity on the utility functions, type independence on the utility functions and endowments. We show that if one relaxes any of those assumptions, Theorem 1 may fail to hold. In particular, Proposition 3 shows that if the strong (conditional) independence is relaxed to a slightly weaker condition of mutual independence, then there exists a large asymmetric information economy in which every allocation has an essentially equivalent version that is not incentive compatible. Proposition 4 shows the importance of the strict concavity condition on the utility functions while Propositions 5 and 6 demonstrate that Theorem 1 cannot be extended to allow either initial endowments or utility functions to depend on private signals.

The paper is organized as follows. Sections 2 and 3 introduce, respectively the information structure and an economy with negligible asymmetric information. The main result is presented in Sect. 4. The relationship between ex ante and ex post efficiency is studied in Sect. 5. In particular, it is shown that ex ante efficiency implies ex post efficiency while the latter does not imply incentive compatibility. Section 6 shows that the main result may fail to hold if one relaxes any of the main assumptions.

[1] See, for example (Glycopantis and Yannelis, 2005, p. vi, Example 0.1).

[2] Several other different approaches for related problems have also been proposed by different authors. Prescott and Townsend (1984) introduced a lottery model and showed the existence of incentive compatible, ex ante efficient lottery allocations. Based on a notion of informational smallness, McLean and Postlewaite (2002) showed the existence of incentive compatible, ex post efficient allocations in an approximate sense for a replica economy. Hahn and Yannelis (1997) focused on a second best Pareto efficiency notion which results in incentive compatible allocations.

All the proofs are given in the appendix. Finally, we note that our exact result in Theorem 1 on an atomless economy with asymmetric information have asymptotic analogs for large but finite asymmetric information economies; this is illustrated in Sect. 6 of Sun and Yannelis (2007).

2 The information structure

We fix an atomless probability space[3] $(I, \mathcal{I}, \lambda)$ representing the space of economic agents, and $S = \{s_1, s_2, \ldots, s_K\}$ the space of **true states** of nature (its power set denoted by $\mathcal{S}$), which are not known to the agents. Let $T^0 = \{q_1, q_2, \ldots, q_L\}$ be the space of all the possible signals (types) for individual agents, $(T, \mathcal{T})$ a measurable space that models the private signal profiles for all the agents, and therefore T is a space of functions from I to T^0.[4] Thus, $t \in T$, as a function from I to T^0, represents a private signal profile for all agents in I. For agent $i \in I$, $t(i)$ (also denoted by t_i) is the **private signal** of agent i while t_{-i} the restriction of the signal profile t to the set $I \setminus \{i\}$ of agents different from i; let T_{-i} be the set of all such t_{-i}. For simplicity, we shall assume that $(T, \mathcal{T})$ has a product structure so that T is a product of T_{-i} and T^0, while $\mathcal{T}$ is the product algebra of the power set $\mathcal{T}^0$ on T^0 with a σ-algebra $\mathcal{T}_{-i}$ on T_{-i}. For $t \in T$ and $t_i' \in T^0$, we shall adopt the usual notation (t_{-i}, t_i') to denote the signal profile whose value is t_i' for agent i, and the same as t for other agents.

Let $(\Omega, \mathcal{F}, P)$ be a probability space representing all the **uncertainty** on the true states as well as on the signals for all the agents, where $(\Omega, \mathcal{F})$ is the product measurable space $(S \times T, \mathcal{S} \otimes \mathcal{T})$. Let P^S and P^T be the marginal probability measures of P, respectively on $(S, \mathcal{S})$ and on $(T, \mathcal{T})$. Let $\tilde{s}$ and $\tilde{t}_i$, $i \in I$ be the respective projection mappings from Ω to S and from Ω to T^0 with $\tilde{t}_i(s, t) = t_i$.[5] For each true state $s \in S$, we assume without loss of generality that the state is non-redundant in the sense that $\pi_s = P^S(\{s\}) > 0$; let P_s^T be the conditional probability measure on $(T, \mathcal{T})$ when the random variable $\tilde{s}$ takes value s. Thus, for each $B \in \mathcal{T}$, $P_s^T(B) = P(\{s\} \times B)/\pi_s$. It is obvious that $P^T = \sum_{s \in S} \pi_s P_s^T$. Note that the conditional probability measure P_s^T is often denoted as $P(\cdot|s)$ in the literature.

For $i \in I$, let τ_i be the signal distribution of agent i on the space T^0,[6] and $P^{S \times T_{-i}}(\cdot|t_i)$ the conditional probability measure on the product measurable space $(S \times T_{-i}, \mathcal{S} \otimes \mathcal{T}_{-i})$ when the signal of agent i is $t_i \in T^0$. If $\tau_i(\{t_i\}) > 0$, then it is clear that for $D \in \mathcal{S} \otimes \mathcal{T}_{-i}$, $P^{S \times T_{-i}}(D|t_i) = P(D \times \{t_i\})/\tau_i(\{t_i\})$.

For $s \in S$, let $P_s^{T_{-i}}$ and τ_{is} be the marginal probability measures of P_s^T, respectively on $(T_{-i}, \mathcal{T}_{-i})$ and $(T^0, \mathcal{T}^0)$. Since redundant signals are allowed for agent $i \in I$ ($q \in T^0$ is a redundant signal for agent i if $\tau_i(\{q\}) = 0$), we shall impose the

[3] We use the convention that all probability spaces are countably additive.

[4] In the literature, one usually assumes that different agents have possibly different sets of signals and require that the agents take all their own signals with positive probability. For notational simplicity, we choose to work with a common set T^0 of signals, but allow zero probability for some of the redundant signals. There is no loss of generality in this latter approach.

[5] $\tilde{t}_i$ can also be viewed as a projection from T to T^0.

[6] For $q \in T^0$, $\tau_i(\{q\})$ is the probability $P(\tilde{t}_i = q)$.

assumption that for any $q \in T^0$, if $\tau_i(\{q\}) > 0$, then $\tau_{is}(\{q\}) > 0$ for all $s \in S$, which means that any non-redundant signal has positive probability conditioned on any given true state.

Let F be the private signal process from $I \times T$ to T^0 such that $F(i,t) = t_i$ for any $(i,t) \in I \times T$. In this paper, we need to work with F that is independent conditioned on the true states $s \in S$. However, an immediate technical difficulty arises, which is the so-called measurability problem of independent processes. In our context, a signal process that is essentially independent, conditioned on the true states of nature is never jointly measurable in the usual sense except for trivial cases.[7] Hence, we need to work with a joint agent-probability space $(I \times T, \mathcal{I} \boxtimes \mathcal{T}, \lambda \boxtimes P_s^T)$ that extends the usual measure-theoretic product $(I \times T, \mathcal{I} \otimes \mathcal{T}, \lambda \otimes P_s^T)$ of the agent space $(I, \mathcal{I}, \lambda)$ and the probability space $(T, \mathcal{T}, P_s^T)$, and retains the Fubini property.[8] Its formal definition is given in Definition 2 of the Appendix.

Let $\mathcal{I} \boxtimes \mathcal{F}$ be the collection of all subsets A of $I \times \Omega$ such that there are sets $A_s \in \mathcal{I} \boxtimes \mathcal{T}$ for $s \in S$ such that $A = \cup_{s \in S}\{(i,s,t) \in I \times \Omega : (i,t) \in A_s\}$. By abusing the notation, we can denote $\mathcal{I} \boxtimes \mathcal{F}$ by $(\mathcal{I} \boxtimes \mathcal{T}) \otimes \mathcal{S}$. Define $\lambda \boxtimes P$ on $\mathcal{I} \boxtimes \mathcal{F}$ by letting $\lambda \boxtimes P(A) = \sum_{s \in S} \pi_s (\lambda \boxtimes P_s^T)(A_s)$. Thus, one can view $\lambda \boxtimes P_s^T$ as the conditional probability measure on $I \times T$, given $\tilde{s} = s$.

We shall assume that F is a measurable process from $(I \times T, \mathcal{I} \boxtimes \mathcal{T})$ to T^0. When the true state is s, the signal distribution of agent i conditioned on the true state is $P_s^T F_i^{-1}$, i.e., the probability for agent i to have $q \in T^0$ as her signal is $P_s^T(F_i^{-1}(\{q\}))$, where $F_i = F(i, \cdot)$. Let μ_s be the agents' **average signal distribution** conditioned on the true state s, i.e.,

$$\mu_s(\{q\}) = \int_I P_s^T(F_i^{-1}(\{q\}))d\lambda = \int_I \int_T 1_{\{q\}}(F(i,t))dP_s^T d\lambda, \tag{1}$$

where $1_{\{q\}}$ is the indicator function of the singleton set $\{q\}$. We shall impose the following **non-triviality assumption** on the process F:

$$\forall s, \quad s' \in S, \quad s \neq s' \Rightarrow \mu_s \neq \mu_{s'}.$$

This says that different true states of nature correspond to different average conditional distributions of agents' signals.

3 The large private information economy

We shall now follow the definition and notation in Sect. 2. We consider a large economy with asymmetric information. The **space of agents** is the atomless probability space $(I, \mathcal{I}, \lambda)$. In this economy, agents $i \in I$ are informed with their **private signals** $t_i \in T^0$

[7] See Sun (2006) and Sun and Yannelis (2007), and their references for detailed discussion of the measurability problem.

[8] $\mathcal{I} \boxtimes \mathcal{T}$ is a σ-algebra that contains the usual product σ-algebra $\mathcal{I} \otimes \mathcal{T}$, and the restriction of the countably additive probability measure $\lambda \boxtimes P_s^T$ to $\mathcal{I} \otimes \mathcal{T}$ is $\lambda \otimes P_s^T$.

but not the true state, and they can have contingent consumptions based on the signal profiles $t \in T$ announced by all the agents. Decisions are made at the ex ante level. The common **consumption set** is the positive orthant $\mathbb{R}^m_+$. In the sequel, we shall state several assumptions on the economy.

A1. *The utility function of each agent depends on her consumption $x \in \mathbb{R}^m_+$ and the true state $s \in S$ but not on the private signals of the agents in the economy. Thus, we can let u be a function from $I \times \mathbb{R}^m_+ \times S$ to $\mathbb{R}_+$ such that for any given $i \in I$, $u(i, x, s)$ is the* **utility** *of agent i at consumption bundle $x \in \mathbb{R}^m_+$ and true state $s \in S$.*

A2. *For any given $s \in S$, $u(i, x, s)$, (also denoted by $u_s(i, x)$),*[9] *is $\mathcal{I}$-measurable in $i \in I$, continuous,* **strictly concave** *and monotonic*[10] *in $x \in \mathbb{R}^m_+$.*

A3. *Let e be a λ-integrable function from I to $\mathbb{R}^m_+$ with $e(i)$ as the* **initial endowment** *of agent i.*[11]

A4. *The private signal process F is a measurable process from $(I \times T, \mathcal{I} \boxtimes \mathcal{T})$ to T^0 that is* **strongly conditionally independent***, given $\tilde{s}$ in the sense that for any $i \in I$, agent i's signal $F(i, \cdot)$ is independent of all the events in the signal space $(T_{-i}, \mathcal{T}_{-i})$, conditioned on the true states of nature.*[12] *In other words, for each $s \in S$, the probability space $(T, \mathcal{T}, P^T_s)$ is the product of its marginal probability spaces $(T_{-i}, \mathcal{T}_{-i}, P^{T_{-i}}_s)$ and $(T^0, \mathcal{T}^0, \tau_{is})$.*

The Assumption **A4** says that conditioned on the true states of nature, the private signal of an individual agent has strictly no influence over any others. Thus, perfect competition prevails in this economy in the sense that agents have negligible initial endowments and negligible private information.

We shall now consider an economy where the agents are informed with their signals but not the true state. Formally, the collection $\mathcal{E}^p = \{(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P), u, e, F, \tilde{s}\}$ is called a **private information economy**.

The space of consumption plans for the economy $\mathcal{E}^p$ is the space $L^1(P^T, \mathbb{R}^m_+)$ of integrable functions from $(T, \mathcal{T}, P^T)$ to $\mathbb{R}^m_+$, which is infinite dimensional. Fix an agent $i \in I$. For a consumption plan $z \in L^1(P^T, \mathbb{R}^m_+)$, let

$$U^p_i(z) = \int_{\Omega} u(i, z(t), s)\, dP \tag{2}$$

be the **ex ante expected utility** of agent i for the consumption plan z.[13]

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[9] In the sequel, we shall often use subscripts to denote some variable of a function that is viewed as a parameter in a particular context.

[10] The utility function $u(i, \cdot, s)$ is monotonic if for any $x, y \in \mathbb{R}^m_+$ with $x \leq y$ and $x \neq y$, $u(i, x, s) < u(i, y, s)$.

[11] Since the true state $s \in S$ is not known to the agents, the agents' endowments cannot depend on s. However, as in McLean and Postlewaite (2002) and Sun and Yannelis (2007), here we also assume that the endowments do not depend on the private signals of agents.

[12] For a general justification of using conditional independence, see Hammond and Sun (2003).

[13] Fix $i \in I$. Since $u(i, \cdot, s)$ is strictly concave for each $s \in S$, there are constants $c, d \in \mathbb{R}_+$ such that $u(i, x, s) \leq c\|x\| + d$ for any $x \in \mathbb{R}^m_+$, where $\|\cdot\|$ is the Euclidean norm. From this condition, it is clear that $\int_{\Omega} u(i, z(t), s)dP$ is finite.

Definition 1 1. An **allocation** for the economy $\mathcal{E}^p$ is an integrable function x^p from $(I \times T, \mathcal{I} \boxtimes \mathcal{T}, \lambda \boxtimes P^T)$ to $\mathbb{R}^m_+$; agent i's consumption plan is $x^p(i, \cdot)$ (also denoted by x_i^p).

2. An allocation x^p is **feasible** if for P^T-almost all $t \in T$, $\int_I x^p(i,t)d\lambda(i) = \int_I e(i)d\lambda(i)$.
3. A feasible allocation x^p is said to be **ex ante efficient** if there does not exist a feasible allocation y^p such that for λ-almost all $i \in I$, $U_i^p(y_i^p) > U_i^p(x_i^p)$.
4. A feasible allocation x^p is said to be **ex post efficient** if for P^T-almost all $t \in T$, x_t^p is efficient in the large deterministic (ex post) economy $\mathcal{E}_t^p = \{(I, \mathcal{I}, \lambda), U(\cdot,\cdot,t), e\}$, where $U(i,x,t) = \sum_{s \in S} u_i(x,s) P^S(\{s\}|t)$ is the ex post utility of agent i (also denoted by $U_i(x|t)$) for her consumption bundle $x \in \mathbb{R}^m_+$ with the given signal profile t.
5. For an allocation x^p, an agent $i \in I$, private signals $t_i, t_i' \in T^0$, let

$$U_i(x_i^p, t_i'|t_i) = \int\limits_{S \times T_{-i}} u_i(x_i^p(t_{-i}, t_i'), s) dP^{S \times T_{-i}}(\cdot|t_i), \tag{3}$$

be the interim expected utility of agent i when she receives private signal t_i but mis-reports as t_i'. The allocation x^p is said to be **incentive compatible** if λ-almost all $i \in I$,

$$U_i(x_i^p, t_i|t_i) \geq U_i(x_i^p, t_i'|t_i)$$

holds for all the non-redundant signals $t_i, t_i' \in T^0$ of agent i (i.e., $\tau_i(\{t_i\}) > 0$ and $\tau_i(\{t_i'\}) > 0$).

6. A feasible allocation x^p is said to be an **ex ante Walrasian allocation** (ex ante competitive equilibrium allocation) if there is a bounded measurable price function p from $(T, \mathcal{T})$ to $\mathbb{R}^m_+ \setminus \{0\}$ such that for λ-almost all $i \in I$, $x^p(i)$ is a maximal element in the budget set

$$\left\{ z \in L^1(P^T, \mathbb{R}^m_+) : \int\limits_T p(t) \cdot z(t) dP^T \leq \int\limits_T p(t) \cdot e(i) dP^T \right.$$
$$\left. = \left(\int\limits_T p(t) dP^T \right) \cdot e(i) \right\}$$

under the expected utility function $U_i^p(\cdot)$.

7. A coalition A (i.e., a set in $\mathcal{I}$ with $\lambda(A) > 0$) is said to ex ante block an allocation x^p in $\mathcal{E}^p$ if there exists an allocation y^p such that $\int_A y^p(i,t)d\lambda(i) = \int_A e(i)d\lambda(i)$ for P^T-almost all $t \in T$, and for λ-almost all $i \in A$, $U_i^p(y_i^p)) > U_i^p(x_i^p)$.[14]

[14] One can also only define the allocation y^p on $A \times T$ instead of $I \times T$. However, there is no loss of generality since one can always extend a function defined on $A \times T$ to $I \times T$ to keep its integrability.

A feasible allocation x^p is said to be in the **ex ante core** of $\mathcal{E}^p$, or simply an ex ante core allocation in $\mathcal{E}^p$, if there is no coalition that ex ante blocks x^p.

4 The main theorem

We are now ready to state the main result of this paper. Its proof will be given in Subsect. 7.2 of the appendix.

Theorem 1 *Under Assumptions* **A1–A4***, any ex ante efficient allocation is incentive compatible.*

It is obvious that any ex ante core allocation is ex ante efficient. It is also easy to check that any ex ante Walrasian allocation is ex ante efficient. Hence the following two corollaries are clear consequences of Theorem 1.

Corollary 1 *Under Assumptions* **A1–A4***, any ex ante core allocation is incentive compatible.*

Corollary 2 *Under Assumptions* **A1–A4***, any ex ante Walrasian allocation is incentive compatible.*

5 Ex ante versus ex post efficiency

In this section, we adopt the economic model specified in Sects. 2 and 3. We assume that Assumptions **A1–A4** are satisfied.

As noted in Holmström and Myerson (1983), ex ante efficiency implies ex post efficiency for a finite-agent economy. The ex post efficiency in Holmström and Myerson (1983) requires all the states to be revealed, and thus incentive compatibility is not an issue. In contrast, the notion of ex post efficiency as considered in McLean and Postlewaite (2002) for a countable replica economy and Sun and Yannelis (2007) for an atomless private information economy requires only the signals to be revealed but not the true states. Nevertheless, the following proposition, whose proof will be given in Subsect. 7.3 of the appendix, shows that ex ante efficiency does imply ex post efficiency in our setting.

Proposition 1 *Every ex ante efficient allocation is ex post efficient.*

Theorem 1 above shows that incentive compatibility follows form ex ante efficiency. However, this is not the case for ex post efficiency. In particular, the following proposition shows that there exists an ex post efficient allocation that is not incentive compatible;[15] by Theorem 1, such an allocation is not ex ante efficient. This means that the converse of Proposition 1 is not true.

Proposition 2 *There exists a large private information economy such that* (1) *it satisfies all the conditions on the information structure in Sect. 2 as well as Assumptions* **A1–A4** *in Sect. 3;* (2) *it has an ex post efficient allocation that is not incentive compatible.*

[15] The details of the construction can be found in Subsect. 7.4 of the appendix.

6 Relaxation of Assumptions A1–A4

Theorem 1 above shows that under Assumptions **A1–A4**, incentive compatibility follows form ex ante efficiency. The purpose of this section is to show that if any of the assumptions is weakened, the result may fail to hold.[16] For simplicity, we only consider the case that the true state space S is a singleton set.

We first consider **A4**. The following proposition shows that if the strong (conditional) independence in **A4** is relaxed to a slightly weaker condition of mutual independence, then every allocation has an essentially equivalent version that is not incentive compatible.

Proposition 3 *There exists a large private information economy $\mathcal{E}^p$ such that* (1) *it satisfies all the conditions on the information structure in Sect. 2 as well as Assumptions* **A1–A3** *in Sect. 3;* (2) *the private signal process F is mutually independent in the sense that for any l different agents, $i_1, \ldots, i_l$, their random private signals $F_{i_1}, \ldots, F_{i_l}$ are mutually independent;* (3) *for any allocation x^p in $\mathcal{E}^p$, there exists an essentially equivalent allocation y^p that is not incentive compatible.*

The condition of strict concavity in **A2** plays a key role in the proof of Theorem 1. The following result shows that if strict concavity is relaxed to just concavity, Theorem 1 may fail.

Proposition 4 *There exists a large private information economy $\mathcal{E}^p$ such that* (1) *it satisfies all the conditions on the information structure in Sect. 2 as well as Assumptions* **A1, A3, A4** *in Sect. 3;* (2) *the relevant utility functions are measurable, concave and monotonic;*[17] (3) *the economy has an ex ante efficient allocation that is not incentive compatible.*[18]

Next, we relax Assumption **A1** by considering the case that for each $i \in I$, agent i's utility function depends on her private signal t_i. Since we take the true state space S to a singleton set in this section, we can simply ignore its existence in the utility function by considering u as a function from $I \times \mathbb{R}^m_+ \times T^0$ to $\mathbb{R}_+$ such that for any given $i \in I$, $u(i, x, q)$ is the **utility** of agent i at consumption bundle $x \in \mathbb{R}^m_+$ and private signal $q \in T^0$. For a given private signal profile $t \in T$, agent i's utility function is $u(i, \cdot, t_i)$.

For a PIE allocation x^p, the expected utility of agent i when she receives private signal t_i but mis-reports as t'_i is

$$U_i(x^p_i, t'_i|t_i) = \int_{S\times T_{-i}} u_i(x^p_i(t_{-i}, t'_i), s, t_i) dP^{S\times T_{-i}}(\cdot|t_i). \tag{4}$$

[16] For the convenience of the reader, the results are stated in this section, using common measure-theoretic terms, while the details of the constructions, which use nonstandard analysis, will be given in Subsect. 7.4 of the appendix.

[17] That is, the strict concavity condition in **A2** is replaced by the concavity condition while the measurability and monotonicity conditions are kept as in **A2**.

[18] The general idea for this result was suggested by Eric Maskin.

The incentive compatibility can be defined in the same way as in Definition 1. For a consumption plan $z \in L^1(P^T, \mathbb{R}^m_+)$, the ex ante utility can be defined as in equation (2) by

$$U_i^p(z) = \int_{\Omega} u(i, z(t), s, t_i)dP \tag{5}$$

Proposition 5 below shows that if Assumption **A1** is relaxed to allow type dependent utilities, then Theorem 1 can fail strongly so that every ex ante efficient allocation in some large private information economy is not incentive compatible. On the other hand, the economy in Proposition 5 does have an incentive compatible, ex post efficient allocation as shown by Theorem 2 of Sun and Yannelis (2007).

Proposition 5 *There exists a large private information economy $\mathcal{E}^p$ such that (1) it satisfies all the conditions on the information structure in Sect. 2 as well as Assumptions* **A2–A4** *in Sect. 3; (2) agent i's utility function depends on her private signal t_i as described above; (3) every ex ante efficient allocation in the economy is not incentive compatible.*

Finally, we relax Assumption **A3** by considering the case that for each $i \in I$, agent i's endowment depends on her private signals. Let e be a function from $I \times T^0$ to $\mathbb{R}^m_+$ with $e(i, t_i)$ as the **initial endowment** of agent i with private signal t_i. We assume that for each $q \in T^0$, $e(i, q)$ is λ-integrable.

Fix an allocation x^p. When agents' endowments do not depend on their private signals, Eq. (3) defines the expected utility of agent i when she receives private signal t_i but mis-reports as t_i'. If we consider the case that the endowment of every agent depends on her private signal as above, then the expected utility of agent i when she receives private signal t_i but mis-reports as t_i' is[19]

$$U_i(x_i^p, t_i'|t_i) = \int_{S\times T_{-i}} u_i(x_i^p(t_{-i}, t_i') - e(t_i') + e(t_i), s)dP^{S\times T_{-i}}(\cdot|t_i). \tag{6}$$

The incentive compatibility can be defined in the same way as in Definition 1.

Proposition 6 *There exists a large private information economy $\mathcal{E}^p$ such that* (1) *it satisfies all the conditions on the information structure in Sect. 2 as well as Assumptions* **A1, A2, A4** *in Sect. 3;* (2) *agent i's endowment depends on her private signal t_i as described above;* (3) *every ex ante efficient allocation in the economy is not incentive compatible.*

[19] The idea is that when agent i receives private signal t_i but mis-reports as t_i', her total consumption is her endowment $e(t_i)$ plus her net trade $x_i^p(t_{-i}, t_i') - e(t_i')$.

7 Appendix

7.1 The exact law of large numbers

In order to work with independent processes constructed from signal profiles, we need to work with an extension of the usual measure-theoretic product that retains the Fubini property. Below is a formal definition of the Fubini extension in Definition 2.2 of Sun (2006).

Definition 2 A probability space $(I \times \Omega, \mathcal{W}, Q)$ extending the usual product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ is said to be a *Fubini extension* of $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ if for any real-valued Q-integrable function f on $(I \times \Omega, \mathcal{W})$,

(1) the two functions f_i and f_ω are integrable, respectively on $(\Omega, \mathcal{F}, P)$ for λ-almost all $i \in I$, and on $(I, \mathcal{I}, \lambda)$ for P-almost all $\omega \in \Omega$;

(2) $\int_\Omega f_i dP$ and $\int_I f_\omega dP$ are integrable, respectively on $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$, with $\int_{I\times\Omega} f dQ = \int_I \left(\int_\Omega f_i dP\right) d\lambda = \int_\Omega \left(\int_I f_\omega d\lambda\right) dP$.[20]

To reflect the fact that the probability space $(I \times \Omega, \mathcal{W}, Q)$ has $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ as its marginal spaces, as required by the Fubini property, it will be denoted by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

We shall now follow the notation of Sect. 2. When the probability space $(I \times T, \mathcal{I} \boxtimes \mathcal{T}, \lambda \boxtimes P_s^T)$ is a Fubini extension of the usual product space $(I \times T, \mathcal{I} \otimes \mathcal{T}, \lambda \otimes P_s^T)$, for each $s \in S$, it can be checked that $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, defined in the last paragraph of Sect. 2, is a Fubini extension of the usual product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$.

The following is an exact law of large numbers for a continuum of independent random variables shown in Sun (2006), which is stated here as a lemma using our notation for the convenience of the reader.[21]

Lemma 1 *If a $\mathcal{I} \boxtimes \mathcal{T}$-measurable process G from $I \times T$ to a complete separable metric space X is essentially pairwise independent conditioned on $\tilde{s}$ in the sense that for λ-almost all $i \in I$, the random variables G_i and G_j from $(T, \mathcal{T}, P_s^T)$ to X are independent for λ-almost all $j \in I$, then for each $s \in S$, the cross-sectional distribution λG_t^{-1} of the sample function $G_t(\cdot) = G(t, \cdot)$ is the same as the distribution $(\lambda \boxtimes P_s^T)G^{-1}$ of the process G viewed as a random variable on $(I \times T, \mathcal{I} \boxtimes \mathcal{T}, \lambda \boxtimes P_s^T)$ for P_s^T-almost all $t \in T$. In addition, for each $s \in S$, when X is the real line $\mathbb{R}$ and G is $(\lambda \boxtimes P_s^T)$-integrable, $\int_I G_t d\lambda = \int_{I\times T} Gd(\lambda \boxtimes P_s^T)$ holds for P_s^T-almost all $t \in T$.*

7.2 Proof of Theorem 1

Let x^p be any ex ante efficient allocation. Then the Fubini property implies that there is a set $A^* \in \mathcal{I}$ with $\lambda(A^*) = 1$ such that for any $i \in A^*$ and any $s \in S$, the integral

[20] The classical Fubini Theorem is only stated for the usual product measure spaces. It does not apply to integrable functions on $(I \times \Omega, \mathcal{W}, Q)$ since these functions may not be $\mathcal{I} \otimes \mathcal{F}$-measurable. However, the conclusions of that theorem do hold for processes on the enriched product space $(I \times \Omega, \mathcal{W}, Q)$ that extends the usual product.

[21] See Corollaries 2.9 and 2.10 in Sun (2006). We state the result using a complete separable metric space X for the sake of generality. In particular, a finite space or an Euclidean space is a complete separable metric space.

$\int_T x^p(i,t)dP_s^T$ is finite.[22] We can define an allocation $\bar{x}^p$ by letting $\bar{x}_i^p(\cdot) = x_i^p(\cdot)$ for $i \in A^*$ and $\bar{x}_i^p(\cdot) \equiv e_i$ for $i \notin A^*$. Then, the integral $\int_T \bar{x}^p(i,t)dP_s^T$ is finite for all $i \in I$ and $s \in S$. It is obvious that $\bar{x}^p$ is feasible and ex ante efficient.

Define the following sets

$$\forall s \in S,\ L_s = \{t \in T : \lambda F_t^{-1} = \mu_s\};\ \ L_0 = T - \cup_{s \in S} L_s.$$

The non-triviality assumption implies that for any $s,\ s' \in S$ with $s \neq s'$, $L_s \cap L_{s'} = \emptyset$. The measurability of the sets $L_s,\ s \in S$ and L_0 follows from the measurability of F. Thus, the collection $\{L_0\} \cup \{L_s,\ s \in S\}$ forms a measurable partition of T.

By Eq. (1) and the Fubini property for $(I \times T, \mathcal{I} \boxtimes \mathcal{T}, \lambda \boxtimes P_s^T)$, we have $\mu_s(\{q\}) = \int_{I \times T} 1_{\{q\}}(F(i,t))d(\lambda \boxtimes P_s^T)$ for any $q \in T^0$. Thus, μ_s is actually the distribution $(\lambda \boxtimes P_s^T)F^{-1}$ of F, viewed as a random variable on the product space $I \times T$. Since the signal process F satisfies the condition of conditional independence, it certainly satisfies the condition of essential pairwise conditional independence, the exact law of large numbers in Lemma 1 implies that $P_s^T(L_s) = 1$ for each $s \in S$.

For each $(i,t) \in I \times T$, let

$$y^p(i,t) = \begin{cases} \int_I e(i)d\lambda & \text{if } t \in L_0, \\ \int_T \bar{x}^p(i,t')dP_s^T(t') & \text{if } t \in L_s,\ s \in S. \end{cases} \tag{7}$$

Since the Fubini property implies that $\int_T \bar{x}^p(\cdot,t)dP_s^T(t)$ is $\mathcal{I}$-measurable on I, it is clear that y^p is $\mathcal{I} \otimes \mathcal{T}$-measurable and hence $\mathcal{I} \boxtimes \mathcal{T}$-measurable. For $t \in L_0$, $y^p(\cdot,t)$ is the constant $\int_I e(i)d\lambda$ and thus $\int_I y^p(i,t)d\lambda = \int_I e(i)d\lambda$; for $s \in S$ and $t \in L_s$, $y^p(i,t)$ is $\int_T \bar{x}^p(i,t)dP_s^T$, and hence

$$\begin{aligned} \int_I y^p(i,t)d\lambda(i) &= \int_I \int_T \bar{x}^p(i,t')dP_s^T(t')d\lambda(i) \\ &= \int_T \int_I \bar{x}^p(i,t')d\lambda(i)dP_s^T(t') = \int_I e(i)d\lambda(i), \end{aligned}$$

where the last identity follows from the feasibility of $\bar{x}^p$. Therefore, y^p is a feasible allocation in $\mathcal{E}^p$ that satisfies the feasibility condition for all $t \in T$.

We shall prove that $\bar{x}^p(i,t) = y^p(i,t)$ for $\lambda \boxtimes P^T$-almost all $(i,t) \in I \times T$. Suppose not; then there exist $s_0 \in S$ and coalition $A \in \mathcal{I}$ (with $\lambda(A) > 0$) such that for each $i \in A$, $P_{s_0}^T(\{t \in T : \bar{x}^p(i,t) \neq y^p(i,t)\}) > 0$, i.e., the random variable $\bar{x}_i^p(\cdot)$ is not essentially constant under the probability measure $P_{s_0}^T$. For each $i \in A$, since

[22] The Fubini property only implies the integrability of $x^p(i,\cdot)$ for λ-almost $i \in I$. It also means that $x^p(i,\cdot)$ may not be integrable for i in a λ-null subset of I.

$u(i, \cdot, s_0)$ is strictly concave, Jensen's inequality implies that

$$\int_T u_i(\bar{x}_i^p(t), s_0) dP_{s_0}^T(t) < u_i\left(\int_T \bar{x}_i^p(t) dP_{s_0}^T, s_0\right). \tag{8}$$

The assumption of monotonicity implies that for each $i \in A$, $u_i(0, s_0) \leq u_i(\bar{x}_i^p(t), s_0)$ for all $t \in T$, which implies that $u_i(0, s_0) \leq \int_T u_i(\bar{x}_i^p(t), s_0) dP_{s_0}^T(t)$. By Eq. (8), we have for each $i \in A$, $u_i(0, s_0) < u_i\left(\int_T \bar{x}_i^p(t) dP_{s_0}^T, s_0\right)$, and hence $\int_T \bar{x}_i^p(t) dP_{s_0}^T$ must have positive components. By the continuity of the utility functions $u(i, \cdot, s_0)$, one can choose a coalition $A_0 \subseteq A$ (with $0 < \lambda(A_0) < 1$), a positive number ϵ_0 and a vector $e_0 \in \mathbb{R}_+^m$ with $e_0 \neq 0$ such that for any $i \in A_0$, $\int_T \bar{x}_i^p(t) dP_{s_0}^T \geq e_0$, and

$$\begin{aligned}\int_T u_i(\bar{x}_i^p(t), s_0) dP_{s_0}^T(t) + \epsilon_0 &< u_i\left(\int_T \bar{x}_i^p(t) dP_{s_0}^T, s_0\right) \\ &< u_i\left(\int_T \bar{x}_i^p(t) dP_{s_0}^T - e_0, s_0\right) + \epsilon_0. \end{aligned} \tag{9}$$

For each $(i, t) \in I \times T$, let

$$z^p(i, t) = \begin{cases} y^p(i, t) & \text{for } (i, t) \in I \times (T \setminus L_{s_0}), \\ y^p(i, t) - e_0 & \text{for } (i, t) \in A_0 \times L_{s_0}, \\ y^p(i, t) + \frac{\lambda(A_0)}{1 - \lambda(A_0)} e_0 & \text{for } (i, t) \in (I \setminus A_0) \times L_{s_0}. \end{cases} \tag{10}$$

It is obvious that z^p is $\mathcal{I} \boxtimes \mathcal{T}$-measurable. When $(i, t) \in A_0 \times L_{s_0}$, $z^p(i, t) = y^p(i, t) - e_0 = \int_T \bar{x}_i^p(t') dP_{s_0}^T(t') - e_0 \geq 0$. It is thus clear that z^p takes values in $\mathbb{R}_+^m$. The feasibility of y^p for all $t \in T$ implies immediately that for $t \in (T \setminus L_{s_0})$, $\int_I z^p(i, t) d\lambda = \int_I e(i) d\lambda$. For $t \in L_{s_0}$,

$$\begin{aligned}\int_I z^p(i, t) d\lambda &= \int_{A_0} (y^p(i, t) - e_0) d\lambda + \int_{I \setminus A_0} \left(y^p(i, t) + \frac{\lambda(A_0)}{1 - \lambda(A_0)} e_0\right) d\lambda \\ &= \int_I y^p(i, t) d\lambda - \lambda(A_0) e_0 + \lambda(I \setminus A_0) \frac{\lambda(A_0)}{1 - \lambda(A_0)} e_0 \\ &= \int_I y^p(i, t) d\lambda = \int_I e(i) d\lambda. \end{aligned}$$

Therefore, z^p is a feasible allocation.

By Eq. (9), we have for any $i \in A_0$,

$$\int_T u_i(\bar{x}_i^p(t), s_0) dP_{s_0}^T(t) < u_i\left(\int_T \bar{x}_i^p(t') dP_{s_0}^T(t') - e_0, s_0\right)$$
$$= \int_T u_i\left(\int_T \bar{x}_i^p(t') dP_{s_0}^T(t') - e_0, s_0\right) P_{s_0}^T(t). \qquad (11)$$

Hence, Eqs. (7), (10) and (11) together with the fact that $P_{s_0}^T(L_{s_0}) = 1$ imply that for any $i \in A_0$,

$$\int_T u_i(\bar{x}_i^p(t), s_0) dP_{s_0}^T(t) < \int_T u_i\left(\int_T \bar{x}_i^p(t') dP_{s_0}^T(t') - e_0, s_0\right) P_{s_0}^T(t)$$
$$= \int_{L_{s_0}} u_i\left(y^p(i,t) - e_0, s_0\right) P_{s_0}^T(t) = \int_{L_{s_0}} u_i\left(z^p(i,t), s_0\right) P_{s_0}^T(t)$$
$$= \int_T u_i(z_i^p(t), s_0) dP_{s_0}^T(t). \qquad (12)$$

For any $i \notin A_0$, Jensen's inequality together with the monotonicity of $u_i(\cdot, s_0)$, Eqs. (7) and (10) and the fact that $P_{s_0}^T(L_{s_0}) = 1$ imply that

$$\int_T u_i(\bar{x}_i^p(t), s_0) dP_{s_0}^T(t) \leq u_i\left(\int_T \bar{x}_i^p(t') dP_{s_0}^T(t'), s_0\right)$$
$$= \int_{L_{s_0}} u_i\left(y^p(i,t), s_0\right) P_{s_0}^T(t)$$
$$< \int_{L_{s_0}} u_i\left(y^p(i,t) + \frac{\lambda(A_0)}{1-\lambda(A_0)} e_0, s_0\right) P_{s_0}^T(t)$$
$$= \int_{L_{s_0}} u_i\left(z^p(i,t), s_0\right) P_{s_0}^T(t)$$
$$= \int_T u_i(z_i^p(t), s_0) dP_{s_0}^T(t). \qquad (13)$$

Similarly, Jensen's inequality together with Eqs. (7) and (10) and the fact that $P_s^T(L_s) = 1$ imply that for any $i \in I$, $s \neq s_0$,

$$\int\limits_T u_i(\bar{x}_i^p(t), s) dP_s^T(t) \leq u_i \left(\int\limits_T \bar{x}_i^p(t') dP_s^T(t'), s \right) = \int\limits_{L_s} u_i \left(y^p(i,t), s \right) P_s^T(t)$$
$$= \int\limits_{L_s} u_i \left(z^p(i,t), s \right) P_s^T(t)$$
$$= \int\limits_T u_i(z_i^p(t), s) dP_s^T(t). \tag{14}$$

Hence, we obtain from Eqs. (12), (13) and (14) that for all $i \in I$, $s \in S$, $\int_T u_i(\bar{x}_i^p(t), s)$ $dP_s^T(t) \leq \int_T u_i(z_i^p(t), s) dP_s^T(t)$ with the strict inequality for $s = s_0$. Since $\pi_{s_0} > 0$, we obtain that for all $i \in I$,

$$\sum_{s \in S} \pi_s \int\limits_T u_i(\bar{x}_i^p(t), s) dP_s^T(t) < \sum_{s \in S} \pi_s \int\limits_T u_i(z_i^p(t), s) dP_s^T(t). \tag{15}$$

Now, by the definition of expected utility in Eq. (2), Eq. (15) implies that

$$U_i^p(\bar{x}_i^p) = \int\limits_\Omega u_i(\bar{x}_i^p(t), s) dP = \sum_{s \in S} \pi_s \int\limits_T u_i(\bar{x}_i^p(t), s) dP_s^T$$
$$< \sum_{s \in S} \pi_s \int\limits_T u_i(z_i^p(t), s) dP_s^T = \int\limits_\Omega u_i(z_i^p(t), s) dP = U_i^p(z_i^p).$$

This contradicts the ex ante efficiency of $\bar{x}^p$. Therefore, y^p must agree with $\bar{x}^p$ except on a $\lambda \boxtimes P^T$-null set.

Since $x_i^p(\cdot) = \bar{x}_i^p(\cdot)$ for $i \in A^*$ with $\lambda(A^*) = 1$, x^p must agree with y^p except on a $\lambda \boxtimes P^T$-null set. Hence, by the Fubini property, there is a $\mathcal{I}$-measurable subset A^{**} of A^* with $\lambda(A^{**}) = 1$ such that for any $i \in A^{**}$, $x_i^p(t) = y^p(t)$ for P^T-almost all $t \in T$.

It remains to show that x^p is incentive compatible. Fix any $i \in A^{**}$ and $s \in S$. Let D^i be a P^T-null set such that $x_i^p(t) = y_i^p(t)$ for any $t \notin D^i$. Since $P^T = \sum_{s' \in S} \pi_{s'} P_{s'}^T$ and $\pi_s > 0$, we also have $P_s^T(D^i) = 0$. Since $P_s^T(L_s) = 1$, we obtain that $P_s^T(D^i \cup (T \setminus L_s)) = 0$. Denote the set $D^i \cup (T \setminus L_s)$ by E^{is}. Let $E_q^{is} = \{t_{-i} \in T_{-i} : (t_{-i}, q) \in E^{is}\}$ for any $q \in T^0$.

Let $P_s^{T_{-i}}$ and τ_{is} be the marginal probability measures of P_s^T, respectively on $(T_{-i}, \mathcal{T}_{-i})$ and $(T^0, \mathcal{T}^0)$. The conditional independence condition on F in Sect. 3 says that $(T, \mathcal{T}, P_s^T)$ is the product of $(T_{-i}, \mathcal{T}_{-i}, P_s^{T_{-i}})$ and $(T^0, \mathcal{T}^0, \tau_{is})$.

For any fixed $t_i, t_i' \in T^0$ with $\tau_i(\{t_i\}) > 0$ and $\tau_i(\{t_i'\}) > 0$, our assumption on non-redundant signals in Sect. 2 implies that $\tau_{is}(\{t_i\}) > 0$ and $\tau_{is}(\{t_i'\}) > 0$. Since

$$P_s^T(E^{is}) = P_s^T(\cup_{q\in T^0} E_q^{is} \times \{q\}) = \sum_{q\in T^0} P_s^{T_{-i}}(E_q^{is}) \cdot \tau_{is}(\{q\}) = 0,$$

we have $P_s^{T_{-i}}(E_{t_i}^{is}) = P_s^{T_{-i}}(E_{t_i'}^{is}) = 0$. Therefore, $P_s^{T_{-i}}(E_{t_i}^{is} \cup E_{t_i'}^{is}) = 0$.

For any fixed $t_{-i} \notin E_{t_i}^{is} \cup E_{t_i'}^{is}$, we have $(t_{-i}, t_i) \notin E^{is}$ and $(t_{-i}, t_i') \notin E^{is}$, which means that $(t_{-i}, t_i) \in L_s \setminus D^i$ and $(t_{-i}, t_i') \in L_s \setminus D^i$. Since $(t_{-i}, t_i), (t_{-i}, t_i') \notin D^i$, the property of the set D^i implies that

$$x_i^p(t_{-i}, t_i) = y_i^p(t_{-i}, t_i), \text{ and } x_i^p(t_{-i}, t_i') = y_i^p(t_{-i}, t_i'). \tag{16}$$

Since $(t_{-i}, t_i), (t_{-i}, t_i') \in L_s$, the definition of y^p implies that

$$y_i^p(t_{-i}, t_i) = y_i^p(t_{-i}, t_i') = \int_{t''\in T} \bar{x}^p(i, t'')dP_s^T(t''), \tag{17}$$

which also equals $\int_{t''\in T} x^p(i, t'')dP_s^T(t'')$ (since i also belongs to A^*). Hence, for any $t_{-i} \notin E_{t_i}^{is} \cup E_{t_i'}^{is}$, Eqs. (16) and (17) imply that $x_i^p(t_{-i}, t_i) = x_i^p(t_{-i}, t_i')$.

The above identity and the fact that $E_{t_i}^{is} \cup E_{t_i'}^{is}$ is a $P_s^{T_{-i}}$-null set imply that

$$\int_{T_{-i}} u_i(x_i^p(t_{-i}, t_i), s)P_s^{T_{-i}} = \int_{T_{-i}} u_i(x_i^p(t_{-i}, t_i'), s)P_s^{T_{-i}}. \tag{18}$$

It is easy to see that for any $q \in T^0$,

$$\int_{S\times T_{-i}} u_i(x_i^p(t_{-i}, q), s)dP^{S\times T_{-i}}(\cdot|t_i) = \sum_{s\in S} \frac{\pi_s \tau_{is}(\{t_i\})}{\tau_i(\{t_i\})} \int_{T_{-i}} u_i(x_i^p(t_{-i}, q), s)P_s^{T_{-i}}. \tag{19}$$

By taking q to be t_i or t_i' in Eq. (19), Eqs. (3) and (18) then imply that $U_i(x_i^p, t_i|t_i) = U_i(x_i^p, t_i'|t_i)$. Therefore, the condition of incentive compatibility is satisfied by an arbitrary ex ante efficient allocation x^p.

7.3 Proof of Proposition 1

Let x^p be an ex ante efficient allocation. We follow the proof of Theorem 1 to show that x^p and y^p differ only on a $\lambda \boxtimes P^T$-null set, where y^p is defined in Eq. (7).

Suppose that x^p is not ex post efficient. Since x^p and y^p differ only on a $\lambda \boxtimes P^T$-null set, the Fubini property implies that for P^T-almost all $t \in T$, x_t^p and y_t^p differ only on a λ-null set. Thus, y^p is ex ante efficient, but not ex post efficient.

Let C be the set of $t \in T$ such that y_t^p is not efficient in the ex post large deterministic economy $\mathcal{E}_t^p = \{(I, \mathcal{I}, \lambda), U(\cdot, \cdot, t), e\}$. Then, C is not a null set under the measure P^T, which also means that there exists $s_0 \in S$ such that C is not a null set under the measure $P_{s_0}^T$.

As shown in the proof of Theorem 1 in Sun and Yannelis (2007), we have for P^T-almost all $t \in L_{s_0}$, $U(i, \cdot, t) = \sum_{s \in S} u(i, \cdot, s) P^S(\{s\}|t) = u(i, \cdot, s_0)$ for all $i \in I$. Hence, there is $t_0 \in C \cap L_{s_0}$ such that $U(\cdot, \cdot, t_0) = u(\cdot, \cdot, s_0)$. Thus, $y_{t_0}^p$ is not efficient in the large deterministic economy $\mathcal{E}_{t_0}^p = \{(I, \mathcal{I}, \lambda), u(\cdot, \cdot, s_0), e\}$, which means that there exists a feasible allocation α for $\mathcal{E}_{t_0}^p$ such that for λ-almost all $i \in I$,

$$u_i(\alpha(i), s_0) > u_i(y_{t_0}^p(i), s_0) = u_i(y_t^p(i), s_0) \tag{20}$$

for all $t \in L_{s_0}$.

Define a feasible allocation β^p such that

$$\beta^p(i, t) = \begin{cases} y^p(i, t) & \text{if } t \notin L_{s_0}, \\ \alpha(i) & \text{if } t \in L_{s_0}. \end{cases} \tag{21}$$

Then, it is easy to see that $U_i^p(\beta_i^p) > U_i^p(y_i^p)$ for λ-almost all $i \in I$, which contradicts the ex ante efficiency of y^p. Therefore, x^p must be ex post efficient.

7.4 Proof of Propositions 2–6

In all the constructions of this subsection, we take S to be a singleton set. Thus, we can identify Ω with T and P with P^T. The constructions will use nonstandard analysis. One can pick up some background knowledge on nonstandard analysis from the first three chapters of the book Loeb and Wolff (2000).

We shall fix some notations first for this subsection. Fix $n \in {}^*\mathbb{N}_\infty$. Let $I = \{1, 2, \ldots, n\}$ with its internal power set $\mathcal{I}_0$ and internal counting probability measure λ_0 on $\mathcal{I}_0$ with $\lambda_0(A) = |A|/|I|$ for any $A \in \mathcal{I}_0$, where $|A|$ is the internal cardinality of A. Let $(I, \mathcal{I}, \lambda)$ be the Loeb space of the internal probability space $(I, \mathcal{I}_0, \lambda_0)$, which will serve as the space of agents for the large private information economies considered in various constructions below.

Let $T^0 = \{0, 1\}$ be the signals for individual agents, and T the set of all the internal functions from I to T^0 (the space of signal profiles). Let $\mathcal{T}_0$ be the internal power set on T, P_0 an internal probability measure on $(T, \mathcal{T}_0)$, and $(T, \mathcal{T}, P)$ the corresponding Loeb space. Except in the proof of Proposition 3 below, P_0 will be taken to be the internal counting probability measure on $\mathcal{T}_0$ in this subsection, i.e., the probability weight for each $t = (t_1, t_2, \ldots, t_n) \in T$ under P_0 is $1/2^n$; and in this case it is obvious that condition **(A4)** is satisfied.

Let $(I \times T, \mathcal{I}_0 \otimes \mathcal{T}_0, \lambda_0 \otimes P_0)$ be the internal product probability space of $(I, \mathcal{I}_0, \lambda_0)$ and $(T, \mathcal{T}_0, P_0)$. Let $(I \times T, \mathcal{I} \boxtimes \mathcal{T}, \lambda \boxtimes P)$ be the Loeb space of the internal product

$(I \times T, \mathcal{I}_0 \otimes \mathcal{T}_0, \lambda_0 \otimes P_0)$, which is indeed a Fubini extension of the usual product probability space by Keisler's Fubini Theorem (see, for example Loeb and Wolff (2000)).

Proof of Proposition 2: We consider a one-good economy with strictly concave and monotonic utility functions $u_i : \mathbb{R}_+ \to \mathbb{R}_+$ and constant endowments $e_i = 1$ for all the agents $i \in I$. Thus, condition (1) of Proposition 2 is satisfied. Note that any feasible allocation $x_i \geq 0, i \in I$ (i.e., $\int_I x_i d\lambda = 1$) is efficient in the relevant deterministic economy $\{(I, \mathcal{I}, \lambda), (u_i, e_i)_{i \in I}\}$.

Next, define an allocation $x^p(i,t) = 2t_i$ in the private information economy. Since Assumption **(A4)** is satisfied, one can apply Lemma 1 to claim that for P-almost all $t \in T$, $\int_I x_t^p(i) d\lambda = \int_I \int_T 2t_i dP d\lambda = 1$, which means that x^p is feasible and ex post efficient. It is easy to see that

$$\begin{aligned} U_i(x_i^p, 1|0) &= \int\limits_{T_{-i}} u_i(x_i^p(t_{-i}, 1)) dP^{T_{-i}} = u_i(2) \\ &> u_i(0) = \int\limits_{T_{-i}} u_i(x_i^p(t_{-i}, 0)) dP^{T_{-i}} = U_i(x_i^p, 0|0). \end{aligned} \tag{22}$$

Hence, x^p is not incentive compatible.

Proof of Proposition 3: We need to work with an internal probability measure P_0 so that Assumption **(A4)** is violated. Define an internal probability measure P_0 on $(T, \mathcal{T}_0)$ such that for any $t = (t_1, t_2, \ldots, t_n) \in T$, its probability weight is defined by[23]

$$P_0(\{t\}) = \begin{cases} \frac{1}{2^{n-1}} & \text{if } \sum_{j=1}^n t_j \text{ is odd,} \\ 0 & \text{if } \sum_{j=1}^n t_j \text{ is even.} \end{cases} \tag{23}$$

Fix any ℓ different agents $i_1, \ldots, i_\ell \in I$. For any $k_1, \ldots, k_\ell \in \{0, 1\}$, if $\sum_{j=1}^\ell k_j$ is odd, then

$$\begin{aligned} & P_0\left(t \in T : t_{i_1} = k_1, \ldots, t_{i_\ell} = k_\ell\right) \\ & = P_0\left(t \in T : \sum_{i \neq i_1, \ldots, i_\ell} t_i \text{ is even, and } t_{i_1} = k_1, \ldots, t_{i_\ell} = k_\ell\right) \\ & = 2^{n-\ell-1} \cdot \frac{1}{2^{n-1}} = \frac{1}{2^\ell}; \end{aligned}$$

similarly, if $\sum_{j=1}^\ell k_j$ is even, one can also obtain that $P_0(t \in T : t_{i_1} = k_1, \ldots, t_{i_\ell} = k_\ell) = \frac{1}{2^\ell}$. Hence, the random private signals $F_{i_1}, \ldots, F_{i_\ell}$ are mutually independent, which means that the private signal process F is mutually independent.

[23] This definition is motivated by a classical example of Bernstein in Feller (1968, p. 126) and its generalization in Wang (1979).

Define a set A in T by letting $A = \{t \in T : \sum_{j=1}^{n} t_j \text{ is odd.}\}$; then $P(A) = 1$. Let $\bar{e}$ be the vector $(1, \ldots, 1)$ in $\mathbb{R}_+^m$. For $i \in I$, let u_i and e_i be the utility function and endowment of agent i that satisfy Assumptions **(A1)–(A3)**. For any allocation x^p in the private information economy, define another allocation y^p in the private information economy such that for any $(i, t) \in I \times T$,

$$y^p(i,t) = \begin{cases} x^p(i,t) & \text{if } t \in A, \\ x^p(i, t_{-i}, 1 - t_i) + \bar{e} & \text{if } t \notin A. \end{cases} \tag{24}$$

It is clear that y^p is essentially equivalent to x^p.

For any fixed agent $i \in I$, let $A_{-i} = \{t_{-i} \in T_{-i} : \sum_{j \neq i} t_j \text{ is odd.}\}$. Then it is easy to see that $P^{T_{-i}}(A_{-i}|t_i = 0) = 1$ and $P^{T_{-i}}(A_{-i}|t_i = 1) = 0$, which means that Assumption **(A4)** is not satisfied. By equation (3) in the definition of incentive compatibility,

$$U_i(y_i^p, t_i'|t_i) = \int_{S \times T_{-i}} u_i(y_i^p(t_{-i}, t_i'))dP^{T_{-i}}(\cdot|t_i),$$

and the monotonicity of the utility function u_i, we can obtain that

$$\begin{aligned} U_i(y_i^p, 1|0) &= \int_{A_{-i}} u_i(y_i^p(t_{-i}, 1))dP^{T_{-i}}(\cdot|t_i = 0) \\ &= \int_{A_{-i}} u_i(x_i^p(t_{-i}, 0) + \bar{e})dP^{T_{-i}}(\cdot|t_i = 0) \\ &> \int_{A_{-i}} u_i(x_i^p(t_{-i}, 0))dP^{T_{-i}}(\cdot|t_i = 0) \\ &= \int_{A_{-i}} u_i(y_i^p(t_{-i}, 0))dP^{T_{-i}}(\cdot|t_i = 0) = U_i(y_i^p, 0|0), \end{aligned} \tag{25}$$

which means that the allocation y^p is not incentive compatible.

Proof of Proposition 4: We consider a one-good economy with utility functions $u_i(x) = x$ and constant endowments $e_i = 1$ for all the agents $i \in I$. (1) and (2) of Proposition 4 are satisfied.

As in the proof of Proposition 2, define a feasible allocation $x^p(i,t) = 2t_i$ in the private information economy. It is then easy to check that for all $i \in I$, $U_i^p(x_i^p) = 1$.

Suppose that there is a feasible allocation y^p such that $U_i^p(y_i^p) > U_i^p(x_i^p) = 1$ for λ-almost all $i \in I$. Then, $\int_I U_i^p(y_i^p)d\lambda > 1$. On the other hand, by the feasibility of y^p, we have $\int_I y^p(i,t)d\lambda = 1$ for P-almost all $t \in T$. The Fubini property implies that $\int_I U_i^p(y_i^p)d\lambda = \int_I \int_T y^p(i,t)dPd\lambda = \int_T \int_I y^p(i,t)d\lambda dP = 1$. This is a contradiction. Therefore, x^p is ex ante efficient. Equation (22) shows that x^p is not incentive compatible.

Proof of Proposition 5: We consider a one-good economy with utility functions $u_i(x, t_i) = (1 + t_i)\sqrt{x}$ and constant endowments $e_i = 1$ for all the agents $i \in I$. (1) and (2) of Proposition 5 are satisfied.

Let x^p be an ex ante efficient allocation. Then, Jensen's inequality implies that

$$\begin{aligned} U_i^p(x_i^p) &= \int\limits_T (1 + t_i)\sqrt{x_i^p(t_{-i}, t_i)}dP \\ &= \frac{1}{2}\int\limits_{T_{-i}} \sqrt{x_i^p(t_{-i}, 0)}dP^{T_{-i}} + \int\limits_{T_{-i}} \sqrt{x_i^p(t_{-i}, 1)}dP^{T_{-i}} \\ &\leq \frac{1}{2}\sqrt{\int\limits_{T_{-i}} x_i^p(t_{-i}, 0)dP^{T_{-i}}} + \sqrt{\int\limits_{T_{-i}} x_i^p(t_{-i}, 1)dP^{T_{-i}}}, \end{aligned} \tag{26}$$

with equality only when for $P^{T_{-i}}$-almost all $t_{-i} \in T_{-i}$, $x_i^p(t_{-i}, 0) = \int_{T_{-i}} x_i^p(t_{-i}, 0) dP^{T_{-i}}$ and $x_i^p(t_{-i}, 1) = \int_{T_{-i}} x_i^p(t_{-i}, 1)dP^{T_{-i}}$.

Fix a constant $c \geq 0$. The function $\frac{1}{2}\sqrt{x_1} + \sqrt{x_2}$, subject to the constraints $x_1 + x_2 = c$, $x_1, x_2 \geq 0$, achieves the maximum value $\sqrt{5\,c}/2$ only at $x_1 = c/5$ and $x_2 = 4c/5$.

Let $a_i = \int_T x_i^p(t)dP$. Then, $\int_{T_{-i}} x_i^p(t_{-i}, 0)dP^{T_{-i}} + \int_{T_{-i}} x_i^p(t_{-i}, 1)dP^{T_{-i}} = 2a_i$. Therefore,

$$\frac{1}{2}\sqrt{\int\limits_{T_{-i}} x_i^p(t_{-i}, 0)dP^{T_{-i}}} + \sqrt{\int\limits_{T_{-i}} x_i^p(t_{-i}, 1)dP^{T_{-i}}} \leq \sqrt{\frac{5\,a_i}{2}} \tag{27}$$

with equality only when $\int_{T_{-i}} x_i^p(t_{-i}, 0)dP^{T_{-i}} = 2a_i/5$ and $\int_{T_{-i}} x_i^p(t_{-i}, 1)dP^{T_{-i}} = 8a_i/5$.

Define an allocation y^p by letting $y^p(i, t) = \frac{2}{5}(1 + 3t_i)a_i$. By the exact law of large numbers, we have

$$\int\limits_I y^p(i, t)d\lambda = \int\limits_I \int\limits_T \frac{2}{5}(1 + 3t_i)a_i dP d\lambda = \int\limits_I \frac{2}{5}(1 + 3/2)a_i d\lambda = \int\limits_I a_i d\lambda = 1$$

for P-almost all $t \in T$. That is, y^p is a feasible allocation. It is also easy to check that $U_i^p(y_i^p) = \sqrt{\frac{5\,a_i}{2}}$. Hence, $U_i^p(y_i^p) \geq U_i^p(x_i^p)$ with equality only when for $P^{T_{-i}}$-almost all $t_{-i} \in T_{-i}$, $x_i^p(t_{-i}, 0) = 2a_i/5$ and $x_i^p(t_{-i}, 1) = 8a_i/5$.

By the ex ante efficiency of x^p, there exists a set A in I with $\lambda(A) = 1$ such that for any $i \in A$, $x_i^p(t_{-i}, 0) = 2a_i/5$ and $x_i^p(t_{-i}, 1) = 8a_i/5$ hold for $P^{T_{-i}}$-almost all $t_{-i} \in T_{-i}$. Let $B = \{i \in A : a_i > 0\}$. Since $\int_I a_i d\lambda = 1$, we have $\lambda(B) > 0$. By the definition of incentive compatibility, we have, for any agent $i \in B$,

$$U_i(x_i^p, 1|0) = \int_{T_{-i}} u_i(x_i^p(t_{-i}, 1), 0) dP^{T_{-i}} = \sqrt{\frac{8a_i}{5}}$$
$$> \sqrt{\frac{2a_i}{5}} = \int_{T_{-i}} u_i(x_i^p(t_{-i}, 0), 0) dP^{T_{-i}} = U_i(x_i^p, 0|0). \qquad (28)$$

Hence, x^p is not incentive compatible.

Proof of Proposition 6: We consider a one-good economy with strictly concave and monotonic utility functions $u_i : \mathbb{R}_+ \to \mathbb{R}_+$, and endowments $e_i(t_i) = 2t_i$ for all $i \in I$. Then, as in the proof of Proposition 2, the exact law of large numbers in Lemma 1 implies that $\int_I e_i(t_i) d\lambda = 1$ for P-almost all $t \in T$.

Let x^p be an ex ante efficient allocation. As in the proof of Theorem 1, Jensen's inequality implies that for λ-almost all $i \in I$, $x_i^p(t) = \int_T x_i^p(t) dP(t)$ for P-almost all $t \in T$.

By the definition of incentive compatibility in Eq. (6), we have, for λ-almost all agents $i \in B$,

$$U_i(x_i^p, 0|1) = \int_{T_{-i}} u_i(x_i^p(t_{-i}, 0) - e(0) + e(1)) dP^{T_{-i}}$$
$$= \int_{T_{-i}} u_i(x_i^p(t_{-i}, 1) + 2) dP^{T_{-i}}$$
$$> \int_{T_{-i}} u_i(x_i^p(t_{-i}, 1)) dP^{T_{-i}} = U_i(x_i^p, 1|1). \qquad (29)$$

Hence, x^p is not incentive compatible.

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