Equilibrium theory with satiable and non-ordered preferences☆

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A R T I C L E   I N F O

A B S T R A C T

This paper investigates the existence of equilibrium in an economy where preferences may be non-ordered and possibly satiable. Remarkably, satiation is allowed to occur only inside the set of feasible and individually rational allocations. One important class of its applications is new developments of asset pricing models where Knightian uncertainty makes preferences incomplete while the absence of a riskless asset makes them satiable. Thus, the result of the paper extends Won et al. (2008) to the case that preferences need be neither complete nor transitive.

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1. Introduction

The importance of establishing the existence of equilibrium with satiation is already alluded to by Mas-Colell (1992). Since Walrasian equilibrium may not exist without nonsatiation, Mas-Colell (1992) introduces a weaker equilibrium notion than the Walrasian equilibrium. This poses the following question. Is it possible to drop the assumption of nonsatiation of preferences and still obtain the existence of Walrasian equilibrium? We provide an affirmative answer to this question. In particular, we introduce a new assumption on preferences which subsumes the existing conditions of satiable preferences as a special case and is automatically fulfilled whenever agents have nonsatiable preferences. As a consequence of this new assumption, we are able to generalize all classical equilibrium results to allow for possibly satiable preferences.

This paper shows the existence of equilibrium in an economy where preferences may be non-ordered and possibly satiable. This result enriches the literature in two important respects. First, it extends to the case of satiable preferences both the classical existence results with non-ordered preferences (e.g., Mas-Colell, 1974; Shafer, 1976; Gale and Mas-Colell, 1975 among others) and the arbitrage-based equilibrium theorems (e.g., Hart, 1974; Hammond, 1983; Page, 1987; Chichilnisky, 1995; Page et al., 2000; Dana et al., 1999; Allouch, 2002, among others). Second, the result of the paper substantially extends the literature with satiable preferences such as Nielsen (1990), Allingham (1991), and Won et al. (2008) to the case of non-ordered preferences. The latter literature examines the equilibrium existence problem with asset pricing models where preferences are represented by a differentiable expected utility function and the absence of riskless assets makes them satiable. As demonstrated in Rigotti and Shannon (2005), preferences with Knightian uncertainty need be neither complete nor differentiable.[1] One important coverage of this paper is asset

[1] Uncertainty is called Knightian if the probabilities of risky events are not known. For the impact of Knightian uncertainty or ambiguity aversion on asset pricing and risk-sharing, we refer the readers to Rigotti and Shannon (2005) and the literature therein.

pricing models with Knightian uncertainty formulated in Bewley (2002) and Rigotti and Shannon (2005), and model uncertainty discussed in Hansen and Sargent (2001), Kogan and Wang (2003), and Cao et al. (2005). As illustrated later in Example 3.1, Knightian uncertainty makes preferences incomplete while the absence of riskless assets makes them satiable. In this example, incompleteness of preferences lead to a continuum of satiation portfolios even when the vNM utility functions are strictly concave. Moreover, satiation portfolios are not comparable to each other.

As shown in the literature, satiation poses no problem to the existence of equilibrium if it does not arise in the set of feasible and individually rational allocations. In this paper, satiation is allowed to occur anywhere in the consumption set. As discussed in Won et al. (2008), the CAPM without riskless assets may have satiation inside the set of feasible and individually rational allocations and thus, the standard existence theorems do no apply to the CAPM without riskless assets. Nielsen (1990) and Allingham (1991) initiated the research into the equilibrium existence theorem for the classical framework of the CAPM. However, their results do not apply to the recent developments of the CAPM which take into account the effects on risk sharing of heterogeneous beliefs or Knightian uncertainty. Won et al. (2008) extend the existence theorems of Nielsen (1990) and Allingham (1991) to the case of heterogeneous expectations about the means and the covariances of asset returns but fail to cover the CAPM with Knightian uncertainty because of incompleteness of preferences. The occurrence of satiation is rather a rule in the case where choice sets are compact. Indeed if consumption sets are compact and preferences are lower semicontinuous, convex, and irreflexive, then there always exists a maximal element (e.g., Yannelis and Prabhakar, 1983), and thus, agents are satiated. This is the case examined in Mas-Colell (1992). However, Mas-Colell (1992) introduces a weak notion of equilibrium called 'equilibrium with slack' and he does not provide conditions under which Walrasian equilibrium exists in the presence of satiation consumptions.

Werner (1987) examines the case with possibly satiated preferences when the set of useful and useless commodity bundles satisfies a uniformity condition. In particular, the uniformity condition requires that the nonempty set of satiation points be unbounded. Allouch et al. (2006) extend the results of Werner (1987) by relaxing the uniformity condition in a certain way. Allouch and Le Van (2009) show that satiation is hamrless to the existence of equilibrium in the economy with the compact utility set, provided that the set of satiation points is bounded and stretches outside the set of feasible and individually rational allocations. Martins-da-Rocha and Monteiro (2009) illustrate that the compactness of the utility set for all agents may not be sufficient for the existence of equilibrium when the set of satiation points is unbounded, and provide an additional condition under which the existence of equilibrium is reinstated. Sato (2000) goes further by taking into account the case where preferences allow satiation to occur on some boundary of the set of feasible and individually rational consumptions. The results of the general equilibrium literature, however, do not apply to the case that satiation occurs only inside the set of feasible and individually rational consumptions. Moreover, they do not cover the case with Knightian uncertainty which leads to incompleteness of preferences.

2. Economies

We follow the notation, definitions, discussions of assumptions, and the preliminary results introduced in Section 2 of Won and Yannelis (2008). Thus, this paper should be read in conjunction with Won and Yannelis (2008). The basic difference between the current paper and Won and Yannelis (2008) lies in the possibility that preferences for agent $i \in I$ can reach satiation in the set H of feasible and individually rational allocations. Specifically, Assumptions B1–B4 and B6 in Won and Yannelis (2008) will be kept here, and Assumption B5 will be dropped and replaced by new conditions which are required to characterize satiation. For the convenience of the reader, we repeat Assumptions B1–B4 and B6 of Won and Yannelis (2008) which are made for all $i \in I$ in the economy $E = \{(X_i, e_i, P_i) : i \in I\}$.

B1. X_i is a closed, nonempty and convex set in $\mathbb{R}^\ell$.
B2. e_i is in the interior of X_i.
B3. P_i is lower semi-continuous.
B4. For all $x \in X$, $x_i \notin co\, P_i(x)$.
B6. Let x be a point in H. Then for each $z_i \in co\, P_i(x)$ and $v_i \in X_i$, there exists $\lambda \in (0, 1)$ such that $\lambda z_i + (1 - \lambda)v_i \in co\, P_i(x)$.
B7. H is bounded.

The condition B4 implies that $R(e) \neq \varnothing$ and thus, $H \neq \varnothing$.[2] The Assumption B7 states that feasible and individually rational allocations form a bounded set.[3] Notice that we deliberately skip the identifier B5 for later use to keep the notational match between the current paper and Won and Yannelis (2008).

Preferences for agent i are *satiated* at $x \in X$ if $P_i(x) = \varnothing$. For each $i \in I$, let S_i denote the set $\{x \in X : P_i(x) = \varnothing\}$ and $\hat{S}_i$ the projection of S_i onto X_i. The set S_i contains the set of choices which lead to satiation of agent i. The following result is immediate from the lower semi-continuity of P_i.

Lemma 2.1. *The set S_i is closed in $\mathbb{R}^\ell$ for each $i \in I$.*

Proof. It is clear that

$$X \setminus S_i \;=\; \{x \in X : P_i(x) \neq \varnothing\}$$
$$= \{x \in X : P_i(x) \cap X_i \neq \varnothing\}.$$

Since P_i is lower semi-continuous, $X \setminus S_i$ is open and therefore, S_i is closed. □

The set H is bounded in many interesting cases. For example, if P_i is represented by a concave utility function u_i on X_i, and $\hat{S}_i$ is nonempty and bounded for each $i \in I$, then H is bounded.

Lemma 2.2. *Suppose that for each $i \in I$, u_i is concave on X_i and $\hat{S}_i$ is nonempty and bounded. Then H is bounded.*

Proof. The preferred set $R^i(x_i) = \{z_i \in X_i : u_i(z_i) \geq u_i(x_i)\}$ is convex for each $x_i \in X_i$. Let C_i denote the recession cone of $R^i(e_i)$. By applying Theorem 8.7 of Rockafellar (1970) to concave functions, the recession of $R^i(x_i)$ for each $x_i \in X_i$ is equal to C_i. We claim that each $R^i(e_i)$ is bounded. Suppose otherwise. Then by Theorem 8.4 of Rockafellar (1970), there is a nonzero vector $v_i \in C_i$ for some $i \in I$. Let s_i denote a satiation point in $\hat{S}_i$. Then for all $\lambda > 0$, $u_i(s_i + \lambda v_i) \geq u_i(s_i)$. This implies that $s_i + \lambda v_i \in \hat{S}_i$ for all $\lambda > 0$ and therefore, $\hat{S}_i$ is unbounded, which contradicts the boundedness of $\hat{S}_i$. Thus, the set $R(e) = \prod_{i \in I} R^i(e_i)$ is bounded and therefore, H is bounded. □

A notable example where H is bounded is the classical CAPM of Nielsen (1990) and Allingham (1991) where the absence of riskless assets makes preferences possess a single satiation portfolio. To be discussed later in Example 3.1, another interesting case is the CAPM where Knightian uncertainty makes preferences incomplete. In this example, incompleteness of preferences yield a continuum

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[2] Suppose that $R(e) = \varnothing$. Then for each $x \in X$, there exists $i \in I$ such that $e_i \in co\, P_i(x)$. In particular, we have $e_i \in co\, P_i(e)$ for some $i \in I$, which contradicts Assumption B4.
[3] The case that H is unbounded is referred to Won and Yannelis (2009).

of satiation portfolios even if the vNM utility function is strictly concave.

For each $x \in X$, we define the set $I^s(x) = \{i \in I : P_i(x) = \varnothing\}$ and $I(x) = I \setminus I^s(x)$. If $S_i \cap H = \varnothing$ for all $i \in I$, this is done in Won and Yannelis (2008) (Assumption B5). Thus, without loss of generality, we may suppose that that $S_i \cap H \neq \varnothing$ for some $i \in I$. We also add the following assumption.

B8. For each $x \in H$, $I^s(x) \neq I$.

Assumption B8 excludes the relatively uninteresting case that $I^s(y) = I$ for some $y \in H$. To make it clear, we consider an economy where agent i's preferences are represented by a utility function u_i on X_i which has a unique satiation point y_i in X_i^H, i.e., $\hat{S}_i = \{y_i\}$ for all $i \in I$, where X_i^H denotes the projection of $cl\, H$ onto X_i. In this case, either y is the unique equilibrium allocation or no equilibrium exists. To show this, let Z denote the $\#I \times \ell$ matrix whose j th row equals $y_j - e_j$ for all $j \in I$, where $\#I$ indicates the number of agents in I. There will be two cases: the rank of Z is either (a) less than ℓ or (b) equal to ℓ. In case (a), there exists a nonzero $p \in \mathbb{R}^\ell$ such that $p \cdot (y_i - e_i) = 0$ for all $i \in I$. Since every agent is satiated at the allocation $y \in H$, (p, y) is an equilibrium. We claim that y is the unique equilibrium allocation. Suppose that there exists an equilibrium (q, x) with $x \neq y$. Then for some $i \in I$ with $x_i \neq y_i$, we have $q \cdot y_i > q \cdot x_i = q \cdot e_i$ and $q \cdot y_j \geq q \cdot x_j = q \cdot e_j$ for all $j \neq i$. This implies $\sum_{j \in I} q \cdot y_j > \sum_{j \in I} q \cdot e_j$, which contradicts the feasibility of y. In case (b), y cannot be an equilibrium allocation because the fact that $p \cdot (y_i - e_i) = 0$ for all $i \in I$ implies $p = 0$. By the previous arguments, we can also show that there exists no equilibrium (q, x) with $x \neq y$.

3. Possibly satiated preferences

Equilibrium may fail to exist in economies where satiation occurs to X_i^H for some $i \in I$. Thus, we impose the following condition on satiable preferences.

S5. For all $x \in H$, $x_i - e_i \in cl[\sum_{j \in I(x)} con\,(P_j(x) - \{x_j\})]$ for each $i \in I^s(x)$.

This condition is related to the conditions of Nielsen (1990), Allingham (1991), and Won et al. (2008).[4]

To motivate the current research, we provide an example based on the recent developments of the asset pricing literature. Specifically, we consider the CAPM without riskless assets where Knightian uncertainty makes preferences incomplete. As in Rigotti and Shannon (2005), we take the approach of Bewley (2002) to incorporate Knightian uncertainty into preferences. In this example, incompleteness of preferences allows agents to possess a continuum of satiation portfolios even if the vNM utility function is strictly concave. Moreover, the size of the satiation set gets bigger as agents have more Knightian uncertainty. No existing literature covers the equilibrium existence issue of the following CAPM because preferences are incomplete and satiable. It is also shown that the Assumption S5 is fulfilled in the CAPM.

Example 3.1. To examine the effect of Knightian uncertainty on the capital asset pricing model without riskless assets, we consider an economy with two agents and two risky assets. Agents are allowed to take unlimited short sales, i.e., $X_i = \mathbb{R}^\ell$ for each $i = 1, 2$. For each $j = 1, 2$, let $\tilde{r}^j$ denote the return of asset j and $\tilde{r}$ the random vector $(\tilde{r}^1, \tilde{r}^2)$.

We suppose that agents know that $\tilde{r}$ is jointly normally distributed but do not know the exact form of the normal distribution. They have ambiguity about the true distribution in that they believe it will be within an 'admissible distance' from the reference normal distribution π_i characterized by the mean vector m_i and the covariance matrix Ω_i. Let ϕ_i be an admissible normal distribution. Then there is a Radon–Nikodym derivative ξ_i such that $d\phi_i = \xi_i d\pi_i$. To describe Knightian uncertainty, for some number $\eta_i > 0$, we define the admissible set of normal distributions

$$\Pi_i = \{\phi_i : E^{\pi_i}(\xi_i \ln \xi_i) \leq \eta_i\}$$

where E^{π_i} indicates the expectation operator with respect to π_i. The number $E^{\pi_i}(\xi_i \ln \xi_i)$ is the relative entropy index which measures the closeness between ϕ_i and π_i.[5] The set Π_i contains normal distributions which are in less distance from π_i than η_i in terms of the relative entropy. We assume that agent i's preferences are represented by the utility function

$$v_i(x_i) = \min_{\phi_i \in \Pi_i} E^{\phi_i}[v_i(x_i \cdot \tilde{r})]$$

where v_i is a strictly increasing, strictly concave utility function of wealth.

Since a normal distribution is determined by the mean vector and the covariance matrix, here Knightian uncertainty can be characterized by parametric uncertainty about them. For tractability, following Kogan and Wang (2003) and Cao et al. (2005), we assume that for each $i = 1, 2$, agent i knows about the covariance matrix of π_i but feels uncertain about the mean vector. Let Ω denote the common covariance matrix agents have in mind. Thus, distributions in Π_i have the covariance matrix Ω but may have distinct mean vectors. Kogan and Wang (2003) show that the utility function of agent i is rewritten as

$$v_i(x_i) = \min_{m_i' \in \Lambda_i} E^{\pi_i}[\xi_i v_i(x_i \cdot \tilde{r})]$$

$$\Lambda_i = \{m_i' \in \mathbb{R}^\ell : \tfrac{1}{2}(m_i' - m_i)^\top \Omega^{-1}(m_i' - m_i) \leq \eta_i\} \text{ and } \xi_i$$

$$\exp\left\{ \frac{1}{2}(m_i' - m_i)^\top \Omega^{-1}(m_i' - m_i) - (m_i' - m_i)^\top \Omega^{-1}(m_i' - \tilde{r}) \right\}.$$

For a normal distribution $\phi_i \in \Pi_i$, let g_i denote the mean–variance utility function derived from $E^{\phi_i}[v_i(\cdot)]$ such that for each $x_i \in X_i$,

$$g_i(\mu_i(x_i; \phi_i), \sigma_i(x_i; \phi_i)) = E^{\phi_i}[v_i(x_i \cdot \tilde{r})]$$

where $\mu_i(x_i; \phi_i)$ and $\sigma_i(x_i; \phi_i)$ is the mean and the variance of $x_i \cdot \tilde{r}$ under the distribution ϕ_i. Since v_i is strictly increasing and strictly concave, g_i is increasing in the first argument and strictly decreasing in the second argument.

Following Bewley (2002), the strict preference ordering $\succ_i$ is defined on $\mathbb{R}^\ell$ such that for any x_i, y_i in $\mathbb{R}^\ell$, $x_i \succ_i y_i$ if and only if

$$E^{\phi_i}(v_i(x_i \cdot \tilde{r})) > E^{\phi_i}(v_i(y_i \cdot \tilde{r})) \quad \text{for all} \quad \phi_i \in \Pi_i.$$

The preference ordering $\succ_i$ is incomplete and transitive. For each $x_i \in \mathbb{R}^\ell$, let $P^i(x_i)$ denote the set $\{y_i \in \mathbb{R}^\ell : y_i \succ_i x_i\}$. Since v_i is concave, $P^i(x_i)$ is convex.

To illustrate that S5 is fulfilled, we consider the more specific case that $m_1 = m_2 = (1.1, 1.1)$, Ω equals the 2×2 identity matrix, and $\eta_i = \varepsilon/2$ for some $\varepsilon > 0$ and each $i = 1, 2$. For simplicity, we assume that agent 1 has ambiguity only about the mean return of the

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[4] For more details on how they are related to S5, see Won and Yannelis (2009).

[5] The relative entropy index indicates how uncertain agent i feels about the true distribution around the reference distribution π_i. For details, see Hansen and Sargent (2001). An interesting example of model uncertainty in static asset pricing models is found in Kogan and Wang (2003) and Cao et al. (2005).

first asset while agent 2 has ambiguity only about the mean of the second asset. This assumption implies that for each $\phi_i \in \Pi_i$, there exist $m_i' \in \mathbb{R}^2$ for each $i = 1, 2$ such that $m_1' = (1.1 + \mu_1, 1.1)$ and $m_2' = (1.1, 1.1 + \mu_2)$ for some μ_1 and μ_2 in $\mathbb{R}$. Then we see that for each $i = 1, 2$,

$$\Lambda_1 = \{(\mu_1, 0) \in \mathbb{R}^2 : -\varepsilon \leq \mu_1 \leq \varepsilon\} \quad \text{and}$$

$$\Lambda_2 = \{(0, \mu_2) \in \mathbb{R}^2 : -\varepsilon \leq \mu_2 \leq \varepsilon\}.$$

We assume that the mean–variance utility function is linear as follows.[6]

$$g_i(\mu, \sigma) = \mu - \frac{1}{2}\sigma^2.$$

For each $w_1 = (x_1, y_1)$ and $w_2 = (x_2, y_2)$ in $\mathbb{R}^2$, we set $u_i(w_i; \phi_i) = g_i(\mu_i(w_i; \phi_i), \sigma_i(w_i; \phi_i))$. Then it follows that

$$
\begin{aligned}
u_1(w_1; \phi_1) &= w_1 \cdot m_1' - \frac{1}{2}w_1^\top \Omega w_1 \\
&= (1.1 + \mu_1)x_1 + 1.1 y_1 - \frac{1}{2}(x_1^2 + y_1^2) \\
u_2(w_2; \phi_2) &= w_2 \cdot m_2' - \frac{1}{2}w_2^\top \Omega w_2 \\
&= 1.1 x_2 + (1.1 + \mu_2)y_2 - \frac{1}{2}(x_2^2 + y_2^2)
\end{aligned}
$$

Then the previous relation is rewritten as

$$
\begin{aligned}
u_1(w_1; \phi_1) &= -\frac{1}{2}\|(x_1, y_1) - (1.1 + \mu_1, 1.1)\|^2 \\
&\quad + \frac{1}{2}[(1.1 + \mu_1)^2 + 1.1^2] \\
u_2(w_2; \phi_2) &= -\frac{1}{2}\|(x_2, y_2) - (1.1, 1.1 + \mu_2)\|^2 \\
&\quad + \frac{1}{2}[(1.1 + \mu_2)^2 + 1.1^2].
\end{aligned}
$$

Thus, the preferred sets $P^1(w_1)$ and $P^2(w_2)$ are given by

$$
\begin{aligned}
P^1(w_1) &= \{\tilde{w}_1 \in \mathbb{R}^2 : \|(\tilde{x}_1, \tilde{y}_1) - (1.1 + \mu_1, 1.1)\|^2 \\
&\quad < \|(x_1, y_1) - (1.1 + \mu_1, 1.1)\|^2 \text{ for all } \mu_1 \in [-\varepsilon, \varepsilon]\} \\
P^2(w_2) &= \{\tilde{w}_2 \in \mathbb{R}^2 : \|(\tilde{x}_2, \tilde{y}_2) - (1.1, 1.1 + \mu_2)\|^2 \\
&\quad < \|(x_2, y_2) - (1.1, 1.1 + \mu_2)\|^2 \text{ for all } \mu_2 \in [-\varepsilon, \varepsilon]\}.
\end{aligned}
$$

It is easy to see that for each $i = 1, 2$, the set of satiation portfolios is compact as shown below.

$$
\begin{aligned}
\hat{S}_1 &= \{(x_1, y_1) \in \mathbb{R}^2 : x_1 \in [1.1 - \varepsilon, 1.1 + \varepsilon], y_1 = 1.1\} \\
\hat{S}_2 &= \{(x_2, y_2) \in \mathbb{R}^2 : x_2 = 1.1, y_2 \in [1.1 - \varepsilon, 1.1 + \varepsilon]\}.
\end{aligned}
$$

In particular, the size of $\hat{S}_i$ relies on that of Λ_i and thus, gets bigger as agent i has more Knightian uncertainty. Moreover, satiation portfolios are not comparable to each other. To show this, let $\delta^n = (0, 1/n)$ for each n. Clearly, the satiation point $\bar{w}_1^0 \equiv (1.1, 1.1)$ for agent 1 belongs to $P^1(\bar{w}_1^0 + \delta^n)$ for each n. It is easy to see that $\cap_{n=1}^\infty P^1(\bar{w}_1^0 + \delta^n) = \{\bar{w}_1^0\}$. Let $\bar{w}_1$ be a point in $\hat{S}_1$ with $\bar{w}_1 \neq \bar{w}_1^0$.
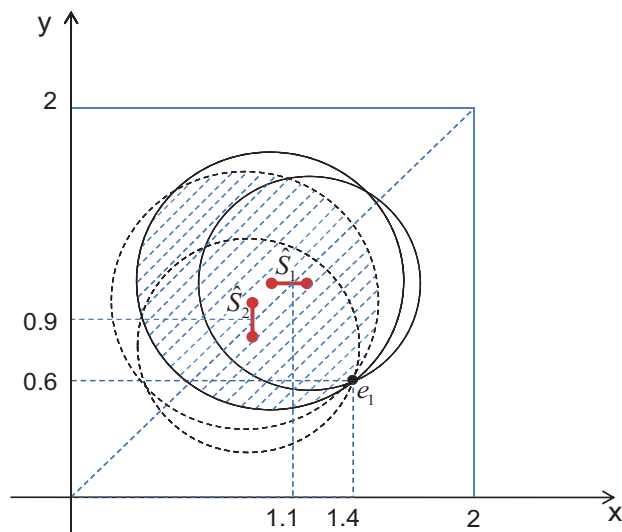

Fig. 1. Preferred sets and satiation

Then there exists n such that $\bar{w}_1 \notin P^1(\bar{w}_1^0 + \delta^n)$.[7] Similarly, we have $\bar{w}_1^0 \notin P^1(\bar{w}_1 + \delta^n)$ for some n. Thus, $\bar{w}_1$ is incomparable to $\bar{w}_1^0$.

Now we check the validity of the Assumption S5 here. To do this, we suppose that $e_1 = (1.4, 0.6)$ and $e_2 = (0.6, 1.4)$ and $\varepsilon = 0.05$. Take a feasible allocation (w_1, w_2) such that w_1 is a point in $\hat{S}_1$, i.e., $w_1 = (1.1 + \mu_1, 1.1)$ for some $\mu_1 \in [-0.05, 0.05]$, and $w_2 = (2, 2) - w_1 = (0.9 - \mu_1, 0.9) \notin \hat{S}_2$. Then we get $w_1 - e_1 = (-0.3 + \mu_1, 0.5)$. Recalling that $|\mu_1| \leq 0.05$ and $|\mu_2| \leq 0.05$, for any $\alpha \in (0, 0.01]$, it is easy to see that

$$
\begin{aligned}
&\|w_2 + \alpha(-0.3 + \mu_1, 0.5) - (1.1, 1.1 + \mu_2)\|^2 \\
&= (0.2 + \mu_1 + 0.3\alpha - \alpha\mu_1)^2 + (0.2 + \mu_2 - 0.5\alpha)^2 \\
&< (0.2 + \mu_1)^2 + (0.2 + \mu_2)^2 \\
&= [(0.9 - \mu_1) - 1.1]^2 + [0.9 - (1.1 + \mu_2)]^2 \\
&= \|w_2 - (1.1, 1.1 + \mu_2)\|^2
\end{aligned}
$$

This implies that $w_2 + \alpha(-0.3 + \mu_1, 0.5)$ is in $P^2(w_2)$ and thus, $w_2 + \alpha(w_1 - e_1) \in P^2(w_2)$. In other words, we have $w_1 - e_1 \in con(P^2(w_2) - \{w_2\})$. Therefore, the condition of S5 holds true for agent 1. Similarly, we can check the condition of S5 for agent 2. Thus, the current example satisfies S5.

Moreover, we see that $\hat{S}_i \subset X_i^H$ for each $i = 1, 2$. In fact, as shown in Fig. 1, $P^1(e_1)$ is the open set in $\mathbb{R}^2$ surrounded by two circles in solid line through e_1 where the small one is centered at the rightmost point $(1.15, 1.1)$ of $\hat{S}_1$ and the larger one is centered at its leftmost point $(1.05, 1.1)$. The set $P^2(e_2)$ is the open set in $\mathbb{R}^2$ surrounded by two circles in dotted line through e_1 where the small one is centered at the lowest point $(1.1, 1.05)$ of $\hat{S}_2$ and the larger one is centered at its highest point $(1.1, 1.15)$. On the other hand, $R^1(e_1)$ is the closed ball with the larger solid-line circle as its boundary while $R^2(e_2)$ is the larger dotted-line circle as its boundary. Thus, X_1^H and X_2^H are represented by the hatched region, the intersection of the two closed balls. Since the hatched region includes both $\hat{S}_1$ and $\hat{S}_2$, satiation occurs only inside X_i^H for each $i = 1, 2$.

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[6] This condition holds when the vNM utility function displays constant absolute risk aversion.

[7] This is because for all n, $P^1(\bar{w}_1^0 + \delta^n)$ is in the ball centered at $\bar{w}_1^0$ with radius $1/n$.

4. The existence of equilibrium

We provide the existence of equilibrium for economies where H is bounded. The proof of the theorem will freely borrow notation and preliminary results from Won and Yannelis (2008).

Theorem 4.1. *Suppose that E satisfies the Assumptions B1–B4, S5 and B6–B8. Then there exists an equilibrium of the economy E.*

Proof. The proof of the theorem is broken into several steps. The first three steps of the proof will be exactly the same as the first three steps of the proof of Theorem 2.3.1 of Won and Yannelis (2008). Thus the readers are referred to the proof of Theorem 2.3.1 of Won and Yannelis (2008) for the overlapped portion of the proof and mathematical notation used therein. As in Won and Yannelis (2008), without loss of generality, B4 and B6 are replaced by the following conditions B4$'$ and B6$'$.[8]

B4$'$. For all $x \in X$ with $P_i(x) \neq \varnothing$, $P_i(x)$ is convex, $x_i \notin P_i(x)$, and for each $y_i \in P_i(x)$, $(x_i, y_i]$ is in $P_i(x)$.

B6$'$. Let x be a point in H. Then for each $z_i \in P_i(x)$ and $v_i \in X_i$, there exists $\lambda \in (0, 1)$ such that $\lambda z_i + (1 - \lambda)v_i \in P_i(x)$.

First, we summarize the preparatory part and Step 1 of the proof of Theorem 1 of Won and Yannelis (2008) to remind the reader of notation and definitions.

Step 1: By B7, we can choose a closed and bounded ball K centered at the origin in $\mathbb{R}^\ell$ which contains X_i^H and e_i in its interior for all $i \in I$. We introduce the truncated economy $\hat{E} = (\hat{X}_i, e_i, \hat{P}_i)$ where for all $i \in I$,

$$\hat{X}_i = X_i \cap K, \quad \hat{X} = \prod_{i \in I} \hat{X}_i \text{ and } \hat{P}_i(x) = P_i(x) \cap K \text{ for all } x \in \hat{X}.$$

We introduce the sets Δ and Δ_1 in $\mathbb{R}^\ell$ defined by

$$\Delta \quad = \{p \in \mathbb{R}^\ell : \|p\| \leq 1\}$$
$$\Delta_1 \quad = \{p \in \mathbb{R}^\ell : \|p\| = 1\}.$$

We consider the abstract economy $\Gamma = (\hat{X}_i, A_i, G_i)_{i \in I'}$ where $I' = I \cup \{0\}$ by adding the agent 0 as follows; if $i = 0$, we set $\hat{X}_0 = \Delta$ and define

$$G_0(p, x) \quad = \{q \in \Delta : q \cdot \left(\sum_{i \in I}(x_i - e_i)\right) > p \cdot \left(\sum_{i \in I}(x_i - e_i)\right)\},$$
$$A_0(p, x) \quad = \Delta \text{ for all } (p, x) \in \Delta \times \hat{X},$$

and if $i \in I$, for all $(p, x) \in \Delta \times \hat{X}$ we set

$$G_i(p, x) \quad = \hat{P}_i(x), \text{ and}$$
$$A_i(p, x) \quad = \{x_i \in X_i : p \cdot x_i < p \cdot e_i + 1 - \|p\|\} \cap K.$$

Step 1 of Theorem 2.3.1 of Won and Yannelis (2008) shows that Γ has a quasi-equilibrium, i.e., there exists $(\hat{p}, \hat{x}) \in \Delta \times \hat{X}$ such that

(a) $\hat{p} \in cl\, A_0(\hat{p}, \hat{x}) = \Delta$ and $G_0(\hat{p}, \hat{x}) \cap \Delta = \varnothing$

and for all $i \in I$,

(b) $\hat{x}_i \in cl\, A_i(\hat{p}, \hat{x})$, i.e., $\hat{p} \cdot \hat{x}_i \leq \hat{p} \cdot e_i + 1 - \|\hat{p}\|$, and
(c) $G_i(\hat{p}, \hat{x}) \cap A_i(\hat{p}, \hat{x}) = \varnothing$, i.e., $\hat{P}_i(\hat{x}) \cap A_i(\hat{p}, \hat{x}) = \varnothing$.

Step 2: This step is exactly the same as Step 2 of Theorem 2.3.1 of Won and Yannelis (2008). In particular, we obtain $\hat{x} \in F$.

Step 3: This step is also exactly the same as Step 3 of Theorem 2.3.1 of Won and Yannelis (2008). In particular, we obtain $\hat{x} \in R(e)$.

Step 4: By Steps 2 and 3, we have $\hat{x} \in H = F \cap R(e)$ and therefore, $\hat{x}_i \in X_i^H$ for each $i \in I$. Recalling that $X_i^H \subset int\, K$, we have $\hat{x}_i \in int\, K$ for all $i \in I$.

To analyze the impact of satiation on equilibrium of the truncated economy $\hat{E}$, we set $\hat{I}^s(\hat{x}) = \{i \in I : \hat{P}_i(\hat{x}) = \varnothing\}$ and $\hat{I}(\hat{x}) = I \setminus \hat{I}^s(\hat{x})$. We claim that $\hat{I}^s(\hat{x}) = I^s(\hat{x})$. Since $P_i(\hat{x}) = \varnothing$ implies $\hat{P}_i(\hat{x}) = \varnothing$, we have $I^s(\hat{x}) \subset \hat{I}^s(\hat{x})$. Let i be a point in I with $P_i(\hat{x}) \neq \varnothing$. Then we can choose $z_i \in P_i(\hat{x})$. Since $\hat{x}_i$ is in the interior of K, there exists $\alpha' \in (0, 1)$ such that $\alpha'\hat{x}_i + (1 - \alpha')z_i \in K$. By B4$'$ we have $\alpha'\hat{x}_i + (1 - \alpha')z_i \in \hat{P}_i(\hat{x})$, and thus, $i \in \hat{I}(\hat{x})$. This implies that $\hat{I}^s(\hat{x}) \subset I^s(\hat{x})$.

Satiation at $\hat{x}$ may occur either to some agent, or to no agent, or to all agents in the truncated economy $\hat{E}$. Thus, the proof will be divided into the three cases; (i) $\hat{I}(\hat{x}) \neq \varnothing$ and $\hat{I}^s(\hat{x}) \neq \varnothing$, and (ii) $\hat{I}(\hat{x}) = I$.[9] The arguments made below will take advantage of the fact that $\hat{I}(\hat{x}) = I(\hat{x})$.

Now we verify the following claims.

Claim 1: *For each $i \in I(\hat{x})$,*

$$\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i + 1 - \|\hat{p}\|.$$

Proof. By condition (b) we have $\hat{p} \cdot \hat{x}_i \leq \hat{p} \cdot e_i + 1 - \|\hat{p}\|$ for all $i \in I$. In particular, it holds for all $i \in I(\hat{x})$. By applying the arguments of Step 4 of Theorem 2.3.1 of Won and Yannelis (2008) to each $i \in I(\hat{x})$, we have $\hat{p} \cdot \hat{x}_i \geq \hat{p} \cdot e_i + 1 - \|\hat{p}\|$ for all $i \in I(\hat{x})$. □

Claim 2: *For each $j \in I^s(\hat{x})$,*

$$\hat{p} \cdot \hat{x}_j \geq \hat{p} \cdot e_j.$$

Proof. First we show that for each $i \in I(\hat{x})$ and all $z_i \in P_i(\hat{x})$,

$$\hat{p} \cdot z_i \geq \hat{p} \cdot e_i + 1 - \|\hat{p}\|. \tag{1}$$

Suppose otherwise. Then there exists $z_i' \in P_i(\hat{x})$ such that $\hat{p} \cdot z_i' < \hat{p} \cdot e_i + 1 - \|\hat{p}\|$. By B4$'$ we have that $\alpha\hat{x}_i + (1 - \alpha)z_i'$ is in $P_i(\hat{x})$ for all $\alpha \in [0, 1)$. On the other hand, by Step 4, $\hat{x}_i$ is in the interior of K. Hence there exists $\alpha' \in (0, 1)$ such that $\alpha'\hat{x}_i + (1 - \alpha')z_i' \in K$. Since $\hat{P}_i(\hat{x}) = P_i(\hat{x}) \cap K$, it follows that $\alpha'\hat{x}_i + (1 - \alpha')z_i' \in \hat{P}_i(\hat{x})$. On the other hand, we have

$$\hat{p} \cdot (\alpha'\hat{x}_i + (1 - \alpha')z_i')$$
$$= \alpha'\hat{p} \cdot \hat{x}_i + (1 - \alpha')\hat{p} \cdot z_i'$$
$$\leq \alpha'(\hat{p} \cdot e_i + 1 - \|\hat{p}\|) + (1 - \alpha')\hat{p} \cdot z_i', \quad (\text{since}\,\hat{x}_i \in cl\, A_i(\hat{p}, \hat{x}))$$
$$< \hat{p} \cdot e_i + 1 - \|\hat{p}\|, \quad (\text{since}\,\hat{p} \cdot z_i' < \hat{p} \cdot e_i + 1 - \|\hat{p}\|).$$

Thus, $\alpha'\hat{x}_i + (1 - \alpha')z_i' \in \hat{P}_i(\hat{x}) \cap A_i(\hat{p}, \hat{x})$ for each $i \in \hat{I}(\hat{x})$, which contradicts the fact that $\hat{P}_i(\hat{x}) \cap A_i(\hat{p}, \hat{x}) = \varnothing$, and this proves that (1) holds.

It follows from (1) and Claim 1 that for each $i \in I(\hat{x})$, all $z_i \in P_i(\hat{x})$ satisfy

$$\hat{p} \cdot z_i \geq \hat{p} \cdot e_i + 1 - \|\hat{p}\| = \hat{p} \cdot \hat{x}_i.$$

Again by Claim 1, it holds that $\hat{p} \cdot \hat{x}_i \geq \hat{p} \cdot e_i$ for each $i \in I(\hat{x})$. Thus, the condition S5 allows us to have $\hat{p} \cdot \hat{x}_j \geq \hat{p} \cdot e_j$ for each $j \in I^s(\hat{x})$ and this completes the proof of the claim.

Step 5: Suppose that (i) holds, i.e., $I(\hat{x}) \neq \varnothing$ and $I^s(\hat{x}) \neq \varnothing$. We show that

$$\|\hat{p}\| = 1 \text{ and } \hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i \text{ for all } i \in I.$$

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[8] For details, see Appendix B of Won and Yannelis (2008).

[9] The case with $\hat{I}^s(\hat{x}) = I$ is excluded by the Assumption B8.

By summing up the relations in Claim 1 and 2 above, we see that

$$\sum_{i \in I} \hat{p} \cdot (\hat{x}_i - e_i) \geq \sum_{i \in I(\hat{x})} (1 - \|\hat{p}\|).$$

Recalling that $\sum_{i \in I}(\hat{x}_i - e_i) = 0$, we obtain $\|\hat{p}\| \geq 1$. Since $\hat{p} \in \Delta$, we conclude that $\|\hat{p}\| = 1$ and therefore, $\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i$ for all $i \in I$.

Step 6: We consider the case (ii), i.e., $I(\hat{x}) = I$. In this case, the arguments made for each $i \in I(\hat{x})$ in Steps 4 and 5 can be repeated here to show that $\|\hat{p}\| = 1$ and $\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i$ for all $i \in I$.

Step 7: In summary, by Steps 4–6, $\hat{p} \neq 0$, $\hat{p} \cdot \hat{x}_i = \hat{p} \cdot e_i$ and $P_i(\hat{x}) \cap \beta_i(\hat{p}) = \varnothing$ for all $i \in I$. By Step 2, $\hat{x}$ is in F. Hence, $(\hat{p}, \hat{x})$ is a quasi-equilibrium of the economy E. All it remains is to show that $P_i(\hat{x}) \cap B_i(\hat{p}) = \varnothing$ for all $i \in I$. Fix i. If $\hat{x} \in S_i$, then $P_i(\hat{x}) = \varnothing$ and therefore, $P_i(\hat{x}) \cap B_i(\hat{p}) = \varnothing$. Suppose that $\hat{x} \in X \setminus S_i$. By the same arguments of Step 5 of the proof of Theorem 2.3.1 of Won et al. (2008), the fact that $P_i(\hat{x}) \cap \beta_i(\hat{p}) = \varnothing$ implies that $P_i(\hat{x}) \cap B_i(\hat{p}) = \varnothing$. Thus, we conclude that $(\hat{p}, \hat{x})$ is an equilibrium of the economy E. $\quad\square$

Remark 4.2. It should be noted that Theorem 4.1 does not follow from Theorem 2.3.1 of Won et al. (2008) because the preferences of each agent i are allowed to reach satiation in X_i^H. Suppose that X_i is compact for each $i \in I$. Then preferences are satiable and H is bounded. In this case, Theorem 4.1 is applicable. However, Won et al. (2008) may not work because of satiation. On the other hand, Assumption S5 is vacuously true whenever $I(x) = I$ for all $x \in cl\, H$. If this is the case, Theorem 4.1 reduces to Theorem 2.3.1 of Won and Yannelis (2008). Thus, the former subsumes the latter as a special case.

5. Concluding remarks

This paper extends the results of Won et al. (2008) to the case where preferences may be non-ordered and satiable. The classical literature of general equilibrium theory allows preferences to reach satiation outside the set H of individually rational and feasible allocations. In this case, satiation has no impact on the existence of equilibrium. Equilibrium may not exist, however, when satiation occurs only inside the set H. To address the problem with satiable preferences, we have introduced Assumption S5. This condition allows us to provide a unified approach to the economy with possibly satiated preferences. In particular, the paper encompasses as a special case the literature including Nielsen (1990), Allingham (1991), and Won et al. (2008).

The outcomes of the paper are particularly useful in addressing the equilibrium existence issue for asset pricing models without riskless assets. One interesting application is found in Won et al. (2008) which are concerned about the capital asset pricing model (CAPM) with heterogeneous expectations. More intriguing examples are recent developments of the CAPM which investigate the effect on asset pricing of mean-preserving-spread risk,[10] model uncertainty or Knightian uncertainty.

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[10] Boyle and Ma (2006) examine the case that preferences display mean-preserving-spread risk aversion and thus, need not be transitive.