Equilibrium Points of Non-Cooperative Random and Bayesian Games

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We provide random equilibrium existence theorems for non-cooperative random games with a countable number of players. Our results yield as corollaries generalized random versions of the ordinary equilibrium existence result of J. Nash [22]. Moreover, they can be used to obtain equilibrium existence results for games with incomplete information, and in particular Bayesian games. In view of recent work on applications of Bayesian games and Bayesian equilibria, the latter results seem to be quite useful since they delineate conditions under which such equilibria exist.

1. Introduction

A finite game consists of a set of players $I = \{1, 2, \ldots, n\}$ each of whom is characterized by a strategy set X_i and a payoff (utility) function

$$u_i\colon \prod_{j \in I} X_j \to R.$$

An equilibrium for this game is a strategy vector such that no player can increase his/her payoff by deviating from his/her equilibrium strategy, given that the other players use their equilibrium strategies, i.e., $x^* \in \Pi_{i \in I} X_i$ is an equilibrium if

$$u_i(x^*) = \max_{y_i \in X_i} u_i(x_1^*, \ldots, x_{i-1}^*, y_i, x_{i+1}^*, \ldots, x_n^*)$$

for all $i \in I$. The above game form and the notion of equilibrium were both introduced in a seminal paper by J. Nash [22].[1] In that same paper Nash proved by means of the Brouwer Fixed Point Theorem, the existence of an equilibrium for the above game, where strategy sets were subsets of R^ℓ, i.e., the ℓ-fold Cartesian product of the set of real numbers R. The work of Nash has found very

*We would like to thank Roko Aliprantis, Erik Balder, Kim Border, Taesung Kim, M. Ali Khan and David Schmeidler for comments, discussions and suggestions. Of course we are responsible for any remaining shortcomings.

[1]Notice that this notion of equilibrium is non-cooperative. No communication between players is allowed.

interesting applications in game theory and mathematical economics (see for instance K. J. Arrow and G. Debreu [2] or G. Debreu [10]). Generalizations of Nash's equilibrium existence theorem to games where strategy sets were subsets of arbitrary Hausdorff linear topological spaces, were obtained by K. Fan [13] and F. E. Browder [8] among others. The results of Fan and Browder were proved by means of infinite dimensional fixed point theorems. Subsequently to the above work, research in economics (see for instance W. J. Shafer and H. F. Sonnenschein [31]) necessitated further generalizations of Nash's equilibrium existence result, to games where each player is equipped with a preference correspondence (instead of a payoff function), which need not be transitive or complete and therefore need not be representable by a utility function. The latter work was motivated by empirical results which indicated that in many instances agents' behavior is not necessarily transitive.

A common characteristic of all the above results is that they are deterministic, i.e., players cannot accommodate any kind of uncertainty or randomness in their responses to potential changes in their primitive environment. In reality, however, there are many factors which go beyond the control of players and cannot be influenced by their actions. In that sense, it seems natural to assume that players' payoff functions depend not only on the strategies, but on the states of nature of the world as well. In other words, payoff functions can be random. This is the type of the game we shall consider in this paper. Of course with the random payoff functions the equilibrium strategy vector will be random as well, and therefore the equilibrium will change from one state of the environment to another.

It is the purpose of this paper to prove random equilibrium existence results for quite general random games. In particular, as in W. J. Shafer and H. F. Sonnenschein [31] and N. C. Yannelis and N. D. Prabhakar [32], instead of assigning each player a random payoff or utility function, we equip each player with a random preference correspondence which need not be representable by a random utility function. It should be noted, however, that our random equilibrium results, provide as corollaries random versions of the theorems of Nash, Fan, and Browder. Moreover, we show that these random equilibrium theorems can be used to obtain equilibrium existence results for games with incomplete information, and in particular, for Bayesian games. The main reference for the latter type of games is J. C. Harsanyi's seminal paper [14]. Recently there is a growing literature on this subject. In particular, Bayesian games have found very interesting applications in economic theory, e.g., R. J. Aumann [3], R. Myerson [21], T. Palfrey and S. Srivastava [23, 24], J. Peck and K. Shell [25] and A. Postlewaite and D. Schmeidler [26] among others.[2]

As in [3, 14, 21, 23, 24, 25] by the term "Bayesian games" we mean games, where each player i is characterized by a strategy set X_i, a random utility function u_i defined on the product space $\Omega \times X$ (where Ω is the set of states of

[2]However, no equilibrium existence results are contained in these papers. E. J. Balder [6], A. Mas-Colell [18], P. Milgrom and R. Weber [20] and R. Radner and R. Rosenthal [27] have provided existence of equilibrium theorems for games with incomplete information, but their approach is different from ours. We shall discuss the work of these authors in Section 3.

the world and $X = \Pi_{i \in I} X_i$), an information set S_i (where S_i is a partition of Ω), and a prior q_i (i.e., a probability measure on Ω). In this setting the corresponding natural extension of Nash's equilibrium concept is that of a Bayesian equilibrium. In particular, if we denote by $E_i(\omega)$ the event in S_i containing the realized state of nature $\omega \in \Omega$, then each agent will choose a strategy which maximizes expected utility conditional on his/her own event $E_i(\omega)$.

Note that in this Bayesian game the conditional expected utility of each player is a random function, i.e., depends on the states of nature of the world and on the strategies. Hence, in essence the problem of the existence of a Bayesian equilibrium is converted to a random equilibrium problem, simply by thinking of the conditional expected utility of each player as his/her random payoff function of some random game. It is exactly for this reason that in certain cases the existence of a Bayesian equilibrium for a Bayesian game follows directly from the existence of a random equilibrium for a random game. The latter result seems to be quite interesting. Specifically, in view of recent work mentioned above, it is important to delineate conditions under which such equilibria exist.

As the deterministic results of Nash, Fan and Browder are based on deterministic fixed point theorems, the proofs of our random equilibrium existence results are based on random fixed point theorems. The idea behind the need of a random fixed point can be intuitively grasped simply by noting that with random payoff functions the best reply correspondence becomes random as well, and therefore a random extension of the Kakutani–Fan–Glicksberg Fixed Point Theorem seems to be required. To this end, we prove a random version of K. Fan's Coincidence Theorem [12, Theorem 6, p. 238], which gives as corollary a random version of the Kakutani–Fan–Glicksberg Fixed Point Theorem. In addition, we employ Aumann-type measurable selection theorems and some recent Carathéodory-type selection results proved in [16, 17].

The paper is organized as follows: Section 2 contains several preliminary results of measure theoretic character. Moreover, a random version of Fan's Coincidence Theorem is established. The main results of the paper are stated in Section 3, 4, and 5. Section 6 contains a discussion of the related literature on games with incomplete information. Finally, concluding remarks are given in Section 7.

2. Mathematical preliminaries

Let X and Y be sets. The graph G_φ of the set-valued function (or correspondence) $\varphi: X \to 2^Y$ is the set $G_\varphi = \{(x,y) \in X \times Y : y \in \varphi(x)\}$. If X and Y are topological spaces, a correspondence $\varphi: X \to 2^Y$ is said to have an *open graph* if the set G_φ is open in $X \times Y$. A correspondence $\varphi: X \to 2^Y$ is said to be *lower semicontinuous* (l.s.c.) if the set $\{x \in X : \varphi(x) \cap V \neq \varnothing\}$ is open in X for every open subset V of Y; and *upper semicontinuous* (u.s.c.) if the set $\{x \in X : \varphi(x) \subset V\}$ is open in X for every open subset V of Y. It can be easily checked that if a correspondence has an open graph, then it is l.s.c., but the converse is not true; see [32, p. 237].

In our discussion, we shall need several results that will be listed in this section.

Theorem 2.1 *Let X be a topological space and Y be a linear topological space. If the correspondence $\varphi: X \to 2^Y$ is l.s.c., then the convex hull correspondence $\psi: X \to 2^Y$, defined by $\psi(x) = \operatorname{co}\varphi(x)$, is also l.s.c.*

Proof: See [19, Proposition 3.6, p. 366]. ∎

Theorem 2.2 *Let X be a topological space and let $\{Y_i : i \in I\}$ (where the set I can be finite or infinite) be a family of compact topological spaces. Let $Y = \Pi_{i \in I} Y_i$. If for each $i \in I$, the correspondence $F_i: X \to 2^{Y_i}$ is u.s.c. and closed valued, then the correspondence $F: X \to 2^Y$, defined by $F(x) = \Pi_{i \in I} F_i(x)$, is also u.s.c.*

Proof: See [11, Lemma 3, p. 124]. ∎

We now turn our attention to some measure theoretic facts. Let X and Y be topological spaces and let $\varphi: X \to 2^Y$ be a nonempty valued correspondence. A *continuous selection* for φ is a continuous function $f: X \to Y$ such that $f(x) \in \varphi(x)$ for all $x \in X$.

Let (Ω, α) be a measurable space, Y be a topological space and $\varphi: \Omega \to 2^Y$ be a nonempty-valued correspondence. A *measurable selection* for φ is a measurable function $f: \Omega \to Y$ such that $f(\omega) \in \varphi(\omega)$ for all $\omega \in \Omega$.

We now define the concept of a Carathéodory selection which combines the notion of continuous selection and measurable selection.

Let (X, α) be a measurable space and let Y and Z be topological spaces. Let $\varphi: X \times Z \to 2^Y$ be a (possibly empty-valued) arbitrary correspondence. Let $U = \{(x, z) \in X \times Z : \varphi(x, z) \neq \emptyset\}$. A *Carathéodory selection* for the correspondence φ is a function $f: U \to Y$ such that:

1. $f(x, z) \in \varphi(x, z)$ for all $(x, z) \in U$;
2. the function $f(x, \cdot)$ is continuous on $U^x = \{z \in Z : (x, z) \in U\}$ for each $x \in X$; and
3. the function $f(\cdot, z)$ is measurable on $U^z = \{x \in X : (x, z) \in U\}$ for each $z \in Z$.

If (X, α) and (Y, β) are measurable spaces and $\varphi: X \to 2^Y$ is a correspondence, then φ is said to have a *measurable graph* if G_φ belongs to the product σ-algebra $\alpha \otimes \beta$. We are usually interested in the situation where (X, α) is a measurable space, Y is a topological space and $\beta = \beta(Y)$ is the Borel σ-algebra of Y. For a correspondence φ from a measurable space into a topological space, if we say that φ has a measurable graph, it is understood that the topological space is endowed with its Borel σ-algebra (unless specified otherwise). In the same setting as above, i.e., (X, α) a measurable space and Y a topological space, φ is said to be *lower measurable* if $\{x \in X : \varphi(x) \cap V \neq \emptyset\} \in \alpha$ for every V open in Y. The following facts will be useful in the sequel.

Theorem 2.3 *Let (Ω, α, μ) be a complete finite measure space, X be a separable metric space and $\varphi: \Omega \to 2^Y$ be a nonempty valued correspondence having a measurable graph, i.e., $G_\varphi \in \alpha \otimes \beta(X)$. Then there exists a measurable selection for φ.*

Proof: See [9, Theorem III.22, p. 22] or [15, Theorem 5.2, p. 60]. ∎

Theorem 2.4 *Let (Ω, α, μ) be a complete finite measure space, X be a complete separable metric space and $\varphi: \Omega \times X \to 2^{R^\ell}$ be a convex (possibly empty) valued correspondence such that:*

1. *φ is lower measurable with respect to the σ-algebra $\alpha \otimes \beta(X)$, and*
2. *the set-valued function $\varphi(\omega, \cdot)$ is l.s.c. for each $\omega \in \Omega$.*

Then there exists a Carathéodory selection for φ.

Proof: See [16, Theorem 3.2]. ∎

Theorem 2.5 *The previous fact remains true if φ is a correspondence from $\Omega \times X$ into 2^Y, where Y is a separable Banach space and (1) and (2) are replaced by*

(1′) *$G_\varphi \in \alpha \otimes \beta(X) \otimes \beta(Y)$, and*

(2′) *the set-valued function $\varphi(\omega, \cdot)$ has an open graph for each $\omega \in \Omega$, i.e., for each $\omega \in \Omega$ the set $G_{\varphi(\omega,\cdot)} = \{(x, y) \in X \times Y : y \in \varphi(\omega, x)\}$ is open in $X \times Y$.*

Proof: See [17, Main Theorem]. ∎

Theorem 2.6 *Let Ω be a measurable space, $\{Y_i : i \in I\}$ (where I is a countable set) be a family of second countable topological spaces. Let $Y = \Pi_{i\in I} Y_i$. If for each $i \in I$, the correspondence $F_i: \Omega \to 2^{Y_i}$ is lower measurable, then the correspondence $F: \Omega \to 2^Y$, defined by $F(\omega) = \Pi_{i\in I} F_i(\omega)$, is also lower measurable.*

Proof: See [15, Proposition 2.3, p. 55]. ∎

Theorem 2.7 *Let Ω be a measurable space, X be a separable metric space and for each $i \in I$ (where I is a countable set) $F_i: \Omega \to 2^X$ is a lower measurable and closed-valued correspondence. If for each $\omega \in \Omega$ the set $F_i(\omega)$ is compact for at least one index $i \in I$, then the correspondence $F: \Omega \to 2^X$, defined by $F(\omega) = \bigcap_{i\in I} F_i(\omega)$, is lower measurable.*

Proof: See [15, Theorem 4.1, p. 58]. ∎

If (X, α), (Y, β) and (Z, Σ) are measurable spaces, $U \subset X \times Z$ and $f: U \to Y$, we call f *jointly measurable* if for every $B \in \beta$ we have $f^{-1}(B) = U \cap A$ for some $A \in \alpha \otimes \Sigma$. It is a standard result that if Z is a separable metric space, Y is a metric space and $f: X \times Z \to Y$ is such that for each fixed $x \in X$ the function

$f(x, \cdot)$ is continuous and for each fixed $z \in Z$ the function $f(\cdot, z)$ is measurable, then f is jointly measurable (where $\beta = \beta(Y)$ and $\Sigma = \beta(Z)$). It turns out that in several instances U is a proper subset of $X \times Z$, and this situation is more delicate. However, in this more delicate situation it can be shown that f is still jointly measurable. In particular, we have the following fact.

Theorem 2.8 *Let (Ω, α) be a measurable space, X be a separable metric space, Y a metric space and $U \subset \Omega \times X$ be such that:*

1. *For each $\omega \in \Omega$ the set $U^\omega = \{x \in X : (\omega, x) \in U\}$ is open in X, and*
2. *for each $x \in X$ the set $U^x = \{\omega \in \Omega : (\omega, x) \in U\}$ belongs to α.*

Let $f: U \to Y$ be a function such that for each $\omega \in \Omega$ the function $f(\omega, \cdot)$ is continuous on U^ω and for each $x \in X$ the function $f(\cdot, x)$ is measurable on U^x. Then f is jointly relatively measurable with respect to the σ-algebra $\alpha \otimes \beta(X)$, i.e., for every open subset V of Y we have

$$\{(\omega, x) \in U : f(\omega, x) \in V\} = U \cap A$$

for some $A \in \alpha \otimes \beta(X)$.

Proof: See [16, Lemma 4.12]. ∎

Theorem 2.9 *Let (Ω, α, μ) be a complete measure space and X be a complete separable metric space. If a set A belongs to $\alpha \otimes \beta(X)$, then its projection $\mathrm{proj}_\Omega(A)$ belongs to α.*

Proof: See [9, Theorem III.23, p. 75]. ∎

The next result is a random version of Fan's Coincidence Theorem. (See [12, Theorem 6, p. 238], and also [7, Theorem 17.1, p. 78].)

Theorem 2.10 *Let X be a nonempty, compact and convex subset of a locally convex separable and metrizable linear topological space Y and let (Ω, Σ, ν) be a complete finite measure space. Let $\gamma: \Omega \times X \to 2^Y$ and $\delta: \Omega \times X \to 2^Y$ be two nonempty, convex, closed and at least one of them compact valued correspondences such that:*

1. *γ and δ are both lower measurable,*
2. *for each $\omega \in \Omega$, the correspondences $\gamma(\omega, \cdot): X \to 2^Y$ and $\delta(\omega, \cdot): X \to 2^Y$ are both u.s.c., and*
3. *for every $(\omega, x) \in \Omega \times X$ there exist three points $y \in X$, $u \in \gamma(\omega, x)$, $z \in \delta(\omega, x)$ and a real number $\lambda > 0$ such that $y - x = \lambda(u - z)$.*

Then there exists a measurable function $x^*: \Omega \to X$ such that

$$\gamma(\omega, x^*(\omega)) \cap \delta(\omega, x^*(\omega)) \neq \emptyset$$

for almost all $\omega \in \Omega$.

Proof: Define the correspondence $W: \Omega \times X \to 2^Y$ by

$$W(\omega, x) = \gamma(\omega, x) \cap \delta(\omega, x).$$

Since γ and δ are closed valued and lower measurable and at least one of them is compact valued, it follows from Theorem 2.7 that W is lower measurable. Define the correspondence $\varphi: \Omega \to 2^X$ by $\varphi(\omega) = \{x \in X : W(\omega, x) \neq \emptyset\}$. Observe that

$$\begin{aligned} G_\varphi &= \{(\omega, x) \in \Omega \times X : x \in \varphi(\omega)\} \\ &= \{(\omega, x) \in \Omega \times X : W(\omega, x) \neq \emptyset\} \\ &= \{(\omega, x) \in \Omega \times X : W(\omega, x) \cap Y \neq \emptyset\}, \end{aligned}$$

and the latter set belongs to $\Sigma \otimes \beta(X)$ since W is lower measurable. Consequently, $G_\varphi \in \Sigma \otimes \beta(X)$. It follows from Fan's Coincidence Theorem [12, Theorem 6, p. 238] that for each $\omega \in \Omega$ we have $\varphi(\omega) \neq \emptyset$. Thus, the correspondence $\varphi: \Omega \to 2^X$ satisfies all the conditions of Theorem 2.3 (the Aumann Measurable Selection Theorem) and consequently, there exists a measurable function $x^*: \Omega \to X$ such that $x^*(\omega) \in \varphi(\omega)$ for almost all ω in Ω, i.e., $\gamma(\omega, x^*(\omega)) \cap \delta(\omega, x^*(\omega)) \neq \emptyset$ for almost all ω in Ω. This completes the proof of the theorem. ∎

An immediate corollary of the above theorem is a random version of the Kakutani–Fan–Glicksberg Fixed Point Theorem [11, Theorem 1, p. 122].

Corollary 2.11 *Let X be a nonempty, compact and convex, subset of a locally convex separable and metrizable linear topological space Y and let (Ω, Σ, ν) be a complete finite measure space. Let $\gamma: \Omega \times X \to 2^X$ be a nonempty, closed, convex valued correspondence such that for each fixed $\omega \in \Omega$ the function γ is u.s.c. and γ is lower measurable. Then γ has a random fixed point, i.e., there exists a measurable function $x^*: \Omega \to X$ such that $x^*(\omega) \in \gamma(\omega, x^*(\omega))$ for almost all $\omega \in \Omega$.*

Proof: Define the correspondence $\delta: \Omega \times X \to 2^X$ by $\delta(\omega, x) = \{x\}$. Clearly, for each fixed $\omega \in \Omega$ the function $\delta(\omega, \cdot)$ is u.s.c. and δ is nonempty, convex, and compact valued and lower measurable. Let $x \in X$ and $\omega \in \Omega$. By choosing $u \in \gamma(\omega, x)$, $z = x \in \delta(\omega, x)$, and $\lambda \in (0, 1)$, the assumption (3) of Theorem 2.10 is satisfied. (Simply notice that $y = x + \lambda(u - z) = \lambda u + (1 - \lambda)x \in X$, since X is convex.) Hence, by the previous corollary, there exists a measurable function $x^*: \Omega \to X$ such that $\gamma(\omega, x^*(\omega)) \cap \delta(\omega, x^*(\omega)) \neq \emptyset$ for almost all $\omega \in \Omega$, i.e., $x^*(\omega) \in \gamma(\omega, x^*(\omega))$ for almost all $\omega \in \Omega$. ∎

Remarks. Theorem 2.10 and Corollary 2.11 remain true if we replace the assumption that (Ω, Σ, ν) is a complete finite (or σ-finite) measure space, by the fact that (Ω, Σ) is simply a measurable space. In this case one only needs to observe that in the proof of Theorem 2.10 for each fixed $\omega \in \Omega$ the function $W(\omega, \cdot)$ is u.s.c. (as it is the intersection of two u.s.c. correspondences) and therefore, the correspondence $\varphi: \Omega \to 2^X$ is closed-valued. Since φ is closed-valued and it has a measurable graph by [15, Theorem 3.3, p. 56], φ is lower measurable. One can now appeal to the Kuratowski and Ryll–Nardzewski Measurable Selection Theorem [15, p. 60] to complete the proof of Theorem 2.10.

We continue with two more results that will be needed later.

Lemma 2.12 *Let (S, α, μ) be a complete measure space, let X and Y be separable metric spaces, and let $\varphi: S \times X \to 2^Y$ be a lower measurable (possibly empty valued) correspondence. Suppose that for each fixed $s \in S$ the function $\varphi(s, \cdot)$ is l.s.c. Put $O = \{(s,x) \in S \times X : \varphi(s,x) \neq \varnothing\}$ and let $f: O \to Y$ be a Carathéodory selection for φ. Then the correspondence $\theta: S \times X \to 2^Y$, defined by*

$$\theta(s,x) = \begin{cases} \{f(s,x)\}, & \text{if } (s,x) \in O_i; \\ Y, & \text{if } (s,x) \notin O_i\,, \end{cases}$$

is lower measurable.

Proof: We begin by making a couple of observations. First notice that since φ is lower measurable, the set $O = \{(s,x) \in S \times X : \varphi(s,x) \cap Y \neq \varnothing\}$ belongs to $\alpha \otimes \beta(X)$. By Theorem 2.9 for each $x \in X$ the set

$$\begin{aligned} O^x &= \{s \in S : (x,s) \in O\} \\ &= \operatorname{proj}_S\big[\{(s,x) \in S \times X : \varphi(s,x) \neq \varnothing\} \cap (S \times \{x\})\big] \\ &= \operatorname{proj}_S\big[O \cap (S \times \{x\})\big], \end{aligned}$$

belongs to α. Moreover, note that since for each fixed $s \in S$ the function $\varphi(s,\cdot)$ is l.s.c., it follows that for each $s \in S$ the set $O^s = \{x \in X : (s,x) \in O\}$ is open in X. Since for each fixed $s \in S$ the function $f(s,\cdot)$ is continuous on O^s and for each $x \in X$ the function $f(\cdot, x)$ is measurable on O^x, by Theorem 2.8 the function f is jointly measurable. Now it can be easily seen that for every open subset V of Y the set $A = \{(s,x) \in S \times X : \theta(s,x) \cap V \neq \varnothing\} = B \cup C$, where $B = \{(s,x) \in O : f(s,x) \in V\}$ and $C = \{(s,x) \in S \times X \setminus O : Y \cap V \neq \varnothing\}$. Clearly, $B \in \alpha \otimes \beta(X)$ and $C \in \alpha \otimes \beta(X)$ and therefore $A = B \cup C$ belongs to $\alpha \otimes \beta(X)$. Consequently, θ is lower measurable, as claimed. ∎

Lemma 2.13 *Let (S, α) be a measurable space, Z be a separable metric space and R^* be the extended real line. Let $g: S \times Z \to R^*$ be a function such that for each fixed $s \in S$ the function $g(s,\cdot)$ is continuous and for each fixed $z \in Z$ the function $g(\cdot, z)$ is measurable. If $K: S \to 2^Z$ is the correspondence defined by*

$$K(s) = \{z \in Z : g(s,z) > 0\},$$

then we have:

a. *$G_K \in \alpha \otimes \beta(Z)$, i.e., K has a measurable graph, and*

b. *K is lower measurable.*

Proof: (a) Since for each fixed $s \in S$ the function $g(s,\cdot)$ is continuous and for each fixed $z \in Z$ the function $g(\cdot,z)$ is measurable, it follows from a standard result that g is jointly measurable. Observe that

$$\begin{aligned} g^{-1}((0,\infty)) &= \{(s,z) \in S \times Z : g(s,z) > 0\} \\ &= \{(s,z) \in S \times Z : z \in K(s)\} \\ &= G_K, \end{aligned}$$

and the latter set belongs to $\alpha \otimes \beta(Z)$ since g is jointly continuous. (b) We must show that the set $\{s \in S : K(s) \cap V \neq \varnothing\}$ belongs to α for every open subset V of Z. As it was remarked above, g is jointly measurable, i.e., g is measurable with respect to the product σ-algebra $\alpha \otimes \beta(Z)$. Let D be a countable dense subset of Z, and let $U = (0,\infty)$. Observe that

$$\begin{aligned} \{s \in S : K(s) \cap V \neq \varnothing\} &= \{s \in S : g(s,z) \in U \text{ for some } z \in V\} \\ &= \{s \in S : g(s,d) \in U \text{ for some } d \in D\} \\ &= \bigcup_{d \in D} \{s \in S : g(s,d) \in U\}, \end{aligned}$$

and the latter set belongs to α since for each fixed $z \in Z$ the function $g(\cdot,z)$ is measurable. This completes the proof of the lemma. ∎

The notions that will be introduced next are quite standard (see for instance N. C. Yannelis [36]) but we briefly outline them for the sake of completeness.

We begin by defining the notion of a Bochner integrable function. Let (T,Σ,μ) be a finite measure space and Y be a Banach space. A function $f: T \to Y$ is called *simple* if there exist $y_1, y_2, \ldots, y_n$ in Y and $A_1, A_2, \ldots, A_n$ in Σ such that $f = \sum_{i=1}^{n} y_i \chi_{A_i}$, where χ_{A_i} denotes the characteristic function of the set A_i. A function $f: T \to Y$ is said to be μ-*measurable* if there exists a sequence of simple functions $\{f_n\}$ such that $\lim_{n\to\infty} \|f_n(t) - f(t)\| = 0$ for almost all $t \in T$. A μ-measurable function $f: T \to Y$ is said to be *Bochner integrable* if there exists a sequence of simple functions $\{f_n\}$ such that

$$\lim_{n\to\infty} \int_T \|f_n(t) - f(t)\| \, d\mu(t) = 0.$$

In this case, the integral of f over a set $E \in \Sigma$ is defined by

$$\int_E f(t)\, d\mu(t) = \lim_{n\to\infty} \int_E f_n(t)\, d\mu(t).$$

It can be shown that if $f: T \to Y$ is a μ-measurable function, then f is Bochner integrable if and only if $\int_T \|f(t)\|\, d\mu(t) < \infty$. We denote by $L_1(\mu, Y)$ the space of equivalence classes of Y-valued Bochner integrable functions $x: T \to Y$ normed by $\|x\| = \int_T \|x(t)\|\, d\mu(t)$. It can be easily shown that $L_1(\mu, Y)$ under the norm $\|\cdot\|$ is a Banach space.

A Banach space Y has the *Radon–Nikodym Property with respect to the measure space* (T,Σ,μ) if for each μ-continuous vector measure $G: \Sigma \to Y$ of bounded variation there exists some $g \in L_1(\mu, Y)$ such that $G(E) = \int_E g(t)\,d\mu(t)$ for all $E \in \Sigma$. A Banach space Y has the *Radon–Nikodym Property* (RNP) if Y has the RNP with respect to every finite measure space. It is a standard result that if Y^* (the norm dual of Y) has the RNP, then $\left(L_1(\mu, Y)\right)^* = L_\infty(\mu, Y^*)$.

The correspondence $\varphi: T \to 2^Y$ is said to be *integrably bounded* if there exists a map $g \in L_1(\mu)$ such that $\sup\{\|x\| : x \in \varphi(t)\} \leq g(t)$ holds for almost all $t \in T$. We denote by L_φ the set of all Y-valued Bochner integrable selections of $\varphi: T \to 2^Y$, i.e.,

$$L_\varphi = \{x \in L_1(\mu, Y) : x(t) \in \varphi(t) \text{ for almost all } t \text{ in } T\}.$$

Define the *integral* of the correspondence φ by

$$\int_T \varphi(t)\,d\mu(t) = \left\{\int_T x(t)\,d\mu(t) : x \in L_\varphi\right\}.$$

By Theorem 2.3 if T is a complete finite measure space, Y is a separable Banach space, and $\varphi: T \to 2^Y$ is a nonempty valued correspondence with a measurable graph (or equivalently φ is lower measurable and closed valued), then φ admits a measurable selection, i.e., there exists a measurable function $f: T \to Y$ such that $f(t) \in \varphi(t)$ for almost all $t \in T$. By virtue of this result (and provided that φ is integrably bounded), we can conclude that $L_\varphi \neq \emptyset$ and therefore $\int_T \varphi(t)\,d\mu(t) \neq \emptyset$.

Finally, we wish to note that *Diestel's Theorem* (see for instance [36, Theorem 3.1]) asserts that if F is an arbitrary nonempty, weakly compact, convex subset of a separable Banach space Y (or more generally if $F: T \to 2^Y$ is a nonempty, integrably bounded, and weakly compact convex valued correspondence) then L_F is a weakly compact subset of $L_1(\mu, F)$. With all these preliminary results out of the way, we can turn to our contributions.

3. Random games and equilibria

Let (Ω, Σ, μ) be a complete finite measure space. We interpret Ω as the states of nature of the world and assume that Ω is large enough to include all events that we consider to be interesting. As usual, Σ denotes the σ-algebra of events. Denote by I the set of players. The set I may be finite or countable.

Definition 3.1 *A* random game *is a set* $\mathcal{E} = \{(X_i, P_i) : i \in I\}$ *of ordered pairs, where*

1. X_i *is the* strategy set *of player* i*, and*
2. $P_i: \Omega \times X \to 2^{X_i}$ *(where* $X = \Pi_{i \in I} X_i$*) is the* random preference (or choice) correspondence *of player* i*.*

We read $y_i \in P_i(\omega, x)$ *as player* i *strictly prefers* y_i *to* x_i *at the state of nature* ω*, if the (given) components of the other players are fixed.*

A random equilibrium *for the game* E *is a measurable function* $x^*: \Omega \to X$ *such that for all* $i \in I$ *we have* $P_i(\omega, x^*(\omega)) = \emptyset$ *for almost all* $\omega \in \Omega$*.*

Notice that each player in the game described above is characterized by a strategy set and a random preference correspondence. We now follow the original formulation by J. Nash [22] (and its generalizations by Fan [12] and Browder [8] among others) where random preference correspondences are replaced by random payoff functions, i.e., real valued functions defined on $\Omega \times X$.

Let $\Gamma = \{(X_i, u_i) : i \in I\}$ be a *Nash-type random game*, i.e.,

1. X_i is the *strategy set* of player i, and
2. $u_i : \Omega \times X \to R$ (where $X = \Pi_{i \in I} X_i$) is the *random payoff function* of player i.

Let $\hat{X}_i = \Pi_{j \neq i} X_j$ and denote the elements of $\hat{X}_i$ by $\hat{x}_i$. A *random Nash equilibrium* for Γ is a measurable function $x^*: \Omega \to X$ such that for all i and almost all $\omega \in \Omega$ we have

$$u_i(\omega, x^*(\omega)) = \max_{y_i \in X_i} u_i(\omega, y_i, \hat{x}_i^*(\omega)).$$

We now state our first random equilibrium existence result.

Theorem 3.2 *Let $\mathcal{E} = \{(X_i, P_i) : i \in I\}$ be a random game satisfying for each i the following properties:*

1. *each X_i is a nonempty, compact, and convex subset of R^ℓ,*
2. *each $\operatorname{co} P_i$ is lower measurable, i.e., for every open subset V of X_i the set*
$$\{(\omega, x) \in \Omega \times X : \operatorname{co} P_i(\omega, x) \cap V \neq \emptyset\}$$
belongs to $\Sigma \otimes \beta(X)$,
3. *for every measurable function $x: \Omega \to X$ we have $x_i(\omega) \notin \operatorname{co} P_i(\omega, x(\omega))$ for almost all $\omega \in \Omega$, and*
4. *for each $\omega \in \Omega$ the set-valued function $P_i(\omega, \cdot)$ is l.s.c.*

Then there exists a random equilibrium for E.

Proof: For each $i \in I$ define the correspondence $\varphi_i : \Omega \times X \to 2^{X_i}$ by $\varphi_i(\omega, x) = \operatorname{co} P_i(\omega, x)$. Since by assumption (4) each $P_i(\omega, \cdot)$ is l.s.c., it follows from Theorem 2.1 that for each fixed $\omega \in \Omega$ the function $\varphi_i(\omega, \cdot)$ is l.s.c. Furthermore, by assumption (2), the function φ_i is lower measurable and clearly convex valued. Next, let

$$O_i = \{(\omega, x) \in \Omega \times X : \varphi(\omega, x) \neq \emptyset\},$$

and

$$O_i^\omega = \{x \in X : (\omega, x) \in O_i\} \quad \text{and} \quad O_i^x = \{\omega \in \Omega : (\omega, x) \in O_i\}.$$

From Theorem 2.4, it follows that there exists a Carathéodory selection f_i for φ_i. I.e., there exists a function $f_i : O_i \to X_i$ such that

- $f_i(\omega, x) \in \varphi_i(\omega, x)$ for all $(\omega, x) \in O_i$,
- for each $x \in X$ the function $f_i(\cdot, x)$ is measurable on O_i^x, and
- for each $\omega \in \Omega$ the function $f_i(\omega, \cdot)$ is continuous on O_i^ω.

Now for each $i \in I$ define the correspondence $F_i: \Omega \times X \to 2^{X_i}$ by

$$F_i(\omega, x) = \begin{cases} \{f_i(\omega, x)\}, & \text{if } (\omega, x) \in O_i; \\ X_i, & \text{if } (\omega, x) \notin O_i \,. \end{cases}$$

Clearly, F is nonempty, closed, and convex valued and by Lemma 2.12 it is also lower measurable. Since for each fixed $\omega \in \Omega$ the function $\varphi_i(\omega, \cdot)$ is l.s.c., the set

$$O_i^\omega = \{x \in X : \varphi_i(\omega, x) \neq \varnothing\} = \{x \in X : \varphi_i(\omega, x) \cap X_i \neq \varnothing\}$$

is open in the relative topology of X, and consequently for each fixed $\omega \in \Omega$ the function $F_i(\omega, \cdot)$ is u.s.c.; see [32, Lemma 6.1].

Next, define the correspondence $F: \Omega \times X \to 2^X$ by $F(\omega, x) = \Pi_{i \in I} F_i(\omega, x)$. Clearly, F is nonempty, closed and convex valued. Since each F_i is lower measurable, it follows from Theorem 2.6 that F is lower measurable as well. By Theorem 2.2, the correspondence $F(\omega, \cdot): X \to 2^X$ is u.s.c. for each $\omega \in \Omega$. Furthermore, F satisfies the hypotheses of Corollary 2.11 and consequently there exists a random fixed point, i.e., there exists a measurable function $x^*: \Omega \to X$ such that $x^*(\omega) \in F(\omega, x^*(\omega))$ for almost all $\omega \in \Omega$.

Finally, we shall show that the random fixed point is by construction a random equilibrium for the game E. Notice that if $(\omega, x^*(\omega)) \in O_i$ for all $\omega \in S$ with $\mu(S) > 0$, then by the definition of F_i, we have $x_i^*(\omega) = f_i(\omega, x^*(\omega)) \in \operatorname{co} P_i(\omega, x^*(\omega))$, contrary to assumption (3). Thus, $(\omega, x^*(\omega)) \notin O_i$ holds for almost all $\omega \in \Omega$ and all $i \in I$. In other words, we have $\operatorname{co} P_i(\omega, x^*(\omega)) = \varnothing$ for almost all $\omega \in \Omega$ and all i, which in turn implies that $P_i(\omega, x^*(\omega)) = \varnothing$ for almost all $\omega \in \Omega$ and all $i \in I$. That is, $x^*: \Omega \to X$ is a random equilibrium for E, and the proof of the theorem is finished. ∎

As a corollary of Theorem 3.2 we obtain a generalized random version of Nash's equilibrium existence result [22, Theorem 1, p. 288].

Corollary 3.3 *Let $\Gamma = \{(X_i, u_i) : i \in I\}$ be a Nash-type random game satisfying for each i the following assumptions:*

1. *each X_i is a nonempty, compact, and convex subset of R^ℓ,*
2. *for each fixed $\omega \in \Omega$ the function $u_i(\omega, \cdot)$ is continuous,*
3. *for each fixed $x \in X$ the function $u_i(\cdot, x)$ is measurable, and*
4. *for each $\omega \in \Omega$ and each $\hat{x}_i \in \hat{X}_i$ the function $u_i(\omega, x_i, \hat{x}_i)$ is quasiconcave in x_i.*

Then there exists a random Nash equilibrium for Γ.

Proof: For each $i \in I$, define the correspondence $Q_i : \Omega \times X \to 2^{X_i}$ by

$$Q_i(\omega, x) = \{y_i \in X_i : h_i(\omega, x, y_i) > 0\},$$

where $h_i(\omega, x, y_i) = u_i(\omega, y_i, \hat{x}_i) - u_i(\omega, x)$. Letting $S = \Omega \times X$, $Z = X_i$, $\alpha = \Sigma \otimes \beta(X)$, $g(s, z) = h_i(\omega, x, y_i)$, $K(s) = Q_i(\omega, x)$ for $s = (\omega, x)$ in Lemma 2.13(b), we can conclude that each Q_i is lower measurable. It follows from assumption (4) that each Q_i is convex valued, and clearly for any measurable function $x : \Omega \to X$ we have $x_i(\omega) \notin \operatorname{co} Q_i(\omega, x(\omega)) = Q_i(\omega, x(\omega))$ for almost all $\omega \in \Omega$. Moreover, it follows from assumption (2) that each set-valued function $Q_i(\omega, \cdot)$ has an open graph in $X \times X_i$. Hence, the random game $E = \{(X_i, Q_i) : i \in I\}$ satisfies the assumptions of Theorem 3.2 and therefore E has a random equilibrium. That is, there exists a measurable function $x^* : \Omega \to X$ such that $Q_i(\omega, x^*(\omega)) = \varnothing$ for almost all $\omega \in \Omega$ and all $i \in I$. But this implies

$$u_i(\omega, x^*(\omega)) = \max_{y_i \in X_i} u_i(\omega, x_1^*(\omega), \ldots, x_{i-1}^*(\omega), y_i, x_{i+1}^*(\omega), \ldots),$$

for almost all $\omega \in \Omega$ and all $i \in I$, i.e., x^* is a random Nash equilibrium for the game $\Gamma = \{(X_i, u_i) : i \in I\}$, as claimed. ◻

We now provide an extension of Theorem 3.2 to strategy sets which may be subsets of a separable Banach space.

Theorem 3.4 *Let $\mathcal{E} = \{(X_i, P_i) : i \in I\}$ be a random game satisfying for each i the following assumptions:*

1. *each X_i is a nonempty, compact, and convex subset of a separable Banach space,*

2. *each $\operatorname{co} P_i$ has a measurable graph, i.e.,*

$$\{(\omega, x, y_i) \in \Omega \times X \times X_i : y_i \in \operatorname{co} P_i(\omega, x)\} \in \Sigma \otimes \beta(X) \otimes \beta(X_i),$$

3. *for every measurable function $x : \Omega \to X$ we have $x_i(\omega) \notin \operatorname{co} P_i(\omega, x(\omega))$ for almost all $\omega \in \Omega$, and*

4. *for each $\omega \in \Omega$ the set-valued function $P_i(\omega, \cdot)$ has an open graph in $X \times X_i$.*

Then $\mathcal{E}$ has a random equilibrium.

Proof: For each $i \in I$ define the correspondence $\varphi_i : \Omega \times X \to 2^{X_i}$ by $\varphi_i(\omega, x) = \operatorname{co} P_i(\omega, x)$. Since, by assumption (4), each $P_i(\omega, \cdot)$ has an open graph in $X \times X_i$, it can be easily checked (see [33, Lemma 4.1]) that so does $\varphi_i(\omega, \cdot)$ for each $\omega \in \Omega$.

Let $O_i = \{(\omega, x) \in \Omega \times X : \varphi_i(\omega, x) \neq \varnothing\}$. Since φ_i has a measurable graph (by hypothesis (2)) and it is convex valued, Theorem 2.5 guarantees the existence of a Carathéodory selection φ_i. To complete the proof now proceed as in the proof of Theorem 3.2. ∎

The following corollary of Theorem 3.4 extends Corollary 3.3 to strategy sets which may be subsets of arbitrary separable Banach spaces. We thus have a random version of Nash's result [22, Theorem 1, p. 288] in separable Banach spaces. It should be noted that Corollary 3.5 below may be seen as a random generalization of the deterministic equilibrium existence results of K. Fan [13, Theorem 4, p. 192] and F. E. Browder [8, Theorem 14, p. 277], but only if the underlying strategy space is separable. Note the latter assumption is needed in order to make the Aumann Measurable Selection Theorem applicable. It is worth noting that Fan and Browder allow only for a finite number of players whereas in our setting the set of players may be countable.

Corollary 3.5 *Replace assumption* (1) *in Corollary 3.3 by*

(1′) X_i *is a nonempty, compact, and convex subset of a separable Banach space.*

Then the conclusion of Corollary 3.3 remains true.

Proof: The proof is identical with that of Corollary 3.3 taking into account that one now has to use Lemma 2.13(a) to show that Q_i has a measurable graph, and appeal to Theorem 3.4 instead of Theorem 3.2. ∎

A couple of comments are in order. Notice that the continuity assumption (4) in Theorem 3.2 is weaker than the continuity assumption (4) of Theorem 3.4. The reason we need a weaker continuity assumption is that the proof of Theorem 3.2 makes use of Theorem 2.4 which is a Carathéodory selection result for a correspondence that is lower measurable in one variable and l.s.c. in the other. However, in the proof of Theorem 3.4 a different Carathéodory selection result is used (Theorem 2.5) which requires a stronger continuity assumption. Moreover, observe that Corollary 3.3 follows directly from Corollary 3.5. Nevertheless, we choose to state Corollary 2.11 since its proof by means of Theorem 3.2 is slightly different than the proof of Corollary 3.3 which follows from Theorem 3.4. Finally, it is important to note that the proofs of Theorems 3.2 and 3.4 do not use any deterministic equilibrium existence results. To the contrary, our arguments start from scratch and provide alternative ways to prove the equilibrium results of Nash, Fan, and Browder.[3]

4. Bayesian games and equilibria

We now turn to the problem of the existence of equilibrium points for Bayesian games. Again, let (Ω, Σ, μ) be a complete finite measure space. We still denote by I the set of players, where I can be finite or countable.

Definition 4.1 *A* Bayesian game *on the complete finite measure space* (Ω, Σ, μ) *is a set* $\mathcal{G} = \{(X_i, h_i, S_i, q_i) : i \in I\}$ *of quadruples such that*

1. *each* X_i *is the strategy set of player* i,

[3] An alternative proof of a version of Theorem 3.4 will be given in Section 7.

2. $h_i: \Omega \times X \to R$ *(where* $X = \Pi_{i \in I} X_i$*) is the random payoff function of player* i,

3. S_i *is a measurable partition of* (Ω, Σ) *denoting the (private) information available to player* i*, and*

4. $q_i: \Omega \to (0, \infty)$ *is the prior probability density of player* i*, i.e.,* q_i *is a measurable function having the property that* $\int_\Omega q_i(t)\, d\mu(t) = 1$.

As in R. J. Aumann [3] or R. Myerson [21] it is assumed that the game $\mathcal{G} = \{(X_i, h_i, S_i, q_i) : i \in I\}$ is common knowledge, i.e., every player knows $\mathcal{G}$, every player knows that every player knows $\mathcal{G}$, every player knows that every player knows that every player knows $\mathcal{G}$, and so on.

We first consider the case where the information set of each player i is the same, i.e., $S_i = S$ for each $i \in I$. Denote by $E(\omega)$ the event in S which contains the realized state of nature $\omega \in \Omega$, and suppose that $q_i(E(\omega)) > 0$ for all $i \in I$. Given $E(\omega)$ in S the conditional expected utility of player i is the function $v_i: \Omega \times X \to R$ defined by

$$v_i(\omega, x) = \int_{E(\omega)} q_i\big(t|E(\omega)\big) h_i(t, x)\, d\mu(t), \tag{4.1}$$

where

$$q_i\big(t|E(\omega)\big) = \begin{cases} 0, & \text{if } t \notin E(\omega); \\ \frac{q_i(t)}{\int_{E(\omega)} q_i(s)\, d\mu(s)}, & \text{if } t \in E(\omega)\,. \end{cases}$$

A *Bayesian equilibrium* for a Bayesian game

$$\mathcal{G} = \{(X_i, h_i, S_i, q_i) : i \in I\}$$

is a function $x^*: \Omega \to X$ such that each $x_i^*(\cdot)$ is S-measurable and for each $i \in I$ we have

$$v_i\big(\omega, x^*(\omega)\big) = \max_{y_i \in X_i} v_i\big(\omega, x_1^*(\omega), \ldots, x_{i-1}^*(\omega), y_i, x_{i+1}^*(\omega), \ldots\big)$$

for almost all $\omega \in \Omega$, where v_i is given by (4.1).

We are now ready to state our fist Bayesian equilibrium existence theorem.

Theorem 4.2 *Let* $\mathcal{G} = \{(X_i, h_i, S_i, q_i) : i \in I\}$ *be a Bayesian game satisfying for each* $i \in I$ *the following properties.*

1. *each* X_i *is a nonempty, compact and convex subset of a separable Banach space* Y,

2. *for each fixed* $\omega \in \Omega$ *the function* $h_i(\omega, \cdot)$ *is continuous,*

3. *for each fixed* $x \in X$ *the function* $h_i(\cdot, x)$ *is measurable,*

4. *for each* $\omega \in \Omega$ *and each* $\hat{x} \in \hat{X}_i (= \Pi_{j \neq i} X_j)$ *the function* $h_i(\omega, x_i, \hat{x}_i)$ *is concave in* x_i*, and*

5. *each h_i is integrably bounded.*

Then the game $\mathcal{G}$ has a Bayesian equilibrium.

Proof: The result follows directly from Corollary 3.5. To see this, note that since each $h_i(\omega,\cdot)$ is continuous and h_i is integrably bounded by virtue of the Lebesgue Dominated Convergence Theorem, we can automatically conclude that the function

$$v_i(\omega,\cdot) = \int_{E(\omega)} q_i\big(t|E(\omega)\big)h(t,\cdot)\,d\mu(t)$$

is continuous, where

$$q_i\big(t|E(\omega)\big) = \begin{cases} 0, & \text{if } t \notin E(\omega); \\ \frac{q_i(t)}{\int_{E(\omega)} q_i(s)\,d\mu(s)}, & \text{if } t \in E(\omega). \end{cases}$$

Furthermore, it can be easily seen that each function $v_i(\cdot,x)$ is S-measurable. Finally, it follows from (4) that for each $\omega \in \Omega$ and each $\hat{x}_i \in \hat{X}_i$ that the function $v_i(\omega,x_i,\hat{x}_i)$ is concave in x_i. We consider the Bayesian game $\mathcal{G} = \{(X_i,h_i,S,q_i) : i \in I\}$ as a random game $\mathcal{E} = \{(X_i,v_i) : i \in I\}$. Obviously, the existence of a random Nash equilibrium for the game $\mathcal{E}$ implies the existence of a Bayesian equilibrium for the game $\mathcal{G}$. It can be easily seen that the random game $\mathcal{E}$, satisfies the assumptions of Corollary 3.5 and consequently, the game $\mathcal{E}$ has a random Nash equilibrium.[4] Hence, there exists an S-measurable function $x^*:\Omega \to X$ such that

$$v_i\big(\omega,x^*(\omega)\big) = \max_{y_i \in X_i} v_i\big(\omega,x_1^*(\omega),\ldots,x_{i-1}^*(\omega),y_i,x_{i+1}^*(\omega),\ldots\big)$$

for almost all $\omega \in \Omega$ and all $i \in I$. In other words, x^* is a Bayesian equilibrium for the game $\mathcal{G} = \{(X_i,h_i,S,q_i) : i \in I\}$, and the proof of the theorem is finished. ∎

5. Asymmetric Bayesian games

We now turn our attention to the rather more interesting case where the information set of each player is different.

Let $\mathcal{G} = \{(X_i,h_i,S,q_i) : i \in I\}$ be a Bayesian game as described before. Denote by L_{X_i} the set of all Bochner integrable and S_i-measurable selections from the strategy set X_i of player i, i.e.,

$$L_{X_i} = \{x_i \in L_1(\mu,Y) : x_i \text{ is } S_i\text{-measurable and } x_i(\omega) \in X_i \text{ for } \mu\text{-a.e. } \omega\}.$$

[4] Note that the proofs of Theorems 3.2 and 3.4 and Corollaries 3.3 and 3.5 remain unchanged if the measurability assumptions on either the preference correspondence P_i or the payoff function u_i of each player are made with respect to the algebra generated by the partition S_i instead of Σ.

Let $L_X = \Pi_{i \in I} L_{X_i}$. Denote by $E_i(\omega)$ the event in S_i containing the true state of nature $\omega \in \Omega$ and suppose that $q_i(E_i(\omega)) > 0$ for all $i \in I$. Given $E_i(\omega)$ in S_i define the conditional expected utility function $v_i: \Omega \times L_X \to R$ of player i by

$$v_i(\omega, x) = \int_{E_i(\omega)} q_i\big(t|E_i(\omega)\big) h_i\big(t, x(t)\big)\, d\mu(t). \tag{5.1}$$

A *Bayesian equilibrium* for $\mathcal{G} = \{(X_i, h_i, S, q_i) : i \in I\}$ is an element $x^* \in L_X$ such that

$$v_i(\omega, x^*) = \max_{y_i \in L_{X_i}} v_i\big(\omega, x_1^*(\omega), \ldots, x_{i-1}^*(\omega), y_i, x_{i+1}^*(\omega), \ldots\big),$$

where v_i is given by formula (5.1).

We now state the following result.

Theorem 5.1 *Let $\mathcal{G} = \{(X_i, h_i, S, q_i) : i = 1, 2, \ldots, n\}$ be a Bayesian game satisfying the properties:*

1. *the measure space (Ω, Σ, μ) is finite, separable and complete,*
2. *each X_i is a nonempty, convex, and weakly compact subset of a separable Banach space Y whose dual Y^* has the RNP,*
3. *each function $h_i(\omega, \cdot)$ is weakly continuous,*
4. *each function $h_i(\cdot, x)$ is measurable,*
5. *for each $\omega \in \Omega$ and each $\hat{x} \in \hat{X}$ the function $h_i(\omega, x_i, \hat{x}_i)$ is concave in x_i, and*
6. *each h_i is integrably bounded.*

Then $\mathcal{G}$ has a Bayesian equilibrium.

Proof: For each $i \in I$ define the correspondence $\varphi_i: L_{\hat{X}_i} \to 2^{L_{X_i}}$ by

$$\varphi(\hat{x}_i) = \Big\{y_i \in L_{X_i} : v_i(\omega, y_i, \hat{x}_i) = \max_{x_i \in L_{X_i}} v_i(\omega, x_i, \hat{x}_i) \text{ for almost all } \omega \in \Omega\Big\}.$$

Also, define the correspondence $F: L_X \to 2^{L_X}$ by

$$F(x) = \prod_{i \in I} \varphi_i(\hat{x}_i).$$

We shall show that the correspondence F satisfies the hypotheses of the Fan–Glicksberg Fixed Point Theorem (see for instance [12]). It can be easily seen that a fixed point of the correspondence F is by construction a Bayesian equilibrium for the game $\mathcal{G}$. We shall complete the proof by several steps.

I. *L_X is nonempty, convex, weakly compact and metrizable.*

The proof of this claim is similar to that of Theorem 3.1 in [36] but we provide an outline for the sake of completeness. First note that since (Ω, Σ, μ) is separable and Y is separable, $L_1(\mu, Y)$ is a separable Banach space. Since by assumption each X_i is nonempty, convex and weakly compact, it follows from Diestel's Theorem that each L_{X_i} is a weakly compact subset of $L_1(\mu, Y)$. Obviously, each L_{X_i} is convex since each X_i is convex and by virtue of the Aumann Measurable Selection Theorem, we can conclude that each L_{X_i} is also nonempty. Furthermore, since each L_{X_i} is a weakly compact subset of the separable Banach space $L_1(\mu, Y)$, it is also metrizable; see [1, Theorem 10.11, p. 154]. Clearly, $L_X = \Pi_{i \in I} L_{X_i}$ is nonempty, convex, weakly compact and metrizable as well.

II. *The function $v_i(\omega, \cdot)$ is weakly continuous for each $\omega \in \Omega$.*

Fix $i \in I = \{1, 2, \ldots, n\}$, $\omega \in \Omega$ and $E_i(\omega) \in S_i$. Let $\{x_n\}$ be a sequence of L_X converging weakly[5] to $x \in L_X$, i.e., the sequence $\{x_n^i : n = 1, 2, \ldots\}$ of L_{X_i} converges weakly to $x^i \in L_{X_i}$ for each $i \in I$. We must show that the sequence $\{x_n^i \chi_{E_i(\omega)}\}$ converges pointwise in the weak topology of X_i to $x^i \chi_{E(\omega)}$ for each i. Then in view of (3) and (6) the result will follow from the Lebesgue Dominated Convergence Theorem.

Now if $S_i = \{E_i^1, E_i^2, \ldots\}$ is a partition of player i, then the fact that x_n^i and x^i belong to L_{X_i} implies that

$$x_n^i = \sum_{k=1}^{\infty} x_n^{i,k} \chi_{E_i^k} \quad \text{and} \quad x^i = \sum_{k=1}^{\infty} x^{i,k} \chi_{E_i^k},$$

with $x_n^{i,k}$, $x^{i,k} \in X_i$, and therefore we can conclude that

$$x_n^i \chi_{E_i(\omega)} = \sum_{k=1}^{\infty} x_n^{i,k} \chi_{E_i^k \cap E_i(\omega)}$$

converges weakly to $x^i \chi_{E_i(\omega)} = \sum_{k=1}^{\infty} x^{i,k} \chi_{E_i^k \cap E_i(\omega)}$.

III. *Each correspondence $\varphi_i : L_{\hat{X}_i} \to 2^{L_{x_i}}$ is nonempty, convex valued and weakly u.s.c.*

It follows from assumption (5) that for each $\omega \in \Omega$ and for each $\hat{x} \in L_{\hat{X}_i}$ that $v_i(\omega, x_i, \hat{x}_i)$ is a concave function of x_i on L_{X_i}, and therefore we can conclude that φ_i is convex valued. By virtue of Berge's Maximum Theorem (see for instance [7, Theorem 12.1]), we see that φ_i is weakly u.s.c. Finally, an appeal to the Weierstrass' Theorem guarantees that φ_i is also a nonempty valued correspondence.

Now since each φ_i is nonempty, closed, convex valued and weakly u.s.c., it follows from Theorem 2.2 that likewise is $F: L_X \to 2^{L_X}$ (and L_X is weakly

[5] Let $\{f_n\}$ be a sequence in $L_1(\mu, Y)$. Then $\{f_n\}$ converges weakly to f if and only if $\langle f_n, p \rangle$ (the value of f_n at p) converges to $\langle f, p \rangle$ for any $p \in L_\infty(\mu, Y^*)$ (recall that Y^* has the RNP). The latter is equivalent to saying that $\langle f_n \chi_A, p \rangle = \langle f_n, \chi_A p \rangle$ converges to $\langle f, \chi_A p \rangle = \langle f_n \chi_A, p \rangle$ for each $p \in L_\infty(\mu, Y^*)$ and each $A \in \Sigma$. Each condition above implies that $\langle f_n \chi_A, x^* \rangle = \langle f_n, \chi_A x^* \rangle$ converges to $\langle f \chi_A, x^* \rangle = \langle f, \chi_A x^* \rangle$ for each $x^* \in Y^*$ and each $A \in \Sigma$.

compact). Hence, the correspondence F satisfies all the conditions of the Fan-Glicksberg Fixed Point Theorem. Consequently, there exists some $x^* \in L_X$ such that $x^* \in F(x^*)$. Now it is a routine matter to verify that x^* is a Bayesian equilibrium for $\mathcal{G} = \{(X_i, h_i, S, q_i) : i = 1, 2, \ldots, n\}$. This completes the proof of the theorem. ∎

Remarks. The proof of Theorem 5.1 remains unchanged if (2) is replaced by:

(2′) $X_i : \Omega \to 2^Y$ is a nonempty, weakly compact and convex valued correspondence having a measurable graph.

We now indicate how one can prove the existence of a pure strategy Bayesian equilibrium. Denote by X_i^e the set of all extreme points of X_i. A *pure strategy Bayesian equilibrium* for $\mathcal{G} = \{(X_i, h_i, S_i, q_i) : i \in I\}$ is some element x^* in the set

$$\prod_{i=1}^{n} \int X_i^e(\omega)\, d\mu(\omega) =$$

$$\prod_{i=1}^{n} \Big\{ \int z_i(\omega)\, d\mu(\omega) : z_i(\cdot) \text{ is } S_i\text{-measurable and } z_i(\omega) \in X_i^e(\omega)\ \mu\text{-a.e.} \Big\}$$

such that for all i, we have

$$v_i(\omega, x^*) = \max_{y_i \in \int X_i^e} v_i(\omega, x_1^*, \ldots, x_{i-1}^*, y_i, x_{i+1}^*, \ldots, x_n^*)$$

for almost all $\omega \in \Omega$ (where v_i is defined as in (5.1)).

For the next result, we will assume that S_i is a σ-subalgebra of Σ.

Theorem 5.2 *Let $\mathcal{G} = \{(X_i, h_i, S_i, q_i) : i = 1, 2, \ldots, n\}$ be a Bayesian game satisfying assumptions* (1), (4), (6), *and* (7) *of Theorem 5.1 in addition to the following conditions:*

1. *(Ω, S_i, μ) is an atomless measure space,*

2. *$X_i : \Omega \to 2^{R^\ell}$ is a nonempty, closed, convex and integrably bounded correspondence with a measurable graph and for each $\omega \in \Omega$ the function $h_i(\omega, \cdot)$ is linear and continuous on X.*

Then there exists a pure strategy Bayesian equilibrium for $\mathcal{G}$.

Proof: First note that since for each fixed $\omega \in \Omega$ the function $h_i(\omega, \cdot)$ is linear and continuous on X the domain of v_i is now $\Omega \times \Pi_{i=1}^n \int X_i$. Let $\int X = \Pi_{i=1}^n \int X_i$. The set $\int X$ will turn out to be equal to $\int X^e = \Pi_{i=1}^n \int X_i^e$ as we shall show below.

For each i define the correspondence $\varphi_i: \int \hat{X}_i^e \to 2^{\int X_i^e}$ by

$$\varphi(\hat{x}_i) = \left\{ y_i \in \int X_i^e : v_i(\omega, y_i, \hat{x}_i) = \max_{x_i \in \int X_i^e} v_i(\omega, x_i, \hat{x}_i) \text{ for almost all } \omega \in \Omega \right\}.$$

Also, define $F: \int X^e \to 2^{\int X^e}$ by $F(x) = \Pi_{i=1}^n \varphi_i(\hat{x}_i)$. Note that by repeating the proof of Theorem 5.1, we can see that each φ_i is nonempty, closed, and convex valued and u.s.c. Clearly, a fixed point of F is a pure strategy Bayesian equilibrium for the game $\mathcal{G}$. If we establish that $\int X = \int X^e$ and that $\int X$ is compact, convex and nonempty, then we are done.

Since X_i is a compact convex set, it follows from the Krein–Milman Theorem that $\mathrm{co}(X_i^e) = X_i$. By [4, Theorem 3, p. 2], we have $\int \mathrm{co}(X_i^e) = \int X_i^e = \int X_i$, and therefore $\int X = \int X^e$. Moreover, by [4, Theorem 4, p. 2], $\int X_i$ is compact. Hence $\int X_i$ is compact, convex and nonempty (the nonemptiness follows from the Measurable Selection Theorem), and so is $\int X$. Now by the Kakutani Fixed Point Theorem there exists $x^* \in \int X^e$ such that $x^* \in F(x^*)$, i.e., x^* is a pure strategy equilibrium for the game $\mathcal{G}$. ∎

If assumption (2) of Theorem 5.2 is replaced by

(2′) $X_i: \Omega \to 2^Y$ (where Y is a separable Banach space whose dual has the RNP) is a nonempty, weakly compact, convex and integrably bounded correspondence having a measurable graph,

then only an approximate pure strategy Bayesian equilibrium can be obtained. The reason is that Aumann's Theorem 3 in [4] is no longer true. (See, for instance, [29] or [36] for a counterexample.) In particular, in this case we have only that $\int X = \int \overline{\mathrm{co}}(X^e) = \overline{\int X^e}$. Moreover, by [35, Lemma 31, p. 307], $\int X$ is weakly compact. Carrying out now the argument outlined in the proof of Theorem 5.2 one can easily prove the existence of an approximate pure strategy Bayesian equilibrium. For other results on approximate purification of mixed strategies see [5, 20, 27, 28].

We close the section by mentioning that all the equilibrium existence results in Sections 3 and 4 can be easily extended to abstract economies as defined in [10], [31], and [32]. Moreover, one can use the equilibrium results for abstract economies to obtain equilibrium existence theorems for random exchange economies or Bayesian exchange economies. In particular, in this setting of incomplete information the appropriate equilibrium notion is that of a rational expectations equilibrium. We hope to take up these details, however, in a subsequent paper.

6. Related literature

The equilibrium existence results for games with incomplete information which are related to Theorems 4.2 and 5.1 that we know of, are those in E. J. Balder [6],

P. Milgrom and R. Weber [20] and R. Radner and R. Rosenthal [27].[6]

Their approach is based on distributional strategies and it is entirely different than ours, which is based on measurable functions. For purposes of comparison, it may be instructive to briefly outline their approach. Following [20] a game is a sextuple $\mathcal{G} = (N, \{T_i\}_{i\in N}, \{A_i\}_{i\in N}, T_0, \{u_i\}_{i\in N}, \zeta)$, where

1. $N = \{1, 2, \ldots, n\}$ is the set of *players*.
2. $\{T_i : i \in N\}$ is the set of *types* for each player. Each T_i is a complete and separable metric space.
3. $\{A_i : i \in N\}$ is the set of *actions* for each player. Each A_i is a compact metric space.
4. T_0 is the set of possible *states*; T_0 is a complete and separable metric space.
5. $u_i : T \times A \to R$ (where $T = T_0 \times \cdots \times T_n$ and $A = A_1 \times \cdots \times A_n$) is the *payoff function* of player i. Each u_i is bounded and measurable.
6. ζ is the *information structure* and is a probability measure on the Borel subsets of T. Denote by ζ_i the marginal distribution on each T_i.

A *distributional strategy* for player i is a probability measure μ_i on the Borel subsets of $T_i \times A_i$ such that the marginal distribution of T_i is ζ_i. The expected payoff of player i is:

$$V_i(\mu_1, \ldots, \mu_n) = \int u_i(t, a)\, \mu_1(da_1|t_1) \ldots \mu_n(da_n|t_n)\, \zeta(dt). \qquad (6.1)$$

The two basic assumptions that P. Milgrom and R. Weber [20] make are:

a. Payoffs are equicontinuous; and

b. The information structure is absolutely continuous.

Conditions which imply either (a) or (b) are given in [20, p. 625]. Balder has succeeded in generalizing their results by relaxing (a), but he still needs (b).[7] For the proof of Theorem 4.2 we did not make use of any of these assumptions and no equicontinuity assumption was needed for the proof of Theorem 5.1. It is important to note that assumption (b) allows the above authors to express the expected utility (6.1) in a convenient way (see [20, p. 625] or [6]). In particular, once distributional strategies are topologized with the weak convergence, the strategy sets are compact metric spaces, the expected utility is continuous and linear and therefore the standard results of either Glicksberg, Fan, or Browder (see [20] or [6]) can be directly applied to prove the existence of an equilibrium.

[6]Since the connection between [20] and [27] has already been discussed by P. Milgrom and R. Weber elsewhere (see [20] for an exact reference), we shall focus on the mixed strategy equilibrium existence results given in [6] and [20].

[7]It should also be mentioned that Balder does not impose any topological structure on the type spaces T_i.

We would like to note that in our framework the expected utility is required to be only quasiconcave in each player's own strategy. Moreover, our expected utility is random, i.e., depends on the state of the world. The latter is quite important since with random expected payoffs the Fan–Glicksberg result is not directly applicable and the use of measurable selection theorems seems to be needed.

Although it is not obvious how one from the approach of Milgrom–Weber and Balder can obtain versions of our Theorems 4.2 and 5.1, it is very clear that these theorems are not subsumed by any of their results. In particular, no assumption of equicontinuity of payoffs is needed and the set of players in Theorem 4.2 is not necessarily finite. It may be instructive to note that our approach, i.e., working with strategies which are measurable functions, seems to be quite natural to analyze economies with incomplete information as recently by T. Palfrey and S. Srivastava [23, 24] and A. Postlewaite and D. Schmeidler [26] or uncertainty in market games examined in J. Peck and K. Shell [25]. In fact, our approach as well as Theorems 4.2 and 5.1 have been motivated from the work of the above authors.

Finally, we would like to note that A. Mas-Colell [18], viewing a game as a probability measure on the space of utility functions, has proved Nash equilibrium existence theorems. He also indicates that his existence results may be useful to obtain results for games with incomplete information.

7. Concluding remarks

We now show how a version of Theorem 3.4 can be easily obtained by combining the deterministic equilibrium result of N. C. Yannelis and N. D. Prabhakar [32] with the Aumann Measurable Selection Theorem.

Theorem 7.1 *The conclusion of Theorem 3.4 remains true if one replaces assumptions* (2) *and* (4) *by:*

(2′) $\operatorname{co} P_i$ *is lower measurable, i.e., for every open set* V *in* X_i *the set*

$$\{(\omega, x) : \operatorname{co} P_i(\omega, x) \cap V \neq \varnothing\}$$

belongs to $\Sigma \otimes \beta(X)$*; and*

(4′) *For each* $\omega \in \Omega$ *the function* $P_i(\omega, \cdot)$ *has open lower sections, i.e., for each* $\omega \in \Omega$ *and for each* $y_i \in X_i$ *the set* $P^{-1}(\omega, y_i) = \{x \in X : y_i \in P_i(\omega, x)\}$ *is open in* X.

Proof: For each $i \in I$ define $\varphi_i : \Omega \times X \to 2^{X_i}$ by $\varphi_i(\omega, x) = \operatorname{co} P_i(\omega, x)$. By assumption (a) each function φ_i is lower measurable. Define the correspondence $F: \Omega \times X \to 2^X$ by $F(\omega, x) = \Pi_{i \in I} \varphi_i(\omega, x)$. By virtue of Theorem 2.6 the function F is lower measurable. Define the correspondence $\Gamma: \Omega \to 2^X$ by

$$\Gamma(\omega) = \{x \in X : F(\omega, x) = \varnothing\}.$$

We shall show that there exists a measurable selection for Γ which will turn out to be a random equilibrium for the random game $\mathcal{E} = \{(X_i, P_i) : i \in I\}$.

In order to apply the Aumann Measurable Selection Theorem 2.3, we need to show that Γ has a measurable graph and is nonempty valued. Since F is lower measurable, the set

$$K = \{(\omega, x) \in \Omega \times X : F(\omega, x) \neq \emptyset\} = \{(\omega, x) \in \Omega \times X : F(\omega, x) \cap X \neq \emptyset\}$$

belongs to $\Sigma \otimes \beta(X)$ and so does its complement K^c. Now observe that

$$\begin{aligned} G_\Gamma &= \{(\omega, x) \in \Omega \times X : x \in \Gamma(\omega)\} \\ &= \{(\omega, x) \in \Omega \times X : F(\omega, x) = \emptyset\} \\ &= \{(\omega, x) \in \Omega \times X : F(\omega, x) \neq \emptyset\}^c \\ &= K^c, \end{aligned}$$

and the latter set belongs to $\Sigma \otimes \beta(X)$ as it was noted above. Therefore, Γ has a measurable graph. Moreover, an appeal to [32, Theorem 6.1, p. 242] (where in [32] for each $i \in I$ and for each $x \in X$ we let $A_i(x) = \overline{A}_i(x) = X_i$) shows that for each $\omega \in \Omega$ we have $\Gamma(\omega) \neq \emptyset$. Therefore, by the Aumann Measurable Selection Theorem there exists a measurable function $x^*: \Omega \to X$ such that $x^*(\omega) \in \Gamma(\omega)$ for almost all $\omega \in \Omega$, i.e., $F(\omega, x^*(\omega)) = \emptyset$ for almost all $\omega \in \Omega$. The latter implies that for each $i \in I$ we have $P_i(\omega, x^*(\omega)) = \emptyset$ for almost all $\omega \in \Omega$, i.e., x^* is a random equilibrium for the game $\mathcal{E}$. ∎

Note that in Theorem 7.1 the assumption that (Ω, Σ, μ) is a complete finite measure space can be replaced by the fact that (Ω, Σ) is a measurable space. The proof remains the same. In particular, since for each fixed $\omega \in \Omega$ the correspondence $F(\omega, \cdot): X \to 2^X$ has open lower sections, it is also l.s.c. ([32, Proposition 4.1, p. 237]) and therefore Γ is closed valued. Since Γ has a measurable graph and is closed valued, it is also lower measurable [15, Theorem 3.3, p. 60]. One can now appeal to the Kuratowski and Ryll–Nardzewski Measurable Selection Theorem to complete the proof of Theorem 7.1.

Finally, note that assumption (4′) of Theorem 7.1 is weaker than assumption (4) of Theorem 3.4 and assumption (2′) is different from assumption (2) of the same theorem. Hence, neither result implies the other. However, the methods of proof are different. It can be easily seen that Corollary 3.5 follows directly from Theorem 7.1. The idea of the proof is identical with the one used to prove Corollary 3.3.

Remarks. (A) The form of the Bayesian game defined in Section 4 can be generalized by replacing each player's random payoff function $h_i: \Omega \times X \to R$ by a random preference correspondence $P_i: \Omega \times X \to 2^{X_i}$. Following the notation of Section 4, in this new setting the conditional expected payoff $F_i(\omega, x)$ of each player is the integral of the correspondence P_i, i.e.,

$$F_i(\omega, x) = \int_{E(\omega)} q_i(t|E(\omega)) P_i(t, x)\, d\mu(t).$$

By replacing assumptions (2) through (5) in Theorem 4.2 by

(2′) for each fixed $\omega \in \Omega$ the function $P_i(\omega, \cdot)$ is l.s.c.,

(3′) $\mathrm{co}\, F_i$ is lower measurable,

(4′) for each measurable function $x: \Omega \to X$ we have $x_i(\omega) \notin \mathrm{co}\, F_i(\omega, x(\omega))$ for almost all $\omega \in \Omega$, and

(5′) P_i is integrably bounded and has a measurable graph,

and invoking [34, Theorem 3.2] (which asserts that the integral of a l.s.c. correspondence which is integrably bounded and has a measurable graph is also l.s.c.), we can guarantee that for each fixed $\omega \in \Omega$ the function $F_i(\omega, \cdot)$ is l.s.c. Therefore, by appealing to Theorem 3.2 one can prove the existence of a Bayesian equilibrium for this more general form of a Bayesian game.

(B) In Section 5 we remarked that if the dimensionality of the strategy space is infinite, then $\overline{\int X^e} = \int X = \int \overline{\mathrm{co}}(X^e)$ and consequently only an approximate pure strategy equilibrium could be found. However, by assuming that there are "many more players than strategies," i.e., if the "dimension" of the measure space of players is larger than the "dimension" of the strategy space, one can remove the norm closure and obtain $\int X^e = \int X = \int \overline{\mathrm{co}}(X^e)$. Hence an exact pure strategy equilibrium can be obtained. Of course, the concept of dimension has to be given a rigorous formulation. See [30] for a further discussion.

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