Equilibrium concepts in differential information economies ⋆

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Summary. We summarize here basic cooperative and noncooperative equilibrium concepts, in the context of differential information economies with a finite number of agents. These, on the one hand, game theoretic, and, on the other hand, Walrasian equilibrium type concepts are explained, and their relation is pointed out, in the context of specific economies with one or two goods and two or three agents. We analyze the incentive compatibility of several cooperative and noncooperative concepts, and also we discuss briefly the possible implementation of these concepts as perfect Bayesian equilibria through the construction of relevant game trees. This possibility is related to whether the allocation is incentive compatible. This depends on whether there is free disposal or not.

Keywords and Phrases: Differential information economy, Walrasian expectations or Radner equilibrium, Rational expectations equilibria, Free disposal, Weak fine core, Private core, Weak fine value, Private value, Coalitional Bayesian incentive compatibility, Game trees, Perfect Bayesian equilibrium, Sequential equilibrium.

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1 Introduction

The classical Walrasian equilibrium model as formalized by Arrow - Debreu (1954) and McKenzie (1954) consists of a finite set of agents each of which is characterized

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by her preferences and initial endowments. The Walrasian model captures in a deterministic way the trade or contract (redistribution of initial endowments) among the agents and has played a central role in all aspects of economics. For this model significant results have been obtained, i.e. existence and Pareto optimality of the Walrasian equilibrium, equivalence of the Walrasian equilibrium with the core, (see Debreu and Scarf, 1963), and the relation between the core and the Shapley value, (see Emmons and Scafuri, 1985). These results have also been extended in infinite dimensional spaces (see for example Aumann, 1964; and the books of Hildenbrand, 1974; Khan and Yannelis, 1991).

Although infinite dimensional commodity spaces do capture uncertainty, they do not capture trade under asymmetric (or differential) information. On the other hand, it should be noted that most trades in an economy are made by agents who are asymmetrically informed and the need to introduce differential information into the Cournot - Nash model and the Arrow- Debreu - McKenzie model was evident in the seminal works of Harsanyi (1967) and Radner (1968). Their equilibrium concepts are noncooperative and have found extensive applications. In seminal papers, Wilson (1978) and Myerson (1982) introduced private information in the cooperative concepts of the core and the Shapley value respectively.

Briefly, the purpose of this paper is to survey the basic equilibrium concepts in economies with differential information. We employ a set of examples of finite economies which enable us to compare the outcomes that different equilibrium concepts generate. Also, we examine the implementation and the incentive compatibility of different equilibrium concepts.

Our survey differs from the two recent ones by Forges (1998), Forges et al. (2000) and Ichiishi and Yamazaki (2002). These papers follow the Harsanyi type model and focus on the devolopment of cooperative, core concepts. In contrast, we focus on the partition model, examine in detail additional concepts such as the Shapley value and provide an extensive form foundation for the concepts we examine. Furthermore we analyze the incentive compatibility of the different equilibrium concepts and consider their implementation as a perfect Bayesian equilibrium (PBE). These considerations can help us to decide how to choose among the available equilibrium concepts the most appropriate one. We also provide several illuminating examples which enable one to contrast and compare the different equilibrium notions. These examples could be especially useful to those who start work in the area.

A finite economy with differential information consists of a finite set of agents and states of nature. Each agent is characterized by a random utility function, a random consumption set, random initial endowments, a private information set which is a partition of the set of the states of nature, and a prior probability distribution on these states. For such an economy a number of cooperative and non-cooperative equilibrium concepts have been developed.

We believe that the natural and intuitive way to proceed is to analyse concepts in terms of measurability of allocations (Yannelis, 1991). In particular, as it is well known, (e.g. Prescott and Townsend, 1984; Allen, 2003), without measurability, the set of feasible and incentive compatible allocations is not convex and therefore the existence of an incentive compatible core becomes a serious problem. On the other hand certain measurability conditions imply incentive compatibility and they help us to narrow down the set of admissible allocations to a more manageable equilibrium set which is not only incentive compatibility but also exists. It is precisely for this reason that we follow the measurability approach.

We concentrate here mainly on cooperative concepts which allow for different types of measurability of the proposed allocations, i.e. for alternative forms of information sharing among the agents. In particular we consider the private core, (Yannelis, 1991), which is the set of all state-wise feasible and private information measurable allocations that cannot be dominated, in terms of expected utility, by any coalition's feasible and private information measurable net trades, the weak fine core (WFC), defined in Yannelis (1991) and Koutsougeras and Yannelis (1993), and the concepts of private value and the weak fine value (WFV), (Krasa and Yannelis, 1994), which employ the Shapley value.[1]

On the other hand we discuss the noncooperative concepts of the generalized Walrasian equilibrium type ideas of Radner equilibrium, defined in Radner (1968), and rational expectations equilibrium (REE), which is discussed in Radner (1979), Allen (1981), Einy et al. (2000), Kreps (1977) and Laffont (1985) and Grossman (1981), among others. Unlike the Walrasian equilibrium, Radner equilibrium with positive prices or REE may not exist in well behaved economies.

The paper is organized as follows. Section 2 contains the definition of a differentiable information economy. Section 3 defines cooperative equilibrium concepts. Section 4 defines noncooperative equilibrium concepts and makes some comparisons between the various ideas. Section 5 applies the equilibrium ideas in the context of one-good and Section 6 in that of two-good examples. Section 7 visits the incentive compatibility idea and Section 8 discusses implementation or non-implementation properties, in terms of PBE, of various equilibrium notions. Section 9 pays special attention to the relation between REE and weak core concepts and Section 10 concludes the discussion with some remarks. Finally Appendix I discusses some relations between core concepts.

2 Differential information economy (DIE)

In this section we define the notion of a finite-agent economy with differential information for the case where the set of states of nature, Ω and the number of goods, l, per state are finite. I is a set of n players and $\mathbb{R}^l_+$ will denote the set of positive real numbers.

A differential information exchange economy $\mathcal{E}$ is a set

$$\{((\Omega, \mathcal{F}), X_i, \mathcal{F}_i, u_i, e_i, q_i) : i = 1, \ldots, n\}$$

where

1. $\mathcal{F}$ is a σ-algebra generated by a partition of Ω;
2. $X_i : \Omega \to 2^{\mathbb{R}^l_+}$ is the set-valued function giving the *random consumption set* of Agent (Player) i, who is denoted by Pi;

[1] See also Allen and Yannelis (2001) for additional references.

3. $\mathcal{F}_i$ is a partition of Ω generating a sub-σ-algebra of $\mathcal{F}$, denoting the *private information*[2] of Pi; $\mathcal{F}_i$ is a partition of Ω generating a sub-σ-algebra of $\mathcal{F}$, denoting the *private information*[3] of Pi;
4. $u_i : \Omega \times \mathbb{R}^l_+ \to \mathbb{R}$ is the *random utility* function of Pi; for each $\omega \in \Omega,\ u_i(\omega, .)$ is continuous, concave and monotone;
5. $e_i : \Omega \to \mathbb{R}^l_+$ is the *random initial endowment* of Pi, assumed to be $\mathcal{F}_i$-measurable, with $e_i(\omega) \in X_i(\omega)$ for all $\omega \in \Omega$;
6. q_i is an $\mathcal{F}$-measurable probability function on Ω giving the *prior* of Pi. It is assumed that on all elements of $\mathcal{F}_i$ the aggregate q_i is strictly positive. If a common prior is assumed on $\mathcal{F}$, it will be denoted by μ.

We will refer to a function with domain Ω, constant on elements of $\mathcal{F}_i$, as $\mathcal{F}_i$*-measurable*, although, strictly speaking, measurability is with respect to the σ-algebra generated by the partition.

In the first period agents make contracts in the ex ante stage. In the interim stage, i.e., after they have received a signal[4] as to what is the event containing the realized state of nature, they consider the incentive compatibility of the contract.

For any $x_i : \Omega \to \mathbb{R}^l_+$, the *ex ante expected utility* of Pi is given by

$$v_i(x_i) = \sum_{\Omega} u_i(\omega, x_i(\omega)) q_i(\omega).$$

Let $\mathcal{G}$ be a partition of (or σ-algebra on) Ω, belonging to Pi. For $\omega \in \Omega$ denote by $E_i^{\mathcal{G}}(\omega)$ the element of $\mathcal{G}$ containing ω; in the particular case where $\mathcal{G} = \mathcal{F}_i$ denote this just by $E_i(\omega)$. Pi's conditional probability for the state of nature being ω', given that it is actually ω, is then

$$q_i\big(\omega' | E_i^{\mathcal{G}}(\omega)\big) = \begin{cases} 0 & : \quad \omega' \notin E_i^{\mathcal{G}}(\omega) \\ \frac{q_i(\omega')}{q_i\left(E_i^{\mathcal{G}}(\omega)\right)} & : \quad \omega' \in E_i^{\mathcal{G}}(\omega). \end{cases}$$

The *interim expected utility* function of Pi, $v_i(x|\mathcal{G})$, is given by

$$v_i(x|\mathcal{G})(\omega) = \sum_{\omega'} u_i(\omega', x_i(\omega')) q_i\big(\omega' | E_i^{\mathcal{G}}(\omega)\big),$$

which defines a $\mathcal{G}$-measurable random variable.

Denote by $L_1(q_i, \mathbb{R}^l)$ the space of all equivalence classes of $\mathcal{F}$-measurable functions $f_i : \Omega \to \mathbb{R}^l$; when a common prior μ is assumed $L_1(q_i, \mathbb{R}^l)$ will be replaced by $L_1(\mu, \mathbb{R}^l)$. L_{X_i} is the set of all $\mathcal{F}_i$-measurable selections from the random consumption set of Agent i, i.e.,

[2] Following Aumann (1987) we assume that the players' information partitions are common knowledge.

[3] Sometimes $\mathcal{F}_i$ will denote the σ-algebra generated by the partition, as will be clear from the context.

[4] A *s*ignal to Pi is an $\mathcal{F}_i$-measurable function to all of the possible distinct observations specific to the player; that is, it induces the partition $\mathcal{F}_i$, and so gives the finest discrimination of states of nature directly available to Pi.

$$L_{X_i} = \{x_i \in L_1(q_i, \mathbb{R}^l) : \; x_i : \Omega \;\to\; \mathbb{R}^l \text{ is } \mathcal{F}_i\text{-measurable and } x_i(\omega) \in X_i(\omega) \; q_i\text{-}a.e.\}$$

and let $L_X = \prod_{i=1}^n L_{X_i}$.

Also let

$$\bar{L}_{X_i} = \{x_i \in L_1(q_i, \mathbb{R}^l) : \; x_i(\omega) \in X_i(\omega) \; q_i\text{-}a.e.\}$$

and let $\bar{L}_X = \prod_{i=1}^n \bar{L}_{X_i}$.

An element $x = (x_1, \ldots, x_n) \in \bar{L}_X$ will be called an *allocation*. For any subset of players S, an element $(y_i)_{i\in S} \in \prod_{i\in S} \bar{L}_{X_i}$ will also be called an allocation, although strictly speaking it is an allocation to S.

In case there is only one good, we shall use the notation $L^1_{X_i}$, L^1_X etc. When a common prior is also assumed $L_1(q_i, \mathbb{R}^l)$ will be replaced by $L_1(\mu, \mathbb{R}^l)$.

Finally, suppose we have a coalition S, with members denoted by i. Their pooled information $\bigvee_{i\in S} \mathcal{F}_i$ will be denoted by $\mathcal{F}_S$ [5]. We assume that $\mathcal{F}_I = \mathcal{F}$.

3 Cooperative equilibrium concepts: Core and Shapley value

We discuss here certain fundamental concepts.[6] First we define the notion of the private core (Yannelis, 1991).

Definition 3.1. An allocation $x \in L_X$ is said to be a *private core allocation* if

(i) $\sum_{i=1}^n x_i = \sum_{i=1}^n e_i$ and
(ii) there do not exist coalition S and allocation $(y_i)_{i\in S} \in \prod_{i\in S} L_{X_i}$ such that $\sum_{i\in S} y_i = \sum_{i\in S} e_i$ and $v_i(y_i) > v_i(x_i)$ for all $i \in S$.

The private core is an ex ante concept and under mild conditions it is not empty, as shown in Yannelis (1991) and Glycopantis et al. (2001). If the feasibility condition (i) is replaced by (i)$'$ $\sum_{i=1}^n x_i \leq \sum_{i=1}^n e_i$ then free disposal is allowed.

Next we define the *weak fine core* (WFC) (Yannelis, 1991; Koutsougeras and Yannelis, 1993). This is a refinement of the fine core concept of Wilson (1978) or Srivastava (1984). The fine core notion of Wilson as well as that in Koutsougeras and Yannelis may be empty in well behaved economies. This is why we are working with a different concept.

Definition 3.2. An allocation $x = (x_1, \ldots, x_n) \in \bar{L}_X$ is said to be a *WFC allocation* if

(i) each $x_i(\omega)$ is F_I-measurable;
(ii) $\sum_{i=1}^n x_i(\omega) = \sum_{i=1}^n e_i(\omega)$, for all $\omega \in \Omega$;
(iii) there do not exist coalition S and allocation $(y_i)_{i\in S} \in \prod_{i\in S} \bar{L}_{X_i}$ such that $y_i(\cdot) - e_i(\cdot)$ is $\mathcal{F}_S$-measurable for all $i \in S$, $\sum_{i\in S} y_i = \sum_{i\in S} e_i$ and $v_i(y_i) > v_i(x_i)$ for all $i \in S$.

[5] The "join" $\bigvee_{i\in S} \mathcal{F}_i$ denotes the smallest σ-algebra containing all $\mathcal{F}_i$, for $i \in S$.
[6] The interim weak fine core (IWFC) is discussed in a later section.

As comparisons are made on the basis of expected utility, the weak fine core is also an ex ante concept. It captures the idea of an allocation which is ex ante "full information" Pareto optimal. As with the private core the feasibility condition can be relaxed to (ii)′ $\sum_{i=1}^{n} x_i(\omega) \leq \sum_{i=1}^{n} e_i(\omega)$, for all $\omega \in \Omega$.

Finally we define the concept of *weak fine value* (WFV) (see Krasa and Yannelis, 1994, 1996). We must first define a transferable utility (TU) game in which each agent's utility is weighted by a non-negative factor $\lambda_i,\ (i = 1, ..., n)$, which allows for interpersonal comparisons. In a TU-game an outcome can be realized through transfers of payoffs among the agents. On the other hand a (weak) fine value allocation is more specific. It is realizable through a redistribution of payoffs among the agents and, following this, no side payments are necessary.[7] The WFV set is also non-empty.

A game with side payments is defined as follows.

Definition 3.3. A game with side payments $\Gamma = (I, V)$ consists of a finite set of agents $I = \{1, ..., n\}$ and a superadditive[8] , real valued function V defined on 2^I such that $V(\emptyset) = 0$. Each $S \subseteq I$ is called a coalition and $V(S)$ is the 'worth' of the coalition S.

The Shapley value of the game Γ (Shapley, 1953) is a rule that assigns to each Agent i a payoff, $Sh_i(V)$, given by the formula[9]

$$Sh_i(V) = \sum_{\substack{S \subseteq I \\ S \supseteq \{i\}}} \frac{(\mid S \mid -1)!(\mid I \mid - \mid S \mid)!}{\mid I \mid!} [V(S) - V(S \backslash \{i\})]. \qquad (1)$$

The Shapley value has the property that $\sum_{i \in I} Sh_i(V) = V(I)$, i.e. the implied allocation of payoffs is Pareto efficient.

We now define for each DIE, $\mathcal{E}$, with common prior μ, which is assumed for simplicity, and for each set of weights, $\lambda = \{\lambda_i \geq 0 : i = 1, \ldots, n\}$, the associated game with side payments (I, V_λ). We also refer to this as a *transferable utility* (TU) game.

Definition 3.4. Given $\{\mathcal{E}, \lambda\}$ an associated game $\Gamma_\lambda = (I, V_\lambda)$ is defined as follows: For every coalition $S \subset I$ let

$$V_\lambda(S) = \max_x \sum_{i \in S} \lambda_i \sum_{\omega \in \Omega} u_i(\omega, x_i(\omega))\mu(\omega) \qquad (2)$$

subject to

(i) $\sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega)$, for all $\omega \in \Omega$, and
(ii) $x_i - e_i$ is $\bigvee_{i \in S} \mathcal{F}_i-$measurable.

We are now ready to define the WFV allocation.

[7] See Emmons and Scafuri (1985, p. 60) and the examples in Section 6 below for further discussion.
[8] This means that given disjoint $S, T \subset I$ then $V(S) + V(T) \leq V(S \cup T)$.
[9] The Shapley value measure is the sum of the expected marginal contributions an agent can make to all the coalitions of which he/she can be a member (see Shapley, 1953).

Definition 3.5. An allocation $x = (x_1, \ldots, x_n) \in \bar{L}_X$ is said to be a *WFV allocation* of the differential information economy, $\mathcal{E}$, if the following conditions hold

(i) Each net trade $x_i - e_i$ is $\bigvee_{i=1}^{n} \mathcal{F}_i$-measurable,
(ii) $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} e_i$ and
(iii) There exist $\lambda_i \geq 0$, for every $i = 1, ..., n$, which are not all equal to zero, with $\sum_{\omega \in \Omega} \lambda_i u_i(\omega, x_i(\omega))\mu(\omega) = Sh_i(V_\lambda)$ for all i, where $Sh_i(V_\lambda)$ is the Shapley value of Agent i derived from the game (I, V_λ), defined in (2) above.[10]

Condition (i) requires the pooled information measurability of net trades. Condition (ii) is the market clearing condition and (iii) says that the expected utility of each agent multiplied by his/her weight, λ_i, must be equal to his/her Shapley value derived from the TU game (I, V_λ). Obviously for the actual utility that the agent will obtain the weight must not be taken into account. Therefore an agent could obtain the utility of a positive allocation even if λ_i were zero.

If condition (ii) in Definitions 3.4 and (i) in 3.5 are replaced by $x_i - e_i$ is $\mathcal{F}_i$-measurable, for all i, then we obtain the definition of the *private value allocation*.

An immediate consequence of Definition 3.4 is that $Sh_i(V_\lambda) \geq \lambda_i \sum_{\omega \in \Omega} u_i(\omega, e_i(\omega))\mu(\omega)$ for every i, i.e. the value allocation is individually rational. This follows immediately from the fact that the game (V_λ, I) is superadditive for all weights λ. Similarly, efficiency of the Shapley value implies that the weak-fine (private) value allocation is weak-fine (private) Pareto efficient.

Note 3.1. The core of an economy with differential information was first defined by Wilson (1978) and the Shapley value with differential information by Myerson (1982). The above analysis is based on the measurability approach introduced by Yannelis (1991). This approach enables one to prove readily the existence of alternative core and value concepts. Furthermore, as we will see in subsequent sections, certain measurability restrictions, as for example the private information measurability of an allocation, ensure incentive compatibility. General existence results for the core and value can be found in Yannelis (1991), Allen (1991a, 1991b), Krasa - Yannelis (1994), Lefebvre (2001) and Glycopantis et al. (2001). The reader is referred to the Appendix for a more complete list of core concepts.

4 Noncooperative equilibrium concepts: Walrasian expectations (or Radner) equilibrium and REE

In order to define a competitive equilibrium in the sense of Radner we need the following. A *price system* is an $\mathcal{F}$-measurable, non-zero function $p : \Omega \to \mathbb{R}^l_+$ and the *budget set* of Agent i is given by

$$B_i(p) = \Big\{ x_i : \; x_i : \Omega \to \mathbb{R}^l \text{ is } \mathcal{F}_i\text{-measurable } \; x_i(\omega) \in X_i(\omega)$$

[10] By replacing the join measurability with private information measurability we can define the private value allocation.

$$\text{and} \sum_{\omega \in \Omega} p(\omega) x_i(\omega) \leq \sum_{\omega \in \Omega} p(\omega) e_i(\omega) \Big\}.$$

Notice that the budget constraint is across states of nature.

Definition 4.1. A pair (p, x), where p is a price system and $x = (x_1, \ldots, x_n) \in L_X$ is an allocation, is a *Walrasian expectations or Radner equilibrium* if

(i) for all i the consumption function maximizes v_i on $B_i(p)$
(ii) $\sum_{i=1}^n x_i \leq \sum_{i=1}^n e_i$ (free disposal), and
(iii) $\sum_{\omega \in \Omega} p(\omega) \sum_{i=1}^n x_i(\omega) = \sum_{\omega \in \Omega} p(\omega) \sum_{i=1}^n e_i(\omega)$.

This is an ex ante concept. We allow for free disposal, because otherwise a Radner equilibrium with positive prices might not exist. This is demonstrated below through Example 5.2 in which a price becomes negative. In general, for purposes of comparison we consider also the case with $\sum_{i=1}^n x_i = \sum_{i=1}^n e_i$.

Proposition 4.1. A (free disposal) Radner equilibrium is in the (free disposal) private core.

The proof parallels the usual one of the complete information case.

We note that a Radner equilibrium with free disposal may not be in the non-free disposal private core. The point can be made using Example 5.2 below, in which the Radner equilibrium with free disposal and private core without free disposal consist of completely different allocations. The question arises why the proposition immediately above fails. The argument cannot be pushed through because under different free disposal assumptions the feasibility condition is different.

Next we turn our attention to the notion of REE. We shall need the following. Let $\sigma(p)$ be the smallest sub-σ-algebra of $\mathcal{F}$ for which a price system $p : \Omega \to \mathbb{R}^l_+$ is measurable and let $\mathcal{G}_i = \sigma(p) \vee \mathcal{F}_i$ denote the smallest σ-algebra containing both $\sigma(p)$ and $\mathcal{F}_i$. We shall also condition the expected utility of the agents on $\mathcal{G}$ which produces a random variable.

Definition 4.2. A pair (p, x), where p is a price system and $x = (x_1, \ldots, x_n) \in \bar{L}_X$ is an allocation, is a *REE* if

(i) for all i the consumption function $x_i(\omega)$ is $\mathcal{G}_i$-measurable;
(ii) for all i and for all ω the consumption function maximizes $v_i(x_i|\mathcal{G}_i)(\omega)$ subject to the budget constraint at state ω,

$$p(\omega) x_i(\omega) \leq p(\omega) e_i(\omega);$$

(iii) $\sum_{i=1}^n x_i(\omega) = \sum_{i=1}^n e_i(\omega)$ for all $\omega \in \Omega$.

REE is an interim concept because we condition on information from prices as well. An REE is said to be *fully revealing* if $\mathcal{G}_i = \mathcal{F} = \bigvee_{i \in I} \mathcal{F}_i$ for all $i \in I$. Although in the definition we do not allow for free disposal, we comment briefly on such an assumption in the context of Example 5.2.

Note 4.1. The concept of Radner equilibrium is due to Radner (1968) and it extends the Arrow-Debreu contingent claims model, (see Debreu, 1959, Ch. 7), to allow for differential information. The existence of a free disposal Radner equilibrium can be found in Radner (1968). The definition of REE is taken from Radner (1979) and Allen (1981). The REE does not exist always, may not be fully Pareto optimal, or incentive compatible and may not be implementable as a PBE (Glycopantis et al., 2003b). The Radner equilibrium without free disposal is always incentive compatible, as it is contained in the private core. Moreover, under standard assumptions, it exists, as shown by Radner (1968). An example illustrating both concepts can be found below.

5 Illustrations of equilibrium concepts and comparisons to each other: One-good examples

We now offer some comments on and make comparisons between the various equilibrium notions. In many instances we will use the same example to compute different equilibrium concepts. Hence the outcomes that different equilibrium concepts generate will become clear.

As we saw in Proposition 4.1 the Radner equilibrium allocations are a subset of the corresponding private core allocations. Of course it is possible that a Radner equilibrium allocation with positive prices might not exist. In the two-agent economy of Example 5.2 below, assuming non-free disposal the unique private core is the initial endowments allocation while no Radner equilibrium exists. On the other hand, assuming free disposal the REE coincides with the initial endowments allocation which does not belong to the private core. It follows that *the REE allocations need not be in the private core*. Therefore *a REE need not be a Radner equilibrium* either. In Example 5.1 below, without free disposal no Radner equilibrium with positive prices exists but REE does. It is unique and it implies no-trade.

As for *the comparison between private core and WFC allocations* the two sets could intersect but there is no definite relation. Indeed the measurability requirement of the private core allocations separates the two concepts. Finally notice that no allocation which does not distribute the total resource could be in the WFC.

For $n = 2$ one can easily verify that the WFV belongs to the weak fine core. However it is known (see for example Scafuri and Yannelis, 1984) that for $n \geq 3$ *a value allocation may not be a core allocation, and therefore may not be a Radner equilibrium*. Also a value allocation might not belong to any Walrasian type set.

In a later section we shall discuss whether core and Walrasian type allocations have certain desirable properties, from the point of view of incentive compatibility. We shall then turn our attention to the implementation of such allocations.

In this and the following sections we indicate, by putting dates, whether we have already discussed in Glycopantis et al. (2001, 2003a, 2003b), at least partly, the various examples. Where both types are calculated we find it more convenient to start with the non-cooperative concepts.

Example 5.1. (2001, 2003a) Consider the following three agents economy, $I = \{1, 2, 3\}$ with one commodity, i.e. $X_i = \mathbb{R}_+$ for each i, and three states of nature $\Omega = \{a, b, c\}$.

The endowments and information partitions of the agents are given by

$$\begin{aligned} e_1 &= (5,5,0), & \mathcal{F}_1 &= \{\{a,b\},\{c\}\}; \\ e_2 &= (5,0,5), & \mathcal{F}_2 &= \{\{a,c\},\{b\}\}; \\ e_3 &= (0,0,0), & \mathcal{F}_3 &= \{\{a\},\{b\},\{c\}\}. \end{aligned}$$

$u_i(\omega, x_i(\omega)) = x_i^{\frac{1}{2}}$ and every player has the same prior distribution $\mu(\{\omega\}) = \frac{1}{3}$, for $\omega \in \Omega$.

It was shown in Appendix II of Glycopantis et al. (2001) that, without free disposal, the redistribution

$$\begin{pmatrix} 4 & 4 & 1 \\ 4 & 1 & 4 \\ 2 & 0 & 0 \end{pmatrix}$$

is a private core allocation, where the ith line refers to Player i and the columns from left to right to states a, b and c.

If the private information set of Agent 3 is the trivial partition, i.e., $\mathcal{F}_3^{'} = \{a,b,c\}$, then no trade takes place and clearly in this case he gets zero utility. Thus the private core is sensitive to information asymmetries. On the other hand in a Radner equilibrium or a REE, Agent 3 will always receive zero quantities as he has no initial endowments, irrespective of whether his private information partition is the full one or the trivial one.

Example 5.2. (2001, 2003a) We now consider Example 5.1 without Agent 3.

For the various types of allocations below, we distinguish between the cases without and with free disposal. We denote the prices by $p(a) = p_1, p(b) = p_2, p(c) = p_3$. Throughout $\varepsilon, \delta \geq 0$.

A. REE

Now, a price function, $p(\omega)$, known to both agents, is defined on Ω. Apart from his own private $E_i \subseteq \mathcal{F}_i$, each agent also receives a price signal which is a value in the range of the price function. Combining the two types of signals he deduces the event from Ω that has been realized, $E_{p,E_i} = \{\omega : p(\omega) = p \text{ and } \omega \in E_i\}$. He then chooses a constant consumption on E_{p,E_i} which maximizes his interim expected utility subject to the budget set at state ω.

We now distinguish between:

Case 1. All prices positive and $p_1 \neq p_2 \neq p_3$.
Then, as soon as the price signal is announced every agent knows the exact state of nature and simply demands his initial endowment in that state.

Case 2. All prices positive and $p_1 = p_2 \neq p_3$.
Then Agent 2 will always realize which is the state of nature and will demand his initial endowment. On the other hand Agent 1 will not be able to distinguish between states a and b. However given the fact that his utility function is the same across states, he will also demand his initial endowment in all states of nature.

Case 3. All prices positive and $p_1 = p_3 \neq p_2$.
This is identical to Case 2 with the roles of the two agents interchanged.

Case 4. The positive prices are constant on Ω and hence non-revealing. Each agent relies exclusively on his private information and will demand in each state his initial endowment.
In all cases the rational expectations price function can be any such that its range of values is a positive vector and it will confirm the initial endowments as equilibrium allocation. Furthermore it makes no difference to the above reasoning whether free disposal is allowed or not.

We can also argue in general that with one good per state and monotonic utility functions, the measurability of the allocations implies that REE, fully revealing or not, simply confirms the initial endowments.

B. Radner equilibrium

The measurability of allocations implies that we require consumptions $x_1(a) = x_2(b) = x$ and $x_1(c)$ for Agent 1, and $x_2(a) = x_2(c) = y$ and $x_2(b)$ for Agent 2. We can also write $x = 5 - \varepsilon, x_1(c) = \delta, y = 5 - \delta$ and $x_2(b) = \varepsilon$.

We now consider,

Case 1. Without free disposal
There is no Radner equilibrium with prices in $\mathbb{R}^3_+$.

Case 2. With free disposal.
The prices are $p_1 = 0, p_2 = p_3 > 0$ and the allocation is

$$\begin{pmatrix} 4 & 4 & 1 \\ 4 & 1 & 4 \end{pmatrix}.$$

It corresponds to $\varepsilon, \delta = 1$ which means that in state a each of the agents throws away one unit of the good.

C. WFC

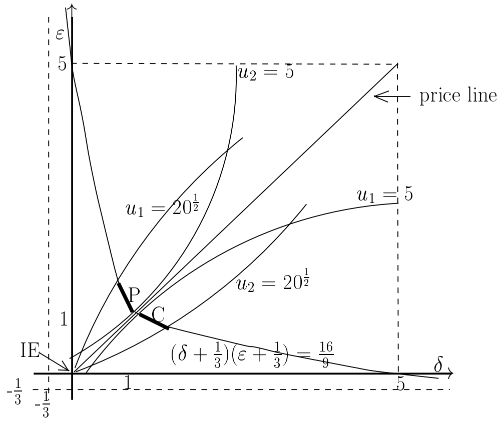
The agents pool their information and therefore any feasible consumption vector to either agent will be measurable. Hence we do not need to distinguish between free disposal and non-free disposal. All WFC allocations will exhaust the resource in each state of nature.

There are uncountably many such allocations, as for example

$$\begin{pmatrix} 5 & 2.5 & 2.5 \\ 5 & 2.5 & 2.5 \end{pmatrix}.$$

This allocation is $\bigvee_{i=1}^{2} \mathcal{F}_i$-measurable and cannot be dominated by any coalition of agents using their pooled information.

Referring back to Example 5.1 we can note that a private core allocation is not necessarily a WFC allocation. For example the division (4, 4, 1), (4, 1, 4) and (2, 0, 0), to Agents 1, 2 and 3 respectively, is a private core but not a weak fine core allocation. The first two agents can get together, pool their information and do



PC: private core; IE: Initial Endowments

Figure 1

better. They can realize the WFC allocation, (5, 2.5, 2.5), (5, 2.5, 2.5) and (0, 0, 0) which does not belong to the private core because of lack of measurability.

D. Private core

Case 1. Without free disposal.
No individual can increase his allocation and retain measurability. Therefore, in this case the only allocation in the core is the initial endowments.

Case 2. With free disposal.
Free disposal can take the form:

$$\begin{pmatrix} 5-\varepsilon & 5-\varepsilon & \delta \\ 5-\delta & \varepsilon & 5-\delta \end{pmatrix}$$

where $\varepsilon, \delta > 0$.

The private core is the section of the curve $(\delta + \frac{1}{3})(\varepsilon + \frac{1}{3}) = \frac{16}{9}$ between the indifference curves corresponding to $\mathcal{U}_1 = 20^{\frac{1}{2}}$ and $\mathcal{U}_2 = 20^{\frac{1}{2}}$. Notice that the allocation

$$\begin{pmatrix} 4 & 4 & 1 \\ 4 & 1 & 4 \end{pmatrix}$$

corresponds to $\delta, \varepsilon = 1$ and is in the private core. The private core and the Radner equilibrium are shown in Figure 1.

E. WFV

Here we shall show that $x_1 = x_2 = (5, 2.5, 2.5)$ is a weak fine value allocation. First we note that the "join" $\mathcal{F}_1 \vee \mathcal{F}_2 = \{\{a\}\{b\}\{c\}\}$. So every allocation is $\mathcal{F}_1 \vee \mathcal{F}_2$-measurable and condition (i) of Definition 3.5 is satisfied. Condition (ii) is also immediately satisfied.

First V_λ is calculated to be

$$V_\lambda(\{1\}) = \frac{2\times 5^{\frac{1}{2}}}{3}\lambda_1, V_\lambda(\{2\}) = \frac{2\times 5^{\frac{1}{2}}}{3}\lambda_2 \quad \text{and}$$
$$V_\lambda(\{1,2\}) = \frac{10^{\frac{1}{2}}+2\times 5^{\frac{1}{2}}}{3}\left(\lambda_1^2+\lambda_2^2\right)^{\frac{1}{2}}$$

It is easy to see that

$$Sh_1(V_\lambda) = \frac{1}{2}\left\{\frac{2\times 5^{\frac{1}{2}}}{3}\lambda_1 + \frac{10^{\frac{1}{2}}+2\times 5^{\frac{1}{2}}}{3}\left(\lambda_1^2+\lambda_2^2\right)^{\frac{1}{2}} - \frac{2\times 5^{\frac{1}{2}}}{3}\lambda_2\right\}. \quad (3)$$

Definition 3.5 gives

$$2(2.5)^{\frac{1}{2}}\lambda_1 = \frac{10^{\frac{1}{2}}+2\times 5^{\frac{1}{2}}}{2}(\lambda_1^2+\lambda_1^2)^{\frac{1}{2}} - 5^{\frac{1}{2}}\lambda_2. \quad (4)$$

Similarly the condition on player 2's allocation gives

$$2(2.5)^{\frac{1}{2}}\lambda_2 = \frac{10^{\frac{1}{2}}+2\times 5^{\frac{1}{2}}}{2}(\lambda_1^2+\lambda_1^2)^{\frac{1}{2}} - 5^{\frac{1}{2}}\lambda_2. \quad (5)$$

Subtracting we get $2\times 2^{\frac{1}{2}}(\lambda_1-\lambda_2) = 5^{\frac{1}{2}}(\lambda_1-\lambda_2)$.

It follows that $\lambda_1=\lambda_2$. Substituting this common value λ not equal to 0 back into one of the conditions, λ cancels leaving $2(2.5)^{\frac{1}{2}} = \frac{10^{\frac{1}{2}}+2\times 5^{\frac{1}{2}}}{2}\times 2^{\frac{1}{2}} - 5^{\frac{1}{2}}$ which is an identity. It follows that Definition 3.5 is satisfied.

Next we investigate whether there are any other WFV. The conditions are $\lambda_1\left[x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}}\right] = 5^{\frac{1}{2}}(\lambda_1-\lambda_2)+k(\lambda_1^2+\lambda_2^2)^{\frac{1}{2}}$ and $\lambda_2\left[(10-x)^{\frac{1}{2}}+(5-y)^{\frac{1}{2}}+(5-z)^{\frac{1}{2}}\right] = 5^{\frac{1}{2}}(\lambda_2-\lambda_1)+k(\lambda_1^2+\lambda_2^2)^{\frac{1}{2}}$ where $k=\frac{10^{\frac{1}{2}}+2\times 5^{\frac{1}{2}}}{2}$.

There is an obvious symmetry here: if $\lambda_1,\lambda_2,x,y,z$ is a solution then so is $\lambda_2,\lambda_1,10-x,5-y,5-z$, so that we may assume, without loss of generality, that λ_2 is different from zero, since both λ's cannot be zero, and write $\theta=\frac{\lambda_1}{\lambda_2}$. Subtracting the two equations we obtain $\theta S_1 - S_2 = 2\times 5^{\frac{1}{2}}(\theta-1)$, where $S_1 = (x)^{\frac{1}{2}}+(y)^{\frac{1}{2}}+(z)^{\frac{1}{2}},\ S_2=(10-x)^{\frac{1}{2}}+(5-y)^{\frac{1}{2}}+(5-z)^{\frac{1}{2}}$, which implies $\theta = \frac{S_2-2\times 5^{\frac{1}{2}}}{S_1-2\times 5^{\frac{1}{2}}}$.

We also have $\theta S_1 = 5^{\frac{1}{2}}(\theta-1)+k(\theta^2+1)^{\frac{1}{2}}$ which implies $[\theta(S_1-5^{\frac{1}{2}})+5^{\frac{1}{2}}]^2 = k(\theta^2+1)^{\frac{1}{2}}$. This in turn implies $\{(S_1-5^{\frac{1}{2}})^2-k^2\}\theta^2+2\times 5^{\frac{1}{2}}(S_1-5^{\frac{1}{2}})\theta+5-k^2=0$. This has real roots iff $5(S_1-5^{\frac{1}{2}})^2\geq(5-k^2)\{(S_1-5^{\frac{1}{2}})^2-k^2\}$, or, equivalently, $(S_1-5^{\frac{1}{2}})^2\geq k^2-5$, or $S_1\geq 5^{\frac{1}{2}}+(k^2-5)^{\frac{1}{2}}$, which implies the root $S_1=5.32978$. By symmetry we also need $S_2\geq 5.32978$. The symmetric case $\theta=1$ gives $S_1=S_2=2^{\frac{1}{2}}k$ which has an approximate value of 5.39835. It corresponds to $x_1=x_2=(5,2.5,2.5)$.

Clearly there is not much room to move away from the symmetric case. On the other hand if S_1 goes up then S_2 goes down. This follows from the fact that the sum of the payoffs to the players is equal to $V_\lambda(\{1,2\})$. This suggests the problem

Maximize S_1 subject to $S_2=\varrho$.

The First Order Conditions are: $(10-x)^{\frac{1}{2}} = \frac{1}{2}\eta x^{\frac{1}{2}}$, $(5-y)^{\frac{1}{2}} = \frac{1}{2}\eta y^{\frac{1}{2}}$ and $(5-z)^{\frac{1}{2}} = \frac{1}{2}\eta z^{\frac{1}{2}}$.

From these we obtain $\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}} = \frac{(5-y)^{\frac{1}{2}}}{(10-x)^{\frac{1}{2}}}$ and $\frac{z^{\frac{1}{2}}}{x^{\frac{1}{2}}} = \frac{(5-z)^{\frac{1}{2}}}{(10-x)^{\frac{1}{2}}}$, which imply $x=2y=2z$.

Re-substituting in $S_2 = \varrho$ we derive $\varrho = (10-2z)^{\frac{1}{2}} + (5-z)^{\frac{1}{2}} + (5-z)^{\frac{1}{2}} = (2+2^{\frac{1}{2}})(5-z)^{\frac{1}{2}}$ which for $\varrho = 5.32978$ implies, approximately, $y = z = 5 - (\frac{\varrho}{2+2^{\frac{1}{2}}})^2 = 2.56310$, $x = 5.12621$, $S_1 = 5.46605$, and $\theta = 0.86290$. It follows that the WFV allocations correspond to $\theta \in [0.86290, 1.158882837]$, where the two numbers are the inverse of each other.

Example 5.3. The problem is a two-state, $\Omega = \{a, b\}$, three-player game with utilities and initial endowments given by:

$$
\begin{aligned}
u_1(x_{1j}) &= x_{1j}^{\frac{1}{2}}; \quad e_1 = (4,0), \quad F_1 = \{\{a\},\{b\}\} \\
u_2(x_{2j}) &= x_{2j}^{\frac{1}{2}}; \quad e_2 = (0,4), \quad F_2 = \{\{a\},\{b\}\} \\
u_3(x_{3j}) &= x_{3j}^{\frac{1}{2}}; \quad e_3 = (0,0), \quad F_3 = \{a,b\},
\end{aligned}
$$

where x_{ij} denotes consumption of Player i in state j, (a is identified with 1 and b with 2). Every player has the same prior distribution $\mu(\omega) = \frac{1}{2}$ for $\omega \in \Omega$.

The associated TU game has value function

$$
\begin{aligned}
&V_\lambda(\{1\}) = \lambda_1, V_\lambda(\{2\}) = \lambda_2, V_\lambda(\{3\}) = 0, \\
&V_\lambda(\{1,2\}) = 2(\lambda_1^2+\lambda_2^2)^{\frac{1}{2}}, V_\lambda(\{1,3\}) = (\lambda_1^2+\lambda_3^2)^{\frac{1}{2}}, V_\lambda(\{2,3\}) = (\lambda_2^2+\lambda_3^2)^{\frac{1}{2}}, \\
&V_\lambda(\{1,2,3\}) = 2(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{\frac{1}{2}}.
\end{aligned}
$$

The equations for a value allocation are then:

$$
\begin{aligned}
&\frac{2}{3}\lambda_1 + \frac{1}{3}\Big(2(\lambda_1^2+\lambda_2^2)^{\frac{1}{2}} - \lambda_2\Big) + \frac{1}{3}(\lambda_1^2+\lambda_3^2)^{\frac{1}{2}} \\
&\quad + \frac{2}{3}\left(2(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{\frac{1}{2}} - \left(\lambda_2^2+\lambda_3^2\right)^{\frac{1}{2}}\right) = \lambda_1\left(x_{11}^{\frac{1}{2}} + x_{12}^{\frac{1}{2}}\right), \\
&\frac{2}{3}\lambda_2 + \frac{1}{3}\Big(2(\lambda_1^2+\lambda_2^2)^{\frac{1}{2}} - \lambda_1\Big) + \frac{1}{3}(\lambda_2^2+\lambda_3^2)^{\frac{1}{2}} \\
&\quad + \frac{2}{3}\Big(2(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{\frac{1}{2}} - (\lambda_1^2+\lambda_3^2)^{\frac{1}{2}}\Big) = \lambda_2\left(x_{21}^{\frac{1}{2}} + x_{22}^{\frac{1}{2}}\right), \\
&\frac{1}{3}\Big((\lambda_1^2+\lambda_3^2)^{\frac{1}{2}} - \lambda_1\Big) + \frac{1}{3}\Big((\lambda_2^2+\lambda_3^2)^{\frac{1}{2}} - \lambda_2\Big) \\
&\quad + \frac{4}{3}\Big((\lambda_1^2+\lambda_2^2+\lambda_3^2)^{\frac{1}{2}} - (\lambda_1^2+\lambda_2^2)^{\frac{1}{2}}\Big) = \lambda_3\left(x_{31}^{\frac{1}{2}} + x_{32}^{\frac{1}{2}}\right), \\
&\quad \text{subject to } x_{11} + x_{21} + x_{31} = 4. x_{12} + x_{22} + x_{32} = 4.
\end{aligned}
$$

The left-hand side are just numbers which we can calculate. General solution of these equations seems difficult, but we would hope to get a symmetric solution, in the following sense: the economy is symmetric under the interchange of Agent

1 with Agent 2, together with interchange of the good in state 1 and the good in state 2; so we might expect a solution in which

$$x_{11} = x_{22},\ x_{12} = x_{21},\ x_{31} = x_{32},\ \lambda_1 = \lambda_2.$$

We will write, for simplicity

$$x_{11}^{\frac{1}{2}} = x_{22}^{\frac{1}{2}} = x,\ x_{12}^{\frac{1}{2}} = x_{21}^{\frac{1}{2}} = y,$$
$$\text{and hence } x_{31} = x_{32} = (4 - (x^2 + y^2))^{\frac{1}{2}},\quad \lambda_1 = \lambda_2 = \lambda.$$

We will treat two cases. Firstly, if $\lambda_3 = 0$, the last equation is identically satisfied and the first two equations (which are the same) give $2 \times 2^{\frac{1}{2}}\lambda = \lambda(x + y)$. So λ is arbitrary and $x + y = 2 \times 2^{\frac{1}{2}}$. If we suppose $x = 2^{\frac{1}{2}} + \delta,\ y = (-\delta)^{\frac{1}{2}}$, then $x_{11} + x_{21} = 4 + \delta^2$, so we have $\delta = 0$ and hence

$$x_{11} = x_{12} = x_{21} = x_{22} = 2,\ x_{31} = x_{32} = 0,$$

with $\lambda_1 = \lambda_2 > 0$ arbitrary and $\lambda_3 = 0$.

Now consider the possibility that $\lambda_3 > 0$ and we may normalise it to be equal to 1. The first two equations are the same and they state:

$$\frac{1}{3}\Big(2(2)^{\frac{1}{2}} + 1\Big)\lambda - \frac{1}{3}(\lambda^2 + 1)^{\frac{1}{2}} + \frac{4}{3}(2\lambda^2 + 1)^{\frac{1}{2}} = \lambda(x + y). \tag{6}$$

The third equation becomes

$$-\frac{2}{3}\Big(2(2)^{\frac{1}{2}} + 1\Big)\lambda + \frac{2}{3}(\lambda^2 + 1)^{\frac{1}{2}} + \frac{4}{3}(2\lambda^2 + 1)^{\frac{1}{2}} = 2\Big[4 - (x^2 + y^2)\Big]^{\frac{1}{2}}. \tag{7}$$

It is a matter of tedious calculations on equations (16) and (17) to show that there are no value allocations with $\lambda_3 \neq 0$ which are symmetric.

We now consider approximate equilibria, using the random algorithm. First we look into the case where in the equations for a value allocation we insert $\lambda_i = 1,\ \ \forall i$. The system does not perform very well. Approximate values can be found but the total error, the square root of the sum of squares of RHS-LHS of the equations, is 0.21098557 which is rather large. On the other hand variations in the total resource improve the approximation.

If we allow in the system above for the λ_i's also to be chosen then a rather satisfactory approximate solution emerges:

$x_{11} = 1.9999,\ x_{12} = 2.0001,\ x_{21} = 2{,}0000,\ x_{22} = 1.9998,\ x_{31} = x_{32} = 0$ (approximately), $\lambda_1 = 1,\ \lambda_2 = 1.00009,\ \lambda_3 = 0.0129$, with total error 0.000000007.

In the example we have examined Agent 3 has zero endowments and bad information. As a result, when all the λ_i's can be chosen the solution of the equations of the value allocation are approximately the same as when no weight is attached to Agent 3.

6 Two-good examples

We note that with one good per state and monotone utility functions there is a direct relation between allocations and utilities, i.e. $x \geq y$ iff $u(x) \geq u(y)$. This allows one to prove results which do not hold in general. This is the reason why we present also examples with two goods. We also note that in the one good case the unique REE allocation exists always and it coincides with no trade. Thus it exists, it is incentive compatible and Pareto optimal. However, as it is shown below, this is not the case when there are two goods.

Example 6.1. (2003b) We consider a two-agent economy, $I = \{1, 2\}$ with two commodities, i.e. $X_i = \mathbb{R}^2_+$ for each i, and three states of nature $\Omega = \{a, b, c\}$.

The endowments, per state a, b, and c respectively, and information partitions of the agents are given by

$$e_1 = ((7,1),(7,1),(4,1)), \qquad \mathcal{F}_1 = \{\{a,b\},\{c\}\};$$
$$e_2 = ((1,10),(1,7),(1,7)), \qquad \mathcal{F}_2 = \{\{a\},\{b,c\}\}.$$

We shall denote $A_1 = \{a,b\}$, $c_1 = \{c\}$, $a_2 = \{a\}$, $A_2 = \{b,c\}$. $u_i(\omega, x_{i1}(\omega), x_{i2}(\omega)) = x_{i1}^{\frac{1}{2}} x_{i2}^{\frac{1}{2}}$, and for all players $\mu(\{\omega\}) = \frac{1}{3}$, for $\omega \in \Omega$. We have that $u_1(7,1) = 2.65,\ u_1(4,1) = 2,\ u_2(1,10) = 3.16,\ u_2(1,7) = 2.65$ and the expected utilities of the initial allocations, multiplied by 3, are given by $\mathcal{U}_1 = 7.3$ and $\mathcal{U}_2 = 8.46$.

A. REE

Case 1.
First, we are looking for a fully revealing REE. Prices are normalized so that $p_1 = 1$ in each state. In effect we are analyzing an Edgeworth box economy per state.

State a. We find that

$$(p_1, p_2) = \left(1, \frac{8}{11}\right); \ \ x^*_{11} = \frac{85}{22}, \ \ x^*_{12} = \frac{85}{16},$$
$$x^*_{21} = \frac{91}{22}, \ \ x^*_{22} = \frac{91}{16}; \ \ u^*_1 = 4.53, \ \ u^*_2 = 4.85.$$

State b. We find that

$$(p_1, p_2) = (1,1); \ \ x^*_{11} = 4, \ \ x^*_{12} = 4, \ \ x^*_{21} = 4, \ \ x^*_{22} = 4; \ \ u^*_1 = 4, \ \ u^*_2 = 4.$$

State c. We find that

$$(p_1, p_2) = \left(1, \frac{5}{8}\right); \ \ x^*_{11} = \frac{37}{16}, \ \ x^*_{12} = \frac{37}{10}, \ \ x^*_{21} = \frac{43}{16},$$
$$x^*_{22} = \frac{43}{10}; \ \ u^*_1 = 2.93, \ \ u^*_2 = 3.40.$$

The normalized expected utilities of the equilibrium allocations are $\mathcal{U}_1 = 11.46$, $\mathcal{U}_2 = 12.25$. This completes the analysis of the fully revealing REE.

We now look into whether there is a partially revealing or a non-revealing REE as well.

Case 2. Referring to the three states, we consider price vectors $p^1 = p^2 \neq p^3$ or $p^1 \neq p^2 = p^3$ or $p^1 = p^3 \neq p^2$.

We find that in all these cases no REE exists.

Case 3. We consider the price vectors to be equal, i.e. $p^1 = p^2 = p^3$, which means that the Agents get no information from the prices.

We find that no such equilibrium exists.

The above analysis shows that there is only a fully revealing REE. The equilibrium quantities are different in each state and therefore the REE allocations do not belong to either the private core or Radner equilibria.

Next we characterize the Radner equilibria. Apart from the analysis in the context of Example 6.1, (Radner equilibria 1), we also consider a modified model, in Example 6.2, in which every agent can distinguish between all states of nature, (Radner equilibria 2). The calculations in the latter case can be contrasted to the ones for the fully revealing equilibria.

Existence arguments in the case of correspondences can be advanced. However the actual calculation of such equilibria is not always straightforward.

B. Radner equilibria 1

The price vectors are $p(a) = p^1 = (p_1^1, p_2^1)$, $p(b) = p^2 = (p_1^2, p_2^2)$ and $p(c) = p^3 = (p_1^3, p_2^3)$. On the other hand we require measurability of allocations with respect to the private information of the agents.

The problems of the agents are:

Agent 1.
Maximize $\mathcal{U}_1 = 2(AB)^{\frac{1}{2}} + (x_{11}^3 x_{12}^3)^{\frac{1}{2}}$
Subject to

$$A(p_1^1 + p_1^2) + B(p_2^1 + p_2^2) + p_1^3 x_{11}^3 + p_2^3 x_{12}^3 = 7(p_1^1 + p_1^2) + (p_2^1 + p_2^2) + 4p_1^3 + p_2^3$$

and

Agent 2.
Maximize $\mathcal{U}_2 = (x_{21}^1 x_{22}^1)^{\frac{1}{2}} + 2(CD)^{\frac{1}{2}}$
Subject to

$$p_1^1 x_{21}^1 + p_2^1 x_{22}^1 + C(p_1^2 + p_1^3) + D(p_2^2 + p_2^3) = p_1^1 + 10p_2^1 + (p_1^2 + p_1^3) + 7(p_2^2 + p_2^3).$$

Applying a Gorman (1959) type argument we see that the demands of the agents will be of the form: $A = \frac{M_1}{2(p_1^1+p_1^2)}$, $B = \frac{M_1}{2(p_2^1+p_2^2)}$, $x_{11}^3 = \frac{M_2}{2p_1^3}$, $x_{12}^3 = \frac{M_2}{2p_2^3}$, $x_{21}^1 = \frac{m_1}{2p_1^1}$, $x_{22}^1 = \frac{m_1}{2p_2^1}$, $C = \frac{m_2}{2(p_1^2+p_1^3)}$ and $D = \frac{m_2}{2(p_2^2+p_2^3)}$.

It follows that a Radner equilibrium with non-negative prices exists if the following system of equations has a non-negative solution.

$$\frac{2}{((p_1^1 + p_1^2)(p_2^1 + p_2^2))^{\frac{1}{2}}} = \frac{1}{(p_1^3 p_2^3)^{\frac{1}{2}}},$$

$$M_1 + M_2 = 7(p_1^1 + p_1^2) + (p_2^1 + p_2^2) + 4p_1^3 + p_2^3$$
$$\frac{1}{(p_1^1 p_2^1)^{\frac{1}{2}}} = \frac{2}{((p_1^2 + p_1^3)(p_2^2 + p_2^3))^{\frac{1}{2}}}$$
$$m_1 + m_2 = p_1^1 + 10p_2^1 + (p_1^2 + p_1^3) + 7(p_2^2 + p_2^3)$$
$$\frac{M_1}{2(p_1^1 + p_1^2)} + \frac{m_1}{2p_1^1} = 8, \quad \frac{M_1}{2(p_2^1 + p_2^2)} + \frac{m_1}{2p_2^1} = 11$$
$$\frac{M_2}{2p_1^3} + \frac{m_2}{2(p_1^2 + p_1^3)} = 5, \quad \frac{M_2}{2p_2^3} + \frac{m_2}{2(p_2^2 + p_2^3)} = 8$$
$$\frac{M_1}{2(p_1^1 + p_1^2)} + \frac{m_2}{2(p_1^2 + p_1^3)} = 8, \quad \frac{M_1}{2(p_2^1 + p_2^2)} + \frac{m_2}{2(p_2^2 + p_2^3)} = 8.$$

The above system of equations is homogeneous of degree zero in the p_j^i's, the M_i's and the m_i's. Therefore some price, for example, p_1^1 could be fixed which reduces by one the number of unknowns. However the market equilibrium equations have one degree of redundancy as a consequence of Walras' law,

$$p_1^1(A + x_{21}^1 - 8) + p_2^1(B + x_{22}^1 - 11) + p_1^2(A + C - 8) + p_2^2(B + D - 8)$$
$$+p_1^3(x_{11}^3 + C - 5) + p_2^3(x_{12}^3 + D - 8) = 0.$$

One can prove the existence of a Radner equilibrium by modifying the usual argument in general equilibrium theory, to take into account the fact that for Cobb-Douglas utility functions the demands are not defined on the whole boundary of the simplex. It is a rather tedious argument and we do not include it.

Approximate values for the equilibrium were obtained from the application of the random selection algorithm. A succession of random variables was appraised using a criterion consisting of the square root of the sum of squares of errors, the best selection so far being retained at each step. We did not normalize prices and all equations were used.

We obtained $p_1^1 = 1.1566$, $p_2^1 = 0.5876$, $p_1^2 = 0.3979$, $p_2^2 = 1.08597$, $p_1^3 = 1.3272$, $p_2^3 = 0.49009$, $M_1 = 14.1971$, $M_2 = 4.1574$, $m_1 = 7.9433$, and $m_2 = 11.8474$, which satisfy the equations to three decimal places. We have also checked the accuracy to more decimal places. If an error implies infeasibility in the sense that demand is larger than the resource then the implication is that a small quantity is not forthcoming. In the calculations we did not normalize prices, in order to allow for the maximum flexibility in the algorithm.

The same approximate solution can be obtained using Newton's method, starting the iteration from a suitable initial set of values. In order to avoid the problems arising from the need to invert a singular matrix, we normalized $p_1^2 = 1$ and, invoking Walras' law, we left out the 4th market equilibrium equation.

However there are dangers which may be illustrated by leaving out the 6th market equation. For the same initial values we approach a different point, where p_2^2 is essentially zero but the sixth equation is not satisfied. This is possible because in the Walras equation the contribution from the 6th equation has coefficient p_2^2 and thus can take any value. This means that a particular limit point cannot be a Radner equilibrium.

We also note that, of course, approximate solutions are not necessarily near the true solution. Even with continuity of functions the changes in the values corresponding to small changes in the variables might be very large.

We now have a digression the purpose of which is to explain that the full information, deterministic Radner equilibrium is not the same as the fully revealing REE.

C. Radner equilibria 2

Example 6.2. We shall now calculate the Radner equilibrium for the case with $\mathcal{F}_1 = \mathcal{F}_2 = \{\{a\}\{b\}\{c\}\}$. All other data are as in Example 6.1.

The problems of the two agents are:

Agent 1.
Maximize $\mathcal{U}_1 = (x_{11}^1 x_{12}^1)^{\frac{1}{2}} + (x_{11}^2 x_{12}^2)^{\frac{1}{2}} + (x_{11}^3 x_{12}^3)^{\frac{1}{2}}$
Subject to

$$p_1^1 x_{11}^1 + p_2^1 x_{12}^1 + p_1^2 x_{11}^2 + p_2^2 x_{12}^2 + p_1^3 x_{11}^3 + p_2^3 x_{12}^3 = 7(p_1^1 + p_1^2) + (p_2^1 + p_2^2) + 4p_1^3 + p_2^3$$

and

Agent 2.
Maximize $\mathcal{U}_2 = (x_{21}^1 x_{22}^1)^{\frac{1}{2}} + (x_{21}^2 x_{22}^2)^{\frac{1}{2}} + (x_{21}^3 x_{22}^3)^{\frac{1}{2}}$
Subject to

$$p_1^1 x_{21}^1 + p_2^1 x_{22}^1 + p_1^2 x_{21}^2 + p_2^2 x_{22}^2 + p_1^3 x_{21}^3 + p_2^3 x_{22}^3 = p_1^1 + 10p_2^1 + (p_1^2 + p_1^3) + 7(p_2^2 + p_2^3).$$

Applying a Gorman type argument we obtain $x_{1j}^i = \frac{M_i}{2p_j^i}$ and $x_{2j}^i = \frac{m_i}{2p_j^i}$. These demands imply $\mathcal{U}_1 = \frac{1}{2(p_1^1 p_2^1)^{\frac{1}{2}}} M_1 + \frac{1}{2(p_1^2 p_2^2)^{\frac{1}{2}}} M_2 + \frac{1}{2(p_1^3 p_2^3)^{\frac{1}{2}}} M_3$ and $\mathcal{U}_2 = \frac{1}{2(p_1^1 p_2^1)^{\frac{1}{2}}} m_1 + \frac{1}{2(p_1^2 p_2^2)^{\frac{1}{2}}} m_2 + \frac{1}{2(p_1^3 p_2^3)^{\frac{1}{2}}} m_3$.

The above $\mathcal{U}_1$ and $\mathcal{U}_2$ have to be maximized, each subject to the Agent's constraint cast in terms of M_i's for Agent 1 and m_i's for Agent 2, which is done below.

Notice that no price could be zero because both agents would seek infinite utility. Conditions for Radner equilibrium, such that each agent buys every good, are:

$$p_1^1 p_2^1 = p_1^2 p_2^2, \quad M_1 + M_2 + M_3 = 7(p_1^1 + p_1^2) + (p_2^1 + p_2^2) + 4p_1^3 + p_2^3$$

$$p_1^1 p_2^1 = p_1^3 p_2^3, \quad m_1 + m_2 + m_3 = p_1^1 + 10p_2^1 + (p_1^2 + p_1^3) + 7(p_2^2 + p_2^3)$$

$$\frac{M_1}{2p_1^1} + \frac{m_1}{2p_1^1} = 8, \quad \frac{M_1}{2p_2^1} + \frac{m_1}{2p_2^1} = 11$$

$$\frac{M_2}{2p_1^2} + \frac{m_2}{2p_1^2} = 8, \quad \frac{M_2}{2p_2^2} + \frac{m_2}{2p_2^2} = 8$$

$$\frac{M_3}{2p_1^3} + \frac{m_3}{2p_1^3} = 5, \quad \frac{M_3}{2p_2^3} + \frac{m_3}{2p_2^3} = 8.$$

The solution is obtained as follows: We normalize prices by setting $p_1^1 = 1$. From the 5th and 6th equation we obtain $p_2^1 = \frac{8}{11}$ and the 7th and 8th equation imply $p_2^1 = p_2^2$. The 9th and 10th equation imply $p_2^3 = \frac{5}{8} p_1^3$. Putting the last relations into the 1st and 2nd we get the remaining prices. Putting all the information together we have $p_1^1 = 1,\ p_2^1 = \frac{8}{11},\ p_1^2 = p_2^2 = (\frac{8}{11})^{\frac{1}{2}},\ p_1^3 = (\frac{64}{55})^{\frac{1}{2}},$ and $p_2^3 = \frac{5}{8} \times (\frac{64}{55})^{\frac{1}{2}}$.

Employing the above values for p_j^i we obtain for M_i and m_i the following relations:

$$M_1 + M_2 + M_3 = 7 \times \frac{8}{11} + 8 \times \left(\frac{8}{11}\right)^{\frac{1}{2}} + 4\frac{5}{8} \times \left(\frac{64}{55}\right)^{\frac{1}{2}},$$

$$m_1 + m_2 + m_3 = 8\frac{3}{8} + 8 \times \left(\frac{8}{11}\right)^{\frac{1}{2}} + 5\frac{3}{8} \times \left(\frac{64}{55}\right)^{\frac{1}{2}},$$

$$M_1 + m_1 = 16,\quad M_2 + m_2 = 16 \times \left(\frac{8}{11}\right)^{\frac{1}{2}} \quad \text{and } M_3 + m_3 = 10 \times \left(\frac{64}{55}\right)^{\frac{1}{2}},$$

which imply a possible solution $M_1 = 7 \times \frac{8}{11}, m_1 = 8 \times \frac{3}{11}, M_2 = m_2 = 8 \times (\frac{8}{11})^{\frac{1}{2}}$, $M_3 = 4\frac{5}{8} \times (\frac{64}{55})^{\frac{1}{2}}$ and $m_3 = 5\frac{3}{8} \times (\frac{64}{55})^{\frac{1}{2}}$. An obvious solution is $M_1 = 7\frac{8}{11}$, $m_1 = 8\frac{3}{11}, M_2 = m_2 = 8 \times (\frac{8}{11})^{\frac{1}{2}}, M_3 = 4\frac{5}{8} \times (\frac{64}{11})^{\frac{1}{2}}$ and $m_3 = 5\frac{3}{8} \times (\frac{64}{11})^{\frac{1}{2}}$.

However this solution is not unique. For example, we can add to the value for M_1 a small $\epsilon > 0$ and subtract it from m_1, and then adjust in the opposite direction M_2 and m_2. We obtain then a new solution to the system with the same maximum value for the utilities.

It follows that the normalized prices for an interior solution are unique, and so are the maximum utilities, but the M_i's and the m_i's can assume a number of values. The explanation of the last observation is as follows. The product of the two goods to the power $\frac{1}{2}$ becomes one good and given the equilibrium prices the structure of the problem is such that the agents are as well off with $\epsilon > 0$ as with $\epsilon = 0$.

One can ask why is it that the same argument would not apply to the previous formulation of Radner equilibria 1. There we seemed to be getting locally unique values of M_i's and m_i's. The reason was that we did not have the property that rearranging incomes between the agents in Period 1 can be fully compensated by doing so also in, for example, Period 2. In the present case the periods are among themselves separated. This was not the case in the previous formulation.

In that case, if we increase the composite commodity $(AB)^{\frac{1}{2}}$, where the M_i's have been calculated and decrease$(x_{21}^1 x_{22}^1)^{\frac{1}{2}}$, by adjusting M_1's and m_1's, then we have to decrease the commodity $(x_{11}^3 x_{12}^3)^{\frac{1}{2}}$, and increase $(CD)^{\frac{1}{2}}$, which requires a reduction in $(AB)^{\frac{1}{2}}$. Everything was finally balanced there.

There are also approximate equilibria from the random algorithm, which approach the true equilibrium above. Its application gives:

$$p_1^1 = 1,\ p_2^1 = 0.7272,\ p_1^2 = 0.8528,\ p_2^2 = 0.8528,\ p_1^3 = 1.0787,\ p_2^3 = 0.6742$$

and, approximately, $M_1 + m_1$ is 16.000051, $M_2 + m_2$ is 13.6448, and $M_3 + m_3$ is 10.7871. The algorithm also captures the fact that the values of the M_i's and m_i's are not fully determined.

On the basis of the above analysis, we see that full information Radner equilibrium is not the same as fully revealing REE because in the latter case a monotonic, nonlinear transformation can be applied, such as replacing $(x^i_{11}x^i_{12})^{\frac{1}{2}}$ by $(x^i_{11}x^i_{12})$, without affecting the results as the calculations are per period. This is not the case in Radner equilibrium where the calculations are on the sum over all the periods.

We return now to the characterization of equilibrium concepts in Example 6.1.

D. WFC

With respect to the cooperative equilibrium concepts, first we show that in this example the fully revealing REE is in the WFC. These allocations are obtained by solving the following problem, where we use superscripts to characterize the states. Superscripts 1, 2 and 3 correspond to states a, b and c respectively. The WFC is characterized as follows:

Problem

Maximize $\mathcal{U}_1 = (x^1_{11}x^1_{12})^{\frac{1}{2}} + (x^2_{11}x^2_{12})^{\frac{1}{2}} + (x^3_{11}x^3_{12})^{\frac{1}{2}}$
Subject to

$$((8-x^1_{11})(11-x^1_{12}))^{\frac{1}{2}}+((8-x^2_{11})(8-x^2_{12}))^{\frac{1}{2}}+((5-x^3_{11})(8-x^3_{12}))^{\frac{1}{2}}=\mathcal{U}_2 \text{ (fixed)}$$
$$\mathcal{U}_1 \geq 7.3, \quad \mathcal{U}_2 \geq 8.46.$$

The conditions on the utility functions imply that there is a unique interior maximum per $\mathcal{U}_2$. Setting up the Lagrangean function we obtain the first order conditions:

$$\frac{{x^1_{12}}^{\frac{1}{2}}}{{x^1_{11}}^{\frac{1}{2}}} = \ell\frac{(11-x^1_{12})^{\frac{1}{2}}}{(8-x^1_{11})^{\frac{1}{2}}}$$
$$\frac{{x^2_{12}}^{\frac{1}{2}}}{{x^2_{11}}^{\frac{1}{2}}} = \ell\frac{(8-x^2_{12})^{\frac{1}{2}}}{(8-x^2_{11})^{\frac{1}{2}}}$$
$$\frac{{x^3_{12}}^{\frac{1}{2}}}{{x^3_{11}}^{\frac{1}{2}}} = \ell\frac{(8-x^3_{12})^{\frac{1}{2}}}{(5-x^3_{11})^{\frac{1}{2}}}$$
$$\left(\frac{{x^1_{12}}^{\frac{1}{2}}}{{x^1_{11}}^{\frac{1}{2}}}\right)^{-1} = \ell\left(\frac{(11-x^1_{12})^{\frac{1}{2}}}{(8-x^1_{11})^{\frac{1}{2}}}\right)^{-1}$$
$$\left(\frac{{x^2_{12}}^{\frac{1}{2}}}{{x^2_{11}}^{\frac{1}{2}}}\right)^{-1} = \ell\left(\frac{(8-x^2_{12})^{\frac{1}{2}}}{(8-x^2_{11})^{\frac{1}{2}}}\right)^{-1}$$
$$\left(\frac{{x^3_{12}}^{\frac{1}{2}}}{{x^3_{11}}^{\frac{1}{2}}}\right)^{-1} = \ell\left(\frac{(8-x^3_{12})^{\frac{1}{2}}}{(5-x^3_{11})^{\frac{1}{2}}}\right)^{-1}$$
$$\left(\left(8-x^1_{11}\right)\left(11-x^1_{12}\right)\right)^{\frac{1}{2}} + ((8-x^2_{11})(8-x^2_{12}))^{\frac{1}{2}} + \left(\left(5-x^3_{11}\right)\left(8-x^3_{12}\right)\right)^{\frac{1}{2}}$$
$$= \mathcal{U}_2 \text{ (fixed).}$$

It is easy to see that these conditions are satisfied by the REE allocations with the Lagrange multiplier $\ell = 1$.

E. Private core

Next we look at the way we can obtain the private core allocations and then we shall have to find the WFV allocations. We allow for free disposal and see what happens. For the private core allocations we impose private information measurability and solve the following:

Problem

Maximize $\mathcal{U}_1 = 2A^{\frac{1}{2}}B^{\frac{1}{2}} + (x_{11}^3 x_{12}^3)^{\frac{1}{2}}$
Subject to

$$
\begin{aligned}
&(x_{21}^1 x_{22}^1)^{\frac{1}{2}} + 2C^{\frac{1}{2}}D^{\frac{1}{2}} \geq \mathcal{U}_2 \text{ (fixed)}\\
&A + x_{21}^1 \leq 8, \;\; B + x_{22}^1 \leq 8, \;\; A + C \leq 8, \;\; B + D \leq 8,\\
&A + C \leq 8, \;\; B + D \leq 8, \;\; x_{11}^3 + C \leq 5, \;\; x_{12}^3 + D \leq 8,\\
&\mathcal{U}_1 \geq 7.3, \mathcal{U}_2 \geq 8.46.
\end{aligned}
$$

We operate with equality constraints eliminating $x_{21}^1,\ x_{22}^1,\ x_{11}^3,\ x_{12}^3,\ A$ and D and forming the Lagrangean $L = 2(8-C)^{\frac{1}{2}}(B)^{\frac{1}{2}} + \lambda\{(C)^{\frac{1}{2}}(11-B)^{\frac{1}{2}} + 2(C)^{\frac{1}{2}}(8-B)^{\frac{1}{2}} - \mathcal{U}_2\}$.

First order conditions are

$$\left(\frac{8-C}{B}\right)^{\frac{1}{2}} + \frac{1}{2}\left(\frac{5-C}{B}\right)^{\frac{1}{2}} = \ell\left\{\frac{1}{2}\left(\frac{C}{11-B}\right)^{\frac{1}{2}} + \left(\frac{C}{8-B}\right)^{\frac{1}{2}}\right\} \tag{8}$$

and

$$\left(\frac{B}{8-C}\right)^{\frac{1}{2}} + \frac{1}{2}\left(\frac{B}{5-C}\right)^{\frac{1}{2}} = \ell\left\{\frac{1}{2}\left(\frac{11-B}{C}\right)^{\frac{1}{2}} + \left(\frac{8-B}{C}\right)^{\frac{1}{2}}\right\} \tag{9}$$

which we can rewrite as

$$\frac{1}{C^{\frac{1}{2}}}\left\{(8-C)^{\frac{1}{2}} + \frac{1}{2}(5-C)^{\frac{1}{2}}\right\} = \ell B^{\frac{1}{2}}\left\{\frac{1}{2}\left(\frac{1}{11-B}\right)^{\frac{1}{2}} + \left(\frac{1}{8-B}\right)^{\frac{1}{2}}\right\} \tag{10}$$

and

$$C^{\frac{1}{2}}\left\{\left(\frac{1}{8-C}\right)^{\frac{1}{2}} + \frac{1}{2}\left(\frac{1}{5-C}\right)^{\frac{1}{2}}\right\} = \ell\frac{1}{B^{\frac{1}{2}}}\left\{\frac{1}{2}(11-B)^{\frac{1}{2}} + (8-B)^{\frac{1}{2}}\right\}. \tag{11}$$

Dividing gives

$$\frac{1}{C}\left\{\frac{(8-C)^{\frac{1}{2}} + \frac{1}{2}(5-C)^{\frac{1}{2}}}{\frac{1}{2}\frac{1}{(8-C)^{\frac{1}{2}}} + \frac{1}{(5-C)^{\frac{1}{2}}}}\right\} = B\left\{\frac{\frac{1}{2}\frac{1}{(11-B)^{\frac{1}{2}}} + \frac{1}{(8-B)^{\frac{1}{2}}}}{\frac{1}{2}(11-B)^{\frac{1}{2}} + (8-B)^{\frac{1}{2}}}\right\}. \tag{12}$$

It is a matter of routine substitutions to show that the allocation $x_1 = ((5.5, 5.5), (5.5, 5.5), (2.5, 5.5))$, $x_2 = ((2.5, 5.5), (2.5, 2.5), (2.5, 2.5))$ is in the private core, with normalized expected utilities $\mathcal{U}_1 = 14.70$ and $\mathcal{U}_2 = 8.70$.

Next we show that this allocation cannot be obtained as a Radner equilibrium with positive prices. We are looking for equality in all the conditions stated in the section Radner equilibria 1. A corner solution would require some zero quantities.

Substituting into the conditions for the demand functions we obtain $M_1 = 11(p_1^1 + p_1^2)$, $m_1 = 5p_1^1$, $M_1 = 11(p_2^1 + p_2^2)$, $m_1 = 11p_2^1$, $M_2 = 5p_1^3$, $m_2 = 5(p_1^1 + p_1^2)$, $M_2 = 11p_3^2$ and $m_2 = (p_2^2 + p_2^3)$. We normalize and set $p_1^1 = 1$. Then we obtain $m_1 = 5$, $p_2^1 = \frac{5}{11}$, $p_2^3 = 1$, $p_1^3 = \frac{11}{5}$, and we require further that $4p_1^3p_2^3 = (p_1^1 + p_1^2)(p_2^1 + p_2^2)$ and $4p_1^1p_2^1 = (p_1^2 + p_1^3)(p_2^2 + p_2^3)$. These equations cannot be satisfied by nonnegative prices because they imply $-3.890 = \frac{6}{11}p_1^2 + \frac{6}{5}p_2^2$.

Obviously there are measurable allocations which are not in the private core, such as

$$x_1 = ((5,5),(5,5),(2,5)), \quad \text{and} \quad x_2 = ((3,6),(3,3),(3,3)),$$
$$x_1 = ((4,4),(4,4,(1,4)), \qquad x_2 = ((4,7),(4,4),(4,4))$$

as can be seen through routine calculations.

On the other hand we can show directly that a Radner equilibrium is in the private core. Taking into account the constraints for demand to be equal to supply, the first order conditions for the agents' maximization of utilities can be cast as follows.

For Agent 1:

$$\frac{B^{\frac{1}{2}}}{(8-C)^{\frac{1}{2}}} - \ell'(p_1^1 + p_1^2) = 0, \qquad \frac{(8-C)^{\frac{1}{2}}}{B^{\frac{1}{2}}} - \ell'(p_2^1 + p_2^2) = 0, \tag{13}$$

$$\frac{1}{2}\frac{B^{\frac{1}{2}}}{(5-C)^{\frac{1}{2}}} - \ell' p_1^3 = 0, \quad \text{and} \qquad \frac{1}{2}\frac{(5-C)^{\frac{1}{2}}}{B^{\frac{1}{2}}} - \ell' p_2^3 = 0, \tag{14}$$

and for Agent 2:

$$\frac{1}{2}\frac{(11-B)^{\frac{1}{2}}}{C^{\frac{1}{2}}} - \psi p_1^1 = 0, \qquad \frac{1}{2}\frac{C^{\frac{1}{2}}}{(11-B)^{\frac{1}{2}}} - \psi p_2^1 = 0, \tag{15}$$

$$\frac{(8-B)^{\frac{1}{2}}}{C^{\frac{1}{2}}} - \psi(p_1^2 + p_1^3) = 0, \quad \text{and} \quad \frac{C^{\frac{1}{2}}}{(8-B)^{\frac{1}{2}}} - \psi(p_2^2 + p_2^3) = 0. \tag{16}$$

Substituting (14), (15), (16) and (17) into (9) and (10) we obtain in both instances the relation $\ell' = \ell\psi$ which shows that the Radner equilibrium is in the private core.

F. WFV

Routine calculations imply $V_\lambda(\{1\}) = \frac{1}{3}\lambda_1 A$, $V_\lambda(\{2\}) = \frac{1}{3}\lambda_2 B$, where $A = (2(7)^{\frac{1}{2}} + 2)$ and $B = (2(7)^{\frac{1}{2}} + 10^{\frac{1}{2}})$.

Next we have, $V_\lambda(\{1,2\}) = \frac{1}{3}\max_x \{\lambda_1(x_{11}^1 x_{12}^1)^{\frac{1}{2}} + \lambda_2(8 - x_{11}^1)^{\frac{1}{2}}(11 - x_{12}^1)^{\frac{1}{2}} + \lambda_1(x_{11}^2 x_{12}^2)^{\frac{1}{2}} + \lambda_2(8 - x_{11}^2)^{\frac{1}{2}}(8 - x_{12}^2)^{\frac{1}{2}} + \lambda_1(x_{11}^3 x_{12}^3)^{\frac{1}{2}} + \lambda_2(5 - x_{11}^3)^{\frac{1}{2}}(8 - x_{12}^3)^{\frac{1}{2}}\}$.

We define the per period terms of the sum by $\mathcal{U}^1, \mathcal{U}^2$ and $\mathcal{U}^3$. We assume that both λ's are positive. Otherwise all the weight is put on one agent. We can do separate maximization and defining $\Lambda_1 = \lambda_1^2$, $\Lambda_2 = \lambda_2^2$ we obtain the conditions

(i) $\Lambda_1 x_{12}^1(8 - x_{11}^1) = \Lambda_2 x_{11}^1(11 - x_{12}^1)$ and $\Lambda_1 x_{11}^1(11 - x_{12}^1) = \Lambda_2 x_{12}^1(8 - x_{11}^1)$
(ii) $\Lambda_1 x_{12}^2(8 - x_{11}^2) = \Lambda_2 x_{11}^2(8 - x_{12}^1)$ and $\Lambda_1 x_{11}^2(8 - x_{12}^1) = \Lambda_2 x_{12}^2(8 - x_{11}^2)$
(iii) $\Lambda_1 x_{12}^3(5 - x_{11}^3) = \Lambda_2 x_{11}^3(8 - x_{12}^3)$ and $\Lambda_1 x_{11}^3(11 - x_{12}^3) = \Lambda_2 x_{12}^3(5 - x_{11}^3)$

From (i), (ii) and (iii) we obtain, respectively, $x_{12}^1 = \frac{11}{8}x_{11}^1$, $x_{12}^2 = x_{11}^2$ and $x_{12}^3 = \frac{8}{5}x_{11}^3$. Which means that the maximum will be sought on these flats.

From the above we obtain $\mathcal{U}^1 = (\frac{11}{8})^{\frac{1}{2}}(\lambda_1 x_{11}^1 + \lambda_2(8 - x_{11}^1))$, $\mathcal{U}^2 = \lambda_1 x_{11}^2 + \lambda_2(8 - x_{11}^2))$ and $\mathcal{U}^3 = (\frac{8}{5})^{\frac{1}{2}}(\lambda_1 x_{11}^3 + \lambda_2(5 - x_{11}^3))$. It follows, that $V_\lambda(\{1,2\}) = \frac{1}{3}\max_{x_1}[(\frac{11}{8})^{\frac{1}{2}}(\lambda_1 x_{11}^1 + \lambda_2(8 - x_{11}^1)) + (\lambda_1 x_{11}^2 + \lambda_2(8 - x_{11}^2)) + (\frac{8}{5})^{\frac{1}{2}}(\lambda_1 x_{11}^3 + \lambda_2(5 - x_{11}^3))]$. I.e. $V_\lambda(\{1,2\}) = \frac{1}{3}\max_{x_1}[8(\frac{11}{8})^{\frac{1}{2}}\{\max(\lambda_1,\lambda_2)\} + 8\{\max(\lambda_1,\lambda_2)\} + 5(\frac{8}{5})^{\frac{1}{2}}\{\max(\lambda_1,\lambda_2)\}]$, which we can write as $V_\lambda(\{1,2\}) = C\max(\lambda_1,\lambda_2)$, where $C = (88)^{\frac{1}{2}} + 8 + (40)^{\frac{1}{2}}$. The significance of the flats is clear. For maximization the choice from the extreme values of the variable x_1 depends on the values of λ_1 and λ_2. In particular for $\lambda_1 > \lambda_2$ all endowments are allocated to the utility function of Agent 1, for $\lambda_1 < \lambda_2$ the one of Agent 2, and for $\lambda_1 = \lambda_2$ the allocation can be arbitrary. This can be seen by obtaining $V_\lambda(\{1,2\})$ through the per period maximization of the utility of Agent 1 subject to the utility of Agent 2 being fixed.

For WFV allocations we require solutions to

$$\lambda_1 \sum_\omega (x_{11}(\omega)x_{12}(\omega))^{\frac{1}{2}} = \frac{1}{2}\{C\max\{\lambda_1,\lambda_2\} + A\lambda_1 - B\lambda_2\} \tag{17}$$
$$\lambda_2 \sum_\omega (x_{21}(\omega)x_{22}(\omega)) = \frac{1}{2}\{C\max\{\lambda_1,\lambda_2\} - A\lambda_1 + B\lambda_2\}$$

subject to

$$x_1 + x_2 \le e_1 + e_2,$$

relaxing the feasibility condition. The right-hand sides of the equations above are the $Sh_i(V_\lambda)$'s.

The set of WFV allocations is not empty. It can be checked that for $\lambda_1 = \lambda_2$ the allocation in which P1 gets $((4,\frac{11}{2}),(5,5),(\frac{5}{4},2))$ and P2 gets $((4,\frac{11}{2}),(3,3),(\frac{15}{4},6))$ is a WFV allocation. We see this by inserting these allocations and $\lambda_1 = \lambda_2$ into the relations above to obtain

$$(22)^{\frac{1}{2}} + 5 + (2.5)^{\frac{1}{2}} = \frac{1}{2}\left((88)^{\frac{1}{2}} + 8 + (40)^{\frac{1}{2}} + 2(7)^{\frac{1}{2}} + 2 - (10)^{\frac{1}{2}} - 2(7)^{\frac{1}{2}}\right)$$
$$2((22)^{\frac{1}{2}} + 3 + (7.5)^{\frac{1}{2}}) = \frac{1}{2}\left((88)^{\frac{1}{2}} + 8 + (40)^{\frac{1}{2}} - 2(7)^{\frac{1}{2}} - 2 + (10)^{\frac{1}{2}} + 2(7)^{\frac{1}{2}}\right)$$

which can be checked that they are satisfied.

On the other hand, it is a matter of tedious calculations to show that the fully revealing REE is not a WFV allocation although it belongs to the weak fine core.

Performing the calculations we obtain the relations

$$11.46\lambda_1 = \frac{1}{2}\big[23.71\{\max(\lambda_1,\lambda_2)\} + 7.291\lambda_1 - 8.45378\lambda_2\big] \text{ and}$$
$$12.25\lambda_2 = \frac{1}{2}\big[23.71\{\max(\lambda_1,\lambda_2)\} - 7.291\lambda_1 + 8.45378\lambda_2\big].$$

We distinguish between two cases and we examine whether the REE is in a weak fine value allocation.

Case 1. $\lambda_1 \geq \lambda_2$
We require

$$11.46\lambda_1 = \frac{1}{2}\big[23.71\lambda_1 + 7.291\lambda_1 - 8.45378\lambda_2\big] \text{ and}$$
$$12.25\lambda_2 = \frac{1}{2}\big[23.71\lambda_1 - 7.291\lambda_1 + 8.45378\lambda_2\big]$$

which imply $4.04\lambda_1 = 4.23\lambda_2$ and $8.21\lambda_1 = 8.22\lambda_1$ both of which cannot be satisfied.

Case 2. $\lambda_2 \geq \lambda_1$
We require now

$$22.92\lambda_1 = 23.71\lambda_2 + 7.291\lambda_1 - 8.45378\lambda_2 \text{ and}$$
$$24.50\lambda_2 = 23.71\lambda_2 - 7.291\lambda_1 + 8.45378\lambda_2$$

which imply $15.63\lambda_1 = 15.26\lambda_2$ and $7.66\lambda_2 = 7.29\lambda_1$ which again cannot be satisfied.

The question arises why is the set of WFV allocations smaller than the WFC, although this of course is only true in the case of two agents. An intuitive explanation is that for the WFV allocations the conditions are more stringent because of the homogeneity of equations in λ_1, and λ_2. We need to get from both equations in (18) the same $\frac{\lambda_1}{\lambda_2}$, and if we are given $x(\omega)$ this is highly unlikely to happen.

Now we show that a WFV equilibrium exists only for $\lambda_1 = \lambda_2$.

Adding side by side the equations (18), we get on the RHS $C\max\{\lambda_1,\lambda_2\}$ which is equal to $V_\lambda(\{1,2\})$. Therefore the sum on the LHS must be also equal to $V_\lambda(\{1,2\})$ and therefore a maximum, and we have seen how this depends on the weights λ_1 and λ_2.

Putting all the information together leads to the following possibilities. $\lambda_1 > \lambda_2$ requires

$$\lambda_1 C = \frac{1}{2}\left\{\lambda_1 C + A\lambda_1 - B\lambda_2\right\}$$
$$0 = \frac{1}{2}\left\{\lambda_1 C - A\lambda_1 + B\lambda_2\right\}.$$

Either of these leads to

$$B\lambda_2 = (A - C)\lambda_1 < 0$$

which is impossible. Similarly $\lambda_1 < \lambda_2$ is impossible.

Finally with $\lambda_1 = \lambda_2$ the equations for a weak-fine-value become, on writing $y_\alpha = x_{12}(\omega_\alpha)$ and recalling that $x_\alpha = \frac{8}{11} y_\alpha$,

$$\left(\frac{8}{11}\right)^{\frac{1}{2}} (2y_1 - 11) + (2y_2 - 8) + \left(\frac{5}{8}\right)^{\frac{1}{2}} (2y_3 - 8) = 2 - 10^{\frac{1}{2}}$$

which is satisfied by the previous specified allocation.

Example 6.3. The problem is a two-state, $\Omega = \{a, b\}$, three-player, two-good game with utilities and initial endowments given by:

$$\begin{aligned}
u_1(x_{11}^j, x_{12}^j) &= \min(x_{11}^j, x_{12}^j); \quad e_1 = ((1,0),(1,0)), \quad F_1 = \{\{a\},\{b\}\} \\
u_2(x_{21}^j, x_{22}^j) &= \min(x_{21}^j, x_{22}^j); \quad e_2 = ((0,1),(0,1)), \quad F_2 = \{\{a\},\{b\}\} \\
u_3(x_{31}^j, x_{32}^j) &= \frac{x_{31}^j + x_{32}^j}{2}; \qquad e_3 = ((0,0),(0,0)), \quad F_3 = \{\{a,b\}\},
\end{aligned}$$

where x_{ik}^j denotes consumption of Player i of Good k, in state j. Every player has the same prior distribution $\mu(\omega) = \frac{1}{2}$ for $\omega \in \Omega$.

The weights of the agents are λ_i for $i = 1, 2, 3$. First we calculate the characteristic function V_λ.

For $S = \{1\}, \ \{2\}$ or $\{3\}$ we have $e_i = x_i$ and so $u_i = 0$. Therefore $V_\lambda(\{i\}) = 0$. Next consider $S = (\{1, 2\})$. The sum of the weighted utilities

$$\sum_{j \in \Omega} \frac{1}{2} [\lambda_1 \min(x_{11}^j, x_{12}^j) + \lambda_2 \min(x_{21}^j, x_{22}^j)]$$

must be maximized subject to $x_{11}^j + x_{21}^j = 1$ and $x_{12}^j + x_{22}^j = 1$ for $j \in \Omega$. It is straightforward that for a maximum we must have $x_{11} = x_{12}$ and $x_{21} = x_{22}$ and then that $V_\lambda(\{1, 2\}) = \max(\lambda_1, \lambda_2)$. It is also straigtforward that $V_\lambda(\{1, 3\}) = V_\lambda(\{2, 3\}) = \frac{\lambda_3}{2}$.

We now turn our attention to $S = \{1, 2, 3\}$. The expression

$$\sum_{j \in \Omega} \frac{1}{2} \left[\lambda_1 \min(x_{11}^j, x_{12}^j) + \lambda_2 \min(x_{21}^j, x_{22}^j) + \lambda_3 \frac{x_{31}^j + x_{32}^j}{2} \right]$$

must be maximized subject to $x_{11}^j + x_{21}^j + x_{31}^j = 1 \quad x_{12}^j + x_{22}^j + x_{32}^j = 1$, for $j \in \Omega$.

Again from the first two terms we get $\max(\lambda_1, \lambda_2)$ and for the whole constraint sum $V_\lambda(\{1, 2, 3\}) = \max(\lambda_1, \lambda_2, \lambda_3)$.

Consider now the special case $\lambda_i = 1$ for $i = 1, 2, 3$. Replacing in the above λ_i by 1 we obtain

$$\begin{aligned}
&V_\lambda(\{i\}) = 0, \quad \text{for } i = 1, 2, 3 \\
&V_\lambda(\{1, 2\}) = 1, \quad V_\lambda(\{1, 3\}) = V_\lambda(\{2, 3\}) = \frac{1}{2} \ \text{ for } i = 2, 3 \\
&V_\lambda(\{1, 2, 3\}) = 1.
\end{aligned}$$

For this particular case, $\lambda_i = 1$, the Shapley values are given by

$$Sh_1(V) = 0 + \frac{1}{6}(1-0) + \frac{1}{6}\left(\frac{1}{2} - 0\right) + \frac{2}{6}\left(1 - \frac{1}{2}\right) = \frac{5}{12}$$
$$Sh_2(V) = \frac{5}{12}, \text{ and } Sh_3(V) = \frac{2}{12}.$$

Hence the value allocation is, per state,

$$(x_{11}, x_{12}) = (x_{21}, x_{22}) = \left(\frac{5}{12}, \frac{5}{12}\right) \text{ and } (x_{31}, x_{32}) = \left(\frac{2}{12}, \frac{2}{12}\right).$$

On the other hand any Walrasian type allocation will award zero quantities to Player 3, as he has no initial endowments. Therefore the point that this example is making is that with the number of agents $n \geq 3$, it is possible that there is a value allocation which does not belong to a Walrasian type set, (i.e. it is not a REE or Radner equilibrium).

However it can also be used to make one more point that is equally important. It can be seen that for $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 0$ or for the case where there is no third agent and with $\lambda_1 = \lambda_2 = 1$ we have $Sh_1(V) = Sh_2(V) = \frac{1}{2}$. This says that it is possible that a group which includes all the agents can do better for its members than each one of them in isolation, but this is not the end of the story. A sub-group might do even better.

With respect to offering an interpretation of the distinction between side payments and a WFV allocation, we look at the following situation. Two agents have some initial endowments, the same weights, and their utilities are really revenues from selling these quantities in a non-competitive market. We can hand over all quantities to one agent, ask him to sell them on the market, keep his Shapley share and hand the other agent his own. With respect to the weak fine value it corresponds to the case when only a redistribution of the endowments is allowed, in which case we might only be able to do it when specific weights are given to the individuals.

Non-existence of REE:
Finally we discuss a specific version of the well known Kreps (1977) example of a non-existent REE. On the other hand, in the same example, the private core exists which suggests that the latter concept has an advantage over that of REE.[11]

Example 6.4 (2003b). There are two agents $I = \{1, 2\}$, two commodities, i.e. $X_i = \mathbb{R}^2_+$ for each Agent, i, and two states of nature $\Omega = \{\omega_1, \omega_2\}$, considered by the agents as equally probable. In x_{ij} the first index will refer to the agent and the second to the good.

We assume that the endowments, per state ω_1 and ω_2 respectively, and information partitions of the agents are given by

$$e_1 = ((1.5, 1.5), (1.5, 1.5), \qquad \mathcal{F}_1 = \{\{\omega_1\}, \{\omega_2\}\};$$
$$e_2 = ((1.5, 1.5), (1.5, 1.5), \qquad \mathcal{F}_2 = \{\{\omega_1, \omega_2\}\}.$$

[11] Example 6.4 is also discussed in Glycopantis et al. (2003b).

The utility functions of Agents 1 and 2 respectively, are for ω_1 given by $u_1 = \log x_{11} + x_{12}$ and $u_2 = 2 \log x_{21} + x_{22}$ and for state ω_2 by $u_1 = 2 \log x_{11} + x_{12}$ and $u_2 = \log x_{21} + x_{22}$.

We consider first the possibility of REE.

Case 1. Fully revealing REE.
Suppose that there exist, after normalization, prices $(p_1(1), p_2(1)) \neq (p_1(2), p_2(2))$, where $p_i(j)$ denotes the price of good i in state j. The problems that the two agents solve are as follows.

State ω_1.

Agent 1:
Maximize $u_1 = \log x_{11} + x_{12}$
Subject to

$$p_1(1)x_{11} + p_2(1)x_{12} = 1.5(p_1(1) + p_2(1))$$

and

Agent 2:
Maximize $u_2 = 2 \log x_{21} + x_{22}$
Subject to

$$p_1(1)x_{21} + p_2(1)x_{22} = 1.5(p_1(1) + p_2(1)).$$

The agents solve analogous problems in state ω_2. However it is not possible to find $(p_1(1), p_2(1)) \neq (p_1(2), p_2(2))$. In the two problems the demands of the agents are interchanged so that the total demand stays the same while the total supply is fixed. It is also straightforward to check that there is no multiplicity of equilibria per state.

Case 2. Non-revealing REE.
Now we consider the possibility of $p_1(1) = p_1(2) = p_1$ and $p_2(1) = p_2(2) = p_2$. The two agents would act as follows.

Agent 1:
He can tell the states of nature and obtains the demand functions
for $\omega_1, x_{11} = \frac{p_2}{p_1}$ and $x_{12} = \frac{1.5p_1}{p_2} + 0.5$ and for $\omega_2, x_{11} = \frac{2p_2}{p_1}$ and $x_{12} = \frac{1.5p_1}{p_2} - 0.5$ for $3p_1 \geq p_2$.

It is clear that the demands differ per state of nature.

Agent 2:
He sets $x_{21}(\omega_1) = x_{21}(\omega_2) = x_{21}$ and $x_{22}(\omega_1) = x_{22}(\omega_2) = x_{22}$ and solves the problem:

Maximize $u_2 = \frac{1}{2}(2 \log x_{21} + x_{22}) + \frac{1}{2}(\log x_{21} + x_{22}) = 1.5 \log x_{21} + x_{22}$
Subject to

$$p_1x_{21} + p_2x_{22} = 1.5(p_1 + p_2).$$

So the highest indifference curve touches the budget constraint only once. On the other hand the demands of Agent 1 differ per ω. It follows that the markets cannot be cleared with common prices in both states of nature.

The above analysis shows that there is no REE in this model.

Next we consider, in the same example, the existence of private core allocations. These are obtained as solutions of the problem:

Maximize $E_2 = 1.5\ log\ x_{21} + x_{22}$
Subject to

$$\frac{1}{2}(\ log\ x_{11}(\omega_1) + x_{12}(\omega_1)) + \frac{1}{2}(\ log\ x_{11}(\omega_2) + x_{12}(\omega_2)) \geq E_1 \text{ (fixed)},$$
$$x_{1j}(\omega_1), x_{1j}(\omega_2) \geq 0, E_1,\ E_2 \geq 1.5\ log\ 1.5 + 1.5$$
$$x_{21} + x_{11}(\omega_1) \leq 3, \quad x_{21} + x_{11}(\omega_2) \leq 3,$$
$$x_{22} + x_{12}(\omega_1) \leq 3, \quad x_{22} + x_{12}(\omega_2) \leq 3.$$

The structure of the problem, i.e. the continuity of the objective function and the compactness of the feasible set, implies that it has always a solution. In particular, if we set the quantity constraints equal to 3 and $1.5\ log\ 1.5 + 1.5 = E_1$ then the initial allocation is in the private core.

The discussion of Example 6.4 indicates that the REE may not be an appropriate concept to explain trades in DIE. The agents here receive no instructions as to what they should be doing.

7 Incentive compatibility

There are alternative formulations of the notion of incentive compatibility. The basic idea is that an allocation is incentive compatible if no coalition can misreport the realized state of nature and have a distinct possibility of making its members better off.

Suppose we have a coalition S, with members denoted by i, and the complementary set $I \setminus S$ with members j. Let the realized state of nature be ω^*. Each member $i \in S$ sees $E_i(\omega^*)$. Obviously not all $E_i(\omega^*)$ need be the same, however all Agents i know that the actual state of nature could be ω^*.

Consider a state $\omega^{'}$ such that for all $j \in I \setminus S$ we have $\omega^{'} \in E_j(\omega^*)$ and for at least one $i \in S$ we have $\omega^{'} \notin E_i(\omega^*)$. Now the coalition S decides that each member i will announce that she has seen her own set $E_i(\omega^{'})$ which, of course, contains a lie. On the other hand we have that $\omega^{'} \in \bigcap_{j \notin S} E_j(\omega^*)$.

The idea is that if all members of $I \setminus S$ believe the statements of the members of S then each $i \in S$ expects to gain. For *coalitional Bayesian incentive compatibility* (CBIC) of an allocation we require that this is not possible. This is the incentive compatibility condition we used in Glycopantis et al. (2001).[12]

We showed in Example 5.1 that in the three-agent economy without free disposal the private core allocation $x_1 = (4, 4, 1)$, $x_2 = (4, 1, 4)$ and $x_3 = (2, 0, 0)$ is incentive compatible. This follows from the fact that Agent 3 who would potentially cheat in state a has no incentive to do so. It has been shown in Koutsougeras and Yannelis (1993) that if the utility functions are monotone and continuous then private core allocations are *always* CBIC.

[12] See Krasa and Yannelis (1994) and Hahn and Yannelis (1997) for related concepts.

On the other hand the WFC allocations are not always incentive compatible, as the proposed redistribution $x_1 = x_2 = (5, 2.5, 2.5)$ in Example 5.2 shows. Indeed, if Agent 1 observes $\{a, b\}$, he has an incentive to report c and Agent 2 has an incentive to report b when he observes $\{a, c\}$.

CBIC coincides in the case of a two-agent economy with the concept of *Individually Bayesian Incentive Compatibility* (IBIC), which refers to the case when S is a singleton.

We consider here explicitly the concept of *Transfer Coalitionally Bayesian Incentive Compatible* (TCBIC) allocations. This allows for transfers between the members of a coalition, and is therefore a strengthening of the concept of Coalitionally Bayesian Incentive Compatibility (CBIC).

Definition 7.1. An allocation $x = (x_1, \ldots, x_n) \in \bar{L}_X$, with or without free disposal, is said to be TCBIC if it is not true that there exists a coalition S, states ω^* and $\omega^{'}$, with ω^* different from $\omega^{'}$ and $\omega^{'} \in \bigcap_{i \notin S} E_i(\omega^*)$ and a random, net-trade vector, $z = (z_i)_{i \in S}$ among the members of S,

$$(z_i)_{i \in S}, \sum_S z_i = 0$$

such that for all $i \in S$ there exists $\bar{E}_i(\omega^*) \subseteq Z_i(\omega^*) = E_i(\omega^*) \cap (\bigcap_{j \notin S} E_j(\omega^*))$, for which

$$\sum_{\omega \in \bar{E}_i(\omega^*)} u_i(\omega, e_i(\omega) + x_i(\omega^{'}) - e_i(\omega^{'}) + z_i) q_i\left(\omega | \bar{E}_i(\omega^*)\right) \quad (18)$$
$$> \sum_{\omega \in \bar{E}_i(\omega^*)} u_i(\omega, x_i(\omega)) q_i\left(\omega | \bar{E}_i(\omega^*)\right).$$

Notice that $e_i(\omega) + x_i(\omega^{'}) - e_i(\omega^{'}) + z_i(\omega) \in X_i(\omega)$ is not necessarily measurable. The definition implies that no coalition can hope that by misreporting a state, every member will become better off if they are believed by the members of the complementary set.

Returning to Definition 7.1, one can define CBIC to correspond to $z_i = 0$ and then IBIC to the case when S is a singleton. Thus we have (not IBCI) $\Rightarrow$ (not CBIC) $\Rightarrow$ (not TCBIC). It follows that TCBIC $\Rightarrow$ CBIC $\Rightarrow$ IBIC.

We now provide a *characterization* of TCBIC:

Proposition 7.1. Let $\mathcal{E}$ be a one-good DIE, and suppose each agent's utility function, $u_i = u_i(\omega, x_i(\omega))$ is monotone in the elements of the vector of goods x_i, that $u_i(., x_i)$ is $\mathcal{F}_i$-measurable in the first argument, and that an element $x = (x_1, \ldots, x_n) \in \bar{L}_X^1$ is a feasible allocation in the sense that $\sum_{i=1}^n x_i(\omega) = \sum_{i=1}^n e_i(\omega) \; \forall \omega$. Consider the following conditions:

(i) $x \in L_X^1 = \prod_{i=1}^n L_{X_i}^1$. and
(ii) x is TCBIC.

Then (i) is equivalent to (ii).

Proof. See Glycopantis et al. (2003a).

Next we state conditions under which the private core allocation is CBIC.

Proposition 7.2. let $\mathcal{E}$ be an arbitrary differential information economy with monotone and continuous utility functions. The private core and the private value are CBIC.

Proof. See Koutsougeras and Yannelis (1993), Krasa and Yannelis (1994), and Hahn and Yannelis (2001).

Corollary 7.1. A no-free disposal Radner equilibrium is CBIC.[13]

Proof. It can be easily shown that any no-free disposal Radner equilibrium belongs to the private core. Therefore by Proposition 7.2 it follows that the Radner equilibrium is CBIC.

Proposition 7.1 characterizes TCBIC and CBIC in terms of private individual measurability of allocations. It will enable us to conclude whether or not, in case of non-free disposal, any of the solution concepts will be TCBIC, whenever feasible allocations are $\mathcal{F}_i$-measurable.

It follows also that the redistribution

$$\begin{pmatrix} 5 & 2.5 & 2.5 \\ 5 & 2.5 & 2.5 \end{pmatrix}$$

is not CBIC because it is not $\mathcal{F}_i$-measurable.

On the other hand the proposition implies, again in Example 5.1, that the allocation

$$\begin{pmatrix} 5 & 5 & 0 \\ 5 & 0 & 5 \end{pmatrix}$$

is incentive compatible. As we have seen this is a non-free disposal REE, and a private core allocation.

We note that the above propositions are not true if we assume free disposal. In that case $\mathcal{F}_i$-measurability *does not* imply incentive compatibility. In the case with *free disposal, private core and Radner equilibrium need not be incentive compatible*. In order to see this we notice that in Example 5.2 the (free disposal) Radner equilibrium is $x_1 = (4, 4, 1)$ and $x_2 = (4, 1, 4)$. The above allocation is clearly $\mathcal{F}_i$-measurable and it can be checked directly that it belongs to the (free disposal) private core. However it is not TBIC since if state a occurs Agent 1 has an incentive to report state c and become better off.

Next we consider Example 6.1. We define $A_1 = \{a, b\}$ and $A_2 = \{b, c\}$. We assume that P1 acts first and that when P2 is to act he has heard the declaration of P1.

As shown in Section 6 the fully revealing REE allocations and corresponding utilities are:

$$\text{In state } a,\ x^*_{11} = \frac{85}{22},\ x^*_{12} = \frac{85}{16},\ x^*_{21} = \frac{91}{22},\ x^*_{22} = \frac{91}{16};\ u^*_1 = 4.53,\ u^*_2 = 4.85.$$

[13] A direct proof of the CBIC of the non-free disposal Radner equilibrium with infinitely many commodities has been given in Herves et al. (2003).

In state b, $x^*_{11} = 4,\ x^*_{12} = 4,\ x^*_{21} = 4,\ x^*_{22} = 4;\ u^*_1 = 4,\ u^*_2 = 4.$

In state c, $x^*_{11} = \frac{37}{16},\ x^*_{12} = \frac{37}{10},\ x^*_{21} = \frac{43}{16},\ x^*_{22} = \frac{43}{10};\ u^*_1 = 2.93,\ u^*_2 = 3.40.$

The normalized expected utilities of the REE are $\mathcal{U}_1 = 11.46,\ \mathcal{U}_2 = 12.25.$

We can see that the REE redistribution, which belongs also to the WFC, is not CBIC as follows.[14] Suppose that P1 sees $\{a, b\}$ and P2 sees $\{a\}$ but misreports $\{b, c\}$. If P1 believes the lie then state b is believed. So P1 agrees to get the allocation $(4, 4)$. P2 receives the allocation $e_2(a)+x_2(b)-e_2(b) = (1, 10)+(4, 4)-(1, 7) = (4, 7)$ with $u_2(4, 7) = 5.29 > u_2(\frac{91}{22}, \frac{91}{16}) = 4.85$. Hence P2 has a possibility of gaining by misreporting and therefore the REE is not CBIC. On the other hand if P2 sees $\{b, c\}$ and P1 sees $\{c\}$, the latter cannot misreport $\{a, b\}$ and hope to gain if P2 believes it is b.

In employing game trees in the analysis we adopt the definition of IBIC. The game-theoretic equilibrium concept employed will be that of PBE. A play of the game will be a directed path from the initial to a terminal node.

In terms of the game trees, a core allocation will be IBIC if there is a profile of optimal behavioral strategies along which no player misreports the state of nature he has observed. This allows for the possibility that players have an incentive to lie from information sets which are not visited by an optimal play.

In view of the analysis using game trees we comment further on the general idea of CBIC. First we look at it again, in a similar manner to the one in the beginning of Section 4.

Suppose the true state of nature is $\bar{\omega}$. Any coalition can only see together that the state lies in $\bigcap_{i\in S} E_i(\bar{\omega})$. If they decide to lie they must first guess at what is the true state and they will do so at some $\omega^* \in \bigcap_{i\in S} E_i(\bar{\omega})$. Having decided on ω^* as a possible true state, they pick some $\omega^{'} \in \bigcap_{j\notin S} E_j(\omega^*)$ and assuming the system is not CBIC they hope, by each of them announcing $E_i(\omega^{'})$ to secure better payoffs.

This is all contingent on their being believed by $I \setminus S$, which depends on having been correct in guessing that $\omega^* = \bar{\omega}$. If $\omega^* \neq \bar{\omega}$, i.e they guess wrongly, then since $\bigcap_{j\notin S} E_j(\omega^*) \neq \bigcap_{j\notin S} E_j(\bar{\omega})$ the lie may be detected, since possibly $\omega^{'} \notin \bigcap_{j\notin S} E_j(\bar{\omega})$.

Therefore the definition of CBIC can only be about situations where a lie might be beneficial. On the other hand the extensive form forces us to consider the alternative of what happens if the lie is detected. It requires statements concerning earlier decisions by other players to lie or tell the truth and what payoffs will occur whenever a lie is detected, through observations or incompatibility of declarations. Only in this fuller description can players make a decision whether to risk a lie. Such considerations probably open the way to an incentive compatibility definition based on expected gains from lying.

The issue is whether cooperative and noncooperative static solutions can be supported through an appropriate noncooperative solution concept. The analysis below shows that CBIC allocations can be supported by a PBE while absence of

[14] Palfrey and Srivastava (1986) have also shown that the REE may not be incentive compatible.

incentive compatibility implies lack of such support. It is also shown how implementation of allocations becomes possible by introducing an exogenous third party or an endogenous intermediary.

We recall that a PBE consists of a set of players' optimal behavioral strategies, and consistent with these, a set of beliefs which attach a probability distribution to the nodes of each information set (Tirole, 1988). It is a variant of the idea of a sequential equilibrium (Kreps and Wilson, 1982).

Note 7.1. Different notions of incentive compatibility for differential information economies were first introduced by in Krasa and Yannelis (1994). It should be noted that the framework for differential information economies is different than the one in the Harsanyi type models and the notions of incentive compatibility which they use. These models assume that the initial endowments are independent of the state of nature and therefore uncertainty comes only from the utility functions.

Notice that if the initial endowments are assumed to be constant, then most of the examples in this paper cannot be analysed by a Harsanyi type model. A comparison between the DIE model and the Harsanyi type models can be found in Hahn and Yannelis (1997). In particular this paper contains a comparison of some of the Holmström and Myerson (1983) incentive compatibility notions and the ones in the DIE literature.

Finally it is important to notice that in a multilateral contracts model, it appears more appropriate to ensure CBIC rather than IBIC. Obviously CBIC implies IBIC but the reverse is not true, as an example in the preface of this volume demonstrates. Therefore lack of CBIC may make a contract unstable or not viable.

8 Non-implementation of Radner equilibria, of WFC and WFV allocations

We examine here the implementation, as a PBE of different equilibrium concepts. This section is closely related to the previous one. The fundamental issue is to connect, in the context of the partition model, the idea of implementation, in the form of a PBE of an extensive form game, to the CBIC property. Namely, we wish to check whether an allocation can be realized as a PBE in an incomplete information, dynamic game, in the form of a tree, and how this is connected to the CBIC property.

The static concept of the CBIC implies that no agent has an incentive to lie with respect to the state(s) he has observed and the PBE satisfies basic rationality criteria in a game tree in which the agents are asymmetrically informed.

We examine whether cooperative or Walrasian, noncooperative, static equilibrium allocations, can be supported as the outcome of a dynamic, noncooperative solution concept. We also examine the role that a third party can play in supporting an equilibrium.

A general conclusion is that static equilibrium allocations with the CBIC property can be supported, under reasonable rules, as PBE outcomes. This discussion helps us to reach a conclusion as to which equilibrium concept can be considered as appropriate. We find that private core allocations have distinct advantages.[15]

8.1 Non-implementation of Radner equilibria, of WFC and WFV allocations

We consider Example 5.2. We show here that lack of IBIC implies that two agents do not sign a proposed contract because they have an incentive to cheat. Therefore PBE leads to no-trade.

We shall investigate the possible implementation of the allocation

$$\begin{pmatrix} 4 & 4 & 1 \\ 4 & 1 & 4 \end{pmatrix}$$

of Example 5.2, contained in a proposed contract between P1 and P2. As we have seen, with free disposal this is a Radner equilibrium allocation.

This allocation is not IBIC because, as we explained in Section 8, if Agent 1 observes $A_1 = \{a, b\}$, he has an incentive to report c and Agent 2 has an incentive to report b when he observes $A_2 = \{a, c\}$.

We construct a game tree and employ reasonable rules for calculating payoffs. In fact we look at the contract

$$\begin{pmatrix} 5 & 4 & 1 \\ 5 & 1 & 4 \end{pmatrix}.$$

The proposed allocation can be obtained by invoking free disposal in state a. Of course to impose free disposal causes certain problems, because the question arises as to how it will be verified that the agents have actually thrown away 1 unit. However we assume that this is possible. In the analysis below we assume that the players move sequentially.

The *rules* for calculating the payoffs in terms of quantities, i.e. the terms of the contract, are:

(i) If the declarations by the two players are incompatible, that is (c_1, b_2) then no-trade takes place and the players retain their initial endowments.
That is the case when either state c, or state b occurs and Agent 1 reports state c and Agent 2 state b. In state a both agents can lie and the lie cannot be detected by either of them. They are in the events A_1 and A_2 respectively, they get 5 units of the initial endowments and again they are not willing to cooperate. Therefore whenever the declarations are incompatible, no trade takes place and the players retain their initial endowments.

(ii) If the declarations are (A_1, A_2) then even if one of the players is lying, this cannot be detected by his opponent who believes that state a has occurred and both players have received endowment 5. Hence no-trade takes place.

[15] For a thorough analysis in this section the reader is referred also to Glycopantis et al. (2001, 2003a, 2003b).

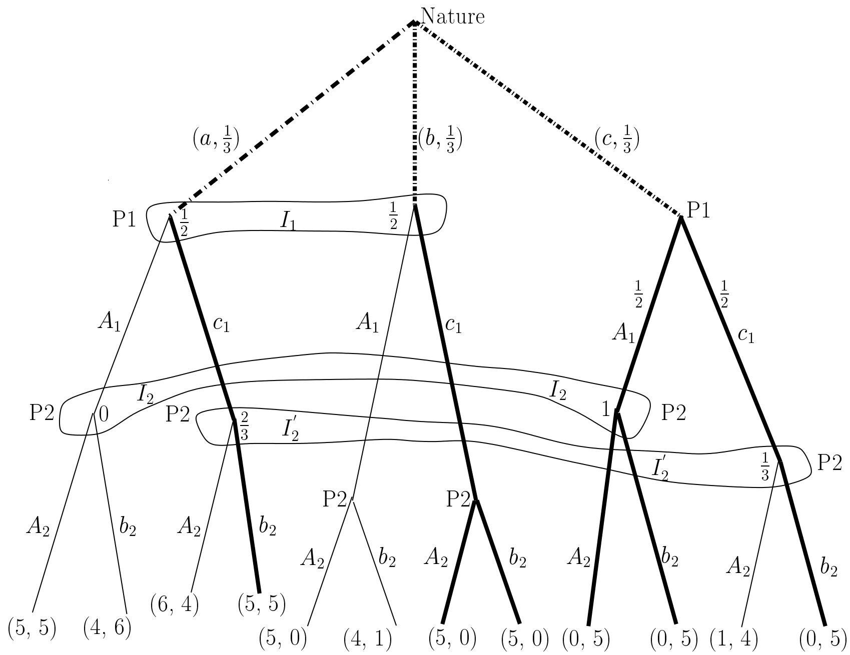


Figure 2

(iii) If the declarations are (A_1, b_2) then a lie can be beneficial and undetected. P1 is trapped and must hand over one unit of his endowment to P2. Obviously if his initial endowment is zero then he has nothing to give.

(iv) If the declarations are (c_1, A_2) then again a lie can be beneficial and undetected. P2 is now trapped and must hand over one unit of his endowment to P1. Obviously if his initial endowment is zero then he has nothing to give.

For the calculations of payoffs the revelation of the actual state of nature is not required. We could specify that a player does not lie if he cannot get a higher payoff by doing so. We assume that each player, given his beliefs, chooses optimally from his information sets.

In Figure 2 we indicate, through heavy lines, plays of the game, obtained through backward induction, which are the outcome of the choices by nature and the optimal behavioral strategies by the players. The interrupted lines signify that nature simply chooses among three alternatives, with equal probabilities. The fractions next to the nodes of the information sets are obtained, whenever possible through Bayesian updating. That is they are consistent with the choice of a state of nature and the optimal behavioral strategies of the players.

For all choices by nature, at least one of the players tells a lie on the optimal play. The players, by lying, avoid the possibility of having to make a payment and the PBE confirms the initial endowments. The decisions to lie imply that the players will not sign the contract $(5, 4, 1)$ and $(5, 1, 4)$. A similar conclusion would have been reached if we investigated directly the allocation (4, 4, 1) and (4, 1, 4).

Finally suppose we were to modify (iii) and (iv) of the *rules* i.e.: (iii) If the declarations are (A_1, b_2) then a lie can be beneficial and undetected, and P1 is trapped and must hand over half of his endowment to P2. Obviously if his endowment is zero then he has nothing to give.
(iv) If the declarations are (c_1, A_2) then again a lie can be beneficial and undetected. P2 is now trapped and must hand over half of his endowment to P1. Obviously if his endowment is zero then he has nothing to give.

The new rules would imply the following changes in the payoffs in Figure 2, from left to right. The second vector would now be (2.5, 7.5), the third vector (7.5, 2.5), the sixth vector (2.5, 2.5) and the eleventh vector (2.5, 2.5). The analysis in Glycopantis et al. (2001) shows that the weak fine core allocation in which both agents receive (5, 2.5, 2.5) cannot be implemented as a PBE. Again this allocation is not IBIC. The same allocation belongs, for equal weights to the agents, also to the WFV.

Finally we note that the PBE implements the initial endowments allocation

$$\begin{pmatrix} 5 & 5 & 0 \\ 5 & 0 & 5 \end{pmatrix}$$

which in the case of non-free disposal, coincides with the REE. However as it is shown in Glycopantis et al. (2003b) a REE is not in general implementable.

8.2 Implementation of Radner equilibria and of WFC allocations through the courts

We shall show briefly that the allocation

$$\begin{pmatrix} 4 & 4 & 1 \\ 4 & 1 & 4 \end{pmatrix}$$

of Example 5.2 can be implemented as a PBE through an exogenous third party. This can be interpreted as a court which imposes penalties when an agent lies.

Nature chooses states a, b and c with equal probabilities. P1 acts first and cannot distinguish between a and b. When P2 is to act we assume that not only he cannot distinguish between a and c but also he does not know what P1 has chosen before him.

The *rules* are:

(i) If a player lies about his observation, then he is penalized by 1 unit of the good. If both players lie then they are both penalized. For example if the declarations are (c_1, b_2) and state a occurs both are penalized. If they choose (c_1, A_2) and state a occurs then the first player is penalized. If a player lies and the other agent has a positive endowment then the court keeps the quantity subtracted for itself. However, if the other agent has no endowment, then the court transfers to him the one unit subtracted from the one who lied.

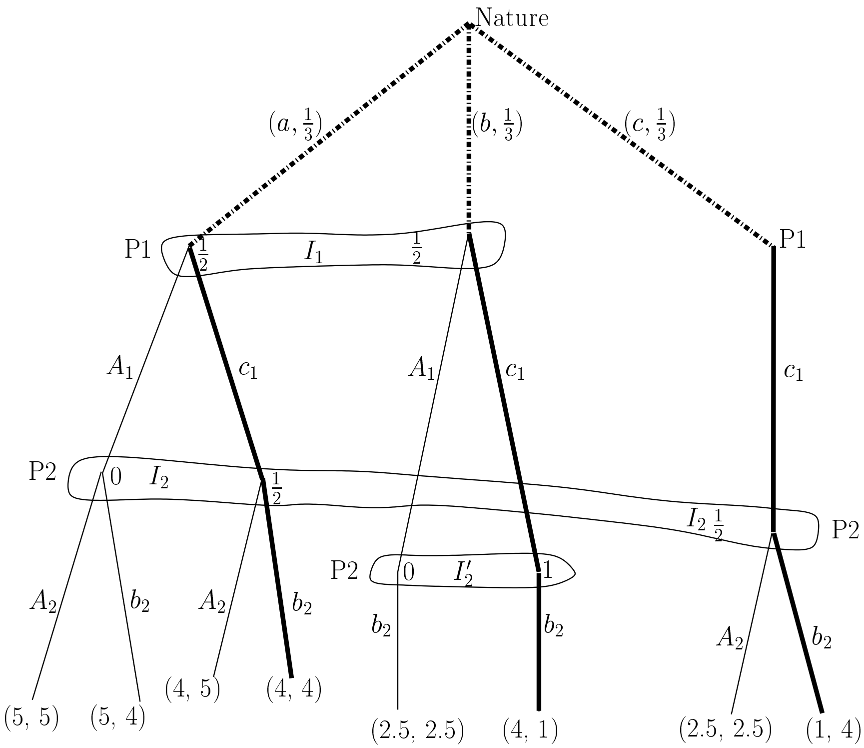


Figure 3

(ii) If the declarations of the two agents are consistent, that is (A_1, A_2) and state a occurs, (A_1, b_2) and state b occurs, (c_1, A_2) and state c occurs, then they divide equally the total endowments in the economy.

We obtain through backward induction the equilibrium strategies by assuming that each player chooses optimally, given his stated beliefs.

Figure 3 indicates, through heavy lines, optimal plays of the game. The fractions next to the nodes of the information sets are obtained through Bayesian updating.

Finally, suppose that the penalties are changed as follows. The court is extremely severe when an agent lies while the other agent has no endowment. It takes all the endowment from the one who is lying and transfers it to the other player.

Now P2 will play A_2 from I_2 and P1 will play A_1 from I_1. Therefore invoking an exogenous agent implies that the PBE will now implement the WFC allocation

$$\begin{pmatrix} 5 & 2.5 & 2.5 \\ 5 & 2.5 & 2.5 \end{pmatrix}.$$

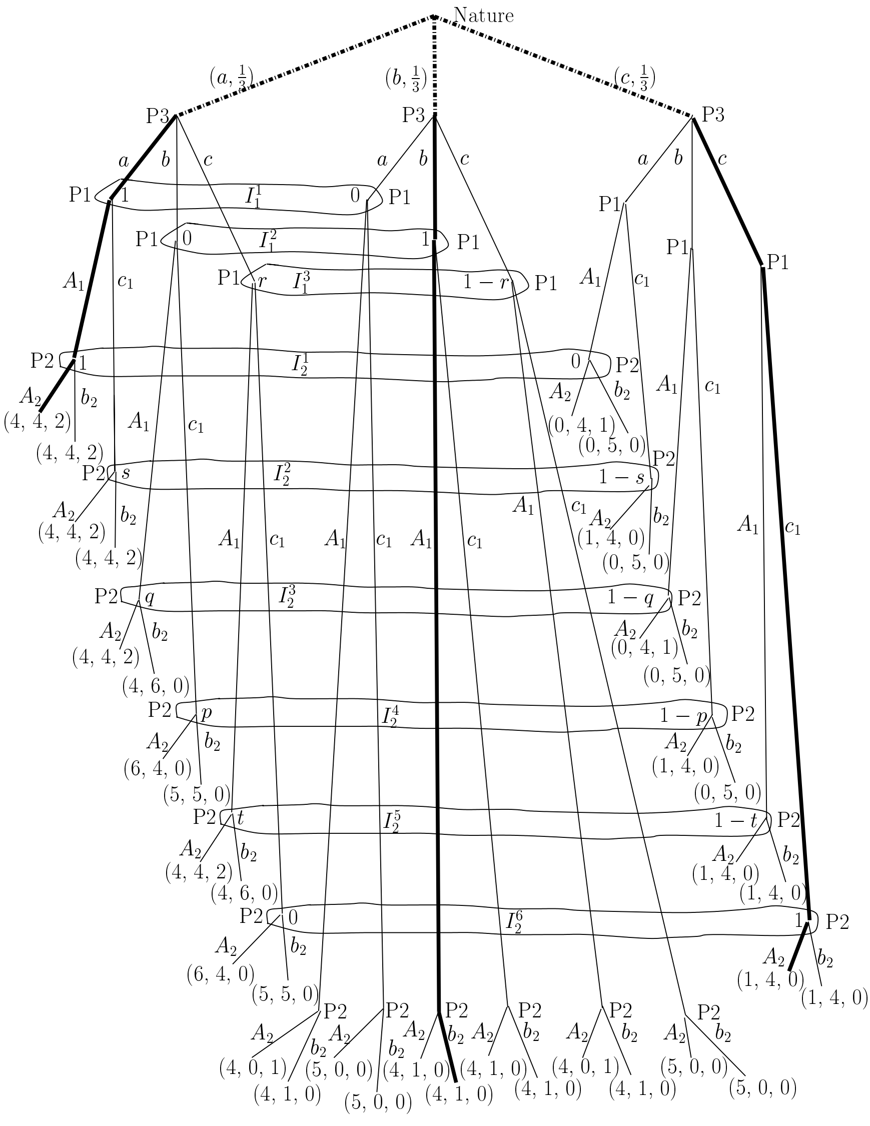


Figure 4

8.3 Implementation of private core allocations

Here we draw upon the discussion in Glycopantis et al. (2001, 2003a). In the case we consider now there is no court and therefore the agents in order to decide must listen to the choices of the other agents before them. P3 is one of the agents and we investigate his role in the implementation of private core allocations. Again we define $A_1 = \{a, b\}$ and $A_2 = \{a, c\}$.

Private core without free disposal seems to be the most satisfactory concept. The third agent, who has superior information, acting as an intermediary, implements the contract and gets rewarded in state a.

We shall consider the private core allocation

$$\begin{pmatrix} 4 & 4 & 1 \\ 4 & 1 & 4 \\ 2 & 0 & 0 \end{pmatrix}$$

of Example 5.1.

We know from Proposition 7.1 that such core allocations are CBIC and we shall show now how they can be supported as PBE of a noncooperative game.

P1 cannot distinguish between states a and b and P2 between a and c. P3 sees on the screen the correct state and moves first. He can either announce exactly what he saw or he can lie. Obviously he can lie in two ways. When P1 comes to decide he has his information from the screen and also he knows what P3 has played. When it is the turn of P2 to decide he has his information from the screen and he also knows what P3 and P1 played before him. Both P1 and P2 can either tell the truth about the information they received from the screen or they can lie.

The *rules* of calculating payoffs, i.e. the terms of the contract, are as follows: If P3 tells the truth we implement the redistribution in the matrix above which is proposed for this particular choice of nature.
If P3 lies then we look into the strategies of P1 and P2 and decide as follows:

(i) If the declaration of P1 and P2 are incompatible we go to the initial endowments and each player keeps his.

(ii) If the declarations are compatible we expect the players to honour their commitments for the state in the overlap, using the endowments of the true state, provided these are positive. If a player's endowment is zero then no transfer from that agent takes place as he has nothing to give.

In Figure 4 we indicate through heavy lines the equilibrium paths. The directed paths (a, a, A_1, A_2) with payoffs (4, 4, 2), (b, b, A_1, b_2) with payoffs (4, 1, 0) and (c, c, c_1, A_2) with payoffs (1, 4, 0) occur, each, with probability $\frac{1}{3}$. It is clear that nobody lies on the optimal paths and that the proposed reallocation is incentive compatible and hence it will be realized.

Further we can show that the PBE in Figure 4 can also be obtained as a sequential equilibrium in the sense of Kreps - Wilson (1982). Now, it is also required that the optimal behavioral strategies, and the beliefs consistent with these, are the limit of a sequence consisting of completely mixed behavioral strategies, and the implied beliefs. Throughout the sequence it is only required that beliefs are consistent with the strategies. The latter are not expected to be optimal.

8.4 Non-implementation of REE

We show here, in the context of an economy with two agents, three states of nature and two goods per state, that a fully revealing REE is not implementable. In fact we consider Example 6.1. We recall that $A_1 = \{a, b\}$, $A_2 = \{b, c\}$, and assume that P1 acts first and that when P2 is to act he has heard the declaration of P1. We have seen in Section 7 that the REE is not CBIC.

Next we show using the sequential decisions approach that the REE is not implementable. We specify the *rules* for calculating payoffs, i.e. the terms of the contract:

(i) If the declarations of the two players are incompatible, that is (c_1, a_2), then this implies that no trade takes place.

(ii) If the declarations of the two players are (A_1, A_2) then this implies that state b is believed. The player who believes it gets his REE allocation (4, 4) and the other player gets the rest. So aA_1A_2 means that P2 has lied but P1 believes it is state b and and gets (4, 4). P2 gets the rest under state a that is (4, 7); bA_1A_2 means that both believe that it is the (actual) state b and each gets (4, 4); cA_1A_2 means that P2 believes it is state b and gets (4, 4) and P1 gains nothing from his lie as he gets (1, 4).

(iii) aA_1a_2, bA_1A_2, cc_1A_2 imply that everybody tells the truth and the contract implements the REE allocation under state a, b, and c respectively. (bA_1A_2 in (ii) and (iii) give of course an identical result).

(iv) ac_1A_2 implies that both lie but their declarations are not incompatible. Each gets his REE under c and there is free disposal.

(v) cA_1a_2 means that both lie and stay with their initial endowments as they cannot get the REE allocations under state a which is the intersection of A_1 and a_2.

(vi) bA_1a_2 implies that P2 misreports and P1 believes and gets his REE under a; P2 gets the rest under b.

(vii) bc_1A_2 means that P1 lies and P2 believes that it is state c. P2 gets his REE allocation under c and P1 gets the rest under b, that is the allocation (5.31, 3.7).

On the game tree of consecutive decisions, the payoffs are translated in terms of utility. The complete optimal paths are shown in Figure 5, through heavy lines. We assume that each player chooses optimally from his information set. Probabilities next to the nodes of the information sets denote the players' beliefs. Strategies and beliefs satisfy the condition of a PBE. Our analysis shows that it is unique[16]. The corresponding normalized expected payoffs of the players are $\mathcal{U}_1 = 10.93$ and $\mathcal{U}_2 = 12.69$.

The equilibrium paths imply that REE is not implementable which matches up with the fact that it is not CBIC. However comparing the normalized expected utilities of the Bayesian equilibrium with those corresponding to the initial allocation we conclude that the proposed contract will be signed. On the other hand P2, because it is not advantageous to him, stops P1 from realizing his normalized REE utility. He ends up with $\mathcal{U}_2 = 12.69$ rather than $\mathcal{U}_2 = 12.25$.

Further, it is shown in Glycopantis et al. (2003b) that if we modify the model into one with simultaneous decisions of the agents again the REE is not implementable.

[16] Notice that as explained in the more detailed analysis, reversing the order of the play between the agents results in more than one PBE.

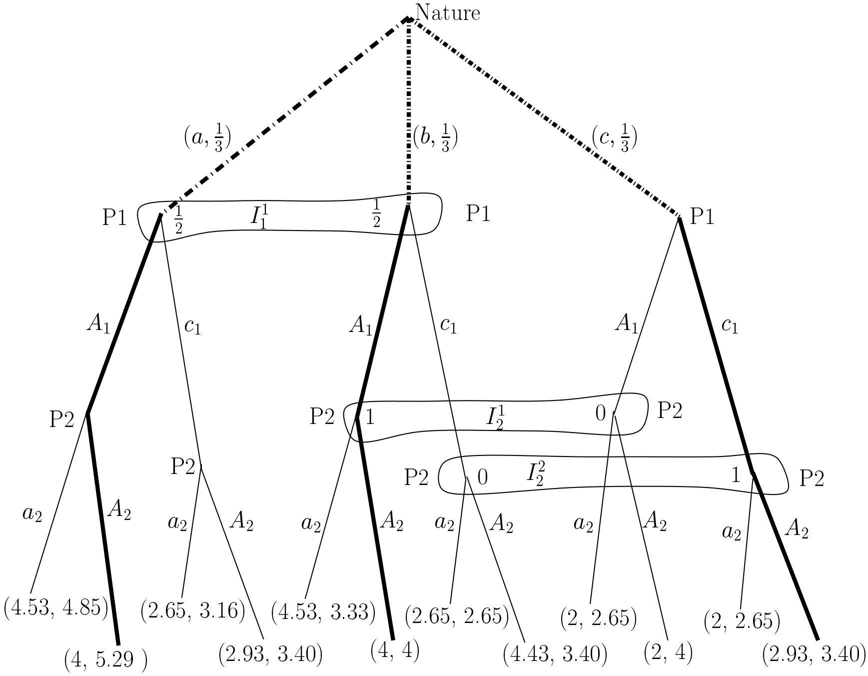


Figure 5

9 REE and weak core concepts

In view of the significance of the REE as an equlibrium concept we look in this section closer at the relation between REE and weak core concepts, which allow for sharing of information among the agents.[17] It is this sharing of information which makes the conditions different and therefore the comparison interesting, as REE is a Walrasian notion. The relation between REE and the private core, in which every agent keeps their own information, has been examined above.

We show here that for state independent utilities, no coalition of agents can block a fully revealing REE. Therefore in this case the REE is always a subset of IWFC and therefore it is interim "fully" Pareto optimal. However for state dependent utility functions the REE is not necessarily in the IWFC as we show below.

We also show that in general a REE does not belong to the WFC. If it so happens that REE does belong to this set then a slight modification of the utility functions implies that the two sets do not overlap anymore.

9.1 REE and IWFC

First we define the cooperative concept of the IWFC concept which is conditional on some information already obtained and shared by coalitions of agents.

[17] This section uses results and statements from Glycopantis et al. (2003b).

Definition 9.1.1. An allocation $x = (x_1, \ldots, x_n) \in \bar{L}_X$ is said to be a IWFC allocation if

(i) each $x_i(\cdot)$ is $\mathcal{F}_I$-measurable;[18]
(ii) $\sum_{i=1}^n x_i(\omega) = \sum_{i=1}^n e_i(\omega)$ for all $\omega \in \Omega$;
(iii) there do not exist state of nature $\omega^* \in \Omega$, coalition S and allocation $(y_i)_{i \in S} \in \prod_{i \in S} \bar{L}_{X_i}$ such that $y_i(\cdot) - e_i(\cdot)$ is $\mathcal{F}_S$-measurable for all $i \in S$, $\sum_{i \in S} y_i(\omega) = \sum_{i \in S} e_i(\omega)$, for all ω and $v_i(y_i|\mathcal{F}_S)(\omega^*) > v_i(x_i|\mathcal{F}_{x_i})(\omega^*)$ for all $i \in S$, where $\mathcal{F}_{x_i}$ denotes the information connected with x_i.

The definition, (see Yannelis, 1991), implies that no coalitions of agents can pool their own information and make each of its members better off.

Proposition 9.1.1. For state independent utility functions, a fully revealing REE allocation belongs to the IWFC.

Proof. Let (x, p) be a fully revealing REE, so that the state of nature that has occurred is known to everybody and x be feasible and measurable with respect to $\mathcal{F}_I$. Suppose now that x is not an element of IWFC. Then there exists $\omega^* \in \Omega$, a coalition S and feasible $(y_i)_{\in S} \in \prod_{i \in S} \bar{L}_{X_i}$ which is $\mathcal{F}_S$-measurable $\forall i \in S$, such that $\sum_{i \in S} y_i(\omega) = \sum_{i \in S} e_i(\omega)$ $\forall \omega \in \Omega$ and

$$v_i(y_i|\mathcal{F}_S)(\omega^*) > v_i(x_i|\mathcal{G}_i)(\omega^*). \tag{19}$$

On the right-hand side of (6) we have that $\mathcal{G}_i = \mathcal{F}$ which in this case is generated by singletons.

We consider the two terms in relation to the Definition 9.1.1. The right-hand side is $v_i(x_i|\mathcal{G}_i)(\omega^*) = u_i(x_i(\omega^*))$, i.e. one single term with probability one. This follows from the fact that x is fully revealing and therefore $E_i^{\mathcal{G}_i}(\omega^*) = \{\omega^*\}$.

On the other hand the left-hand side is

$$v_i(\omega^*, y_i(\omega^*)) = \sum_{\omega'} u_i(y_i(\omega'))q_i(\omega'|E_i^{\mathcal{F}_S}(\omega^*)), \tag{20}$$

where in (7)

$$q_i(\omega'|E_i^{\mathcal{F}_S}(\omega^*)) = \begin{cases} 0 & : \quad \omega' \notin E_i^{\mathcal{F}_S}(\omega^*) \\ \dfrac{q_i(\omega')}{q_i(E_i^{\mathcal{F}_S}(\omega^*))} & : \quad \omega' \in E_i^{\mathcal{F}_S}(\omega^*). \end{cases}$$

and $E_i^{\mathcal{F}_S}(\omega^*)$ is a subset of $\mathcal{F}_S$ on which y_i is constant.

This allows us to take the utility term out of the sum[19] and deduce that $u_i(y_i(\omega^*)) > u_i(x_i(\omega^*))$. This implies that when x_i was chosen y_i was too expensive and therefore $p(\omega^*)y_i(\omega^*) > p(\omega^*)x_i(\omega^*) = p(\omega^*)e_i(\omega^*) \quad \forall\ i \in S$. Then summing up with respect to $i \in S$ we obtain

$$p(\omega^*)\sum_{i \in S} y_i(\omega^*) = \sum_{i \in S} p(\omega^*)y(\omega^*) > \sum_{i \in S} p_i(\omega^*)e_i(\omega^*) = p(\omega^*)\sum_{i \in S} e_i(\omega^*). \tag{21}$$

[18] Recall that for $S \subseteq I$, $\mathcal{F}_S$ denotes the "join" of coalition S, i.e. $\bigvee_{i \in S} \mathcal{F}_i$.

[19] Notice that if $u_i(\omega', x_i(\omega'))$ depended separately on ω' then, in general, it would not have been possible to take $u_i(\omega', y_i(\omega'))$ out of the sum. On the other hand measurability of u_i with respect to its first argument would rescue the proof.

Relation (21) is a contradiction to $\sum_{i\in S} y_i(\omega) = \sum_{i\in S} e_i(\omega)$ because in order to obtain the inequality $p(\omega^*)\sum_{i\in S} y_i(\omega^*) > p(\omega^*)\sum_{i\in S} e_i(\omega^*)$ at least one element of the vector $\sum_{i\in S} y_i(\omega)$ must be larger than the corresponding element of $\sum_{i\in S} e_i(\omega)$.

Remark 9.1.1. With state independent utilities, Proposition 9.1.1 can be proven even if x is a partially revealing or non-revealing REE. It does not matter whether the information of the coalition is finer or not than the one of the REE. Also with state dependent utilities the proposition can be proven for general REE and an appropriately defined WFC concept if coalitions are only allowed to form which have the same information as REE. Then there is no need to take the utility expressions out of the relation $v_i(y_i|\mathcal{F}_S)(\omega^*) > v_i(x_i|\mathcal{G}_i)(\omega^*)$. An interpretation of what the proposition implies is that, under certain conditions, allowing all possible coalitions to share their information will not block the REE allocations.

Kwasnica (1998) has discussed a related result for a different core concept which is not interim fully Pareto optimal.

The conditions under which Proposition 9.1.1 holds are limited. We now construct examples to show that it does not necessarily hold when we have state dependent utilities.

In the examples below the introduction of Agent 3 is done so that the REE satisfy (i) in the definition of the IWFC. Alternatively, without introducing a third agent we can argue that given a REE there exists an IWFC allocation which improves the conditional utility of an agent given some particular state.

Example 9.1.1. There are only two, equally probable, from the point of view of the agents, states of nature, (one can add more states to make the model richer but this is not important), and two goods. Players 1 and 2 cannot distinguish between states a and b. On the other hand their utility functions differ per state. Player 3 can distinguish between all states of nature, has no initial endowments and has some utility function. His role is to ensure that the vector x described below satisfies condition (i) of IWFC. We turn our attention to the other players.

We are assuming the following. In state a: $u_1 = \min\{\epsilon x_{11}, x_{12}\}$, where $\epsilon > 1$, and $e_1{=}(2,0)$; $u_2{=}\min\{x_{21}, x_{22}\}$, and $e_2{=}(0,2)$. In state b: $u_1{=}\min\{x_{11}, x_{12}\}$, and $e_1{=}(2,0)$ $u_2{=}(x_{21}x_{22})^c$, where $c > 0$ will be determined later, and $e_2 = (0,2)$.

We construct two Edgeworth boxes and find the fully revealing REE, and hence our vector x, to be as follows. In state a: $p_1 = 0, p_2 = 1$; Agent 1 gets zero quantities and Agent 2 gets everything; $u_1 = 0$ and $u_2 = 2$. In state b: $p_1 = 1, p_2 = 1$; every agent gets 1 unit of each good; $u_1 = 1$ and $u_2 = 1$. In both states, Player 3 receives no quantities.

We will now show that this REE is not in the IWFC. Since, when the two players share their information, they still cannot distinguish between the two states we still require measurability of the feasible allocation to satisfy condition (iii) of IWFC.

The proposed allocation is that Agent 1 gets $y_1(a) = y_1(b) = (0.75, 0.75)$ and Agent 2 gets $y_2(a) = y_2(b) = (1.25, 1.25)$. The utility levels are as follows. In state a: $u_1 = 0.75$, and $u_2 = 1.25$ and in state b: $u_1 = 0.75$, $u_2 = (1.25 \times 1.25)^c$.

We choose state a for the condition (iii) of IWFC. For agent 1 we have that $v_1(y_1)(a)$ is larger than his REE utility which is zero. Also, for sufficiently large c, we have for agent 2 that $v_2(y_2)(a) = (\frac{1}{2})1.25 + (\frac{1}{2})(1.25 \times 1.25)^c > u_2 = 2$ (REE utility under a).

As for the alternative approach, without introducing a third agent we can argue that, given a REE, there exists an IWFC allocation which does better for some agent. First we use the above y_i allocation to show that it does better under a. Then we can argue that there exists an IWFC allocation which for some agent does even better in terms of utility conditioned on state a.

Example 9.1.2 (2003b). There are two, equally probable, from the point of view of the agents, states $\Omega = \{a, b\}$ and three players $I = \{1, 2, 3\}$. Player 3 can detect all states, but he has no initial endowments; his only role is to ensure that the x_i calculated below satisfy condition (i) of IWFC. Players 1 and 2 cannot distinguish between the states.

We are assuming that in state a: $u_1 = x_{11}^2 x_{12}$, $u_2 = x_{21}^2 x_{22}^2$, $e_1 = (\frac{9}{13}, \frac{9}{13})$, $e_2 = (\frac{4}{13}, \frac{4}{13})$, and in state b: $u_1 = x_{11}^{0.5} x_{12}$, $u_2 = x_{21} x_{22}$, $e_1 = (\frac{9}{13}, \frac{9}{13})$, $e_2 = (\frac{4}{13}, \frac{4}{13})$.

The REE is given by $p(a)=(8, 5)$, $x_1(a)=(0.75, 0.6)$, $x_2(a)=(0.25, 0.4)$, and $p(b)=(5, 8)$, $x_1(b)=(0.6, 0.75)$, $x_2(b)=(0.4, 0.25)$.

In the IWFC definition choose $\omega^*=a$, $S=\{1, 2\}$, $y_1(a)=y_1(b)=(0.6, 0.8)$, and $y_2(a) = y_2(b) = (0.4, 0.2)$.

Then $v_1(y_1)(a)=0.454$, $u_1(a, x_1(a))=0.337$, $v_2(y_2)(a)=0.043$, $u_2(a,x_2(a)) = 0.01$.

9.2 REE and WFC

Next we consider the relation between REE and the WFC in the context of a more general model than Example 6.1 which was considered above. We find that an REE allocation is not necessarily in the WFC.

Example 9.2.1 (2003b). For simplicity, we treat originally a case with two players, two goods and two states. We also assume, in the beginning, that the players are, in all states, endowed with strictly positive endowments of both goods and that for both players all states are equally probable. We assume that all states, $j \in \Omega$, are distinguishable by the two players when they pool their information.

The normalized expected utility functions of the two players are $\mathcal{U}_1 = \sum_j (x_{11}^j)^\alpha (x_{12}^j)^\beta$ and $\mathcal{U}_2 = \sum_j (x_{21}^j)^\alpha (x_{22}^j)^\beta$ where $\alpha, \beta > 0$. Namely we assume that they have identical, state independent utility functions. These assumptions can be relaxed. In summary, the result of the analysis is that in general the REE does not belong to the WFC.

The WFC allocations are characterized through the following problem:

Maximize $\sum_j (x_{11}^j)^\alpha (x_{12}^j)^\beta$
Subject to

$$\sum_j (S_1^j - x_{11}^j)^\alpha (S_2^j - x_{12}^j)^\beta = \bar{\mathcal{U}}_2 \text{ (fixed)},$$
$$0 \le x_{11}^j \le S_1^j, \;\; 0 \le x_{12}^j \le S_2^j \;\; \forall j,$$

where S_i^j denotes the total quantity of Good i in state j. Note that $0 < \mathcal{U}_2 < \sum_j (S_1^j)^\alpha (S_2^j)^\beta$.

Because of the feasibility constraints on quantities, the Lagrange theory cannot be applied in general in order to obtain the solution. However we can comment on the relation between REE and WFC allocations by arguing through another route.

We apply a Gorman type separation argument (see Gorman, 1959). We consider the contract curve per state. First we consider the following problem.

Maximize $(x_{11}^j)^\alpha (x_{12}^j)^\beta$
Subject to

$$(S_1^j - x_{11}^j)^\alpha (S_2 - x_{12}^j)^\beta = {u_2}^j \text{ (fixed)},$$
$$0 \le x_{11}^j \le S_1^j, \;\; 0 \le x_{12}^j \le S_2^j.$$

The solution implies $S_2^j x_{11}^j = S_1^j x_{12}^j$, which is the diagonal of the Edgeworth box. All WFC allocations are on contract curve in each state, for otherwise we can move to a Pareto superior point on the contract curve. It is also true that a REE, fully revealing or not, will be on the diagonal with every agent receiving positive quantities from both goods. This follows from the fact that otherwise, in at least one state, the markets will not clear.

The actual solution is

$$x_{11}^j = \left(\frac{S_1^j}{S_2^j}\right)^{\frac{\beta}{\alpha+\beta}} \left[(S_1^j)^{\frac{\alpha}{\alpha+\beta}} (S_2^j)^{\frac{\beta}{\alpha+\beta}} - (u_2^j)^{\frac{1}{\alpha+\beta}}\right],$$
$$x_{12}^j = \left(\frac{S_2^j}{S_1^j}\right)^{\frac{\alpha}{\alpha+\beta}} \left[(S_1^j)^{\frac{\alpha}{\alpha+\beta}} (S_2^j)^{\frac{\beta}{\alpha+\beta}} - (u_2^j)^{\frac{1}{\alpha+\beta}}\right].$$

We write $(S_1^j)^{\frac{\alpha}{\alpha+\beta}} (S_2^j)^{\frac{\beta}{\alpha+\beta}} = T^j$ and $(u_2^j)^{\frac{1}{\alpha+\beta}} = W^j$, and substitute into the objective function to get $\sum_j [T^j - W^j]^{(\alpha+\beta)}$ which is to be maximized subject to the constraints $\sum_j u_2^j = \bar{\mathcal{U}}_2$ and $u_2^j \ge 0$ which are equivalent to $\sum_j (W^j)^{(\alpha+\beta)} = \bar{\mathcal{U}}_2$ and $W^j \ge 0$. Considering the solution for the $x's$ we also have $0 \le W^j \le T^j$. So in summary we are solving:

Maximize $\sum_j [T^j - W^j]^\gamma$
Subject to

$$\sum_j (W^j)^\gamma = \bar{\mathcal{U}}_2 \text{ (fixed), and}$$
$$0 \le W^j \le T^j.$$

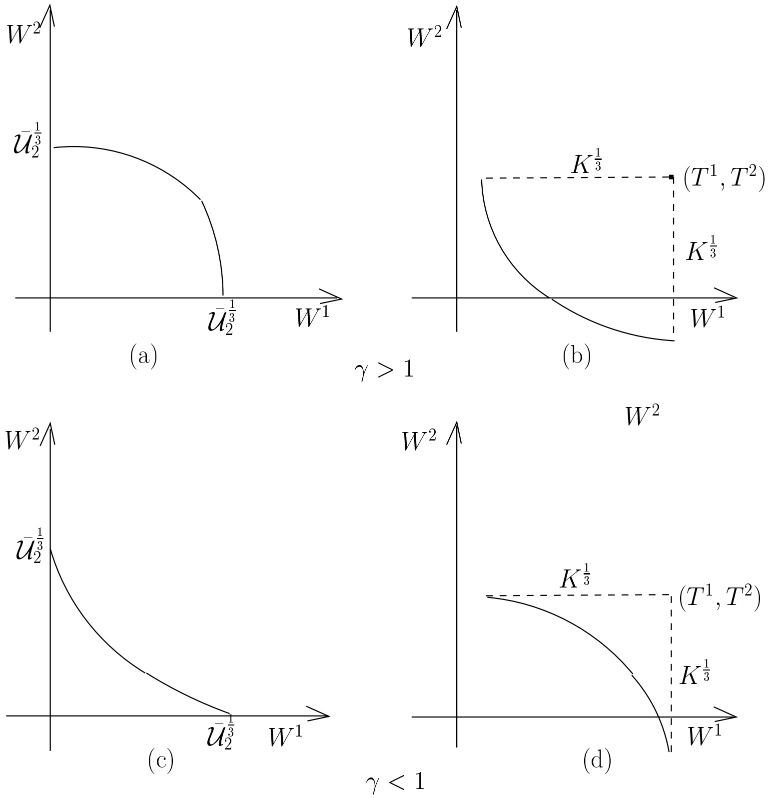


Figure 6

where $\gamma = \alpha + \beta$.

We now look at the form of the functions. Consider $\sum_j (W^j)^\gamma = 1$ for any $\gamma > 0$.
For $\gamma = 1$ this is a hyperplane. In the positive orthant, $\gamma > 1$ causes the surface to bulge away from the hyperplane so as to enclose a convex set including the origin ($\gamma = 2$ is the exemplary case, which is a hypersphere). Conversely for $\gamma < 1$ it produces a surface which bulges in towards the origin. $\sum_j (W^j)^\gamma = \bar{\mathcal{U}}_2$ is similar in shape but scaled by a factor $\bar{\mathcal{U}}_2^{\frac{1}{\gamma}}$.

Finally the shape of $\sum_j [T^j - W^j]^\gamma = K$ (fixed) can be derived from the above. The origin has been shifted to the point with coordinates (T^j) after the surface has been reflected along each coordinate axis.

Now we look at the solution of the overall Gorman problem. We distinguish between:

(i) $\gamma > 1$; the constraint is concave, in the nonnegative area, with perpendicular intersections with the axes. The indifference curves of the objective function are convex, with nonnegative coordinates, (see Fig. 6), and increase in value as we move in the direction of the origin. It follows that the maximum will be at one or both of the corner points. This means that the REE is not in the WFC.

(ii) $\gamma < 1$; in this case the constraint is convex and the indifference curves are concave, (see Fig. 6), and increase in value as we move in the direction of the origin. The solution is away from the corner points at a point of tangency. Even under symmetric conditions there is no reason why the REE should be in the WFC.

(iii) $\gamma = 1$; inspection of the objective function and the constraint shows that the WFC coincides with the linear constraint. It follows that the REE allocation is in the WFC and this is the case in Example 6.1. However, attaching a weight to the utility of Player 1 in one state implies a corner solution and therefore the REE is not in the WFC.

10 Bayesian learning with cooperative solution concepts

As we indicated in the previous sections, the private core and the private value outcomes are sensitive to changes in the private information of the agents. In this section we sketch out how information available to the agents can change through time.

The idea of learning introduces changes in the information structure of the agents. We consider a DIE that extends over many periods. The agents have initially private information which reflects their own personal characteristics, i.e. the random initial endowments and preferences. However, in each period they draw new information from the realized core or value allocation. Hence we consider an economy $\mathcal{E}$ in a dynamic framework.

One way of explaining how the agents refine their private information over time is as follows. Suppose, for example, that the same utility functions and endowments are repeated at each point in time. The chances are that over a long period all states of nature will occur. Suppose now that Agent i knows exactly what this state is, say a, but Agent j observes an element of his information partition with more than one state. Agent j cannot distinguish between the various states in his information set. However he can start slowly associating state a with signals which he originally considered as unimportant or irrelevant and which now he sees coincide with the announcement, through his private core or value allocation, of this state by Agent i. At no stage is it assumed that the agents get together to share their information.

Let $T = \{1, 2, ...\}$ denote the set of time periods and $\sigma(e_i^t, u_i^t)$ the σ-algebra that the random initial endowments and utility function of Agent i generated at time t. At any given point in time $t \in T$, the private information of Agent i is defined as:

$$\mathcal{F}_i^t = \sigma(e_i^t, u_i^t, (x^{t-1}, x^{t-2}, ...)) \tag{22}$$

where x^{t-1}, x^{t-2}, ... are past periods private core or value allocations.

Relation (22) says that at any given point in time t, the private information which becomes available to Agent i is $\sigma(e_i^t, u_i^t)$ together with the information that the private core (value) allocations generated in all previous periods. In this scenario, the private information of Agent i in period $t+1$ will be $\mathcal{F}_i^t$ together with the information the private core (value) allocation generated at period t, i.e. $\sigma(x^t)$. More explicitly, the assumption is that the private information of Agent i at time $t+1$ will be $\mathcal{F}_i^{t+1} = \mathcal{F}_i^t \vee \sigma(x^t)$, which denotes the "join", that is the smallest σ-algebra containing $\mathcal{F}_i^t$ and $\sigma(x^t)$.

Therefore for each Agent i we have that

$$\mathcal{F}_i^t \subseteq \mathcal{F}_i^{t+1} \subseteq F_i^{t+2} \subseteq \dots . \tag{23}$$

Relation (23) represents a learning process for Agent i and it generates a sequence of differential information economies $\{\mathcal{E}^t : t \in T\}$ where now the corresponding private information sets are given by $\{\mathcal{F}_i^t : t \in T\}$.

The agents are myopic, in the sense that they do not form expectations over the entire horizon but only for the current period, i.e. each agent's interim expected utility is based on his/her current period private information. Obviously, since the private information set of each agent becomes finer over time, the interim expected utility of each agent is changing as well. The information gathered at a given time t, will affect the private core (or value) outcome in periods $t+1,\ t+2,\ \dots$. The example below attempts to explain the idea of learning.

Example 10.1. Consider the following DIE with two agents $I = \{1, 2\}$ three states of nature $\Omega = \{a, b, c\}$ and goods, in each state, the quantities of which are denoted by x_{i1}, x_{i2}, where i refers to the agent. The utility functions are given by $u_i(\omega, x_{i1}, x_{i2}) = x_{i1}^{\frac{1}{2}} x_{i2}^{\frac{1}{2}}$, and states are equally probable, i.e. $\mu(\{\omega\}) = \frac{1}{3}$, for $\omega \in \Omega$. Finally the measurable endowments and the private information of the agents is given by

$$\begin{aligned} e_1^t &= ((10,0),(10,0),(0,0)), & \mathcal{F}_1 &= \{\{a,b\},\{c\}\}; \\ e_2^t &= ((10,0),(0,0),(10,0)), & \mathcal{F}_2 &= \{\{a,c\},\{b\}\}. \end{aligned} \tag{24}$$

The structure of the private information of the agents implies that the private core allocation, (x_1^t, x_2^t), in $t = 1$ consists of the initial endowments.

Notice also that the information generated in Period 2 is the full information $\sigma(x_1^t, x_2^t) = \{\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{a,b,c\},\emptyset\}$. It follows that the private information of each agent in periods $t \geq 2$ will be

$$\begin{aligned} \mathcal{F}_1^{t+1} &= F_1^t \vee \sigma(x_1^t, x_2^t) = \{\{a\},\{b\},\{c\}\}; \\ \mathcal{F}_2^{t+1} &= F_2^t \vee \sigma(x_1^t, x_2^t) = \{\{a\},\{b\},\{c\}\}. \end{aligned} \tag{25}$$

Now in $t = 2$ the agents will make contracts on the basis of the private information sets in (25). It is straightforward to show that a private core allocation in period $t \geq 2$ will be

$$\begin{aligned} x_1^{t+1} &= ((5,5),(10,0),(0,0)); \\ x_2^{t+1} &= ((5,5),(0,0),(0,10)). \end{aligned} \tag{26}$$

Notice that the allocation in (26) makes both agents better off than the one given in (24). In other words, by refining their private information using the private core allocation they have observed, the agents realized a Pareto improvement.

Of course, in a generalized model with more than two agents and a continuum of states, unlike the above example, there is no need that the full information private core or value will be reached in two periods. The main objective of learning is to examine the possible convergence of the private core or value in an infinitely repeated DIE. In particular, let us denote the one shot limit full information economy by $\bar{\mathcal{E}} = \{(X_i, u_i, \bar{\mathcal{F}}_i, e_i, q_i : i = 1, 2, ..., n)\}$ where $\bar{\mathcal{F}}_i$ is the pooled information of Agent i over the entire horizon, i.e. $\bar{\mathcal{F}}_i = \bigvee_{i=1}^{\infty} \mathcal{F}_i^t$.

The questions that learning addresses itself to are the following:

(i) If $\left\{\mathcal{E}^t : t \in T\right\}$ is a sequence of DIE and x^t is a corresponding private core or value allocation, can we extract a subsequence which converges to a limit full information private core or value allocation for $\bar{\mathcal{E}}$?

(ii) Is the answer to (i) above affirmative, if we allow for bounded rationality in the sense that x^t is now required to be an approximate, ϵ-private core or ϵ-value allocation for $\mathcal{E}^t$, but nonetheless it converges to an exact private core or value allocation for $\bar{\mathcal{E}}$?

(iii) Given a limit full information private core or value allocation say $\bar{x}$ for $\bar{\mathcal{E}}$, can we construct a sequence of ϵ-private core or ϵ-value allocation x^t in $\mathcal{E}^t$ which converges to $\bar{x}$? In other words, can we construct a sequence of bounded rational plays, such that the corresponding ϵ-private core or ϵ-value allocations converge to the limit full information private core or value allocation.

The above questions have been affirmatively answered in Koutsougeras and Yannelis (1999).

It should be noted that in the above framework it may be the case that in the limit incomplete information may still prevail. In other words, it could be the case that

$$\bar{\mathcal{F}}_i = \bigvee_{i=1}^{\infty} \mathcal{F}_i^t \subset \bigvee_{i=1}^{n} \mathcal{F}_i^t.$$

Hence in the limit a private core or value allocation may not be a fully revealing allocation of the same kind. However, if learning in each period reaches the complete information in the limit, i.e. $\bar{\mathcal{F}}_i \supset \bigvee_{i=1}^{n} \mathcal{F}_i^t$ the private core or value allocation is indeed fully revealing.

Learning applied to cooperative solution concepts was first discussed in Koutsougeras and Yannelis (1999). A generalization of their results to non-myopic learning which allows agents to discount the future can be found in Serfes (2001).

11 Concluding remarks

We have reviewed here relations between some of the main cooperative and noncooperative equilibrium concepts in the area of finite economies with asymmetric information. It is precisely the asymmetry in the information of the agents which leads to a variety of cooperative and noncooperative equilibrium concepts. It is then appropriate that their properties be compared. As explained in Glycopantis and Yannelis in this volume, the example of Wilson (1978) shows that even the list of noncooperative concepts employed is not exhaustive.

Notice that we have not examined large economies or economies with infinite dimensional commodity spaces. There is a growing literature on such economies but we decided to focus mainly on finite economies. This was for the sake of simplicity, and also for focusing on conceptual issues rather than proving powerful theorems.

In modeling a DIE, we followed the partition approach. Alternative concepts are defined depending mainly on whether the calculations are in the ex ante or the interim state, the degree of information sharing among the agents, the free disposability or not of goods.

A number of examples calculate in detail equilibria, which makes their comparison transparent. Relations are obtained and the significance of superior information is brought out.

Given the variety of equilibrium concepts, the question arises which ones have satisfactory properties. Two such properties are the static Bayesian incentive compatibility and the dynamic PBE implementability of an equilibrium. We have also exhibited here some of the results obtained earlier which examined the connection between these ideas.

The discussion considered both cooperative and Walrasian type equilibrium concepts. The presentation here points out the positive association between Bayesian incentive compatibility of a concept and its implementability as a PBE. This investigation is wider than the Nash (1953) programme which concentrates in providing support to cooperative, static concepts through noncooperative, extensive form constructions.

A main conclusion is that equilibrium notions which may not be incentive compatible, cannot easily be supported as a PBE, e.g. REE and Radner equilibrium. On the contrary notions which are incentive compatible can be supported as a PBE, e.g. private core and private value.

We consider the area of incomplete and differential information and its modelling important for the development of economic theory. We believe that the introduction of game trees, which give a dynamic dimension to the analysis by making the individual decisions transparent, helps in the development of ideas. The partition model is, in our view, a natural way to analyze DIE and the use of game trees provides a noncooperative foundation of the equilibrium concepts.

Appendix I: On core concepts

We construct here a table containing a number of core concepts, taken as a starting point Yannelis (1991) and Koutsougeras and Yannelis (1993). We assume non-free disposal and that the utility function with which comparisons will be made is the ex ante one. First we cast the definition of a private core allocation in a form which will facilitate the comparison with other concepts.

Definition I.1. An allocation $x(\omega) = (x_1(\omega), x_2(\omega), ..., x_n(\omega))$ with $x_i(\omega) \in X_i(\omega)$ for all $\omega \in \Omega$ and $i = 1, ..., n$, is a *private core allocation* if

(i) x_i is $\mathcal{F}_i$-measurable, for all i,

(ii) $\sum_{i=1}^{n} x_i(\omega) = \sum_{i=1}^{n} e_i(\omega)$ for all ω, and
(iii) there do not exist coalition S and allocation to S given by $y(\omega) = (y_1(\omega), y_2(\omega), ..., y_n(\omega))$ with $y_i(\omega) \in X_i(\omega)$ for al $\omega \in \Omega$ and $i \in S$ such that
  (a) $y_i - e_i$ is $\mathcal{F}_i$-measurable for all i,
  (b) $\sum_{i \in S} y_i(\omega) = \sum_{i \in S} e_i(\omega)$ for all ω, and
  (c) $v_i(y_i) = \sum_{\omega \in \Omega} u_i(y_i(\omega))\mu(\omega) > v_i(x_i) = \sum_{\omega \in \Omega} u_i(y_i(\omega))u_i(x_i(\omega))\mu(\omega)$ for $i \in S$.

We can now proceed to the following classification:

A1: If in (iii) (a) is replaced by $\bigwedge_{i \in S} \mathcal{F}_i$-measurable[20], it is a *coarse core allocation*
A2: If also (i) is replaced by $\bigwedge_{i \in I} \mathcal{F}_i$-measurable, it is a *strong coarse core allocation*
B1: If in (iii) (a) is replaced by $\bigvee_{i \in S} \mathcal{F}_i$-measurable, it is a *fine core allocation*
B2: If also (i) is replaced by $\bigvee_{i \in I} \mathcal{F}_i$-measurable, it is a *a WFC allocation*.

Therefore when we use the terms *coarse* or *fine* we are referring to the measurability of y_i in (iii) (a). The terms *strong* or *weak* refer to the measurability of x_i in (i).

Next we note that if $\mathcal{F} \subseteq \mathcal{G}$ then x is $\mathcal{F}$-measurable $\Longrightarrow$ x is $\mathcal{G}$-measurable. Thus if we make the σ-algebra in (i) finer, we make it easier to find a core element. Conversely, in (iii), where we ask that a certain function should not exist, making the σ-algebra coarser makes it easier to find a core element.

We note the relation between the sets, Fine Core (possibly $\emptyset$) $\subseteq$ Private Core $\subseteq$ Coarse Core. The latter consists of individually rational Pareto optimal allocations. We also have that the strong coarse core is possibly empty, while the WFC exists.

We have that $\bigwedge_{i \in S} \mathcal{F}_i \subseteq \mathcal{F}_i \subseteq \bigvee_{i \in S} \mathcal{F}_i$. Therefore, theoretically, we could have nine core concepts, shown in the table below.

$(i) \diagdown (iii)$	$\bigwedge \mathcal{F}_i$	$\mathcal{F}_i$	$\bigvee \mathcal{F}_i$
$\bigwedge \mathcal{F}_i$	Strong Coarse	α	β
$\mathcal{F}_i$	Coarse	Private	Fine
$\bigvee \mathcal{F}_i$	γ	δ	Weak Fine

The set inclusion sign $\supseteq$ applies in each row of the table from left to right, and in each column as we go down.

Note also that since WFC exists so do γ and δ. In the context of measurability the private core concept is important. It has good properties: CBIC and it exists. It is the smallest set which exists and is incentive compatible.

Obviously there are classifications as well, such as producing a table for free disposal and one with interim utility functions. Some comparisons between entries across tables can be made.

[20] The "meet" is the largest σ-algebra which is contained in each $\mathcal{F}_i$. It is in a sense the intersection of these algebras.

It is of interest to make a comparison between Definition I.1 of the private core above, (Koutsougeras and Yannelis, 1993), and the definition below, (Yannelis, 1991), which is cast in a positive formulation.

Definition I.2. An allocation $x \in L_X$ is said to be an *interim private core allocation* (IPC) if

(i) $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} e_i$ and
(ii) for all S and all $(y_i)_{i \in S} \in \prod_{i \in S} L_{X_i}$ such that $\sum_{i \in S} y_i = \sum_{i \in S} e_i$, $\exists i \in S$ such that $v_i(\omega, x_i) \geq v_i(\omega, y_i)$ for some ω with $\mu(\omega) > 0$.

Despite the fact that in Definition I.2 interim expected utility functions were used, one can show that IPC contains the ex ante private core in Definition I.1, i.e. PC$\subseteq$ IPC but not the other way round.

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PART 1

CORE NOTIONS, EXISTENCE RESULTS