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Equilibria in Abstract Economies with a Measure Space of Agents and with an Infinite Dimensional Strategy Space

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Khan's result [7, 8] on equilibrium points of non-atomic games is generalized to a setting where agents' preferences need not be ordered.

1. INTRODUCTION

In the mid-seventies the equilibrium existence results of Nash [16] and Debreu [3] were generalized in two main directions: The first generalization was due to Schmeidler [19] and allowed for an atomless measure space of agents. This extension was important as it captured the meaning of "negligible" agents which is an inherent element of cometitive theory. The second generalization was due to Shafer and Sonnenschein [20] and was inspired by a theorem of Mas–Colell [14] which allowed for a more general class of agents' preferences. In particular, in this approach,

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agents' preferences need nt be transitive or complete and therefore may not be representable by utility functions. This extension was of great importance since empirical evidence indicates that agents do not always make transitive choices.

Recently the work of Schmeidler [19] has been generalized by Khan [7, 8]. In particular, Khan has allowed the dimensionality of the strategy space to be infinite. This extension is of great importance since infinite dimensional spaces arise very naturally in several situations. The purpose of this paper is to present an equilibrium existence theorem which simultaneously allows (i) the dimensionality of the strategy space to be infinite, (ii) the set of agents to be a measure space, and (iii) for agents' preferences which need not be ordered.[1]

We believe that our equilibrium result is economically interesting for two main teasons. First it proves the existence of a Nash equilibrium in a quite general setting, and second it may become a useful technical tool in proving the existence of competitive equilibria in economies with a measure space of agents and with infinitely many commodities.[2]

The paper is organized as follows. Section 2 contains notation and definitions. Our main equilibrium existence theorem is stated in Section 3 and Section 4 contains the proof of this theorem.

2. Notation and Definitions

2.1. *Notation*

2^A denotes the set of all nonempty subsets of the set A,

$\mathbb{R}$ denotes the set of real numbers,

con A denotes the convex hull of the set A,

cl A denotes the norm closure of the set A,

$\setminus$ denotes the set theoretic subtraction,

If $\phi: X \to 2^Y$ is a correspondence then $\phi \mid U: U \to 2^Y$ denotes the restriction of ϕ to U.

2.2. *Definitions*

Let X, Y be two topological spaces. A correspondence $\phi: X \to 2^Y$ is said to be *upper-semicontinuous* (usc) if the set $\{x \in X: \phi(x) \subset V\}$ is open in X

[1] Khan and Papageorgiou [10] and Yannelis [22] have proved a related result. However, in both papers the measurability and continuity assumptions are different from ours and none of these results implies the other. It is important, however, to note that these differences in assumptions necessitate intricate mathematical arguments quite different from theirs. For a complete comparison between the above papers as well as related work by others see Yannelis [22].

[2] See Yannelis [22, Remark 6.5, p. 109] for a further discussion of this point.

for every open subset V of Y. The *graph* of the correspondence $\phi: X \to 2^Y$ is denoted by $G_\phi = \{(x, y) \in X \times Y: y \in \phi(x)\}$. The correspondence $\phi: X \to 2^Y$ is said to have an *open graph* if the set G_ϕ is open in $X \times Y$. A correspondence $\phi: X \to 2^Y$ is said to have *open lower sections* if for each $y \in Y$ the set $\phi^{-1}(y) = \{x \in X: y \in \phi(x)\}$ is open in X. Let (T, τ, μ) be a complete finite measure space; i.e., μ is a real-valued, non-negative, countably additive measure defined in a complete σ-field τ of subsets of T such that $\mu(T) < \infty$ and let X be a Banach space. $L_1(\mu, X)$ denotes the space of equivalence classes of X-valued Bochner integrable functions $f: T \to X$ normed by

$$\|f\| = \int_T \|f(t)\| \, d\mu(t).$$

A correspondence $\phi: T \to 2^X$ is said to be *integrably bounded* if there exists a map $g \in L_1(\mu)$ such that for almost all $t \in T$, $\sup\{\|x\| : x \in \phi(t)\} \leqslant g(t)$. The correspondence $\phi: T \to 2^X$ is said to have a *measurable graph* if $G_\phi \in \tau \otimes \mathscr{B}(X)$, where $\mathscr{B}(X)$ denotes the Borel σ-algebra on X and $\otimes$ denotes the σ-product algebra.

We shall also need the notion of a *separable measure space*. To introduce this, recall that the measure algebra M of (T, τ, μ) is the factor algebra of τ modulo the μ-null sets. M is a metric space with the distance given by the measure of the symmetric difference. We call the measure space (T, τ, μ) separable if M is separable. A well-known theorem of Carathéodory states that all separable atomless measure spaces have isomorphic measure algebras (see for instance Royden [18, p. 321, Theorem 2]). It is easy to see that (T, τ, μ) is separable if and only if there is a countable subalgebra $\tilde{\tau}$ of τ ($\tilde{\tau}$ is not a σ-algebra unless finite) such that the factor algebra of $\tilde{\tau}$ modulo the null sets is dense in M. Standard arguments show that functions which are simple relative to $\tilde{\tau}$ and take values in a countable dense subset of a separable Banach space Y are dense in $L_1(\mu, Y)$; thus $L_1(\mu, Y)$ is separable if (T, τ, μ) is (see for instance Kolmogorov and Fomin [13, p. 381]).

Finally, let Z be a topological space and let $\phi: T \times Z \to 2^Y$ be a nonempty valued correspondence. A function $f: T \times Z \to Y$ is said to be a *Carathéodory-type selection* from ϕ if $f(t, z) \in \phi(t, z)$ for all $(t, z) \in T \times Z$ and $f(\cdot, z)$ is measurable for all $z \in Z$ and $f(t, \cdot)$ is continuous for all $t \in T$. A Carathéodory-type selection existence theorem needed for the proof of our main theorem is given in [11], and it is stated below.

CARATHÉODORY-TYPE SELECTION THEOREM. *Let (T, τ, μ) be a complete measure space, let Y be a separable Banach space, and let Z be a complete separable metric space. Let $X: T \to 2^Y$ be a correspondence with a measurable graph, i.e., $G_X \in \tau \otimes \mathscr{B}(Y)$, and let $\phi: T \times Z \to 2^Y$ be a convex* *valued correspondence (possibly empty) with a measurable graph; i.e.,* $G_\phi \in \tau \otimes \mathscr{B}(Z) \otimes \mathscr{B}(Y)$ *where* $\mathscr{B}(Y)$ *and* $\mathscr{B}(Z)$ *are the Borel* σ*-algebras of* Y *and* Z*, respectively. Furthermore suppose that conditions* (i), (ii), *and* (iii) *below hold*:

(i) *for each* $t \in T$, $\phi(t, x) \subset X(t)$ *for all* $x \in Z$.

(ii) *for each* t, $\phi(t, \cdot)$ *has open lower sections in* Z*; i.e., for each* $t \in T$ *and each* $y \in Y$, $\phi^{-1}(t, y) = \{x \in Z: y \in \phi(t, x)\}$ *is open in* Z.

(iii) *for each* $(t, x) \in T \times Z$, *if* $\phi(t, x) \neq \emptyset$, *then* $\phi(t, x)$ *has a non-empty interior in* $X(t)$.

Let $U = \{(t, x) \in T \times Z: \phi(t, x) \neq \emptyset\}$ *and for each* $x \in Z$, $U_x = \{t \in T: (t, x) \in U\}$ *and for each* $t \in T$, $U^t = \{x \in Z: (t, x) \in U\}$. *Then for each* $x \in Z$ U_x *is a measurable set in* T *and there exists a Carathéodory-type selection from* $\phi|_U$*; i.e., there exists a function* $f: U \to Y$ *such that* $f(t, x) \in \phi(t, x)$ *for all* $(t, x) \in U$ *and for each* $x \in Z$, $f(\cdot, x)$ *is measurable on* U_x *and for each* $t \in T$, $f(t, \cdot)$ *is continuous on* U^t. *Moreover,* $f(\cdot, \cdot)$ *is jointly measurable.*

3. The Equilibrium Existence Theorem

3.1. *The Main Result*

Throughout the paper (T, τ, μ) will be a finite, positive, complete, and separable measure space of agents. Let Y be a separable Banach space. For any correspondence $X: T \to 2^Y$, $L_1(\mu, X)$ will denote the set $\{x \in L_1(\mu, Y): x(t) \in X(t)$ for almost all t in $T\}$. We now define the notion of an abstract economy as follows:

An *abstract economy* Γ is a quadruple $[(T, \tau, \mu), X, P, A]$, where

(1) (T, τ, μ) is a measure space of agents;

(2) $X: T \to 2^Y$ is a strategy correspondence;

(3) $P: T \times L_1(\mu, X) \to 2^Y$ is a preference correspondence such that $P(t, x) \subset X(t)$ for all $(t, x) \in T \times L_1(\mu, X)$;

(4) $A: T \times L_1(\mu, X) \to 2^Y$ is a constraint correspondence such that $A(t, x) \subset X(t)$ for all $(t, x) \in T \times L_1(\mu, X)$.

Notice that since P is a mapping from $T \times L_1(\mu, X)$ to 2^Y, we have allowed for interdependent preferences. Roughly speaking the interpretation of these preference correspondences is that $y \in P(t, x)$ means that agent t strictly prefers y to $x(t)$ if the given strategies of other agents are fixed. For a more detailed discussion of the interpretation of these preference correspondences see Khan [6]. Notice that preferences need not be transitive or complete and therefore need not be representable by utlity functions. However, it will be assumed that $x(t) \notin \operatorname{con} P(t, x)$ for all $x \in L_1(\mu, X)$ and for almost all t in T, which implies that $x(t) \notin P(t, x)$ for all $x \in L_1(\mu, X)$ and almost all t in T; i.e., $P(t, \cdot)$ is *irreflexive* for almost all t in T.

An *equilibrium* for Γ is an $x^* \in L_1(\mu, X)$ such that for almost all t in T the following conditions are satisfied:

(i) $x^*(t) \in \operatorname{cl} A(t, x^*)$, and

(ii) $P(t, x^*) \cap \operatorname{cl} A(t, x^*) = \emptyset$.

We can now state the assumptions needed for the proof of the main theorem.

(A.1) $X: T \to 2^Y$ is an integrably bounded correspondence with a measurable graph such that for all $t \in T$, $X(t)$ is a non-empty, convex, and weakly compact subset of Y.

(A.2) $A: T \times L_1(\mu, X) \to 2^Y$ is a correspondence such that:

(a) $\{(t, x, y) \in T \times L_1(\mu, X) \times Y\colon y \in A(t, x)\} \in \tau \otimes \mathscr{B}_w(L_1(\mu, X)) \otimes \mathscr{B}(Y)$ where $\mathscr{B}_w(L_1(\mu, X))$ is the Borel σ-algebra for the weak topology on $L_1(\mu, X)$ and $\mathscr{B}(Y)$ is the Borel σ-algebra for the norm topology on Y;

(b) it has weakly open lower sections, i.e., for each $t \in T$ and for each $y \in Y$, the set $A^{-1}(t, y) = \{x \in L_1(\mu, X)\colon y \in A(t, x)\}$ is weakly open in $L_1(\mu, X)$;

(c) for all $(t, x) \in T \times L_1(\mu, X)$, $A(t, x)$ is convex and has a non-empty interior in the relative norm topology of $X(t)$[3];

(d) for each $t \in T$, the correspondence $\bar{A}(t, \cdot)\colon L_1(\mu, X) \to 2^Y$, defined by $\bar{A}(t, x) = \operatorname{cl} A(t, x)$ for all $(t, x) \in T \times L_1(\mu, X)$, is usc in the sense that the set $\{x \in L_1(\mu, X)\colon \bar{A}(t, x) \subset V\}$ is weakly open in $L_1(\mu, X)$ for every norm open subset V of Y.

(A.3) $P: T \times L_1(\mu, X) \to 2^Y$ is a correspondence such that:

(a) $\{(t, x, y) \in T \times L_1(\mu, X) \times Y\colon y \in \operatorname{con} P(t, x)\} \in \tau \otimes \mathscr{B}_w(L_1(\mu, X)) \otimes \mathscr{B}(Y)$;

(b) it has weakly open lower sections, i.e., for each $t \in T$ and each $y \in Y$, $P^{-1}(t, y) = \{x \in L_1(\mu, X)\colon y \in P(t, x)\}$ is weakly open in $L_1(\mu, X)$;

(c) for all $(t, x) \in T \times L_1(\mu, X)$, $P(t, x)$ is norm open in $X(t)$;

(d) $x(t) \notin \operatorname{con} P(t, x)$ for all $x \in L_1(\mu, X)$ and for almost all t in T.

[3] Observe that assumption (A.2)(c) is quite mild. It is implied for instance by the fact that for all $(t, x) \in T \times L_1(\mu, X)$, $A(t, x)$ is open in the relative norm topology of $X(t)$. It is obvious that this assumption has nothing to do with whether or not the positive cone of Y has a non-empty norm interior.

MAIN EXISTENCE THEOREM. *Let* $\Gamma = [(T, \tau, \mu), X, P, A]$ *be an abstract economy satisfying* (A.1)–(A.3). *Then* Γ *has an equilibrium.*

3.2. *Discussion of the Weak Topology on* $L_1(\mu, X)$

It may be instructive to discuss assumptions (A.3)(b) and (c). In particular, having the weak topology on $L_1(\mu, X)$, which is the set of all joint strategies, signifies a natural form of myopic behavior on the part of the agents. Namely, an agent has to arrive at his decisions on the basis of knowledge of only finitely many (average) numerical characteristics of the joint strategies. However, there is no a priori upper bound on how many of these (average) numerical characteristics of the joint strategies an agent might seek in order to arrive at his decision. On the othe hand, since each agent's strategy set is endowed with the norm topology this may be interpreted as signifying a very high degree of ability to discriminate between his own options. Of course, the agent's decisions depend on both of these observations, i.e., the ones of joint strategies made in the sense of the weak topology, as well as his own options made with reference to the norm topology.

Although our choice of the weak topology on $L_1(\mu, X)$ was disctated by mathematical considerations (this is the only setting in which we are able to obtain a positive result), this setting seems to be more realistic than the one with a norm topology on $L_1(\mu, X)$. This latter setting would correspond to an extremely high degree of knowledge of the joint strategy on the part of each individual agent. However, in this latter setting, by means of a counterexample, we show that one cannot expect an equilibrium to exist.

3.3. *An Example of Non-Existence of Equilibrium*

As was remarked in Section 3.2, having the weak topology on $L_1(\mu, X)$ was the only setting in which we were able to obtain a positive result. We now show that if we relax (A.3)(b) to the assumption that P has norm open lower sections then our existence result fails.

EXAMPLE 3.1. Consider an abstract economy with one agent. Let $Y = l_2$, where l_2 is the space of square summable real sequences. Denote by l_2^+ the positive cone of l_2. Let the strategy set X be equal to $\{z \in l_2^+ : \|z\| \leqslant 1\}$. Obviously X is convex and weakly compact, (recall Alaoglu's theorem).

Let $x = (x_0, x_1, x_2, \ldots) \in X$, and let $f\colon X \to X$ be a norm continuous mapping which does not have the fixed point property (for instance, let $f(x) = (1 - \|x\|, x_0, x_1, x_2, \ldots)$; then $f\colon X \to X$ is a norm continuous function and it can be easily seen that $x \neq f(x)$). Denote by $B(f(x), \|x - f(x)\|/2)$ an open ball in l_2, centered at $f(x)$ with radius $\|x - f(x)\|/2$. For each $x \in X$, let the preference correspondence be $P(x)=B(f(x), \|x-f(x)\|/2) \cap X$. Now, it can be easily checked that P has norm open lower and upper sections, is convex valued, and is irreflexive. Define the constraint correspondence $A: X \to 2^X$ by $A(x)=X$. Observe that for all $x \in X$, $f(x) \in P(x)$; i.e., P has no maximal element in X. Hence, there is no equilibrium in this one person abstract economy.

Of course if there is no equilibrium in this one person economy we cannot expect an equilibrium to exist if the set of agents is an atomless measure space. The above example can be trivially modified to show this. Let (T, τ, μ) be an atomless measure space of agents. Set $X=X(t)$ for all t in T. For $t \in T$ and $x \in L_1(\mu, X)$, let $P(t, x)=B(f(x(t)), \|x(t)-f(x(t))\|/2) \cap X$ and $A(t, x)=X$. As above one can easily see that $f(x(t)) \in P(t, x) \cap A(t, x)$ for all $x \in L_1(\mu, X)$ and all t in T; i.e., P has no maximal element in X.

4. Proof of the Main Existence Theorem

We begin by proving a lifting lemma which is crucial for the proof of our main existence theorem.

Main Lemma. *Let Y be a separable Banach space and let $X: T \to 2^Y$ be an integrably bounded, non-empty convex valued correspondence such that for all $t \in T$, $X(t)$ is a weakly compact subset of Y. Let $\theta: T \times L_1(\mu, X) \to 2^Y$ be a non-empty closed, convex valued correspondence such that $\theta(t, x) \subset X(t)$ for all $(t, x) \in T \times L_1(\mu, X)$, $\theta(\cdot, x)$ has a measurable graph for each $x \in L_1(\mu, X)$, and for each $t \in T$, $\theta(t, \cdot): L_1(\mu, X) \to 2^Y$ is usc in the sense that the set $\{x \in L_1(\mu, X): \theta(t, x) \subset V\}$ is weakly open in $L_1(\mu, X)$ for every norm open subset V of Y. Then the correspondence $F: L_1(\mu, X) \to 2^{L_1(\mu, X)}$ defined by $F(x)=\{y \in L_1(\mu, X)$: for almost all $t \in T$, $y(t) \in \theta(t, x)\}$ is usc in the sense that the set $\{x \in L_1(\mu, X): F(x) \subset V\}$ is relatively weakly open in $L_1(\mu, X)$ for every relatively weakly open subset V of $L_1(\mu, X)$.*

Proof. By Theorem 4.2 in Papageorgiou [17], $L_1(\mu, X)$ as well as $F(x)$, for each $x \in L_1(\mu, X)$, endowed with the weak topology is compact. Since the weak topology of a weakly compact subset of a separable Banach space is metrizable (Dunford and Schwartz [4, p. 434]), $L_1(\mu, X)$ is a compact metrizable space. Thus, it suffices to show that if x and x_n $(n=0, 1, \ldots)$ belong to $L_1(\mu, X)$, $\{x_n\}$ converges weakly to x, and V is a relatively weakly open subset of $L_1(\mu, X)$ containing $F(x)$, then $F(x_n) \subset V$ for all sufficiently large n. For if $U=\{z \in L_1(\mu, X): F(z) \subset V\}$ is not relatively weakly open in $L_1(\mu, X)$, we can pick some $x \in U$ such that every neighborhood of x in the (relative) weak topology of $L_1(\mu, X)$ contains an $\tilde{x} \notin U$. We thus readily construct a sequence $\{x_n\}$ converging weakly to x such that $x_n \in L_1(\mu, X)$ and $F(x_n) \not\subset V$.

Let B and $\tilde{B}$ denote the closed unit balls in Y and $L_1(\mu, Y)$, respectively, and let $\varepsilon > 0$. It will suffice to show that for a suitable n_0, $F(x_n) \subset F(x) + \varepsilon\tilde{B}$ for $n \geqslant n_0$. This is because every weakly open neighborhood of the weakly compact set $F(x)$ contains the norm neighborhood $F(x) + \varepsilon\tilde{B}$ for a suitable $\varepsilon > 0$, as is easy to see.

We begin by finding an n_0 that works. Since for each $t \in T$, $\theta(t. \cdot)$ is usc, we can find a minimal N_t such that

$$\theta(t, x_n) \subset \theta(t, x) + \frac{\varepsilon}{3\mu(T)} B \qquad \text{for all} \quad n \geqslant N_t. \tag{4.1}$$

Let $\varepsilon/3\mu(T) = \delta_1$. By assumption, for fixed x and n, the correspondences $\theta(\cdot, x)\colon T \to 2^Y$ and $\theta(\cdot, x_n)\colon T \to 2^Y$ have measurable graphs. Thus by the projection theorem (Castaing and Valadier [2; Theorem III.23]) the set Q defined below belongs to τ:

$$Q = \operatorname{proj}_T\{(t, y) \in T \times Y\colon (t, y) \in G_{\theta(\cdot, x_n)} \cap (G_{\theta(\cdot, x) + \delta_1 B})^c\} \in \tau$$

(where A^c denotes the complement of the set A). Also note that

$$Q = \{t \in T\colon \theta(t, x_n) \not\subset \theta(t, x) + \delta_1 B\} = \{t \in T\colon \theta(t, x_n)/(\theta(t, x) + \delta_1 B) \neq \varnothing\}.$$

This will enable us to conclude that N_t is a measurable function of t. This is clearly so if we can show that

$$\{t \in T\colon N_t = m\} = \bigcap_{n \geqslant m} \{t \in T\colon \theta(t, x_n) \subset \theta(t, x) + \delta_1 B\}$$
$$\cap \{t \in T\colon \theta(t, x_{m-1}) \not\subset \theta(t, x) + \delta_1 B\}.$$

Let us prove the inclusion "$\subset$," leaving the opposite inclusion for the reader to verify. If $N_t = m$, we then clearly have

$$\theta(t, x_n) \subset \theta(t, x) + \delta_1 B$$

for all $n \geqslant m$. Were

$$\theta(t, x_{m-1}) \subset \theta(t, x) + \delta_1 B$$

also true, we would clearly have $N_t \leqslant m - 1$. Thus $\theta(t, x_{m-1}) \not\subset \theta(t, x) + \delta_1 B$ and the desired conclusion follows.

We are now ready to choose n_0. Since $X(\cdot)$ is integrably bounded, there exists $g \in L_1(\mu)$ such that $\sup\{\|x\|\colon x \in X(t)\} \leqslant g(t)$. Pick δ_2 such that if $\mu(A) < \delta_2$ $(A \subset T)$ then $\int_A g(t)\, dt < \varepsilon/3$. Since N_t is a measurable function of

t, we can choose n_0 such that $\mu(\{t \in T : N_t \geqslant n_0\}) < \delta_2$. This is the desired n_0.

Let $n \geqslant n_0$ and $y \in F(x_n)$. We shall show that $y \in F(x) + \varepsilon \tilde{B}$, completing the proof of the lemma.

Since $\theta(\cdot, x)$ has a measurable graph, there is a measurable selection $z_1 : T \to Y$, $z_1(t) \in \theta(t, x)$ for almost all $t \in T$.

The correspondence

$$\psi(t) = (\{y(t)\} + \delta_1 B) \cap \theta(t, x)$$

likewise has a measurable graph and by (4.1) is non-empty valued for $t \in T_0 = \{t : N_t \leqslant n_0\}$.

Thus there is a measurable function $z_2 : T \to Y$ such that $z_2(t) \in \psi(t)$ for almost all $t \in T_0$. Finally set

$$z(t) = z_1(t) \quad \text{for} \quad t \notin T_0 \qquad \text{and} \qquad z(t) = z_2(t) \quad \text{for} \quad t \in T_0.$$

Then $z(t) \in \theta(t, x)$ for almost all t and thus $z \in F(x)$. We shall now show that $\|z - y\| < \varepsilon$, completing the proof.

We have

$$\begin{aligned}
\|z - y\| &= \int_{T \setminus T_0} \|z_1(t) - y(t)\| \, d\mu(t) + \int_{T_0} \|z_2(t) - y(t)\| \, d\mu(t) \\
&< 2 \int_{T \setminus T_0} g(t) \, d\mu(t) + \int_{T_0} \delta_1 \, d\mu(t) \\
&< \frac{2}{3} \varepsilon + \delta_1 \mu(T) = \frac{2}{3} \varepsilon + \frac{\varepsilon}{3\mu(T)} \cdot \mu(T) = \varepsilon.
\end{aligned}$$

The proof of the main lemma is now complete.

Proof of the Main Existence Theorem. Define $\psi : T \times L_1(\mu, X) \to 2^Y$ by $\psi(t, x) = \operatorname{con} P(t, x)$. By Lemma 5.1 in Yannelis and Prabhakar [23] for each $t \in T$, $\psi(t, \cdot)$ has weakly open lower sections, and it is relatively norm open valued in $X(t)$. Define $\phi : T \times L_1(\mu, X) \to 2^Y$ by $\phi(t, x) = A(t, x) \cap \psi(t, x)$. Then it can be easily checked that ϕ is convex valued, has a non-empty interior in the relative norm topology of $X(t)$, and for each $t \in T$, $\phi(t, \cdot)$ has weakly open lower sections. Moreover by Theorem III.40 in Castaing and Valadier [2], ϕ has a measurable graph. Let $U = \{(t, x) \in T \times L_1(\mu, X) : \phi(t, x) \neq \varnothing\}$. For each $x \in L_1(\mu, X)$, let $U_x = \{t \in T : \phi(t, x) \neq \varnothing\}$ and for each $t \in T$, let $U^t = \{x \in L_1(\mu, X) : \phi(t, x) \neq \varnothing\}$. By the Carathéodory-type selection theorem there exists a function $f : U \to Y$ such that $f(t, x) \in \phi(t, x)$ for all $(t, x) \in U$, and for each $x \in L_1(\mu, X)$, $f(\cdot, x)$ is measurable on U_x and for each $t \in T$, $f(t, \cdot)$ is continuous on U^t. Define $\theta: T \times L_1(\mu, X) \to 2^Y$ by $\theta(t, x) = \{f(t, x)\}$ if $(t, x) \in U$ and $\theta(t, x) = \mathrm{cl}\, A(t, x)$ if $(t, x) \notin U$.

It follows from Theorem III.40 in Castaing and Valadier [2], that for each $x \in L_1(\mu, X)$, the correspondence $\mathrm{cl}\, A(\cdot, x): T \to 2^Y$ has a measurable graph. By Lemma 4.12 in Kim *et al.* [12], $f(\cdot, \cdot)$ is jointly measurable. Hence for each $x \in L_1(\mu, X)$ the correspondence $\theta(\cdot, x): T \to 2^Y$ has a measurable graph. Notice that since for each $t \in T$, $\phi(t, \cdot)$ has weakly open lower sections, for each $t \in T$, the set U^t is weakly open in L_1 (μ, X). Consequently, by Lemma 6.1 in Yannelis and Prabhakar [23], for each $t \in T$, $\theta(t, \cdot)$: $L_1(\mu, X) \to 2^Y$ is usc in the sense that the set $\{x \in L_1(\mu, X)$: $\theta(t, x) \subset V\}$ is weakly open in $L_1(\mu, X)$ for every norm open subset V of Y. Moreover, θ is convex and non-empty valued. Define $F: L_1(\mu, X) \to 2^{L_1(\mu, X)}$ by $F(x) = \{y \in L_1(\mu, X)$: for almost all t in T, $y(t) \in \theta(t, x)\}$. Since for each $x \in L_1(\mu, X)$, $\theta(\cdot, x)$ has a measurable graph, F is non-empty valued as a consequence of the Aumann measurable selection theorem. Since θ is convex valued, so is F. By the main lemma, F is weakly usc. Furthermore, since $X(\cdot)$ is integrably bounded and has a measurable graph, $L_1(\mu, X)$ is non-empty by the Aumann measurable selection theorem, and obviously it is convex since $X(\cdot)$ is so. Therefore, by the Fan fixed point theorem (Fan [5, Theorem 1]), there exists $x^* \in L_1(\mu, X)$ such that $x^* \in F(x^*)$. It can now be easily checked that the fixed point is by construction an equilibrium for Γ. This completes the proof of the main existence theorem.

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