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Existence of an interim and ex-ante minimax point for an asymmetric information game[☆]

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ABSTRACT

We introduce the notions of ex-ante and interim minimax point for an asymmetric information game and prove the existence of such points. Our new results include as a special case the theorem in (Aliprantis et al., 2009).

1. Introduction

In a recent paper Aliprantis et al. (2009) provide an existence proof of a minimax point for a strategic (normal) form game. This theorem is useful as it has found applications in game theory (see for example Fudenberg-Maskin (1986), Myerson (1991), Thomas (1995) among others). Our objective is to extend the above result of Aliprantis et al. (2009) to an asymmetric information game. To this end we introduce the notions of ex-ante and interim minimax points and prove the existence of such points. Although the idea of the proof is the same with that in (Aliprantis et al., 2009) the introduction of asymmetric information necessitates the use of some non trivial theorems. In particular, we make use of Diestel's theorem on weak compactness on the space of Bochner integrable functions and the Kuratowski and Ryll-Nardzewski measurable selection theorem. As the deterministic result proved in (Aliprantis et al., 2009) has found interesting applications in repeated games, we think that our new results will be of interest and applicable to a framework of repeated games with asymmetric information. Obviously, our new general setting includes as a special case the result in (Aliprantis et al., 2009).

2. The strategic game with asymmetric information

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space denoting the states of nature of the world, let I be the set of players, which may be any finite or infinite set, and Y be a separable Banach space denoting the strategy sets. A strategic game with asymmetric information $G=\{(X_i, \mathcal{F}_i, u_i, q_i)_{i\in I}\}$ is a set of quadruples $(X_i, \mathcal{F}_i, u_i, q_i)$, where for each player i,

1. $X_i : \Omega \to 2^Y$ is the random strategy set,
2. $\mathcal{F}_i$ is a measurable partition[1] of $(\Omega, \mathcal{F})$ denoting the private information of player i,
3. $u_i : \Omega \times \prod_{i\in I} X_i(\cdot) \to \mathbb{R}$ is the random payoff function,
4. q_i: $\Omega \to \mathbb{R}_{++}$ is the prior of player i (i.e., q_i is a Radon-Nikodym derivative having the property that $\int_{\omega\in\Omega} q_i(\omega)\, d\mu(\omega)=1$).

As usual if $x \in \prod_{i\in I} X_i(\cdot)$, then for each player i, we write $x=(x_{-i}, x_i)$, where $x_{-i}=(x_1,\ldots, x_{i-1}, x_{i+1},\ldots)$.

The σ-field of events discernable by every player is the "coarse" σ-field $\wedge_{i\in I}\mathcal{F}_i$, which is the largest σ-algebra contained in each $\mathcal{F}_i$. While, players by pooling their information, discern the events in the "fine" σ-field $\vee_{i\in I}\mathcal{F}_i$, which denotes the smallest σ-algebra containing all $\mathcal{F}_i$.

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[1] By an abuse of notation we will still denote by $\mathcal{F}_i$ the σ-algebra that the partition $\mathcal{F}_i$ generates.

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We denote by $L_1(\mu, Y)$ the space of equivalence classes of Y-valued Bochner integrable functions x: $\Omega \to Y$.

For each player $i \in I$ define the set of all Bochner integrable and $\mathcal{F}_i$ measurable selections from the strategy set of player i, i.e.,

$$L_{X_i} = \{x_i \in L_1(\mu, Y):\ x_i(\cdot) \text{ is } \mathcal{F}_i-\text{measurable and } x_i(\omega) \in X_i(\omega) \text{ for almost all } \omega \in \Omega\},$$

and let $L_X = \prod_{i \in I} L_{X_i}$. Similarly, for each player $i \in I$, define $L_{X_{-i}} = \prod_{j \in I \setminus \{i\}} L_{X_j}$ and notice that for each i, $L_X = L_{X_{-i}} \times L_{X_i}$.

For each player i, the ex-ante expected payoff v_i: $L_{X_{-i}} \times L_{X_i} \to \mathbb{R}$ is defined by

$$v_i(x_{-i}, x_i) = \int_{\omega \in \Omega} u_i(\omega, x_{-i}(\omega), x_i(\omega)) q_i(\omega) d\mu(\omega).$$

For each player i, denote by $E^{\mathcal{F}_i}(\omega)$ the event in $\mathcal{F}_i$ containing the realized state of nature ω. Suppose that for all ω, $\int_{\omega' \in E^{\mathcal{F}_i}(\omega)} q_i(\omega') d\mu(\omega') > 0$. For each player i, given $E^{\mathcal{F}_i}(\omega)$, the interim expected payoff V_i: $\Omega \times L_{X_{-i}} \times L_{X_i} \to \mathbb{R}$ is defined by

$$V_i(\omega, x_{-i}, x_i) = \int_{\omega' \in E^{\mathcal{F}_i}(\omega)} u_i(\omega', x_{-i}(\omega'), x_i(\omega')) \frac{q_i(\omega')}{\int_{\omega' \in E^{\mathcal{F}_i}(\omega)} q_i(\omega') d\mu(\omega')} d\mu(\omega').$$

3. Definitions of ex-ante and interim minimax point

We now introduce the definitions of a minimax point in the context of asymmetric information games, by considering the ex-ante as well as the interim case.

The minimax payoff of player i gives the maximal punishment that all the other players can inflict on him. Using the notation above, the maximal punishment that the players $I \setminus \{i\}$ can inflict to player i is represented by actions in the set $L_{X_{-i}}$, i.e., the set containing all the private information strategies of the players $I \setminus \{i\}$. The best player i can do is to maximize her payoff based on her own private information. This leads to the following definitions.

Definition 3.1. The ex-ante minimax point of a strategic game with asymmetric information G is a sequence of extended real numbers $v^* = (v_1^*, v_2^*, \ldots)$, where for each player i we have

$$v_i^* = \inf_{x_{-i} \in L_{X_{-i}}} \sup_{x_i \in L_{X_i}} v_i(x_{-i}, x_i).$$

We shall say that the ex-ante minimax point is attainable, if v_i^* is attained for each player i, i.e.,

$$v_i^* = \min_{x_{-i} \in L_{X_{-i}}} \max_{x_i \in L_{X_i}} v_i(x_{-i}, x_i). \tag{1}$$

Similarly, we define the interim minimax point for which the actions are made after all players have received their own private information, that is, at an interim stage.

Definition 3.2. The interim minimax point of a strategic game with asymmetric information G is a sequence of extended real valued functions $V^*(\cdot) = (V_1^*(\cdot), V_2^*(\cdot), \ldots)$, where for each player i and $\omega \in \Omega$, we have

$$V_i^*(\omega) = \inf_{x_{-i} \in L_{X_{-i}}} \sup_{x_i \in L_{X_i}} V_i(\omega, x_{-i}, x_i).$$

We shall say that the interim minimax point is attainable, if for each player i and $\omega \in \Omega$, we have

$$V_i^*(\omega) = \min_{x_{-i} \in L_{X_{-i}}} \max_{x_i \in L_{X_i}} V_i(\omega, x_{-i}, x_i). \tag{2}$$

Definitions 3.1 and 3.2 do not take into account the fact that players $I \setminus \{i\}$ may cooperate against player i. Even if, an explicit cooperation is not allowed, player i may not know this. Therefore, the worst punishment (most severe punishment) player i may expect to receive is when the others cooperate against him. This idea can be formalized by assuming that players in $I \setminus \{i\}$, pool their own private information. In other words, the strategy vector $x_{-i}(\cdot)$ is assumed to be $\vee_{j \in I \setminus \{i\}} \mathcal{F}_j$-measurable.

On the other hand, the least severe punishment that player i may expect to receive is when all the other players use the common knowledge information strategies, that is the strategy vector $x_{-i}(\cdot)$ is $\wedge_{j \in I \setminus \{i\}} \mathcal{F}_j$-measurable.

To this end, define for each fixed player i and each $j \in I \setminus \{i\}$ the sets[2]

$$L^p_{X^j_{-i}} = \{x_j \in L_1(\mu, Y):\quad x_j(\cdot) \text{ is } \bigvee_{j \in I \setminus \{i\}} \mathcal{F}_j-\text{measurable and } x_j(\omega) \in X_j(\omega) \mu-\text{a.e.}\},$$

$$L^c_{X^j_{-i}} = \{x_j \in L_1(\mu, Y):\quad x_j(\cdot) \text{ is } \bigwedge_{j \in I \setminus \{i\}} \mathcal{F}_j-\text{measurable and } x_j(\omega) \in X_j(\omega) \mu-\text{a.e.}\}.$$

Let $L^p_{X_{-i}} = \prod_{j \in I \setminus \{i\}} L^p_{X^j_{-i}}$ and $L^c_{X_{-i}} = \prod_{j \in I \setminus \{i\}} L^c_{X^j_{-i}}$.

Observe, that if the players had used the common knowledge information strategies or the private information (i.e., $\mathcal{F}_j$), then since $L_{X_{-i}}{}^c$ and $L_{X_{-i}}$ are subsets of $L_{X_{-i}}{}^p$, the punishment inflicted to player i would have been less severe. Indeed, the larger the set, the bigger the punishment. Notice that, since for each $i \in I$, $L^c_{X_{-i}} \subseteq L_{X_{-i}} \subseteq L^p_{X_{-i}}$, it follows that

$$\inf_{x_{-i} \in L^c_{X_{-i}}} \sup_{x_i \in L_{X_i}} v_i(x_{-i}, x_i) \geq \inf_{x_{-i} \in L_{X_{-i}}} \sup_{x_i \in L_{X_i}} v_i(x_{-i}, x_i) \geq \inf_{x_{-i} \in L^p_{X_{-i}}} \sup_{x_i \in L_{X_i}} v_i(x_{-i}, x_i).$$

Definitions 3.1 and 3.2 can be formulated in terms of the sets $L_{X_{-i}}{}^p$ and $L_{X_{-i}}{}^c$ and the existence Theorems 4.1 and 4.2 remain valid.

As noted above, player i maximizes her payoff using her own private information, i.e., $x_i(\cdot)$ is $\mathcal{F}_i$-measurable. For each player $i \in I$, we call an action x satisfying (1) an ex-ante optimizer for the ex-ante minimax value v_i^*. Similarly, we call an action x satisfying (2) an interim optimizer for the interim minimax value V_i^*. Clearly optimizers, if they exist, may be different for different players. Furthermore, if the minimax point is attainable, the sets of ex-ante and interim optimizers may differ, as shown in (Pesce and Yannelis, 2009) (see example 3.3). If the payoff functions are not continuous or if the random strategy sets are not compact, then the set of ex-ante and interim optimizers may be empty (see example 4.1 and example 4.2 in (Pesce and Yannelis, 2009)). By imposing compactness and continuity in the weak topology[3], we will be able to prove the existence of an ex-ante and an interim minimax point in the next section.

4. Existence theorems

4.1. Assumptions

We now list the main assumptions needed to prove that an ex-ante as well as an interim minimax point is attainable.

A.1. For each i, X_i: $\Omega \to 2^Y$ is $\mathcal{F}_i$-lower measurable[4], non-empty, integrably bounded, closed, weakly compact and convex valued correspondence,

A.2. For each $x \in \prod_{i \in I} X_i(\cdot)$ and for each player $i \in I$, $u_i(\cdot, x)$: $\Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable. Moreover, for all ω and for all i, $u_i(\omega, \cdot)$: $\prod_{i \in I} X_i(\cdot) \to \mathbb{R}$ is weakly jointly continuous and integrably bounded.

[2] The apexes "p" and "c" stand respectively for "pooled" and "common" information.

[3] By weak compactness and weak continuity we mean with respect to the weak topology $\sigma(L_1(\mu, X), L^\infty(\mu, X^*))$, where $L^\infty(\mu, X^*)$ is the dual of $(L_1(\mu, X), \|\cdot\|_1)$.

[4] A correspondence ϕ: $X \to 2^Y$ from a measurable space (X, α) into a topological space Y is said to be lower measurable if $\{x \in X$: $\phi(x) \cap V \neq \emptyset\} \in \alpha$ for every V open in Y.

4.2. Theorems

Theorem 4.1. *Assume that (A.1) and (A.2) hold. Then the ex-ante minimax point is attainable.*

Theorem 4.2. *Assume that (A.1) and (A.2) hold. Then the interim minimax point is attainable.*

Clearly, the theorem in the deterministic case, proved by Aliprantis et al. in (2009), can be viewed as a corollary of ours. Indeed, in the special case of full information, i.e., when the private information of each player is singletons, the interim expected payoff function reduces to be the ex-post one. Therefore, Theorem 4.2 includes as a special case a version of the existence of a deterministic minimax point, in (Aliprantis et al., 2009).

4.3. Proof of the Theorem 4.1

Claim 4.3. If (A.1) holds, then L_X is non-empty and weakly compact.

Proof. We first prove that L_X is non-empty. Since, for each fixed i, X_i is $\mathcal{F}_i$-measurable, all the conditions of Kuratowski and Ryll-Nardzewski Measurable Selection Theorem (see (Aliprantis and Border, 2006) p. 600) are satisfied and hence there exists an $\mathcal{F}_i$-measurable function $x_i^*: \Omega \to Y$ such that $x_i^*(\omega) \in X_i(\omega)$ for all $\omega \in \Omega$. Therefore, we just need to show that $x_i^* \in L_1(\mu, Y)$, that is x_i^* is a Bochner integrable function. But this follows directly from the assumption that X_i is integrably bounded. Thus, for all i, L_{X_i} is non-empty, and so is $L_X = \prod_{i \in I} L_{X_i}$.

We are now ready to prove that L_X is weakly compact. First, notice that for all $i \in I$, L_{X_i} is a weakly closed subset of the weakly compact set $\{x_i \in L_1(\mu, Y): x_i(\omega) \in X_i(\omega)$ for almost all $\omega \in \Omega\}$, (recall Diestel's theorem, see (Yannelis, 1991)). Therefore, for each fixed i, L_{X_i} is weakly compact, since it is weakly closed subset of a weakly compact set. Consequently, the set $L_X = \prod_{i \in I} L_{X_i}$ is also weakly compact by Tychonoff's Theorem. □

Claim 4.4. Assume that (A.1) and (A.2) hold, then for each i and ω, the functions $v_i(\cdot)$ and $V_i(\omega, \cdot)$ are weakly continuous.

Proof. See Yannelis (1991, p. 191).

We can now complete the proof of the theorem by applying the Berge Maximum Theorem. For each fixed player i, consider the constant correspondence $\Phi_i : L_{X_{-i}} \to 2^{L_{Xi}}$ defined by $\phi_i(x_{-i}) = L_{X_i}$ for all $x_{-i} \in L_{X_{-i}}$. Obviously $\Phi_i(\cdot)$ is weakly continuous (because it is constant), non-empty and weakly compact-valued correspondence (by Claim 4.3).

It can be easily checked that the graph of Φ_i coincides with L_X. Define the function $f_i : Gr_{\Phi_i} = L_X \to \mathbb{R}$ by $f_i(x_{-i}, x_i) = v_i(x_{-i}, x_i)$. By Claim 4.4 it is weakly continuous.

From Claims 4.3 and 4.4 it follows that all conditions of the Berge Maximum Theorem are satisfied and hence the value function $m_i : L_{X_{-i}} \to \mathbb{R}$ defined by $m_i(x_{-i}) = \max_{x_i \in \Phi_i(x_{-i})} f_i(x_{-i}, x_i) = \max_{x_i \in L_{X_i}} v_i(x_{-i}, x_i)$ is weakly continuous and the correspondence $\mu_i : L_{X_{-i}} \to 2^{L_{Xi}}$ of maximizers, defined by $\mu_i(x_{-i}) = \{x_i \in L_{X_i}: v_i(x_{-i}, x_i) = m_i(x_{-i})\}$, has non-empty and weakly compact values.

Thus, by virtue of the weak compactness of the set $L_{X_{-i}}$, the weakly continuous function m_i attains its minimum over $L_{X_{-i}}$. □

Proof of Theorem 4.2. Similar arguments to the ones used above can be adopted to complete the proof. We refer the reader to (Pesce and Yannelis, 2009) for the details.

Open question: The separability assumption plays an important role to apply the Kuratowski and Ryll-Nardzewski Measurable Selection Theorem. We do not know if the existence theorem can be proved without the separability assumption on the strategy space.

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