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The Core of an Economy Without Ordered Preferences

Nicholas C. Yannelis*

Abstract. Core existence results are proved for exchange economies with an infinite dimensional commodity space. In particular, the commodity space may be any ordered Hausdorff linear topological space, and agents' preferences need not be transitive, complete, monotone or convex; preferences may even be interdependent. Under these assumptions a quasi equilibrium may not exist.

1. Introduction

During the last decade, contributions in consumer theory [e.g., Sonnenschein (1971), Shafer (1974) and Kim-Richter (1986)] and contributions in equilibrium theory [e.g., Mas-Colell (1974), Gale-Mas-Colell (1975), Shafer-Sonnenschein (1975), Borglin-Keiding (1976), McKenzie (1981), and Yannelis-Prabhakar (1983)] have shown that the transitivity axiom is not only a restrictive assumption but unnecessary as well. In fact, very general competitive equilibrium existence results have been obtained for finite economies where agents' preferences need not be ordered, i.e., need not be transitive or complete (therefore, need not be representable by utility functions), and may be interdependent. These existence results for the competitive equilibrium have been further generalized to economies with infinitely many commodities [see for instance Mas-Colell (1986) or Yannelis-Zame (1986) among others]. Thus, significant progress has been made on the task of establishing very general conditions for the existence of a competitive equilibrium.

The core is an alternative solution concept which has been widely used in game theory and by extension in general equilibrium analysis. It is still not known whether or not core existence results can be obtained

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* The results of this paper were obtained in 1984. The present version is virtually identical to the Discussion paper No. 214, June 1985, University of Minnesota. The minor changes are due to suggestions made by Charles Holly to whom I am very thankful. It should be noted that Atsumi Kajii has recently obtained α-core existence results for normal form games without ordered preferences.

at the level of generality established for competitive equilibrium existence theorems.

The first core existence result for an economy was proved in Scarf (1967, 1971). He required agents' preferences to be transitive and complete. Border (1984) recently generalized this result to allow for preferences which need not be transitive or complete. Both authors obtain their results for economies with a finite dimensional commodity space, and follow a common argument: First, they establish that the core of a balanced non-side payment game is nonempty; and second, they show the nonemptiness of the core of an economy by showing that the game derived from an economy is balanced.[1]

Recently, several nonexistence core results have been given for infinite dimensional commodity spaces [see for instance Araujo (1985) and Mas-Colell (1986)]. In particular, these authors have shown by means of counter-examples that in an infinite dimensional commodity space, where agents' preferences are representable by very well-behaved utility functions, one can not necessarily even expect individually rational Pareto optimal allocations to exist. Therefore, the question is raised under what conditions can core existence results be obtained in an infinite dimensional commodity space. The purpose of this paper is to show that in any ordered Hausdorff linear topological space, core allocations exist under very mild assumptions. In particular, agents' preferences need not be ordered, monotone or nonsaturated. Indeed, under these assumptions even a quasi-equilibrium need not exist. Moreover, we show that in any ordered Hausdorff linear topological space, individually rational Pareto optimal allocations exist, even if preferences are interdependent and may not be ordered, monotone or nonsaturated.

It may be instructive to comment on the technical aspects of the

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[1] It should be noted that a different proof of Scarf's result has been given in Shapley (1973). In particular Shapley provides an extension of the Sperner Lemma which is used to obtain a generalized version of the Knaster-Kuratowski-Mazurkiewicz (K-K-M) theorem known in the literature as K-K-M-S. By means of the K-K-M-S theorem Shapley proves that the core of a balanced game is nonempty. Here we must note that an elegant proof of the K-K-M-S theorem was recently given by Ichiichi (1981), by using the coincidence theorem of Fan (1969).

paper. Although the arguments of Scarf (1967, 1971) and Shapley (1973) are based on finite dimensional results, Border's (1984) proof is based on an infinite dimensional fixed point result of Fan (1969). At first glance, it seems that Border's arguments might be extended to cover infinite dimensional commodity spaces; unfortunately, a careful examination of his proof indicates that this is not possible. The problem arises from the fact that the convex hull of an upper-semicontinuous (u.s.c.) correspondence need not be u.s.c. when the dimensionality of the commodity space is infinite [see Schaefer (1971, exercise 27, p. 72)]. Consequently, in order to prove the nonemptiness of the core for an economy with infinitely many commodities and without ordered preferences different arguments than the ones used by Scarf, Shapley and Border seem to be needed. In particular, following Bewley's (1972) ideas we will prove an infinite dimensional core existence result by considering its trace in finite dimensions.

However, a different approach is adopted to prove that with interdependent preferences individually rational Pareto optimal allocations exist. In particular, the main mathematical tool that we use is an existence of maximal elements result which is a corollary of either the Knaster-Kuratowski-Mazurkiewicz (K-K-M) Lemma as extended by Fan (1962) or the Browder (1968) fixed point theorem. In fact, we will show that these two remarkable technical theorems turn out to be equivalent in the sense that each one can be derived from the other. It should be noted that the idea of using maximal elements results to prove optima goes back to Debreu (1959, p. 92). The same idea was also used in Hildenbrand (1974, Theorem 3, p. 230) and Berninghaus (1977, Theorem 1, p. 283). However, the assumption that preferences are transitive and complete and consequently representable by utility functions is crucial to their arguments. It turns out, that allowing simultaneously, preferences to be nonordered and the dimensionality of the commodity space to be infinite, rather powerful fixed point results seem to be needed. It is exactly for this reason that we make use of the theorems of Fan (1962) and Browder (1968).

The paper is organized in the following way. Section 2 contains some notation and definitions. Section 3 shows the equivalence between the K-K-M Lemma as extended by Fan and the Browder fixed point theorem. The main results of the paper, i.e., core existence theorems, are stated in Section 4 and their proofs are given in Section 5. In Section 6 we discuss some pathological examples known in the literature. Finally, some concluding remarks are given in Section 7.

2. Notation and Definitions

2.1 Notation.

2^A denotes the set of all subsets of A.

con A denotes the convex hull of the set A.

$\mathbb{R}^\ell$ denotes the ℓ-fold product of the set of real numbers $\mathbb{R}$.

$|S|$ denotes the number of elements in the set S.

If $\phi : X \to 2^Y$ is a correspondence, $\phi|_A$ denotes the restriction of ϕ to A, i.e., $\phi|_A : A \to 2^Y$.

$\emptyset$ denotes the empty set.

$\setminus$ denotes the set theoretic subtraction.

int A denotes the interior of A.

If X is a linear topological space, its dual is the space X^* of all continuous linear functionals on X.

2.2 Definitions. Let X and Y be two topological spaces. Let $\phi : X \to 2^Y$ be a set-valued function (or correspondence). The set-valued function $\phi^{-1} : Y \to 2^X$ defined by $\phi^{-1}(y) = \{x \in X : y \in \phi(x)\}$ is called the *lower section* of ϕ. We say that $\phi : X \to 2^Y$ has *open lower sections* if for each $y \in Y$ the set $\phi^{-1}(y) = \{x \in X : y \in \phi(x)\}$ is open in X. A binary relation $\mathcal{P}$ on X is a subset of $X \times X$. We read $x\mathcal{P}y$ as "x is strictly preferred to y." Define the correspondence $P : X \to 2^X$ by $P(x) = \{y \in X : y\mathcal{P}x\}$. We call P a *preference correspondence*, and $P(x)$ denotes its *upper section* and $P^{-1}(y)$ its *lower section*. The set-valued function $P : X \to 2^X$ has an *open graph* if the set $\{(x, y) \in X \times X : y \in P(x)\}$ is open in $X \times X$. Moreover, $P : X \to 2^X$ is said to be *lower semicontinuous* if the set $\{x \in X : P(x) \cap V \neq \emptyset\}$ is open in X for every open subset V of X. If there exists $x^* \in X$ such that $P(x^*) = \emptyset$ we say that x^* is a *maximal element* in X.

3. The K-K-M-F Lemma and the Browder Fixed Point Theorem

3.1 Theorems. Fan (1962) extended the powerful Knaster-Kuratowski-Mazurkiewicz (K-K-M) theorem from a Euclidean space to Hausdorff linear topological spaces. Another simple but powerful fixed point theorem was proved by Browder (1968). Both results, in addition to their applications in mathematics, have recently proved extremely useful in economics. In fact, they have become the main technical tools to prove the existence of maximal elements and equilibria in linear topological spaces of arbitrary dimension. As a consequence, generalizations of the results of Debreu (1952), Sonnenschein (1971), Mas-Colell (1974), Gale-Mas-Colell (1975), and Shafer-Sonnenschein (1975) have been obtained [see for instance, Borglin-Keiding (1976), Yannelis-Prabhakar (1983), and Toussaint (1984)]. Since these two theorems will be the main mathematical tools used in the sequel, it is of interest to know the relationship between them. The purpose of this section is to show that Fan's generalization of the K-K-M theorem (called here K-K-M-F) theorem) can be easily derived from Browder's fixed point theorem and that the Browder fixed point theorem can be easily derived from the K-K-M-F theorem. Therefore one may consider these two results as equivalent.

The K-K-M-F theorem proved in Fan (1962) is stated below:

Theorem 3.1 (K-K-M-F). *Let X be an arbitrary convex set in a Hausdorff linear topological space Y. For each $x \in X$, let $F(x)$ be a closed set in Y such that the following two conditions are satisfied:*

(i) the convex hull of any finite subset $\{x_1, \ldots, x_n\}$ of X is contained in $\bigcup_{i=1}^n F(x_i)$, and

(ii) $F(x)$ is compact for at least one $x \in X$.

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

We now state Browder's (1968) fixed point theorem.

Theorem 3.2 (Browder). *Let X be a compact, convex, nonempty subset of a Hausdorff linear topological space Y and $\phi : X \to 2^X$ be a correspondence such that:*

(1) $\phi(x)$ is nonempty for all $x \in X$,

(2) $\phi(x)$ is convex for all $x \in X$,

(3) for each $y \in X$, the set $\phi^{-1}(y) = \{x \in X : y \in \phi(x)\}$ is open in X, i.e., ϕ has open lower sections.

Then there exists $x^* \in X$ such that $x^* \in \phi(x^*)$.

3.2 Proof of the K-K-M-F Theorem via Browder's Fixed Point Theorem. Since $F(x)$ is closed in Y for each $x \in X$ and compact for at least one $x \in X$, it suffices to prove that $\bigcap_{i=1}^{n} F(x_i) \neq \emptyset$ for every subset $\{x_1, \ldots, x_n\}$ of X. Suppose otherwise, i.e., $\bigcap_{i=1}^{n} F(x_i) = \emptyset$ for some finite subset $\{x_1, \ldots, x_n\}$ of X. Let Δ be the finite dimensional simplex spanned by the finite set $\{x_1, \ldots, x_n\}$. Since the topology induced on any finite dimensional subspace of Y by the topology of Y coincides with the Euclidean topology, Δ is homeomorphic to a Euclidean ball (Kelley and Namioka, 1963, Theorem 7.3, p. 59). Define the correspondence $\psi : \Delta \to 2^{\Delta}$ by $\psi(x) = \{y \in \Delta : x \notin F(y)\}$. Then for each $x \in \Delta$, $\psi(x)$ is nonempty. Indeed, at least one x_i, $(1 \leq i \leq n)$ is in $\psi(x)$, for otherwise $x \in \bigcap_{i=1}^{n} F(x_i)$, and so $\bigcap_{i=1}^{n} F(x_i) \neq \emptyset$. Notice that for each $y \in \Delta$, $\psi^{-1}(y) = \{x \in \Delta : y \in \psi(x)\} = \Delta \setminus \{x \in \Delta : x \in F(y)\}$. Observe that $\{x \in \Delta : x \in F(y)\} = \Delta \cap F(y)$, and this is a closed set in Δ. Hence, for each $y \in \Delta$ the set $\psi^{-1}(y)$ is open in Δ. Define the correspondence $\phi : \Delta \to 2^{\Delta}$ by $\phi(x) = \operatorname{con} \psi(x)$ for all $x \in \Delta$. Then, $\phi(x)$ is convex and nonempty for all $x \in \Delta$. Furthermore, by Lemma 5.1 in Yannelis-Prabhakar (1983) the set $\phi^{-1}(y) = \{x \in \Delta : y \in \phi(x)\}$ is open in Δ for each $y \in \Delta$. Consequently, the correspondence $\phi : \Delta \to 2^{\Delta}$ satisfies all the assumptions of Theorem 3.2. Hence, there exists $x^* \in \Delta$ such that $x^* \in \phi(x^*) = \operatorname{con} \psi(x^*)$. But, $x^* \in \operatorname{con} \psi(x^*)$ implies that there exist points $y_1, \ldots, y_m$ in Δ and real numbers $a_1, \ldots, a_m$, $a_j \geq 0$, $(1 \leq j \leq m)$, $\sum_{j=1}^{m} a_j = 1$, such that $x^* = \sum_{j=1}^{m} a_j y_j$ and $y_j \in \psi(x^*)$ for all j, a contradiction to assumption (i). Indeed, by assumption (i), for any arbitrary collection of points $\{y_1, \ldots, y_m\}$ out of X, we have that $\operatorname{con}\{y_1, \ldots, y_m\} \subset \bigcup_{i=1}^{m} F(y_i)$. Thus, if $x^* \in \operatorname{con}\{y_1, \ldots, y_m\}$, then $x^* \in \bigcup_{i=1}^{m} F(y_i)$ which implies that $x^* \in F(y_i)$ for at least one i. The above contradiction establishes that, $\bigcap_{i=1}^{n} F(x_i) \neq \emptyset$, and this completes the proof of the K-K-M-F theorem.

3.3 Proof of Browder's Fixed Point Theorem via the K-K-M-F Theorem. Suppose otherwise, i.e., for all $x \in X$, $x \notin \phi(x)$. Let for each $y \in X$, $F(y) = X \setminus \phi^{-1}(y)$. Since by assumption (3) for each $y \in X$, $\phi^{-1}(y)$ is open in X, it follows that for each $y \in X$, $F(y)$ is closed in X and obviously closed in Y since X is a compact subset of Y. Moreover, $F(y)$ is compact for each $y \in X$. It is easy to see that for any arbitrary set of points $\{y_1, \ldots, y_n\} \subset X$, we have that $\text{con}\{y_1, \dots, y_n\} \subset \bigcup_{i=1}^n F(y_i)$. For otherwise, there exists $x \in \text{con}\{y_1, \dots, y_n\}$ and $x \notin \bigcup_{i=1}^n F(y_i)$ which implies that $x \in \phi^{-1}(y_i)$ for all i or $y_i \in \phi(x)$ for all i and therefore $x \in \text{con}\{y_1, \dots, y_n\} \subset \text{con}\,\phi(x) = \phi(x)$, a contradiction to $x \notin \phi(x)$ for all $x \in X$. Hence, by Theorem 3.1 $\bigcap_{y \in X} F(y) \neq \emptyset$. Let $z \in \bigcap_{y \in X} F(y)$. Then for all $y \in X$, $z \notin \phi^{-1}(y)$ which implies that $\phi(z) = \emptyset$, for some $z \in X$. But this contradicts assumption (1). Therefore there exist $x^* \in X$ such that $x^* \in \phi(x^*)$, and the proof of the Browder theorem is now complete.

3.4 Existence of Maximal Elements.

It is easy to check that Browder's fixed point theorem is equivalent to the following existence of maximal elements result.

Theorem 3.3. *Let X be a nonempty, compact, convex subset of a Hausdorff linear topological space Y and $P : X \to 2^X$ be a preference correspondence such that:*

(i) $x \notin P(x)$ for all $x \in X$
(ii) $P(x)$ is convex for all $x \in X$
(iii) P has open lower sections.

Then there exists $x^* \in X$ such that $P(x^*) = \emptyset$.

Hence, we can reach the following conclusion:

K-K-M-F $\Leftrightarrow$ *Browder Theorem*
$\Leftrightarrow$ *Existence of Maximal Elements Theorem.*

A direct consequence of the K-K-M-F or Browder theorems is the following result, whose proof can be found in Yannelis and Prabhakar (1983 p. 239, Theorem 5.1).

Theorem 3.4. *Let X be a nonempty, compact, convex subset of a Hausdorff linear topological space and $\phi : X \to 2^X$ be a correspondence having open lower section satisfying the condition that $x \notin \text{con}\,\phi(x)$ for all $x \in X$. Then there exists $x^* \in X$ such that $\phi(x^*) = \emptyset$.*

By means of Theorem 3.4 one can obtain the following Corollary [see Yannelis and Prabhakar (1983, p. 240, Corollary 5.1)] which is a generalized version of a result of Borglin-Keiding (1976, Corollary 1, p. 314). We first need to introduce a definition.

Definition 3.1. *Let X be a subset of a linear topological space. A correspondence $\phi : X \to 2^X$ is said to be of class $\mathcal{L}$, if*

(i) $x \notin \operatorname{con} \phi(x)$ *for all* $x \in X$,

(ii) ϕ *has open lower sections.*

Let $\psi : X \to 2^X$ be a correspondence. The correspondence $\phi_x : X \to 2^X$ is an *$\mathcal{L}$-majorant* of ψ at x if ϕ_x is of class $\mathcal{L}$ and there is an open neighborhood of x denoted by N_x such that for all $z \in N_x$, $\psi(z) \subset \phi_x(z)$. The correspondence $\psi : X \to 2^X$ is *$\mathcal{L}$-majorized* if for each $x \in X$ such that $\psi(x) \neq \emptyset$, there is an $\mathcal{L}$-majorant of ψ at x.

Corollary 3.1. *Let X be a nonempty, compact, convex subset of a Hausdorff linear topological space and $\phi : X \to 2^X$ be an $\mathcal{L}$-majorized correspondence. Then there exists $x^* \in X$ such that $\phi(x^*) = \emptyset$.*

By means of the above Corollary we will prove Theorem 4.1. We would like to emphasize the fact that Corollary 3.1 is a consequence of the Theorems of K-K-M-F and Browder. Moreover, it was pointed out in Borglin-Keiding (1976) that Corollary 3.1 yields an extension of the Kakutani fixed point to Hausdorff locally convex linear topological spaces. With those preliminary mathematical results out of the way we now turn to our core existence theorems.

4. The Main Results

4.1 The Economy. We formalize the notion of an exchange economy in the usual way. Let $I = \{1, \ldots, N\}$ be a finite set of agents. For each $i \in I$, let X_i be a nonempty subset of an ordered Hausdorff linear topological space L. By an *exchange economy with N agents and a commodity space L* (or simply *an economy in L*) we mean the set $\mathcal{E} = \{(X_i, P_i, e_i) : i = 1, \ldots, N\}$ of triples where,

(a) X_i is the *consumption set* of agent i;

(b) $P_i : X \to 2^X$ (where $X = \prod_{i \in I} X_i$) is the *preference correspondence* of agent i;

(c) e_i is the *initial endowment* of agent i, where $e_i \in X_i$ for all $i \in I$.

An *allocation* is a vector $x = (x_1, \ldots, x_N) \in X = \prod_{i \in I} X_i$. An allocation x is said to be *feasible* if $\sum_{i \in I} x_i = \sum_{i \in I} e_i$. Denote by F the *set* *of all feasible allocations*, i.e., $F = \{x \in X : \sum_{i \in I} x_i = \sum_{i \in I} e_i\}$. Notice that we have allowed for interdependent preferences. In this framework $y \in P_i(x)$ means that agent i strictly prefers the allocation y to x. More simply one may define $P_i : X \to 2^X$ by $P_i(x_1, \ldots, x_N) = \{y \in X : (y_1, \ldots, y_N)\mathcal{P}_i(x_1, \ldots, x_N)\}$.

4.2 The α-Core. If $S \subset I$ then $(y^S, x^{I \setminus S})$ denotes the vector z in X such that:

$$z_i = \begin{cases} y_i & \text{if } i \in S \\ x_i & \text{if } i \notin S. \end{cases}$$

An *α-core allocation* of $\mathcal{E}$ is a vector $x = (x_1, \ldots, x_N) \in X$ such that:

(i) $x \in F$, and
(ii) it is not true that there exist $S \subset I$ and $(y_i)_{i \in S} \in \prod_{i \in S} X_i$ such that $\sum_{i \in S} y_i = \sum_{i \in S} e_i$, and $(y^S, z^{I \setminus S}) \in P_i(x_1, \ldots, x_N)$ for all $i \in S$ and for any $z \in \prod_{i \notin S} X_i$, $\sum_{i \notin S} z_i = \sum_{i \notin S} e_i$.

In other words an α-core allocation for the economy $\mathcal{E}$ must satisfy two conditions. First it must be feasible and secondly, no coalition of agents can redistribute their initial endowments and make all its members better off, once the complementary coalition chooses to redistribute its initial endowment. For a game in normal form, the notion of α-core was introduced in Aumann (1964). It was also used by Scarf (1971) who proved the nonemptiness of the α-core for an n-person game with a finite dimensional strategy space, where each agent's preferences were assumed to be transitive, complete, and continuous.

The *set of all α-core allocations for the economy* $\mathcal{E}$ is denoted by $\mathcal{C}(\mathcal{E})$.

4.3 The Extreme α-Core. If $i \in I$, then (y_i, z^{I^i}) denotes the vector w in X such that:

$$w_j = \begin{cases} y_j & \text{if } j = i \\ z_j & \text{if } j \neq i. \end{cases}$$

An allocation $x \in X$ is said to be *individually rational* if:

(i) $x \in F$, and
(ii) for all $i \in I$, it is not true that $e \in P_i(x)$.

An allocation $x \in X$ is said to be *Pareto optimal* if:

(i) $x \in F$, and
(ii) there is no $y \in F$ such that $y \in P_i(x)$ for all $i \in I$.

An *extreme α-core allocation* of $\mathcal{E}$ is a vector $x = (x_1, \ldots, x_N) \in \prod_{i \in I} X_i$ which satisfies individual rationality and Pareto optimality.

Denote by $\mathcal{C}_e(\mathcal{E})$ the *set of all extreme α-core allocations for the economy* $\mathcal{E}$. Notice that the concept of extreme α-core allocations takes into account only two extreme coalitions, i.e., the grant coalition and the coalitions of one agent alone. Therefore, it is clear that the set of all extreme α-core allocations for $\mathcal{E}$ is bigger than the set of all α-core allocations for $\mathcal{E}$, i.e., $\mathcal{C}(\mathcal{E}) \subseteq \mathcal{C}_e(\mathcal{E})$. However, *it is easy to see that in a two person economy, i.e., when $|I| = 2$, $\mathcal{C}(\mathcal{E}) = \mathcal{C}_e(\mathcal{E})$.*

Finally, if preferences are "selfish," i.e., $P_i : X_i \to 2^{X_i}$ is defined by $P_i(x_i) = \{y_i \in X_i : y_i \mathcal{P}_i x_i\}$, we will call an individually rational Pareto optimal allocation, an *extreme core allocation.*

4.4 The Selfish Core. Let $\mathcal{E} = \{(X_i, \bar{P}_i, e_i) : i = 1, \ldots, N\}$ be an exchange economy, where $\bar{P}_i : X_i \to 2^{X_i}$ is defined by $\bar{P}_i(x_i) = \{y_i \in X_i : y_i \mathcal{P}_i x_i\}$. Notice that preferences are not interdependent. In this framework we may define the notion of selfish core or simply core as follows:

A *selfish core* (or *core*) *allocation* of $\mathcal{E}$ is a vector $x = (x_1, \ldots, x_N) \in X$ such that:

(i) $x \in F$, and
(ii) it is not true that there exist $S \subset I$ and $(y_i)_{i \in S} \in \prod_{i \in S} X_i$ such that $\sum_{i \in S} y_i = \sum_{i \in S} e_i$ and $y_i \in \bar{P}_i(x_i)$ for all $i \in S$.

The above notion of core is the one extensively used in equilibrium analysis. In fact, this is the notion of core used recently in Border (1984) as well. Denote by $\mathcal{C}_s(\mathcal{E})$ the *set of all selfish core allocations for* $\mathcal{E}$.

4.5 Theorems. Before we state our two main results we will need the following definition.

Definition 4.1. *A Hausdorff topology τ, on an ordered Hausdorff linear topological space L, will be called compatible if:*

(a) τ is weaker than the Hausdorff topology of L;
(b) τ is a vector space topology (i.e., the vector space operations on L are continuous in the topology τ);
(c) all order intervals $[0, y] = \{z \in L : 0 \leq z \leq y\}$ in L are τ-compact.

Theorem 4.1. *Let $\mathcal{E} = \{(X_i, P_i, e_i) : i \in I\}$ be an exchange economy* *in L, where L is endowed with the compatible topology τ, satisfying for each $i \in I$ the following assumptions:*

(4.1) $X_i = L^+$, *(L^+ denotes the positive cone of L),*

(4.2) $x \notin \operatorname{con} P_i(x)$ *for all* $x \in X$,

(4.3) P_i *has τ-open lower sections, i.e., for each $y \in X$ the set* $\{x \in X : y \in P_i(x)\}$ *is τ-open in X.*

Then there exists an extreme α-core allocation of $\mathcal{E}$, i.e., $\mathcal{C}_e(\mathcal{E}) \neq \emptyset$.

Theorem 4.2. *Let $\mathcal{E} = \{(X_i, \bar{P}_i, e_i) : i \in I\}$ be an exchange economy in L, where L is endowed with the compatible topology τ, satisfying for each $i \in I$ the following assumptions:*

(4.4) $X_i = L^+$,

(4.5) $x_i \notin \operatorname{con} \bar{P}_i(x_i)$ *for all* $x_i \in X_i$,

(4.6) $\bar{P}_i$ *has a τ-open graph, i.e., the set* $\{(x_i, y_i) \in X_i \times X_i : y_i \in \bar{P}_i(x_i)\}$ *is τ-open in* $X_i \times X_i$.

Then there exists a selfish core allocation of $\mathcal{E}$, i.e., $\mathcal{C}_s(\mathcal{E}) \neq \emptyset$.

Corollary 4.1. *Let $\mathcal{E} = \{(X_i, P_i, e_i) : i \in I\}$ be an exchange economy in L, satisfying the following assumptions:*

(4.7) X_i *is a nonempty, convex, compact subset of L,*

(4.8) $x \notin \operatorname{con} P(x)$ *for all* $x \in X$,

(4.9) P_i *has open lower sections.*

Then there exists an extreme α-core allocation of $\mathcal{E}$, i.e., $\mathcal{C}_e(\mathcal{E}) \neq \emptyset$.

Corollary 4.2. *Since for $I = \{1, 2\}$, $\mathcal{C}(\mathcal{E}) = \mathcal{C}_e(\mathcal{E})$ it follows from Theorem 4.1 that $\mathcal{C}(\mathcal{E}) \neq \emptyset$.*

Remark 4.1. Notice that if in Theorem 4.1 and Corollary 4.1 we had selfish preferences, i.e., $\bar{P}_i : X_i \to 2^{X_i}$ the arguments in the proofs (see Section 5) remain unaffected. In fact, define $P_i : X \to 2^X$ by $P_i(x) = \bar{P}_i(x_i) \times \prod_{j \neq i} X_j$, then the proofs go through with no modification.

4.6 Discussion of the Assumptions.

Let us now discuss the assumptions in Theorems 4.1 and 4.2.

First notice that (4.1) is identical with (4.4), and (4.2) is essentially the same with (4.5). Assumption (4.1) is quite standard in equilibrium theory and needs no explanation. Assumption (4.2) is a very weak form of convexity of the upper section. It was first introduced by Shafer-Sonnenschein (1975). Notice that $x \notin \operatorname{con} P_i(x)$ for all $x \in X$ implies that $x \notin P_i(x)$ for all $x \in X$, i.e., P_i is *irreflexive* Of course the same conclusion can be obtained for the selfish preference correspondence $\bar{P}_i$. Assumption (4.3) is a quite weak form of continuity. In fact, if P_i has a τ-open graph in $X \times X$ then both sections (upper and lower) are τ-open. Notice that, (4.3) implies [see Yannelis-Prabhakar (1983, Proposition 4.1, p. 237)] that P_i is τ-lower semicontinuous, i.e., the set $\{x \in X : P_i(x) \cap V \neq \emptyset\}$ is τ-open in X for every τ-open subset V of X.

Finally, since any competitive equilibrium allocation is in the selfish core and obviously in the extreme core it is of interest to know whether or not under the assumptions of either Theorems 4.1 or 4.2 or Corollary 4.1 there exists a competitive equilibrium. However, the example of Mas-Colell (1986), indicates that under the assumptions of either Theorems 4.1 or 4.2 or Corollary 4.1 one should not even expect quasi-equilibria to exist.

It is important to note that in a finite dimensional commodity space if one consumer has a concave, monotone, continuous utility function, strictly positive initial endowments and his/her consumption set is compact, there is always an equilibrium and *a fortiori* the core is nonempty. Contrary to the finite dimensional commodity setting, in an infinite dimensional commodity framework, Mas-Colell's example shows that one should not even expect a quasi equilibrium to exist. Therefore, Theorems 4.1, 4.2 and Corollary 4.1 provide existence core results for economies in which quasi equilibria may not exist.

4.7 Concrete Spaces. In Theorems 4.1 and 4.2 the commodity space L was assumed to be any ordered Hausdorff linear topological space endowed with the compatible topology τ. However, in concrete spaces the topology τ will vary according to the underlying ordered Hausdorff linear topological space L. For instance if the commodity space is the Lebesgue space L_p, $1 \leq p < \infty$ the compatible topology will be the weak topology.

This follows from the fact that the spaces L_p, $1 \leq p < \infty$ are normed vector lattices with order continuous norm, i.e., order intervals are weakly compact [see Aliprantis and Burkinshaw (1985, Theorem 12.9) or Schaefer (1974, Theorem 6.6, p. 100 and Example 6, p. 92)]. If the commodity space is L_∞ or ℓ_∞ the compatible topology will be the weak* topology. Recall that Alaoglou's theorem implies that order intervals are weak* compact [see Aliprantis and Burkinshaw (1985, Theorem 9.20)]. Finally, if the commodity space is the space of real sequences ℓ_p, $1 \leq p < \infty$ the compatible topology will be the norm topology. This follows from the standard result that order intervals in ℓ_p, $1 \leq p < \infty$ are norm compact [see for instance Yannelis and Zame (1984, Theorem 10.1, p. 48)].

It may be instructive to compare our continuity assumption (4.3) with that of Araujo (1985) [or Berninghaus (1977)] whose commodity space is ℓ_∞ (or L_∞), with consumption sets $X = \ell_\infty^+$ (or L_∞^+).

In Araujo (1985), preferences are given by a weak preference relation $\succsim$ which is reflexive, transitive, complete. Assume that $\succsim$ satisfies:

(i) the set $\{y \in X : y \succsim x\}$ is Mackey closed in X and convex for each $x \in X$,

(ii) the set $\{x \in X : y \succsim x\}$ is norm closed in X for each $y \in X$.

If we let P be the strict preference relation induced by $\succsim$, then $P(x) = X \setminus \{y \in X : x \succsim y\}$ and $P^{-1}(y) = \{x \in X : y \in P(x)\} = X \setminus \{x \in X : x \succsim y\}$. Therefore, for each $x \in X$, $P(x)$ is norm open in X and for each $y \in X$, $P^{-1}(y)$ is Mackey open in X. However, since by the Mackey-Arens Theorem [see for instance Bewley (1972, p. 352, (8))] the Mackey topology coincides with the weak* topology on closed convex sets, it follows that the set $\{y \in X : y \succsim x\}$ is weak* closed in X and consequently $P^{-1}(y)$ is weak* open in X. Therefore, since in L_∞ (or ℓ_∞) the compatible topology is the weak* topology, the continuity assumption (4.3) in Theorem 4.1, for $L = L_\infty$ is not stronger than the ones of Araujo's (1985) (or Berninghaus' (1977)) continuity assumptions, who require that the set $\{y \in X : y \succsim x\}$ is Mackey (weak*) closed in X for every $x \in X$. Hence, Theorem 4.1 can be considered as a generalization of the existence results of Araujo (1985) and Berninghaus (1977). Specifically, the commodity space can be any arbitrary ordered linear topological space and preferences need not be transitive, complete or convex and may be interdependent.

5. Proof of the Theorems

5.1 Proof of Theorem 4.1. Suppose otherwise, i.e., $\mathcal{C}_e(\mathcal{E}) = \emptyset$, then for all $x \in F$ either

(5.1) there exists $y \in F$ such that $y \in P_i(x)$ for all i, or

(5.2) for at least one agent i, $e \in P_i(x)$.

For each $i \in I$ define $\psi_i : X \to 2^X$ by $\psi_i(x) = \operatorname{con} P_i(x)$. Since by assumption (4.3) P_i has τ-open lower sections it follows from Lemma 5.1 in Yannelis and Prabhakar (1983, p. 239), that ψ_i has τ-open lower sections in X. Let $\psi_{i|F}$ be the restriction of ψ_i to F. It follows from (5.1) that:

(5.3) for all $x \in F$ there exists $y \in F$ such that $y \in P_i(x) \subset \operatorname{con} P_i(x) = \psi_{i|F}(x)$ for all $i \in I$.

For each $i \in I$ define $\Phi_i : F \to 2^F$ by $\Phi_i(x) = \psi_{i|F}(x) \cap F$. Define $A = \{w \in F : \text{there exist } z \in F \text{ such that } z \in P_i(w) \text{ for all } i \in I\}$. It can be easily checked that A is open in F. It follows from (5.3) that:

(5.4) for all $x \in A$, $\Phi_i(x)$ is nonempty for all $i \in I$.

Notice that from assumption (4.2) we have that $x \notin \operatorname{con} \Phi_i(x) = \Phi_i(x)$ for all $x \in F$. Moreover, it can be easily seen that Φ_i has τ-open lower sections in F, i.e., Φ_i is of class $\mathcal{L}$. For $x \in F$, let $S_x = \{i \in I : e \in P_i(x)$. It follows from (5.2) that

(5.5) for all $x \in F$ and all $i \in S_x$, $\Phi_i(x) \neq \emptyset$.

Indeed, from (5.2) we can conclude that for all $x \in F$ and all $i \in S_x$, $e \in P_i(x) \subset \operatorname{con} P_i(x) = \psi_{i|F}(x)$. Consequently, for all $x \in F$ and all $i \in S_x$, $e \in \Phi_i(x)$.

Define the correspondence $\theta : F \to 2^F$ by

$$\theta(x) = \begin{cases} \bigcap_{i \in I} \Phi_i(x) & \text{if } x \in A \\ \bigcap_{i \in S_x} \Phi_i(x) & \text{if } x \in F \setminus A. \end{cases}$$

It follows from (5.4) and (5.5) that

(5.6) for all $x \in F$, $\theta(x) \neq \emptyset$.

Notice that F is nonempty, convex, bounded and τ-closed. Moreover, F lies on the order interval $[0, Ne]^N = \{x \in X : 0 \leq x_j \leq Ne \text{ for all } j \in I\}$ which is τ-compact. Therefore, F is a τ-compact subset of X. If we show that θ is $\mathcal{L}$-majorized we can then appeal to Corollary 3.1. To this end let $x \in F$. Then either (a) $x \in A$, or (b) $x \in F \setminus A$. If (a) holds then $\theta(x) = \bigcap_{i \in I} \Phi_i(x) \neq \emptyset$. Choose $w \in \theta(x)$. Then $w \in \Phi_i(x)$ for all i. Fix an agent j in I. Since A is an open set in F, and Φ_j has τ-open lower sections in F, then $\theta(z) = \bigcap_{i \in I} \Phi_i(z) \subset \Phi_j(z)$ for all $z \in A$. If (b) holds then $\theta(x) = \bigcap_{i \in S_x} \Phi_i(x) \neq \emptyset$. Choose $e \in \theta(x)$. Then $e \in \Phi_i(x)$ for all $i \in S_x$ which implies that $e \in P_i(x)$ for all $i \in S_x$. Fix an agent j in S_x. Since P_j has τ-open lower sections in F there exists a neighborhood of x, N_x such that $e \in P_j(z)$ for all $z \in N_x$. But then $j \in S_z$ for all $z \in N_x$. Consequently, $\theta(z) = \bigcap_{i \in S_z} \Phi_i(z) \subset \Phi_j(z)$ for all $z \in N_x$. Therefore, θ is $\mathcal{L}$-majorized. By Corollary 3.1 there exists $x^* \in F$ such that $\theta(x^*) = \emptyset$, a contradiction to (5.6). Since we have obtained a contradiction to our supposition that $\mathcal{C}_e(\mathcal{E}) = \emptyset$ the proof of the Theorem is complete.

5.2 Proof of Theorem 4.2. Let $\mathcal{F}$ be the set of all finite dimensional subspaces of L containing the initial endowments. For each $f \in \mathcal{F}$ and for each $i \in I$ define the consumption set X_i^f and the preference correspondence $\bar{P}_i^f : X_i^f \to 2^{X_i^f}$ by

$$X_i^f = X_i \cap f$$

$$\bar{P}_i^f(x_i) = \bar{P}_i(x_i) \cap f.$$

We now have an economy $\mathcal{E}^f = \{(X_i^f, \bar{P}_i^f, e_i) : i = 1, \ldots, N\}$, in a finite dimensional commodity space. It can be checked that each economy $\mathcal{E}^f$ satisfies all the conditions of Border's Proposition (1984, p. 1540), and consequently for each $f \in \mathcal{F}$, $\mathcal{C}_s(\mathcal{E}^f) \neq \emptyset$, i.e., there exists $x^f = (x_1^f, \ldots, x_N^f)$ in $\prod_{i \in I} X_i^f$ such that:

(5.7) $\sum_{i \in I} x_i^f = \sum_{i \in I} e_i$, and

(5.8) it is not true that there exist $S \subset I$ and $(y_i)_{i \in S} \in \prod_{i \in S} X_i^f$ such that $\sum_{i \in S} y_i = \sum_{i \in S} e_i$ and $y_i \in \bar{P}_i^f(x_i^f)$ for all $i \in S$.

From (5.7) it follows that for each $f \in \mathcal{F}$

$$0 \leq \sum_{i \in I} x_i^f = \sum_{i \in I} e_i = e \leq Ne.$$

Hence for each $f \in \mathcal{F}$ the vectors x_i^f lie on the order interval $[0, Ne]$, which is τ-compact. Direct the set $\mathcal{F}$ by inclusion so that $\{(x_1^f, \ldots, x_N^f) : f \in \mathcal{F}\}$ forms a net in $L \times L \times \cdots \times L$. Since all the vectors x_i^f belong to the order interval $[0, Ne]$ which is τ-compact, the net $\{(x_1^f, \ldots, x_N^f) : f \in \mathcal{F}\}$ has a subnet which converges in the compatible topology τ, to some vector $x_1^*, \ldots, x_N^*$ in $[0, Ne]$. We must show that $x_1^*, \ldots, x_N^*$ is a core allocation for the economy $\mathcal{E}$.

Denote the convergent subnet by $\{(x_1^{f(m)}, \ldots, x_N^{f(m)}) : m \in M\}$ where M is a set directed by "$\geq$." First of all we know that $\sum_{i\in I} x_i^{f(m)} = \sum_{i\in I} e_i$ for all $m \in M$. Since the vectors $x_i^{f(m)}$ converge to x_i^* in the compatible topology τ, and τ is a vector space topology we conclude that $\sum_{i\in I} x_i^* = \sum_{i\in I} e_i$, i.e., $x^* = (x_1^*, \ldots, x_N^*)$ is a feasible allocation for the economy $\mathcal{E}$. To complete the proof we must show that:

(5.9) it is not true that there exist $S \subset I$ and $(y_i)_{i\in S} \in \prod_{i\in S} X_i$ such that $\sum_{i\in S} y_i = \sum_{i\in S} e_i$ and $y_i \in \bar{P}_i(x_i^*)$ for all $i \in S$.

Suppose otherwise, i.e., there exist $S \subset I$ and $(y_i)_{i\in S} \in \prod_{i\in S} X_i$ such that $\sum_{i\in S} y_i = \sum_{i\in S} e_i$ and $y_i \in \bar{P}_i(x_i^*)$ for all $i \in S$. Since $x_i^{f(m)}$ converges to x_i^* in the compatible topology τ and by assumption (4.6) $\bar{P}_i$ has a τ-open graph, there exists $m_1 \in M$ such that $y_i \in \bar{P}_i(x_i^{f(m)})$ for all $m \geq m_1$ and all $i \in S$. Choose $m_2 \geq m_1$ so that, if $m \geq m_2$, $y_i \in X_i^{f(m)}$ for all $i \in S$. Then $y_i \in \bar{P}_i^{f(m)}(x_i^{f(m)})$ for all $m \geq m_2$, all $i \in S$, and clearly $\sum_{i\in S} y_i = \sum_{i\in S} e_i$. But this contradicts the fact that $x^{f(m)}$ is a core allocation of the economy $\mathcal{E}^{f(m)}$. Hence (5.9) is satisfied and this completes the proof of the theorem.

5.3 Proof of Corollary 4.1. It follows from assumption (4.7) that the set of all feasible allocations F is compact. Therefore, an identical argument with that used in Theorem 4.1 can be adopted to complete the proof of the Corollary.

6. Examples

We can now turn to some known pathological examples in the literature and see what goes wrong in infinite dimensions. In particular, Araujo (1985), Mas-Colell (1986) and Jones (1986) illustrated by means of simple examples the difficulties in obtaining existence of extreme core allocations in an infinite dimensional commodity setting. The following simple example due to Jones (1986) may be used to illustrate these difficulties.

Example 6.1. *Consider an economy with two agents, i.e., $I = \{1, 2\}$. The commodity space is $L = C[0, 1]$, i.e., the space of continuous functions on the interval $[0, 1]$ under the supnorm. The consumption sets* *are* $X_1 = X_2 = C[0,1]^+$, *i.e., the positive cone of* $C[0,1]$. *Their utility functions and their initial endowments are given as follows:*

$$u_1(x) = \int_0^1 tx(t)\,dt,$$

$$u_2(x) = \int_0^1 (1-t)x(t)\,dt,$$

and

$$e_1 = e_2 = \frac{1}{2}.$$

Notice that utility functions are norm continuous, concave, and monotone. However, there is no individually rational Pareto optimal allocation, i.e., the extreme core which coincides with the selfish core is empty. (Of course there are two Pareto optimal allocations which are not individually rational, i.e., give all the initial endowment to either agent 1 or agent 2.) The non-existence of extreme core allocations lies on the fact that the set of all feasible allocations (which is norm closed and bounded) is not norm compact (notice that consumption sets are not norm compact). Thus, the proofs of Theorems 4.1 and 4.2 or Corollary 4.1 do not go through. The same difficulty occurs in the Araujo-Mas-Colell example [see for instance Araujo (1974, Theorem 3)]. In particular there are two agents whose preferences are norm continuous monotone convex but consumption sets are not norm compact. Hence, the above examples have violated assumption (4.7) of Corollary 4.1 and assumptions (4.3) and (4.6) of Theorems 4.1 and 4.2 respectively. Consequently, the conclusion to be drawn is that if the set of all feasible allocations is compact in a topology which is at least as strong as the topology in which preferences are continuous, then extreme α-core allocations always exist.

The intuition behind the above conclusion is quite simple. In particular, the maximal elements result (Theorem 3.3) is used to prove the existence of extreme α-core allocations (Theorem 4.1). However, if in Theorem 3.3 the preference correspondence $P : X \to 2^X$ has open lower sections in a topology which is stronger than the topology in which the set X is compact, then Theorem 3.3 fails and a fortiori Corollary 3.1 fails as well. The following example illustrates this.

Example 6.2. *Let Y be a Banach space. Denote by $\|\cdot\|$ the norm on Y. Let X be equal to the set $\{x \in Y^* : \|x\| \leq 1\}$. Notice that by the Alaoglou theorem [see Aliprantis and Burkinshaw (1985, Theorem 9.20)] X is weak* compact, and it is obviously convex and nonempty. Let $f : X \rightarrow X$ be a norm continuous mapping which does not have the fixed point property, i.e., $x \neq f(x)$ for any $x \in X$. Let $S((f(x), \frac{\|x-f(x)\|}{2})$ be an open ball in Y centered at $f(x)$ with radius $\frac{\|x-f(x)\|}{2}$. Define the preference correspondence $P : X \rightarrow 2^X$ by $P(x) = S(f(x), \frac{\|x-f(x)\|}{2}) \cap X$. It can be easily checked that P has norm open lower sections, is convex valued and irreflexive, i.e., P is of class $\mathfrak{L}$ and consequently $\mathfrak{L}$-majorized. Notice, that for all $x \in X$, $f(x) \in P(x)$, i.e., P has no maximal element in X.*

7. Remarks

Remark 7.1. A careful examination of Theorem 4.1 shows that its proof remains unaffected if the set of agents I is any countable set, provided that we assume that the aggregate initial endowment is finite.

Remark 7.2. Theorem 4.2 can be extended in a straightforward manner to coalition production economies as in the Border (1984) framework. One needs to impose in addition to balanced technology [see for instance Border (1984)], the standard assumptions on the production side of the economy, which guarantee that the set of all feasible allocations is compact in the compatible topology. The proof of Theorem 4.2 remains essentially unchanged.

Remark 7.3. Ichiichi and Schaffer (1983) have obtained core existence results, for games in characteristic function form, with a measure space of agents, and with a strategy space which is L_∞. Although our framework is entirely different than theirs, it is still of interest to know whether Theorems 4.1 and 4.2 can be extended to a measure space of agents. It seems to us that there are serious technical difficulties.

Remark 7.4. Recently the work of Kim and Richter (1986) in consumer and equilibrium theory showed that the strict preference relations which need not be transitive or complete can be replaced by weak preference relations which need not be transitive or complete. Using their methods one can obtain core existence results for weak preference relations which need not be transitive or complete.

Remark 7.5. A rather more natural definition of the core with interdependent preferences is what Aumann (1964) calls strong equilibrium. One may define a *strong equilibrium allocation* of $\mathcal{E} = \{(X_i, P_i, e_i) : i \in I\}$ as a vector $x = (x_1, \ldots, x_N) \in X$ such that

(i) x is feasible, and

(ii) it is not true that there exist $S \subset I$ and $(y_i)_{i \in S} \prod_{i \in S} X_i$ such that for all $i \in S$, $\sum_{i \in S} y_i = \sum_{i \in S} e_i$, $\sum_{i \notin S} x_i = \sum_{i \notin S} e_i$, and $(y^S, x^{I \setminus S}) \in P_i(x_1, \ldots, x_N)$.

However, Scarf (1967, p. 180) showed that even with stronger conditions than those used in Theorem 4.2, the β-core (recall that the set of strong equilibrium allocations is a subset of the β-core) may be empty and therefore the set of strong equilibrium allocations may be empty as well.

Appendix

Fan's (1962) extension of the K-K-M Lemma to Hausdorff linear topological spaces was based on the finite dimensional K-K-M result. This way of proving an infinite dimensional result by considering its trace on finite dimensions, sometimes simplifies the arguments considerably. Indeed this method of proof was adopted by Fan (1952) to extend the Kakutani fixed point to Hausdorff locally convex linear topological spaces. We now provide an alternative proof of the K-K-M-F theorem which is similar in spirit with that of Fan but makes use of the Brouwer fixed point theorem. In addition to the fact that our proof is very intuitive it turns out to be elementary. Notice that in finite dimensions the Brouwer fixed point can be used to derive the K-K-M theorem,[3] the Sperner Lemma and the Kakutani fixed point theorem. In that sense Brouwer's result may be considered as a milestone in Fixed Point Theory.

[3] A proof that the Brouwer fixed point theorem implies the K-K-M Lemma is $\mathbb{R}^\ell$ is given by Ichiichi (1981a) who attributes the argument to K. C. Border and E. Green. Although our proof is more involved than that in Ichiichi (1981) the idea is essentially the same.

Proof of Theorem 3.1. It suffices to prove that $\bigcap_{i=1}^n F(x_i) \neq \emptyset$ for every finite subset $\{x_1, \ldots, x_n\}$ of X. Suppose otherwise; i.e., $\bigcap_{i=1}^n F(x_i) = \emptyset$ for some finite subset $\{x_1, \ldots, x_n\}$ of X. Let Δ be the simplex spanned by the finite set $\{x_1, \ldots, x_n\}$. Since the topology induced on any finite dimensional subspace by the topology of Y is equivalent to the Euclidean topology, Δ is homeomorphic to a Euclidean ball [Kelley and Namioka (1963, Theory 7.3, p. 49)]. Denote by d the Euclidean metric in the finite dimensional subspace spanned by $\{x_1, \ldots, x_n\}$. Define $\psi : \Delta \to 2^\Delta$ by $\psi(x) = \{y \in \Delta : x \notin F(y)\}$. Then for each $x \in \Delta$, $\psi(x) \neq \emptyset$. Indeed at least one x_i, $(1 \leq i \leq n)$ is in $\psi(x)$, for otherwise $x \in \bigcap_{i=1}^n F(x_i)$. For each $y \in \Delta$ let $\psi^{-1}(y) = \{x \in \Delta : y \in \psi(x)\} = \Delta \setminus \{x \in \Delta : x \in F(y)\}$. Since $\{x \in \Delta : x \in F(y)\} = \Delta \cap F(y)$ is closed in Δ, the set $\psi^{-1}(y)$ is open in Δ for each $y \in \Delta$. Define $\phi : \Delta \to 2^\Delta = \operatorname{con} \psi(x)$. Then ϕ is convex and nonempty valued. Moreover, ϕ has open lower sections in Δ [Yannelis and Prabhakar (1983, Lemma 5.1)]. Nonempty valueness of ϕ implies that for every $x \in \Delta$ there is a $y \in \Delta$ such that $x \in \phi^{-1}(y)$. Hence, the collection $\{\phi^{-1}(y) : y \in \Delta\}$ is an open cover of Δ. But Δ compact implies that there exists a finite set of points $\{y_1, \ldots, y_n\}$ such that $\Delta \subseteq \bigcup_{i=1}^n \phi^{-1}(y_i)$. Define $\alpha_i : \Delta \to \mathbb{R}_+$ by $\alpha_i(x) = d(x, \Delta \setminus \phi^{-1}(y_i))$, $1 \leq i \leq n$. Set $g_i(x) = (\alpha_i(x))/(\sum_{j=1}^n \alpha_j(x))$ for all $x \in \Delta$, $1 \leq i \leq n$. Then, $g_i(x) = 0$ for $x \notin \phi^{-1}(y_i)$, $0 \leq g_i(x) \leq 1$ and $\sum_{i=1}^n g_i(x) = 1$ for all $x \in \Delta$. Define $f : \Delta \to \Delta$ by $f(x) = \sum_{i=1}^n g_i(x) y_i$. Clearly f is continuous and for each i such that $g_i(x) \neq 0$, $x \in \phi^{-1}(y_i)$ or $y_i \in \phi(x)$. Hence, $f(x)$ is a convex combination of points y_i in the convex set $\phi(x)$ and so $f(x) \in \phi(x)$ for all $x \in \Delta$. By Brouwer's fixed point theorem there exists $x^* \in \Delta$ such that $x^* = f(x^*) \in \phi(x^*) = \operatorname{con} \psi(x^*)$. But $x^* \in \operatorname{con} \psi(x^*)$ implies that there exist points $y_1, \ldots, y_m$ in Δ and real numbers $a_1, \ldots, a_m$, $a_j \geq 0$, $(1 \leq j \leq m)$, $\sum_{j=1}^m a_j = 1$ such that $x^* = \sum_{j=1}^m a_j y_j$ and $y_j \in \psi(x^*)$ for all j. But $y_j \in \psi(x^*)$ implies that $x^* \notin F(y_j)$ for all j, a contradiction to assumption (i). Therefore $\bigcap_{i=1}^n F(x_i) \neq \emptyset$, and this completes the proof.

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