Volume 6, Number Proof, 2021, 1–

CONVERGENCE AND STABILITY OF WALRASIAN EQUILIBRIUM UNDER ASYMMETRIC INFORMATION

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Dedicated to our long time friend and mentor Ali Khan for his 70th birthday.

Abstract. We study the evolution of Walrasian expectations equilibrium in a sequence of asymmetric information economies. Equilibrium allocations generate additional information that changes the information structures over time. Agents are bounded rational in the sense that a small error within maximization is allowed. We address the following: (i) given a sequence of equilibrium in each period, there is a subsequence that converges to an equilibrium for the limit full information economy, (ii) any equilibrium in the limit full information economy can be reached by a sequence of approximate equilibria.

1. Introduction

The objective of this paper is to provide a dynamic framework to the Walrasian equilibrium notion introduced in [20].

Specifically, we consider a repeated asymmetric information economy where agents from period to period refine their private information by observing the equilibrium outcome and they use this new information in subsequent periods. This repetition leads to a Walrasian equilibrium in the limit full information economy where there might be no additional information to be learned. Agents can reach the best possible Walrasian equilibrium outcome which could even be in the complete information economy (i.e., the Arrow-Debreu state contingent model). It is important to notice that our Walrasian equilibrium notion from period to period is incentive compatible and the limiting equilibrium is also incentive compatible. This should be contrasted with the free disposal equilibrium notion of [21]. In particular, the Radner notion not only it is different from ours but, as shown in [9], it is not incentive compatible (see the discussion of this issue in [20]).

In addition to the above result which shows that, by repetition the asymmetric information, Walrasian equilibrium allocations will converge to a Walrasian equilibrium allocation in the limit information economy, we prove the converse. Specifically, given a Walrasian equilibrim in the limit full information economy one can construct a sequence of approximate Walrasian equilibrium allocations and prices that converge to the limit information Walrasian equilibrium. This can be viewed as a stability result.

2020 *Mathematics Subject Classification.* 91B24, 91B50 .

Key words and phrases. Learning, Walrasian expectations equilibrium, asymmetric information, Stability.

Although the results of this paper are new, the main idea goes back to the works of [15] and [14]. They showed convergence results for non-cooperative Bayesian Nash equilibrium. [16] and [22] established the convergence of private core in myopic and non-myopic setting respectively. However, in those papers game theoretic solution concepts were considered which are free of prices. The introduction of prices complicates the analysis considerably as the use of the weak star topology on the price space plays an important role. It should be also noted that in order to make sure that bilinear form is also jointly continuous new arguments are needed that are far from trivial.

The paper proceeds as follows. In the next section some notation and mathematical preliminaries are introduced. In section 3 the model and the main definition are presented. Section 4 discusses the dynamic and the idea of learning. Section 5 formally introduces the limit full information economy. Section 6 contains two examples that examine the learning process we consider in this paper. Section 7 provides conditions under which a sequence of Walrasian equilibrium allocations converges to an equilibrium in the limit full information economy. Finally section 8 shows that any Walrasian equilibrium in the limit full information economy can be reached as the limit of an approximate sequence of Walrasian equilibria.

2. Notation and mathematical preliminaries

$\mathbb{R}^n$ denotes the n-fold Cartesian product of the set of real numbers $\mathbb{R}$.
2^A denotes the set of all subsets of the set A.
$\emptyset$ denotes the empty set.
$\setminus$ denotes the set of theoretic subtraction.
If $A \subseteq Y$, where Y is a Banach space, clA denotes the norm closure of A. An orderd vector space Y is a real vector space with an order relation $\geq$ that is compatible with the algebraic structure of Y in the sense that it satisfies the following two properties: (1) $x \geq y \implies x + z \geq y + z$ for each $z \in Y$; and (2) $x \geq y \implies \alpha x \geq \alpha y$ for each $\alpha \geq 0$. In an ordered vector space Y the set $\{x \in Y : x \geq 0\}$ is called positive cone of Y and is denoted by Y_+.
If Y is a linear topological space, its dual is the space Y^* of all continuous linear functionals on Y, and if $p \in Y^*$ and $x \in Y$ the value of p at x is denoted by $p \cdot x \in \mathbb{R}$.
Let X and Y be two sets. The *graph* of the correspondence $\phi : X \to 2^Y$ is denoted by $G_\phi = \{(x, y) \in X \times Y : y \in \phi(x)\}$. Let $(\Omega, \mathcal{F}, \mu)$ be a complete, finite measure space and Y be a separable Banach space. The correspondence $\phi : \Omega \to 2^Y$ is said be have a *measurable graph* if $G_\phi \in \mathcal{F} \otimes \mathcal{B}(Y)$, where $\mathcal{B}(Y)$ denotes the Borel σ-field on Y and $\otimes$ denotes the product σ-field.
Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and Y be a Banach space. Following [7], the function $f : \Omega \to Y$ is called *simple* if there exist $y_1, y_2, \ldots, y_n$ in Y and $E_1, E_2, \ldots, E_n$ in $\mathcal{F}$ such that

$$f = \sum_{i=1}^{n} y_i \chi_{E_i},$$

where $\chi_{E_i}(\omega) = 1$ if $\omega \in E_i$ and $\chi_{E_i}(\omega) = 0$ if $\omega \notin E_i$. A function $f : \Omega \to Y$ is called *$\mathcal{F}$-measurable* if there exists a sequence of simple functions $f_n : \Omega \to Y$ such that

$$\lim_{n\to+\infty} \|f_n(\omega) - f(\omega)\| = 0 \text{ for } \mu - a.e.\omega.$$

A function $f : \Omega \to Y$ is called *weakly measurable* if for each $y^* \in Y^*$ the numerical function $y^* \cdot f$ is measurable.

Pettis's Measurability Theorem: A function $f : \Omega \to Y$ is measurable if and only if

(*i*) f is μ-essentially separable valued, i.e., there exists $E \in \mathcal{F}$ with $\mu(E) = 0$ and such that $f(\Omega \setminus E)$ is a (norm) separable subset of Y, and

(*ii*) f is weakly measurable.

An $\mathcal{F}$-measurable function $f : \Omega \to Y$ is said to be *Bochner integrable* if there exists a sequence of simple function $\{f_n : n = 1, 2, \ldots\}$ such that

$$\lim_{n\to+\infty} \int_{\omega\in\Omega} \|f_n(\omega) - f(\omega)\| d\mu(\omega) = 0.$$

In this case, for each $E \in \mathcal{F}$, we define the *integral* of f, denoted by $\int_E f(\omega)d\mu(\omega)$, as

$$\int_E f(\omega)d\mu(\omega) = \lim_{n\to+\infty} \int_E f_n(\omega)d\mu(\omega).$$

It is known (see Theorem 2 p. 45 in [7]) that if $f : \Omega \to Y$ is an $\mathcal{F}$-measurable function, then f is Bochner integrable if and only if

$$\int_\Omega \|f(\omega)\|d\mu(\omega) < \infty.$$

For $\leq p < \infty$, $L_p(\mu, Y)$ stands for the space of equivalence classes of Bochner integrable function $f : \Omega \to Y$ such that

$$\|f\|_p = \left(\int_\Omega \|f\|_Y^p d\mu\right)^{1/p} < \infty.$$

Normed by the functional $\| \cdot \|$ above, $L_p(\mu, Y)$ becomes a Banach space (see [7] p. 50). $L_\infty(\mu, Y)$ stands for the space of equivalence classes of essentially bounded Bochner integrable function $f : \Omega \to Y$. Normed by the functional $\| \cdot \|_\infty$ defined by

$$\|f\|_\infty = \text{ess sup}\|f\|_Y,$$

$L_\infty(\mu, Y)$ is a Banach space. The simbol $L_p(\mu)$ with $1 \leq p \leq \infty$ denoted the space $L_p(\mu, Y)$ with $Y = \mathbb{R}$. A Banach space Y has the *Radon-Nikodym property (RNP)* with respect to the measure space $(\Omega, \mathcal{F}, \mu)$ if for each ν-continuous vector measure $G : \mathcal{F} \to Y$ of bounded variation, there exists some $g \in L_1(\mu, Y)$ such that for all $E \in \mathcal{F}$,

$$G(E) = \int_E g(\omega)d\nu(\omega).$$

It is known that $(L_p(\mu, Y))^* = L_q(\mu, Y^*)$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $1 \leq p < \infty$, if and only if Y^*, the norm dual of Y, has the RNP with respect to $(\Omega, \mathcal{F}, \mu)$ (see Theorem 1 p. 98 in [7]). In particular, $(L_1(\mu, Y))^* = L_\infty(\mu, Y^*)$ iff Y^* has the RNP.

A correspondence $\phi : \Omega \to 2^Y$ is said to be *integrably bounded* if there exists a

function $h \in L_1(\mu)$ such that $\sup\{\|y\| : y \in \phi(\omega)\} \leq h(\omega)$ for $\mu - a.e.\omega$.
Below we state some basic theorems that will be useful in this paper.

Aumann measurable selection theorem: Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite measure space, Y be a complete, separable metric space and $\phi : \Omega \to 2^Y$ be a nonempty valued correspondence with a measurable graph. Then, ϕ admits a measurable selection, i.e., there exists a measurable function $f : \Omega \to Y$ such that $f(\omega) \in \phi(\omega)$ μ-a.e..

Eberlein-Smulian theorem: In the weak topology on a normed space, compactness and sequential compactness coincide. That is, a subset A of a normed space X is weakly compact if and only if every sequence in A has a weakly convergent subsequence in A.

James' theorem: A nonempty weakly closed bounded subset of a Banach space is weakly compact if and only if every continuous linear functional attains a maximum on the set.

Alaoglu's theorem: The closed unit ball of the norm dual of a normed space is weak* compact.

Dominated Convergence Theorem: If $f_n : \Omega \to Y$ $(n = 1, 2, \ldots)$ is a sequence of Bochner integrable functions such that for μ-a.e. ω,

$$\lim_{n \to +\infty} f_n(\omega) = f(\omega) \text{ and } \|f_n(\omega)\| \leq g(\omega),$$

where $g : \Omega \to \mathbb{R}$ is an integrable function, then f is Bochner integrable and

$$\lim_{n \to +\infty} \int_{\omega \in \Omega} \|f_n(\omega) - f(\omega)\| d\mu(\omega) = 0.$$

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\mathcal{F}_1$, $\mathcal{F}_2$ be two σ-fields on Ω. $\mathcal{F}_1$ is *finer* than $\mathcal{F}_2$ and $\mathcal{F}_2$ is *coarser* than $\mathcal{F}_1$ if $\mathcal{F}_2 \subseteq \mathcal{F}_1$. The *join* of $\mathcal{F}_1$ and $\mathcal{F}_2$, denoted by $\mathcal{F}_1 \vee \mathcal{F}_2$, is the smallest σ-field containing both $\mathcal{F}_1$ and $\mathcal{F}_2$. The *meet* of $\mathcal{F}_1$ and $\mathcal{F}_2$, denoted by $\mathcal{F}_1 \wedge \mathcal{F}_2$, is the largest σ-field contained in both $\mathcal{F}_1$ and $\mathcal{F}_2$.
Let B be a sub-σ-field of $\mathcal{F}$ and $f \in L_1(\Omega, Y)$. An element g of $L_1(\mu, Y)$ is called *conditional expectation of f relative to B* if g is B-measurable and

$$\int_E g d\mu = \int_E f d\mu \text{ for all } E \in B.$$

In this case g is denoted by $E[f|B]$ (see Definition 1 p. 121 in [7]). Moreover, $E[f|B]$ exists for every $f \in L_1(\mu, Y)$ because if $f \in L_p(\mu, Y)$, with $1 \leq p < \infty$, then $\|E[f|B]\|_p \leq \|f\|_p$. Consequently $E[\cdot|B]$ is a linear contractive projection on $L_p(\mu, Y)$, $1 \leq p < \infty$ (see Theorem 4 p. 123 in [7]). Notice that if f is B-measurable, then $E[f|B] = f$. Moreover, if $B_1 \subseteq B_2 \subseteq \mathcal{F}$, then $E[E[f|B_1]|B_2] = E[f|B_1]$ and $E[E[f|B_2]|B_1] = E[f|B_1]$, that is the smaller σ-field wins. This property is known as the *tower property.*
A family of sub-σ-fields $(B_t, t \in T)$ is defined to be a filtration if $B_t \subseteq B_s$ whenever $t \leq s$ in T. A net $(f_t, t \in T)$ in $L_p(\mu, Y)$, with $1 \leq p < \infty$, over the same directed

set T, is a *martingale* if $E[f_t|B_s] = f_s$ for all $t \geq s$ (see Definition 5 p. 123 in [7]).

Martingale convergence theorem: A martingale $(f_t, B_t, t \in T)$ in $L_p(\mu, Y)$, with $1 \leq p < \infty$, is convergent in $L_p(\mu, Y)$-norm if and only if there exists $f \in L_p(\mu, Y)$ such that $f_t = E[f|B_t]$ for all $t \in T$ (see Corollary 2 p. 126 in [7]). Furtheremore, if a martingale $(f_t, B_t, t \in T)$ converges in the $L_1(\mu, Y)$-norm to $f \in L_1(\mu, Y)$, then it also converges almost everywhere, i.e., $\lim_t f_t(\omega) = f(\omega)$ for μ-a.e.ω (see Theorem 8 p.129 in [7]).

3. Asymmetric information economy

Let $(\Omega, \mathcal{F}, \mu)$ be a complete, finite, separable measure space, where Ω is the set of states of nature and $\mathcal{F}$ is the σ-field of all the events. Let Y be an ordered separable Banach space whose dual Y^* has the RNP and whose positive cone Y_+ has non-empty interior. Let I be a finite set of agents.
Each agent $i \in I$ is characterized by

(1) $X_i : \Omega \to 2^{Y_+}$, where $X_i(\omega)$ is the consumption set at state ω.
(2) $e(i, \cdot) : \Omega \to Y_+$ is the initial endowment.
(3) $\mathcal{F}_i$ is a sub-σ-field of $\mathcal{F}$ denoting the private information.
(4) $P_i : L_{X_i} \to 2^{L_{X_i}}$ is the (strict) preference correspondence where L_{X_i} is the set of integrable selections of X_i, that is $L_{X_i} = \{x \in L_1(\mu, Y_+) : x(\omega) \in X_i(\omega)\ \mu.a.e.\omega\}$. We call L_{X_i} the set of *random commodity bundles* of agent i. For each $i \in I$ and each $x \in L_{X_i}$, $x \notin P_i(x)$ then $P_i(x)$ is interpreted as the set of random commodity bundles strictly preferred by i to x.

We denote by $L_{X_i}^{\mathcal{F}_i}$ is the set of integrable selections of X_i compatible with the private information $\mathcal{F}_i$, that is

$$L_{X_i}^{\mathcal{F}_i} := \{x \in L_{X_i} : x(\cdot) \,\text{is}\, \mathcal{F}_i - \text{measurable}\}.$$

An asymmetric information economy $\mathcal{E}$ is the collection

$$\mathcal{E} = \{(\Omega, \mathcal{F}, \mu); Y; (X_i, e(i, \cdot), \mathcal{F}_i, P_i)_{i \in I}\}.$$

An *allocation* x for the economy $\mathcal{E}$ is a function $x : I \times \Omega \to Y_+$ such that $x(i, \cdot) \in L_{X_i}^{\mathcal{F}_i}$ for all $i \in I$. We assume that for each $i \in I$, $e(i, \cdot) \in L_{X_i}^{\mathcal{F}_i}$. Denoted by $L_X = \prod_{i \in I} L_{X_i}^{\mathcal{F}_i}$, an allocation x can be viewed as an element of L_X. An allocation x is said to be *feasible* if

$$\sum_{i \in I} x(i, \omega) = \sum_{i \in I} e(i, \omega) \text{ for } \mu - a.e.\omega.$$

We denote by $L_P := \{p \in L_\infty(\mu, Y^*) : ||p||_\infty \leq 1\}$ the *price space*. Remember that $L_\infty(\mu, Y^*) = (L_1(\mu, Y))^*$ whenever Y^* (the norm dual of Y) has the Radom-Nikodym property. Notice that prices are not required to be non negative because, as shown by [8], a Walrasian expectations equilibrium with exact feasibility may not exist with positive prices (see also [3]). The couple $< L_1(\mu, Y), L_\infty(\mu, Y^*) >$ is called *commodity-price duality*. Given a price system $p \in L_P$, the value of a random commodity bundle $x \in L_{X_i}$ is denoted by $p \cdot x$, where $p \cdot x = \int_{\omega \in \Omega} p(\omega) \cdot x(\omega) d\mu(\omega)$.

Assumption 3.1. *We assume that any sub-σ-field $\mathcal{G}$ of $\mathcal{F}$ (i.e. $\mathcal{G} \subseteq \mathcal{F}$) is generated by a finite or countable partition of Ω, denoted by $\Pi_{\mathcal{G}}$. Moreover, for each $\omega \in \Omega$, we denote by $\Pi_{\mathcal{G}}(\omega)$ the unique element of $\Pi_{\mathcal{G}}$ containing ω.*

Definition 3.2. An allocation $x \in L_X$ is a Walrasian expectations equilibrium (WEE) allocation if there exists a price $p \in L_P$ such that $p \neq 0$ and

(i) $\sum_{i\in I} x(i,\omega) = \sum_{i\in I} e(i,\omega)$ for $\mu - a.e.\omega$
(ii) $p \cdot x(i,\cdot) \leq p \cdot e(i,\cdot)$ for all $i \in I$
(iii) $y_i \in P_i(x(i,\cdot)) \Rightarrow p \cdot y_i > p \cdot e(i,\cdot)$.

The pair (x,p) is called Walrasian expectations equilibrium (WEE) and we denote by $W(\mathcal{E})$ the set of all Walrasian expectations equilibrium (WEE) allocations.

Remark 3.3. Observe that if the $\mathcal{F}_i$-measurability assumption is omitted, then the definition of the Walrasian expectations equilibrium reduces to the *state contingent model* of [4]. Recall that there is no asymmetric information in the state contingent model of [4], and therefore no incentives to misreport the information arise. This state contingent model has a two period interpretation, i.e., in the first period agents make contracts contingent on the realized state of nature in period two. For this model to make sense one has to assume that once the state of nature is realized an exogenous enforcer (Government or Court) makes sure that all deliveries are executed. The lack of an exogenous enforcer, may lead people to renege ex-post. The assumption of the private information measurability of allocations in the definition 3.2 not only introduces asymmetric information on the [4] state contingent model, since agents make decisions based on their own private information, but also results in allocations that they are in the private core ([23]) and therefore ex-post coalitional incentive compatible, i.e., these contracts are private efficient, incentive compatible and hence executable, (see [[20], section 4], for a complete discussion of those issues). Obviously, the WEE is an ex-ante concept as agents maximize before the state of nature is realized and there is no signaling.

4. Learning process

Let $T = \{1, 2, \ldots\}$ be the discrete set of time horizon. For each $t \in T$, let $\mathcal{E}^t$ be an asymmetric information economy at time t with initial endowment e^t. At beginning $t = 1$, the information of each individual i is given by σ-field generated by his random initial endowment and his random strategy set, that is $\mathcal{F}_i^1 = \sigma(e^1(i,\cdot), X_i)$. This means that $e^1(i,\cdot)$ is $\mathcal{F}_i^1$-measurable and the graph of X_i, i.e. $Gr_{X_i} = \{(\omega, x) \in \Omega \times Y_+ : x \in X_i(\omega)\}$ is an element of $\mathcal{F}_i^1 \otimes \mathcal{B}(Y)$, where $\mathcal{B}(Y)$ denotes the Borel σ-algebra on Y and $\otimes$ denotes the product σ-algebra. At time t, there is additional information available to agent i acquired by observing past WEE equilibria. We can express each agent's private information recursively by

$$\mathcal{F}_i^{t+1} = \mathcal{F}_i^t \vee \sigma(x^t, p^t),$$

where $\sigma(x^t, p^t)$ is the information that the WEE equilibrium generates at period t, i.e. the smallest σ-field for which the equilibrium allocation and the equilibrium price at period t are measurable, and $\mathcal{F}_i^t \vee \sigma(x^t, p^t)$ is the join (i.e., the smallest

σ-field containing $\mathcal{F}_i^t$ and $\sigma(x^t, p^t)$). Thus, for all $i \in I$,

$$\begin{array}{lllll} \text{if } t=1 & \text{then,} & \mathcal{F}_i^1 & = & \sigma(e^1(i,\cdot), X_i) \\ \text{if } t=2 & \text{then,} & \mathcal{F}_i^2 & = & \mathcal{F}_i^1 \vee \sigma(x^1, p^1) \\ \text{if } t=3 & \text{then,} & \mathcal{F}_i^3 & = & \mathcal{F}_i^2 \vee \sigma(x^2, p^2) \\ & & & = & \mathcal{F}_i^1 \vee \sigma(x^1, p^1) \vee \sigma(x^2, p^2) \\ & & & \vdots & \end{array}$$

Therefore, the private information of agent i at time t is given by

$$\begin{aligned} \mathcal{F}_i^t &= \sigma(e^1(i,\cdot), X_i, (x^{t-1}, p^{t-1}), (x^{t-2}, p^{t-2}) \ldots, (x^{t-(t-1)}, p^{t-(t-1)})) = \\ &= \mathcal{F}_i^1 \vee \left(\bigvee_{k=1}^{t-1} \sigma(x^k, p^k) \right), \end{aligned}$$

where $(x^{t-1}, p^{t-1}), (x^{t-2}, p^{t-2}) \ldots, (x^{t-(t-1)}, p^{t-(t-1)})$ are past WEE equilibria. Clearly, for each agent i and each time period t we have

$$\mathcal{F}_i^t \subseteq \mathcal{F}_i^{t+1} \subseteq \mathcal{F}_i^{t+2} \subseteq \cdots$$

Then the family of σ-field $\{\mathcal{F}_i^t\}_{t\in T}$ constitutes a filtration being $\mathcal{F}_i^t \subseteq \mathcal{F}_i^s$ whenever $t \leq s$.

The above expression represents a *learning process* for agent i and it generates a sequence of asymmetric information exchange economies

$$\mathcal{E}^t = \{(\Omega, \mathcal{F}, \mu); Y; (X_i, e^t(i,\cdot), \mathcal{F}_i^t, P_i) : i \in I\},$$

where $e^t(i,\cdot) \in L_{X_i}^{\mathcal{F}_i^t} = \{x \in L_{X_i} : x(\cdot) \,\text{is}\, \mathcal{F}_i^t - \text{measurable}\}$ and $L_X^t = \prod_{i\in I} L_{X_i}^{\mathcal{F}_i^t}$ denotes the set of allocations in $\mathcal{E}^t$.

Our framework is not a dynamic model because we repeat the economy over time allowing each agent to refine his own private information. However, note that agents' initial endowment varies over time and satisfies the private information measurablity condition in each period. This includes as a particular case the situation in which the initial endowment remains unchanged, i.e. $e^t(i,\cdot) = e^1(i,\cdot)$ for all $i \in I$ and all $t \in T$, as well as the case in which the initial endowment at time t is the equilibrium allocation at the previous period x^{t-1}. On the other hand, the (strict) preference relation P_i as well as the random consumption set X_i do not change.

Notice also that at time of contracting agents do not consider the entire horizon, but only the current period. Obviously, since the private information set of each agent becomes finer over time, the information gathered at a given time t will affect the outcome in periods $t+1, t+2, \ldots$. Furthermore notice that for each agent $i \in I$ and each time $t \in T$,

$$\mathcal{F}_i^t = \mathcal{F}_i^1 \vee \left(\bigvee_{k=1}^{t-1} \sigma(x^k, p^k) \right),$$

that is for all i and for all t, $\mathcal{F}_i^t = \mathcal{F}_i^1 \vee G^t$, where for all $t \in T$, $G^t = \bigvee_{k=1}^{t-1} \sigma(x^k, p^k)$ is the information generated by all past WEE equilibria until time t.

Thus, the information of each player i, after t periods, has two components. The former is given by his initial private information $\mathcal{F}_i^1$ which contributes to the asymmetric part and the latter is given by the information generated by the past period WEE equilibria which creates the common part of information since all agents see the same outcome in each period. In the limit economy, the private information of each agent i, $\mathcal{F}_i^*$, is the initial private information information of i, $\mathcal{F}_i^1$, pooled the information generated by all past period WEE equilibria, that is

$$\mathcal{F}_i^* = \mathcal{F}_i^1 \vee \left(\bigvee_{k=1}^{\infty} \sigma(x^k, p^k) \right) = \mathcal{F}_i^1 \vee G^{\infty}.$$

Notice that for each agent i, the limit information $\mathcal{F}_i^*$ is also given by the pooled information of i over the entire horizon, i.e., $\mathcal{F}_i^* = \bigvee_{t \in T} \mathcal{F}_i^t$.

5. Limit full information economy

We define the *limit full information economy* as follows:

$$\mathcal{E}^* = \{(\Omega, \mathcal{F}, \mu); Y; \, (X_i, e^*(i, \cdot), \mathcal{F}_i^*, P_i) : i \in I\}$$

where, $(\Omega, \mathcal{F}, \mu)$, Y, X_i and P_i are defined as in section 3, whereas the information of each agent i is given by the pooled information of i over the entire horizon, i.e.,

$$\mathcal{F}_i^* = \bigvee_{t \in T} \mathcal{F}_i^t \qquad \text{for all } i \in I.$$

$\mathcal{F}_i^*$ is all the information that agent i can learn over the entire horizon. For this reason we call $\mathcal{E}^*$ the *limit full information economy*.
For each $i \in I$, consistently with section 3, we denote by

$$L_{X_i}^{\mathcal{F}_i^*} = \{x_i \in L_{X_i} : \quad x_i(\cdot) \text{ is } \mathcal{F}_i^*\text{−measurable}\},$$

the set of random consumption bundles of i in $\mathcal{E}^*$ and we assume that $e^*(i, \cdot) \in L_{X_i}^{\mathcal{F}_i^*}$. Let $L_X^* = \prod_{i \in I} L_{X_i}^{\mathcal{F}_i^*}$. The notion of Walrasian expectations equilibrium in the limit full information economy is given by adapting suitably definition 3.2.

In the limit, agents may still have different and partial information. This is the case in which, for example, in each period the information generated by the WEE equilibria is coarser than the information of each individual, that is $\sigma(x^t, p^t) \subseteq \mathcal{F}_i^t$ for any $i \in I$ and any $t \in T$. In such a situation, in each period agents learn nothing because their information do not change and even in the limit economy the information of each agent is the initial one. Therefore, it might be the case that in each period the same economy is replicated and consequently the equilibria remain the same. This makes the convergence results trivial, because a constant sequence of identical equilibria converges to itself and viceversa. On the other hand, it might also be the case that in the limit no informational asymmetry arises and all agents have the same information, although incomplete. Once, in the limit all agents are fully informed, i.e., $\mathcal{F}_i^* = \mathcal{F}$ for all $i \in I$, the resulting equilibrium coincides with a Walrasian equilibrium in the state contingent model of Arrow-Debreu (see [6]). [19] introduce a *non-trivial learning condition*, according to which the pooled information generated by all past equilibrium allocations is at least as fine as agents' initial information, i.e., $\mathcal{F}_i^1 \subseteq G^\infty = \bigvee_{k=1}^\infty \sigma(x^t, p^t)$ for all $i \in I$. This implies that unless already at beginning (i.e., $t = 1$) all agents are equally informed, at least one agent learns something in some period, but it does not mean that all agents learn something. Thus, under the non-trivial learning condition, the information $\mathcal{F}_i^t$ of each agent i converges to G^∞, which is the information generated by all the past equilibria.[1] Therefore, as time goes on, the common part of information becomes prevalent and in the limit all agents will have the same information given by G^∞. However, it still might be the case that in the limit economy agents, although equally informed, are partially informed because $\mathcal{F}_i^* = G^\infty \subset \mathcal{F}$ for all $i \in I$.

It can be proved that a WEE allocation is (ex-ante) efficient according to the notion due to [23] for which agents consider only their own private information within any coalition. Then, in the limit full information economy in which, under the non-trivial learning condition, agents are symmetrically informed the no trade result of [18] applies: an (ex-ante) efficient allocation cannot be improved upon even after agents receive signals and update their knowledge about the state of nature; no further trade arises (see Theorem 1 in [17]).[2]

In the next section we illustrate two asymmetric information economies where, in the first example agents immediately become fully informed whereas, in the second example agents will be never able to discern each state of nature.

6. Examples

Here we present two examples that examines the learning process we consider in this paper.

Example 6.1. Consider an asymmetric information economy with three states of nature, i.e. $\Omega = \{\omega_1, \omega_2, \omega_3\}$, three agents, i.e. $I = \{1, 2, 3\}$, and one commodity, i.e. $X_i(\omega) = \mathbb{R}_+$ for any $i \in I$ and any $\omega \in \Omega$. All agents have the same utility function $u(\omega, x) = \sqrt{x}$ for any $\omega \in \Omega$ and the same prior (i.e. $\mu(\{\omega\}) = \frac{1}{3}$). Given a random bundle $x : \Omega \to \mathbb{R}_+$, the ex-ante expected utility function is given by $U(x) = \sum_{\omega \in \Omega} \frac{1}{3}\sqrt{x(\omega)}$. At time $t = 1$ agents' primitives are given as follows:

$$(e^1(1,\omega_1), e^1(1,\omega_2), e^1(1,\omega_3)) = (5,5,0);\ \mathcal{F}_1^1 = \sigma(\Pi_1^1) \text{ where } \Pi_1^1 = \{\{\omega_1,\omega_2\},\{\omega_3\}\};$$
$$(e^1(2,\omega_1), e^1(2,\omega_2), e^1(2,\omega_3)) = (5,0,5);\ \mathcal{F}_2^1 = \sigma(\Pi_2^1) \text{ where } \Pi_2^1 = \{\{\omega_1,\omega_3\},\{\omega_2\}\};$$
$$(e^1(3,\omega_1), e^1(3,\omega_2), e^1(3,\omega_3)) = (0,0,0);\ \mathcal{F}_3^1 = \sigma(\Pi_3^1) \text{ where } \Pi_3^1 = \{\{\omega_1\},\{\omega_2\},\{\omega_3\}\}.$$

Notice that for each $i \in I$; $\mathcal{F}_i^1 = \sigma(e^1(i,\cdot), X_i, u_i)$. It can be shown that in equilibrium there is no trade, meaning that the unique Walrasian expectations equilibrium allocation at time $t = 1$ is the initial endowment. At the next time period $t = 2$, agents observe the equilibrium allocation and acquire the new information $\sigma(x^1, p^1)$, so that

$$\mathcal{F}_i^2 = \mathcal{F}_i^1 \vee \sigma(x^1, p^1).$$

[1]Actually, the non-trivial learning condition is also necessary for $\mathcal{F}_i^* = G^\infty$ (see [19]).

[2]We thank the Referee for having pointed out this remark to us.

Since $\sigma(x^1) = \bigvee_{i\in I} \sigma(x^1(i,\cdot)) = 2^\Omega$, from period $t = 2$ on, agents are fully informed as they are able to discern each state of nature, i.e. $\Pi_i^t = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}$ for any $t \geq 2$. In this example agents learn in one shot the true state of nature and become completely informed. We remark that nothing changes if agent 3 has the coarsest possibile information at time $t = 1$, i.e. $\Pi_3^1 = \{\{\omega_1, \omega_2, \omega_3\}\}$.

The next example shows that, contrary to the previous example, it might be the case that the true state of nature is never revealed and, even in the limit economy, agents are partially informed.

Example 6.2. Consider an asymmetric information economy with three states of nature, i.e. $\Omega = \{\omega_1, \omega_2, \omega_3\}$, three agents, i.e. $I = \{1, 2, 3\}$, and two commodities, i.e. $X_i(\omega) = \mathbb{R}_+^2$ for any $i \in I$ and any $\omega \in \Omega$. All agents have the same utility function $u(\omega, x, y) = \sqrt{xy}$ for any $\omega \in \Omega$ and the same prior (i.e. $\mu(\{\omega\}) = \frac{1}{3}$). The ex-ante expected utility function is $U(x, y) = \sum_{\omega\in\Omega} \frac{1}{3}\sqrt{x(\omega)y(\omega)}$. At time $t = 1$ agents' primitives are given as follows:

$$\begin{aligned}
(e^1(1,\omega_1), e^1(1,\omega_2), e^1(1,\omega_3)) &= ((1,3),(2,2),(1,3));\ \mathcal{F}_1^1 = \sigma(\{\{\omega_1,\omega_3\},\{\omega_2\}\});\\
(e^1(2,\omega_1), e^1(2,\omega_2), e^1(2,\omega_3)) &= ((3,1),(2,2),(3,1));\ \mathcal{F}_2^1 = \sigma(\{\{\omega_1,\omega_3\},\{\omega_2\}\});\\
(e^1(3,\omega_1), e^1(3,\omega_2), e^1(3,\omega_3)) &= ((2,2),(2,2),(2,2));\ \mathcal{F}_3^1 = \sigma(\{\{\omega_1,\omega_2,\omega_3\}\}).
\end{aligned}$$

Notice that for each $i \in I$; $\mathcal{F}_i^1 = \sigma(e^1(i,\cdot), X_i, u_i)$. It can be shown that the pair $((x, y); (p, q))$ where $(x_i(\omega), y_i(\omega)) = (2, 2)$ and $(p(\omega), q(\omega)) = (p, p)$ for all $i \in I$ and all $\omega \in \Omega$ is a WEE, so that $\sigma((x, y), (p, q)) = \{\emptyset, \Omega\} \subseteq \mathcal{F}_i^1$ for all $i \in I$. Therefore, at time $t = 2$, agents learn nothing since $\mathcal{F}_i^2 = \mathcal{F}_i^1$ for all $i \in I$. Thus, in any subsequent period $t \geq 2$, by considering $e^t(i,\cdot) = e^1(i,\cdot)$ and iterating the same WEE equilibrium, we get that even in the limit economy $\mathcal{E}^*$ agents will be partially and asymmetrically informed, because $\mathcal{F}_i^* = \bigvee_{t\in T} \mathcal{F}_i^t = \mathcal{F}_i^1 \subset \mathcal{F}$. Notice that in this example the *non-trivial learning condition* does not hold.

7. Convergence of equilibria

In this section we provide an interesting convergence property of the WEE. We will prove that any sequence of WEE has a subsequence which converges to a WEE in the limit full information economy. This result suggests that, as the learning process unravels near the limit, the additional information acquired does not change drastically equilibrium outcomes. In order to appreciate the value of the next theorem, one should contemplate the consequence of its failure: it would mean that small perturbation of the information structure would have drastic effects on the equilibrium outcome, which would have implications on the robustness of the equilibrium concept.
The following assumptions will be needed.

Assumption 7.1. *For each $i \in I$, $X_i : \Omega \to 2^Y$ is a nonempty, convex, norm compact valued and integrably bounded correspondence with $\mathcal{F}_i^1$-measurable graph, that is $G_{X_i} \in \mathcal{F}_i^1 \otimes \mathcal{B}(Y)$.*

Assumption 7.2. *(i) $P_i(x) = \{y \in L_{X_i} : U_i(y) > U_i(x)\}$ for any $x \in L_{X_i}$, where $U_i : L_{X_i} \to \mathbb{R}$ is concave and weakly continuous (i.e. if x^t converges weakly to x^* then $U_i(x^t)$ converges to $U_i(x^*)$).*

(*ii*) *For every feasible allocation x, there exists an agent $i \in I$ such that $P_i(x(i,\cdot)) \neq \emptyset$.*

Assumption 7.3. *For each $i \in I$, the initial endowment is such that*

(*i*) $e^t(i,\cdot) = E[e^*(i,\cdot)|\mathcal{F}_i^t]$ *for each $t \in T$; where*
(*ii*) *in the limit economy $\mathcal{E}^*$ the intial endowment $e^*(i,\cdot)$ is such that for all $\omega \in \Omega$ and all $q \in Y^*$, $\{z \in X_i(\omega) : q \cdot z < q \cdot e^*(i,\omega)\} \neq \emptyset$.*

Remark 7.4. Assumption 7.1 is quite standard as it ensures that L_X is nonempty, convex and weakly compact. Assumption 7.2 (i) imposes that agents' preferences are represented by an ex-ante expected utility function $U_i : L_{X_i} \to \mathbb{R}$. Assumption 7.2 ($ii$) can be interpreted as a non-satiation of preferences at feasible allocations and it guarantees that the equilibrium price is non null (see also Theorem 4 in [10] and ($A3$) in [20]). Notice that assumption 7.3 (i) implies that $(e^t(i,\cdot), \mathcal{F}_i^t, t \in T)$ is a martingale. Indeed by the tower property,[3] being $\mathcal{F}_i^t \subseteq \mathcal{F}_i^{t+1}$ for each $t \in T$, we have that $e^t(i,\cdot) = E[e^*(i,\cdot)|\mathcal{F}_i^t] = E[E[e^*(i,\cdot)|\mathcal{F}_i^{t+1}]|\mathcal{F}_i^t] = E[e^{t+1}(i,\cdot)|\mathcal{F}_i^t]$. Then, by the martingale convergence theorem $e^t(i,\cdot)$ converges to $e^*(i,\cdot)$ in $L_1(\mu, Y)$-norm and hence weakly. Therefore, $q \cdot e^t(i,\omega)$ converges to $q \cdot e^*(i,\omega)$ for all $q \in Y^*$, and hence assumption 7.3 (ii) implies that for any $q \in Y^*$ there exist $z \in X_i(\omega)$ and a subsequence $e^{t_n}(i,\omega)$ such that $q \cdot e^{t_n}(i,\omega) > q \cdot z$ for any n.

Theorem 7.5. *Let $\{\mathcal{E}^t : t \in T\}$ be a sequence of asymmetric information economies satisfying assumptions 7.1, 7.2 and 7.3. Let $x^t \in W(\mathcal{E}^t)$ then there exists a subsequence $\{(x^{t_n}) : n = 1, 2, \ldots\}$ of $\{(x^t) : t \in T\}$ that converges weakly to $x^* \in W(\mathcal{E}^*)$.*

7.1. **Proof of Theorem 7.5.** Given a sequence $(x^t)_{t\in T}$ of allocations such that for each period $t \in T$, x^t is a WEE for the asymmetric information economy $\mathcal{E}^t$, we show that there exists a subsequence x^{t_n} converging weakly to an allocation x^*, which is a WEE for the limit full information economy $\mathcal{E}^*$. The proof is split in several lemmata.

Let $Z = \{(x^t) : t \in T\}$. Note that for all $i \in I$ and for all $t \in T$, since $\mathcal{F}_i^t \subseteq \mathcal{F}_i^{t+1} \subseteq \mathcal{F}_i^*$ it follows that $L_{X_i}^{\mathcal{F}_i^t} \subseteq L_{X_i}^{\mathcal{F}_i^{t+1}} \subseteq L_{X_i}^{\mathcal{F}_i^*}$. Hence $x^t(i,\cdot) \in L_{X_i}^{\mathcal{F}_i^*}$ and $x^t \in L_X^*$ for all $i \in I$ and all $t \in T$, implying that $Z \subseteq L_X^*$.
The next lemma is known as *Diestel's theorem* and several alternative proofs can be found in the literature (see [11], Theorem 3.1 in [24] and references therein, Lemma A.3 of [14] and Lemma A.4 in [13] among others). For sake of completeness we provide its demonstration.

Lemma 7.6. *Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite measure space, Y be a separable Banach space and $\phi : \Omega \to 2^Y$ be an integrably bounded, convex, weakly compact and nonempty valued correspondence. Then, the set $L_\phi = \{x \in L_1(\mu, Y) : \, x \,\text{is}\, \mathcal{F}-\text{measurable}\,\text{and}\, x(\omega) \in \phi(\omega)\, \mu - \text{a.e.}\omega\}$ is weakly compact in $L_1(\mu, Y)$.*

Proof. First note that $(L_1(\mu, Y))^* = L_\infty(\mu, Y_{w^*}^*)$, where w^* denotes the weak* topology. In order to apply James' theorem, we consider an arbitrary element p of $L_\infty(\mu, Y_{w^*}^*)$ and we show that p attains its supremum on L_ϕ. Note that

[3] If $B_1 \subseteq B_2 \subseteq \mathcal{F}$, then $E[E[f|B_1]|B_2] = E[f|B_1]$ and $E[E[f|B_2]|B_1] = E[f|B_1]$ (see section 2).

$$\sup_{x\in L_\phi} x\cdot p = \sup_{x\in L_\phi}\int_{\omega\in\Omega}(x(\omega)\cdot p(\omega))d\mu(\omega) = \int_{\omega\in\Omega}\sup_{y\in\phi(\omega)}(y\cdot p(\omega))d\mu(\omega),$$

where the second equality follows from Lemma 1 in [5]. Define the correspondence $\theta:\Omega\to 2^Y$ by

$$\theta(\omega)=\{z\in\phi(\omega): z\cdot p(\omega)=\sup_{y\in\phi(\omega)} y\cdot p(\omega)\},$$

which is nonempty valued because $\phi:\Omega\to 2^Y$ is weakly compact valued. Define the function $F:\Omega\times Y\to[-\infty,+\infty]$ by $F(\omega,z)=z\cdot p(\omega)-\sup_{y\in\phi(\omega)} y\cdot p(\omega)$. Note that for each fixed $\omega\in\Omega$, $F(\omega,\cdot)$ is continuous and for each fixed $z\in Y$, $F(\cdot,z)$ is measurable. Hence F is a Caratheodory function and hence jointly measurable (see Proposition 3.1 p. 41 in [25]). Thus the set

$$F^{-1}(0)=\{(\omega,z)\in\Omega\times Y:\ F(\omega,z)=0\}\ \text{belongs to}\ \mathcal{F}\otimes\mathcal{B}(Y).$$

Notice that $G_\theta=G_\phi\cap F^{-1}(0)$ and, since $F^{-1}(0)\in\mathcal{F}\otimes\mathcal{B}(Y)$ and $\phi(\cdot)$ has a measurable graph, $G_\theta\in\mathcal{F}\otimes\mathcal{B}(Y)$. By Aumann measurable selection theorem, there exists a measurable selection $z:\Omega\to Y$ such that $z(\omega)\in\theta(\omega)$ $\mu-a.e.$ Since ϕ is integrably bounded, $z\in L_\phi$ and

$$\sup_{x\in L_\phi} x\cdot p=\int_{\omega\in\Omega}(z(\omega)\cdot p(\omega))d\mu(\omega)=z\cdot p.$$

Because $p\in L_\infty(\mu,Y^*_{w^*})$ has been chosen arbitrary, we can conclude that every element of $L_1(\mu,Y)^*$ attains its supremum on L_ϕ. This completes the proof. □

From Lemma 7.6 (see assumption 7.1), the sets $L_{X_i}^{\mathcal{F}_i^*}$ are weakly compact and by Tychonoff Product Theorem so is $L_X^*=\prod_{i\in I}L_{X_i}^{\mathcal{F}_i^*}$. Hence the weak closure of Z, $w-clZ$ is weakly compact and by Eberlein-Smulian theorem, it follows that $w-clZ$ is weakly sequentially compact. Then, there exists a subsequence x^{t_n} converging weakly to $x^*\in w-clZ\subseteq L_X^*$. We need to show that x^* is a WEE for $\mathcal{E}^*$, i.e. $x^*\in W(\mathcal{E}^*)$.

Lemma 7.7. *For each $i\in I$, if $x^t(i,\cdot)$ converges weakly to $x^*(i,\cdot)$ in $L_1(\mu,Y_+)$, where for each $t\in T$ $x^t(i,\cdot)\in L_{X_i}^{\mathcal{F}_i^t}$, then for all $\omega\in\Omega$ there exists a subsequence $x^{t_n}(i,\omega)$ converging in norm to $x^*(i,\omega)$.*

Proof. It immediately follows from the fact that for all $t\in T$ and for all $\omega\in\Omega$, $x^t(i,\omega)$ belongs to $X_i(\omega)$ which is norm compact (see Theorem 3.28 p.86 in [1]). □

From assumption 7.3 (i), $(e^t(i,\cdot),\mathcal{F}_i^t,t\in T)$ is a martingale (see remark 7.4). Hence, from the martingale converge theorem the sequence $e^t(i,\cdot)$ converges to $e^*(i,\cdot)$ in $L_1(\mu,Y)$-norm (see Corollary 2 p.126 of Diestel-Uhl) and hence weakly in $L_1(\mu,Y)$. Consequently the same is true for any subsequence of $e^t(i,\cdot)$. By lemma 7.7, for all $\omega\in\Omega$, $x^{t_n}(i,\omega)$ converges to $x^*(i,\omega)$ and $e^{t_n}(i,\omega)$ to $e^*(i,\omega)$. Since for each t, $x^{t_n}(i,\cdot)$ is feasible in $\mathcal{E}^{t_n}$, that is

$$\sum_{i\in I}x^{t_n}(i,\omega)=\sum_{i\in I}e^{t_n}(i,\omega)\ \text{for}\ \mu-a.e.\omega.$$

it follows that

$$\sum_{i\in I} x^*(i,\omega) = \sum_{i\in I} e^*(i,\omega) \text{ for } \mu - a.e.\omega.$$

Hence x^* is feasible (see condition (i) in definition 3.2).

For each $t \in T$, $x^t \in W(\mathcal{E}^t)$ and hence there exists $p^t \in L_P = \{p \in L_\infty(\mu, Y^*_{w^*}) : \|p\|_\infty \leq 1\}$, where $p^t \neq 0$ and L_P is w^*-compact by Alaoglu's Theorem. Since $(\Omega, \mathcal{F}, \mu)$ is separable, so is the space $L_1(\mu, Y)$. Therefore, L_P is metrizable with respect to the weak* topology (see Theorem 10.7 p.153 in [2]) and hence there exists a subsequence $p^{t_n} \in L_P$ of p^t that weakly* converges to $p^* \in L_P$ (see [2] p.160).

Lemma 7.8. *If p^t converges weakly* to p^* in $L_\infty(\mu, Y^*_{w^*})$, then for all $\omega \in \Omega$ the sequence $p^t(\omega)$ converges weakly* to $p^*(\omega)$ in Y^*.*

Proof. Let $\omega \in \Omega$ be a fixed state of nature, we need to show that for all $x \in Y$, $p^t(\omega) \cdot x$ converges to $p^*(\omega) \cdot x$. By assumption 3.1, let $\Pi = \{E^1, E^2, \ldots\}$ be a finite or countable partition of Ω, then p^t and p^* can be written as

$$p^t = \sum_{k=1}^{\infty} p^{t,k} \chi_{E^k} \text{ and } p^* = \sum_{k=1}^{\infty} p^k \chi_{E^k},$$

where $p^{t,k}, p^k \in Y^*$. Note that there exists a unique event $E \in \Pi$ containing ω, that we denote by $\Pi(\omega)$. Let x be arbitrarly taken in Y. Since $p^t(\omega') = p^t(\omega)$ and $p^*(\omega') = p^*(\omega)$ for all $\omega' \in \Pi(\omega)$, we have that

$$p^t(\omega) \cdot x = \int_{\omega' \in \Omega} p^t(\omega) \cdot \frac{x}{\mu(\Pi(\omega))} \chi_{\Pi(\omega)}(\omega') d\mu(\omega') = \int_{\omega' \in \Omega} p^t(\omega') \cdot \frac{x}{\mu(\Pi(\omega))} \chi_{\Pi(\omega)}(\omega') d\mu(\omega'),$$

$$p^*(\omega) \cdot x = \int_{\omega' \in \Omega} p^*(\omega) \cdot \frac{x}{\mu(\Pi(\omega))} \chi_{\Pi(\omega)}(\omega') d\mu(\omega') = \int_{\omega' \in \Omega} p^*(\omega') \cdot \frac{x}{\mu(\Pi(\omega))} \chi_{\Pi(\omega)}(\omega') d\mu(\omega').$$

Note that

$$\frac{x}{\mu(\Pi(\omega))} \chi_{\Pi(\omega)} \in L_1(\mu, Y) \text{ and } p^t \in L_\infty(\mu, Y^*).$$

Since p^t converges weakly* to p^* in $L_\infty(\mu, Y^*)$,

$$\int_{\omega' \in \Omega} p^t(\omega') \cdot \frac{x}{\mu(\Pi(\omega))} \chi_{\Pi(\omega)}(\omega') d\mu(\omega')$$

converges to

$$\int_{\omega' \in \Omega} p^*(\omega') \cdot \frac{x}{\mu(\Pi(\omega))} \chi_{\Pi(\omega)}(\omega') d\mu(\omega'),$$

that is $p^t(\omega) \cdot x$ converges to $p^*(\omega) \cdot x$. Because the choice of x in Y is arbitrary, $p^t(\omega)$ converges weakly* to $p^*(\omega)$. This completes the proof of Lemma 7.8. □

Lemma 7.9. *If for every $i \in I$ and every $\omega \in \Omega$, $x^t(i,\omega)$ converges in norm to $x^*(i,\omega)$ and $p^t(\omega)$ converges weakly* to $p^*(\omega)$, then $\lim_{t\to+\infty} p^t \cdot x^t(i,\cdot) = p^* \cdot x^*(i,\cdot)$, that is*

$$\lim_{t\to+\infty} \int_{\omega' \in \Omega} p^t(\omega') \cdot x^t(i,\omega') d\mu(\omega') = \int_{\omega' \in \Omega} p^*(\omega') \cdot x^*(i,\omega') d\mu(\omega').$$

Proof. Fix $i \in I$ and $\omega \in \Omega$. First notice that since $x^t(i,\omega)$ converges in norm to $x^*(i,\omega)$ and $p^t(\omega)$ converges weakly* to $p^*(\omega)$, then $p^t(\omega) \cdot x^t(i,\omega)$ converges to $p^*(\omega) \cdot x^*(i,\omega)$ (see Lemma A of [27] p.107). Moreover, given $\omega \in \Omega$, the function $\omega \in \Omega \mapsto p^t(\omega) \cdot x^t(i,\omega) \in \mathbb{R}$ is dominated by an integrable function for any $t \in T$. Indeed, being X_i integrably bounded, there exists $h \in L_1(\mu)$ such that for $\mu - a.e.\omega$

$$|p^t(\omega') \cdot x^t(i,\omega')| \leq \|p^t(\omega)\|_{Y^*}\|x^t(i,\omega)\|_Y \leq \|p^t\|_\infty \|x^t(i,\omega)\|_Y \leq \|x^t(i,\omega)\|_Y \leq h(\omega).$$

By the Lebesgue dominated convergence theorem,

$$lim_{t\to+\infty} \int_{\omega'\in\Omega} p^t(\omega') \cdot x^t(i,\omega') d\mu(\omega') = \int_{\omega'\in\Omega} p^*(\omega') \cdot x^*(i,\omega') d\mu(\omega'). \tag{7.1}$$

□

Applying lemma 7.9 to the sequence of i's initial endowment we have that[4]

$$lim_{t\to+\infty} p^t \cdot e^t(i,\cdot) = p^* \cdot e^*(i,\cdot). \tag{7.2}$$

Since $x^t \in W(\mathcal{E}^t)$, $p^t \cdot x^t(i,\cdot) \leq p^t \cdot e^t(i,\cdot)$ for all $t \in T$. This together with (7.1) and (7.2) implies that $p^* \cdot x^*(i,\cdot) \leq p^* \cdot e^*(i,\cdot)$, showing that x^* satisfies condition (ii) of definition 3.2.

We now show that x^* satisfies also condition (iii) of definition 3.2. Assume to the contrary that there exists $y_i \in L_{X_i}$ such that

$$y_i \in P_i(x^*(i,\cdot)), \quad \text{and} \tag{7.3}$$

$$p^* \cdot y_i \leq p^* \cdot e^*(i,\cdot) \tag{7.4}$$

Without loss of generality, we may assume that

$$p^* \cdot y_i < p^* \cdot e^*(i,\cdot). \tag{7.5}$$

Indeed, define the correspondence $K : \Omega \to 2^{Y_+}$ such that for all $\omega \in \Omega$

$$K(\omega) = \{z \in X_i(\omega) : p^*(\omega) \cdot z < p^*(\omega) \cdot e^*(i,\omega)\},$$

which is nonempty valued by assumption 7.3 (ii). Notice that $K(\omega) = X_i(\omega) \cap L(\omega)$, where $L : \Omega \to 2^Y$ is defined as $L(\omega) = \{z \in Y : p^*(\omega) \cdot z < p^*(\omega) \cdot e^*(i,\omega)\}$, and $G_L = \{(\omega, z) \in \Omega \times Y : f(\omega, z) > 0\} = f^{-1}(]0, +\infty[)$, where $f(\omega, z) = p^*(\omega) \cdot e^*(i,\omega) - p^*(\omega) \cdot z$. The function $f(\omega,\cdot)$ is continuous, moreover from Pettis's measurable theorem (see Theorem 2 p.42 in [7]) we have that $f(\cdot, z)$ is $\mathcal{F}$-measurable. Thus $f : \Omega \times Y \to \mathbb{R}$ is a Caratheodory function and therefore jointly measurable (see Proposition 3.1 p. 41 in [25]). This implies that $G_L \in \mathcal{F} \otimes \mathcal{B}(Y)$ and, by assumption 7.1, $G_K = G_{X_i} \cap G_L \in \mathcal{F} \otimes \mathcal{B}(Y)$. Thus we can apply the Aumann measurable selection theorem and get a measurable function $z : \Omega \to Y$ such that $z(\omega) \in K(\omega)$ μ-a.e. Since X_i is integrably bounded by assumption 7.1, z is also integrable, i.e., $z \in L_{X_i}$. Note that L_{X_i} is convex because so is $X_i(\cdot)$. By assumption 7.2 (i) there exists $\alpha \in (0, 1]$ such that $y_\alpha = \alpha z + (1-\alpha) y_i \in P_i(x^*(i,\cdot))$ and $p^* \cdot y_\alpha = \alpha p^* \cdot z + (1-\alpha) p^* \cdot y_i < p^* \cdot e^*(i,\cdot)$.

Define for each $t \in T$, $y_i^t = E[y_i|\mathcal{F}_i^t]$ which converges in $L_1(\mu, Y)$-norm (and hence weakly) to y_i (see Corollary 2 p. 126 in [7]). By lemma 7.9 and (7.5) we have that $p^t \cdot y_i^t \leq p^t \cdot e^t(i,\cdot)$ for infinitely many $t \in T$. Since x^t is a WEE, by condition

[4] Actually, we should consider the subsequences x^{t_n} and e^{t_n} because of lemma 7.7, but we prefer keeping the symbol x^t and e^t for sake of simplicity.

(iii) of definition 3.2 it follows that $U_i(y_i^t) \leq U_i(x^t(i,\cdot))$, and hence, because of assumption 7.2 (i), $U_i(y_i) \leq U_i(x^*(i,\cdot))$ which contradicts (7.3). To conclude the proof we need to show that $p^* \neq 0$. Suppose ortherwise that $p^* = 0$, then for any $z \in L_{X_i}, p^* \cdot z \leq p^* \cdot e^*(i,\cdot)$ and in particular for the random consumption bundle $z \in P_i(x^*(i,\cdot))$ whose existence is ensured by assumption 7.2 (ii). This is impossible since x^* satisfies condition (iii) of definition 3.2.

8. Stability of equilibria

Let us now show the converse of the convergence theorem 7.5, i.e., any WEE in the limit full information economy can be reached by a sequence of approximate equilibrium outcomes. We can view this result as a stability property of the WEE equilibria, in the sense that we can always construct a route to reach the WEE in the limit full information economy. This result on the approximation of equilibria has an important interpretation as theorem 7.5. Here again in order to appreciate the value of this result one should contemplate of its failure: if some WEE in the limit economy failed to be approximated even by approximate outcomes, that would mean that those equilibria are artifacts of the definition and they would be irrelevant for analytical purposes. That would cast doubts on the value of the concept of WEE. We assume that agents' preferences are represented by an ex-ante expected utility function $U_i : L_{X_i} \to \mathbb{R}$ so that $P_i(x) = \{y \in L_{X_i} : U_i(y) > U_i(x)\}$ for any $x \in L_{X_i}$.

Definition 8.1. Given $\varepsilon > 0$, an allocation $x \in L_X$ is an ε-Walrasian expectations equilibrium (ε-WEE) if there exists a price $p \in L_P$ such that $p \neq 0$ and

(i) $\sum_{i\in I} x(i,\omega) = \sum_{i\in I} e(i,\omega)\, \mu.a.e.\omega$,
(ii) $p \cdot x(i,\cdot) \leq p \cdot e(i,\cdot) + \varepsilon$ for all $i \in I$,
(iii) $U_i(y_i) > U_i(x(i,\cdot)) + \varepsilon \Rightarrow p \cdot y_i > p \cdot e(i,\cdot)$.

We denote by $W_\varepsilon(\mathcal{E})$ the set of all ε-WEE allocations.

Remark 8.2. For $\varepsilon = 0$, definition 8.1 coincides with definition 3.2. Clearly, $W(\mathcal{E}) \subseteq W_\varepsilon(\mathcal{E})$ for any $\varepsilon > 0$.

Assumption 8.3. *For each $i \in I$ and each $t \in T$,*
$\sum_{i\in I} e^t(i,\cdot) = E[\sum_{i\in I} e^*(i,\cdot)| \bigwedge_{j\in I} \mathcal{F}_j^t]$.

Remark 8.4. By applying the same arguments used in remark 7.4, we can observe that assumption 8.3 implies that $\left(\sum_{i\in I} e^t(i,\cdot), \bigwedge_{j\in I} \mathcal{F}_j^t, t \in T\right)$ is a martingale. Furthermore, assumptions 7.3 (i) and 8.3 together imply a sort of consistency of individual and aggregate expectations. Indeed, from remark 7.4 we know that $e^t(i,\cdot) = E[e^{t+1}(i,\cdot)|\mathcal{F}_i^t]$ and hence

$$\sum_{i\in I} e^t(i,\cdot) = \sum_{i\in I} E[e^{t+1}(i,\cdot)|\mathcal{F}_i^t]. \tag{8.1}$$

On the other hand, by assumption 8.3 we know that

$$\sum_{i\in I} e^t(i,\cdot) = E\left[\sum_{i\in I} e^{t+1}(i,\cdot)| \bigwedge_{j\in I} \mathcal{F}_j^t\right] = \sum_{i\in I} E\left[e^{t+1}(i,\cdot)| \bigwedge_{j\in I} \mathcal{F}_j^t\right]. \tag{8.2}$$

Thus, from (8.1) and (8.2) we get that

$$\sum_{i\in I}\left(E[e^{t+1}(i,\cdot)|\mathcal{F}_i^t] - E\left[e^{t+1}(i,\cdot)|\bigwedge_{j\in I}\mathcal{F}_j^t\right]\right) = 0,$$

which means that what i believes his initial endowment will be in period $t+1$ (private expectations), that is given by $E[e^{t+1}(i,\cdot)|\mathcal{F}_i^t]$, and what the common belief of all agents about i's initial endowment at period $t+1$ is (common knowledge expectations), that is given by $E\left[e^{t+1}(i,\cdot)|\bigwedge_{j\in I}\mathcal{F}_j^t\right]$, must balance out on aggregate (see also [16]).

Theorem 8.5. *Let $\{\mathcal{E}^t : t \in T\}$ be a sequence of asymmetric information economies satisfying assumptions 7.1, 7.2 (i), 7.3 (i), 8.3 and let $x^* \in W(\mathcal{E}^*)$. Then, for any $\varepsilon > 0$, there exists $\{x^t : t \in T\}$, where $x^t \in W_\varepsilon(\mathcal{E}^t)$ for each $t \in T$, such that x^t converges in $L_1(\mu, Y)$-norm to x^*.*

Proof. Let $\varepsilon > 0$ be arbitrarily fixed. Since $x^* \in W(\mathcal{E}^*)$, there exists $p^* \in L_P$, $p^* \neq 0$, such that all conditions in definition 3.2 hold. For each $i \in I$ and each $t \in T$, let

$$x^t(i,\cdot) = E\left[x^*(i,\cdot)|\bigwedge_{j\in I}\mathcal{F}_j^t\right].$$

Hence, $\left(x^t(i,\cdot), \bigwedge_{j\in I}\mathcal{F}_j^t, t\in T\right)$ is a martingale in $L_{X_i}^{\mathcal{F}_i^t} \subseteq L_1(\mu, Y)$ and by the martingale convergence theorem, for each $i \in I$, $x^t(i,\cdot)$ converges in $L_1(\mu, Y)$-norm to $x^*(i,\cdot)$. We now show that $x^t \in W_\varepsilon(\mathcal{E}^t)$, with respect to the same non-null price $p^* \in L_P$, for infinitely many $t \in T$.

First note that by assumption 8.3 and feasibility of x^* we get that

$$\begin{aligned}\sum_{i\in I} x^t(i,\cdot) &= \sum_{i\in I} E\left[x^*(i,\cdot)|\bigwedge_{j\in I}\mathcal{F}_j^t\right] = E\left[\sum_{i\in I}x^*(i,\cdot)|\bigwedge_{j\in I}\mathcal{F}_j^t\right] = \\ &= E\left[\sum_{i\in I}e^*(i,\cdot)|\bigwedge_{j\in I}\mathcal{F}_j^t\right] = \sum_{i\in I}e^t(i,\cdot),\end{aligned}$$

meaning that x^t is feasible in $\mathcal{E}^t$ for any $t \in T$ and hence x^t satisfies condition (i) of definition 8.1.

Assume by the way of contradiction that for some agent i, $p^*\cdot x^t(i,\cdot) > p^*\cdot e^t(i,\cdot)+\varepsilon$ for infinitely many $t \in T$. Since $x^t(i,\cdot)$ and $e^t(i,\cdot)$ converge in $L_1(\mu, Y)$-norm (and hence weakly) respectivey to $x^*(i,\cdot)$ and $e^*(i,\cdot)$, we have that $p^*\cdot x^*(i,\cdot) \geq p^*\cdot e^*(i,\cdot) + \varepsilon > p^*\cdot e^*(i,\cdot)$, which contradicts condition (ii) of definition 3.2. This means that for any $i \in I$, $x^t(i,\cdot)$ fails condition (ii) of definition 8.1 only for finitely many periods.

To conclude the proof we must show that for any $i \in I$, $x^t(i,\cdot)$ fails condition (iii) of definition 8.1 only for finitely many period t. Suppose to the contrary that for some agent $i \in I$ and infinitely many period $t \in T$, there exists $y_i^t \in L_{X_i}$ such that $U_i(y_i^t) > U_i(x^t(i,\cdot)) + \varepsilon$ and $p^*\cdot y_i^t \leq p^*\cdot e^t(i,\cdot)$. From lemma 7.6, L_{X_i} is weakly compact and by Eberlein-Smulian theorem it is weakly sequentially compact. Then, there exists a subsequence $y_i^{t_n}$ converging weakly to some $y_i \in L_{X_i}$. Thus, by assumption 7.2 (i) we have that $U_i(y_i) \geq U_i(x^*(i,\cdot)) + \varepsilon > U_i(x^*(i,\cdot))$ and by lemma 7.9 $p^* \cdot y_i \leq p^* \cdot e^*(i,\cdot)$, which is an absurd because $x^* \in W(\mathcal{E}^*)$. □

Acknowledgements

We gratefully acknowledge the Editor and the Referee for their comments and suggestions.

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Manuscript received
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