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Erik J. Balder · Nicholas C. Yannelis

Continuity properties of the private core

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Abstract Let $(\mathcal{E}_k)_{k\in\mathbb{N}}$ be a sequence of differential information economies, converging to a limit differential information economy $\mathcal{E}_\infty$ (written as $\mathcal{E}_k \Rightarrow \mathcal{E}_\infty$). Denote by $C_\epsilon(\mathcal{E}_k)$ the set of all ϵ-private core allocations, $\epsilon \geq 0$ (for $\epsilon = 0$ we get the private core of Yannelis (1991), denoted by $C(\mathcal{E}_k)$). Under appropriate conditions, we prove the following stability results:

(1) (upper semicontinuity): if $\mathcal{E}_k \Rightarrow \mathcal{E}_\infty$, $f_k \in C(\mathcal{E}_k)$, and if $f_k \to f_\infty$ L^1-weakly, then $f_\infty \in C(\mathcal{E}_\infty)$.
(2) (lower semicontinuity): if $\mathcal{E}_k \Rightarrow \mathcal{E}_\infty$, $f_\infty \in C(\mathcal{E}_\infty)$, $\epsilon > 0$, then there exist $f_k \in C_\epsilon(\mathcal{E}_k)$, with $f_k \to f_\infty$ L^1-weakly.

Keywords Private core · Upper semicontinuity · Lower semicontinuity · Weak L^1-convergence · Martingales

JEL Classification Numbers D82 · D50 · D83 · C62 · C71 · D46 · D61

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E. J. Balder
Mathematical Institute, University of Utrecht,
P.O. Box 80.010, 3508 TA Utrecht, Netherlands
e-mail: balder@math.uu.nl

N. C. Yannelis (✉)
Department of Economics, University of Illinois at Urbana-Champaign,
Champaign, IL 61820, USA
e-mail: nyanneli@uiuc.edu

1 Introduction

Let $(\Omega, \mathcal{F})$ be an exogenously given measurable space. Here Ω is the set of all states of nature. Let S be a separable and reflexive Banach space. The space S is endowed with the Borel σ-algebra $\mathcal{B}(S)$. Let $I := \{1, \ldots, m\}$ be a set of agents (or players).

We study the following notion of a *differential information economy* $\mathcal{E} := \{(\Sigma^i, u^i, e^i, \mathcal{F}^i, \mu^i) : i \in I\}$. For every $i \in I$ in the above expression $\mathcal{F}^i \subset \mathcal{F}$ denotes agent i's *private information σ-algebra*[1] and μ^i is agent i's *prior* probability measure on $\mathcal{F}^i$, reflecting agent i's *personal beliefs* about the probability distribution of the state of nature. Evidently there exists a finite measure μ on $(\Omega, \mathcal{F})$ such that for every $i \in I$ the probability measure μ^i has a probability density ϕ^i with respect to μ. Thus, we have $\mu^i(A) = \int_A \phi^i \, d\mu$ for all $A \in \mathcal{F}^i$ and $\phi^i : \Omega \to \mathbb{R}_+$ is $\mathcal{F}^i$-measurable and μ-integrable. Also, in the above expression for $\mathcal{E}$, Σ^i is a multifunction $\Sigma^i : \Omega \to 2^S$ with a $\mathcal{F}^i \times \mathcal{B}(S)$-measurable graph, which denotes the *random consumption set* of agent i. Contingent upon the realized state of nature ω in Ω, agent i must select his/her consumption bundle in the subset $\Sigma^i(\omega)$ of S. This leads to the following notion: an allocation for agent i is a function $f^i : \Omega \to S$ that is $\mathcal{F}^i$-measurable, μ-integrable, and such that $f^i(\omega) \in \Sigma^i(\omega)$ for μ-a.e. ω in Ω (and hence also for μ^i-a.e. ω in Ω). Thus, with probability 1 the consumption choice of each agent is contingent upon the state of nature that is realized. The set of all allocations for agent i is denoted by $\mathcal{L}^1_{\Sigma^i} := \mathcal{L}^1_{\Sigma^i}(\Omega, \mathcal{F}^i, \mu)$. An *allocation* for the economy $\mathcal{E}$ is defined to be a vector function $(f^i)_{i\in I}$ such that $f^i \in \mathcal{L}^1_{\Sigma^i}$ for every $i \in I$; thus $\Pi_{i\in I}\mathcal{L}^1_{\Sigma^i}$ is the set of all allocations. For every $i \in I$ let $e^i \in \mathcal{L}^1_{\Sigma^i}$ be agent i's *initial endowment*; we observe that also initial endowment is contingent upon the state of nature in a way that is compatible with the information σ-algebra $\mathcal{F}^i$. An allocation $(f^i)_{i\in I} \in \Pi_{i\in I}\mathcal{L}^1_{\Sigma^i}$ for the pure exchange economy $\mathcal{E}$ is said to be *attainable* or *feasible* if $\sum_{i\in I} f^i = \sum_{i\in I} e^i$ μ-a.e. The graph of the multifunction Σ^i is denoted by gph Σ^i. For every $i \in I$ let u^i : gph $\Sigma^i \to \mathbb{R}$ be a $\mathcal{F}^i \otimes \mathcal{B}(S)$-measurable function. This is agent i's *utility function* and it is state-contingent: if state ω is realized and agent i chooses bundle $s \in \Sigma^i(\omega)$, then his/her resulting payoff is $u^i(\omega, s)$. The *(ex ante) expected utility* of agent i's feasible consumption allocation $f^i \in \mathcal{L}^i_{\Sigma^i}$ is given by

$$U^i(f^i) := \int_\Omega u^i(\omega, f^i(\omega))\mu^i(d\omega) = \int_\Omega u^i(\omega, f^i(\omega))\phi^i(\omega)\mu(d\omega),$$

provided that the integral exists. Here ϕ^i is the probability density defined above.

Following Yannelis (1991), where this notion was defined for $\epsilon = 0$, the *private ϵ-core $C_\epsilon(\mathcal{E})$* of the exchange economy $\mathcal{E}$ is defined as follows for any $\epsilon \geq 0$ (see also Koutsoungeras and Yannelis 1999):

Definition 1.1 *The* private ϵ-core $C_\epsilon(\mathcal{E})$ *of $\mathcal{E}$ is the set of all attainable allocations $(f^i)_{i\in I}$ in $\Pi_{i\in I}\mathcal{L}^1_{\Sigma^i}$ for which the following is* not *true: There exist a nonempty $J \subset I$ and $(g^j)_{j\in J} \in \Pi_{j\in J}\mathcal{L}^1_{\Sigma^j}$ such that*

(1) $\sum_{j\in J} g^j = \sum_{j\in J} e^j$ μ-a.e.

[1] For instance, $\mathcal{F}^i$ could be generated by some finite or countable partition of Ω.

(2) $U^j(g^j) > U^j(f^j) + \epsilon$ for all $j \in J$.

For $\epsilon = 0$, the private 0-core will simply be called the *private core*; it is denoted by $C(\mathcal{E})$ and most of this paper will concentrate on this notion. Clearly, a private core allocation is an attainable allocation, having the property that no coalition of agents can make a pure-exchange-feasible redistribution of their initial endowments, using their own information privately and making each member of that coalition better off.[2]

The main question which this paper addresses is the following: Does a "small" change in the primitives or parameters (i.e., Σ^i, u^i, e^i, $\mathcal{F}^i$) of the model result in a "small" change in the equilibrium (i.e., private core) consumption allocation? Obviously, one would like to have a continuous relationship between all the parameters and the equilibrium consumption allocations. Indeed, we will show that the private core is upper semicontinuous and also lower semicontinuous in some specific sense.

Although we are focusing on the upper and lower semicontinuity of the private core the techniques introduced can be used to cover other core notions (that is, coarse and weak fine core, Koutsougeras and Yannelis 1993). Here we only focus on the private core because it is known to have good properties. At least it exists, it is incentive compatible and it can be implemented as a perfect Bayesian Nash equilibrium of an extensive form game (Allen and Yannelis 2001; Glycopantis et al. 2001, 2003; Koutsougeras and Yannelis 1993; Yannelis 1991). The paper is organized as follows. In section 2 we present our model and our first main result. This is followed by a discussion of the related literature. In section 3 we prove the lower semicontinuity result.

2 Continuity results for the private core

Given $(\Omega, \mathcal{F}, \mu)$ and S as introduced in the previous Section, let $(\mathcal{E}_k)_{k\in\mathbb{N}}$ be a sequence of differential information economies, $\mathcal{E}_k := \{(\Sigma^i_k, u^i_k, e^i_k, \mathcal{F}^i_k, \mu^i) : i \in I\}$, of the kind defined in Section 1, and let $\mathcal{E}_\infty := \{(\Sigma^i_\infty, u^i_\infty, e^i_\infty, \mathcal{F}^i_\infty, \mu^i) : i \in I\}$ be another differential information economy, which plays the role of a "limit" economy. Denote by $C(\mathcal{E}_k)$ the private core of $\mathcal{E}_k$. In view of section 1, the above means, among other things, that for every $i \in I$ and $k \in \mathbb{N}^* := \mathbb{N} \cup \{\infty\}$ the multifunction $\Sigma^i_k : \Omega \to 2^S$ has a $\mathcal{F}^i \times \mathcal{B}(S)$-measurable graph, that the function $u^i_k : \text{gph } \Sigma^i_k \to \mathbb{R}$ is $\mathcal{F}^i_k \times \mathcal{B}(S)$-measurable and that e^i_k belongs to $\mathcal{L}^1_{\Sigma^i_k}(\Omega, \mathcal{F}^i_k, \mu)$. We wish to view the economies $\mathcal{E}_k$ as small perturbations with respect to $\mathcal{E}_\infty$, where "small" has the following meaning.

Definition 2.1 *We say that the sequence of differential economies $(\mathcal{E}_k)_k$* converges *to $\mathcal{E}_\infty$ (notation: $\mathcal{E}_k \Rightarrow \mathcal{E}_\infty$) if all of the following requirements hold:*

[2] Notice that in the above model, despite the fact that it is static, we may give a two-period interpretation as follows: in period one agents make contracts ex-ante (i.e., before the state of nature is realized). In period two, agents carry out the contracts made in the first period and consumption takes place.

(1) (private σ-algebra convergence*): for every* $i \in I$ *the sequence* $(\mathcal{F}_k^i)_k$ *is increasing and has* $\mathcal{F}_\infty^i$ *as its monotone limit,*[3] *i.e.,*

$$\mathcal{F}_k^i \subset \mathcal{F}_{k+1}^i \text{ for every } k \in \mathbb{N},$$

and

$$\mathcal{F}_\infty^i = \vee_k \mathcal{F}_k^i$$

where our standard notation on the right denotes the σ*-algebra that is generated by all* σ*-algebras* $\mathcal{F}_k^i$, $k \in \mathbb{N}$.

(2) (consumption correspondence convergence and consistency*): for every* $i \in I$ *and* $k \in \mathbb{N}^*$ *the correspondence* Σ_k^i *has closed convex and nonempty values (possibly unbounded). Also, for every* $i \in I$ *the sequence* $(\mathrm{dist}(\Sigma_k^i, 0))_k$ *of distance functions to the origin* 0 *of* S *is uniformly* μ*-integrable and for every nonempty* $J \subset I$ *and for* μ*-a.e.* ω

$$E_\mu(\Sigma_\infty^j \mid \cap_{i \in J} \mathcal{F}_k^i)(\omega) \subset \Sigma_k^j(\omega)$$

for every $j \in J$.[4] *Moreover, for every* $i \in I$ *and for every* ω *the sequence* $(\Sigma_k^i(\omega))_k$ *is upper semicontinuous*[5] *in that*

$$w\text{-}\mathrm{Ls}_k \Sigma_k^i(\omega) \subset \Sigma_\infty^i(\omega).$$

(3) (utility function convergence*): for every* $i \in I$ *for every* ω *and for every sequence* $(s_k^i)_k$ *in* S*, with* $s_k^i \in \Sigma_k^i(\omega)$*, which converges to* $\bar{s}^i \in S$*,*

$$\lim_{k \to \infty} u_k^i(\omega, s_k^i) = u_\infty^i(\omega, \bar{s}^i).$$

Observe here that by requirement (2) such $\bar{s}^i$ *can be supposed to belong to* $\Sigma_\infty^i(\omega)$*; in turn, this causes* $u_\infty^i(\omega, \bar{s}^i)$ *to be well-defined. Moreover, for every* $i \in I$ *and* $\omega \in \Omega$ *the function* $u_\infty^i(\omega, \cdot)$ *is concave on* $\Sigma_\infty^i(\omega)$.

(4) (initial endowment consistency*): for every nonempty* $J \subset I$ *and* $k \in \mathbb{N}$

$$\sum_{j \in J} e_k^j(\omega) = E_\mu(\sum_{j \in J} e_\infty^j \mid \cap_{j \in J} \mathcal{F}_k^j)(\omega),$$

for μ*-a.e.* ω*. That is to say,* $\sum_{j \in J} e_k^j$ *is a version of the conditional expectation of* $\sum_{j \in J} e_\infty^j$ *with respect to the* σ*-algebra* $\cap_{j \in J} \mathcal{F}_k^j$*. Moreover, the sum* $\Sigma_{i \in I} e_k^i$ *is required to be* $\vee_k \cap_{i \in I} \mathcal{F}_k^i$*-measurable.*[6]

[3] This assumption is common to models with learning; e.g., see Koutsougeras and Yannelis (1999) or Serfes (2001).

[4] See Appendix A for the definition of this conditional expectation.

[5] The limes superior notion below on the left is recalled in Appendix A.

[6] As follows by the martingale convergence theorem, this implies in particular that $(e_k^i)_k$ L^1-converges to e_∞^i (take $J = \{i\}$).

Especially requirements (2) and (4) are quite stringent, but we point out that they certainly hold in the deterministic case (i.e., when there exist $\bar{S}^i \subset S$ and $\bar{e}^i \in \mathcal{L}^1_{\bar{S}^i}$ such that $\Sigma^i_k(\omega) = \bar{S}^i$ and $e^i_k(\omega) = \bar{e}^i$ for all $i \in I$, $k \in \mathbb{N}^*$ and $\omega \in \Omega$). They also hold in a quite natural form when all the private information σ-algebras $\mathcal{F}^i_k$, $i \in I$, coincide for each k.[7]

In addition, throughout this paper we suppose that for every $i \in I$ there is a μ^i-integrable function $\psi^i : \Omega \mapsto \mathbb{R}$ such that

$$\sup_{s \in \Sigma^i_k(\omega)} |u^i_k(\omega, s)| \leq \psi^i(\omega)$$

for every $k \in \mathbb{N}^*$. Consequently, for every $i \in I$ and $k \in \mathbb{N}^*$ the expected utilities

$$U^i_k(f^i) := \int_\Omega u^i_k(\omega, f^i(\omega))\mu^i(\mathrm{d}\omega),$$

are well-defined for every measurable selection f^i of Σ^i_k, so in particular for every allocation $f^i \in \mathcal{L}^i_{\Sigma^i_k}(\Omega, \mathcal{F}^i_k, \mu)$.

2.1 Upper semicontinuity of the private core

Our first main result is as follows:

Theorem 2.1 *Suppose that $\mathcal{E}_k \Rightarrow \mathcal{E}_\infty$. For every $i \in I$ and $k \in \mathbb{N}^* = \mathbb{N} \cup \{\infty\}$ let $f^i_k \in \mathcal{L}^1_{\Sigma^i_k}(\Omega, \mathcal{F}^i_k, \mu^i)$ be a feasible consumption allocation for agent i. Suppose that the vector function $(f^i_k)_{i\in I}$ belongs to the core $C(\mathcal{E}_k)$ of the economy $\mathcal{E}_k$ for every $i \in I$ and $k \in \mathbb{N}$. Suppose also that for every $i \in I$ the sequence $(f^i_k)_k$ converges weakly in $\mathcal{L}^1_S(\Omega, \mathcal{F}^i, \mu)$ to f^i_∞. Then $(f^i_\infty)_{i\in I}$ belongs to the core $C(\mathcal{E}_\infty)$.*

Proof Notice that $(f^i_\infty)_{i\in I}$ satisfies

$$\limsup_k U^i_k(f^i_k) \leq U^i_\infty(f^i_\infty) \text{ for all } i \in I$$

by Lemma A.1 (see Appendix A). Also, by Definition 2.1(4) the attainability of $(f^i_k)_{i\in I}$ for every k gives

$$\sum_{i\in I} f^i_k = \sum_{i\in I} e^i_k = E_\mu(\sum_{i\in I} e^i_\infty \mid \cap_{i\in I}\mathcal{F}^i_k).$$

The expression on the left side converges weakly to $\sum_{i\in I} f^i_\infty$ (observe that this implies in particular the uniform integrability of $(\sum_{i\in I} f^i_k)_k$). Also, the expression on the right converges μ-a.e. to $E_\mu(\sum_{i\in I} e^i_\infty \mid \vee_k \cap_{i\in I} \mathcal{F}^i_k)$ by a well-known martingale convergence result, recalled in Proposition A.1. Again by Definition 2.1(4) we have $E_\mu(\sum_{i\in I} e^i_\infty \mid \vee_k \cap_{i\in I} \mathcal{F}^i_k) = \sum_{i\in I} e^i_\infty$ μ-a.e. So by Vitali's extension of the Lebesgue dominated convergence theorem (Neveu 1965, Proposition II.5.4)

[7] Another example is for instance for $m = 2$ the case where $\mathcal{F}^1_k \subset \mathcal{F}^2_k$ for each k.

it follows that $(\sum_{i\in I} f_k^i)_k$ converges in L^1-seminorm to $\sum_{i\in I} e_\infty^i$. Then a fortiori $(\sum_{i\in I} f_k^i)_k$ converges weakly to $\sum_{i\in I} e_\infty^i$, so the two weak limits $\sum_{i\in I} f_\infty^i$ and $\sum_{i\in I} e_\infty^i$ coincide μ-a.e. We finish the proof by showing that the following *cannot* occur: there is a nonempty $J \subset I$ and a corresponding vector function $(g^j)_{j\in J}$ in $\Pi_{j\in J}\mathcal{L}^1_{\Sigma^j_\infty}(\Omega, \mathcal{F}^j, \mu)$ such that

(1) $\sum_{j\in J} g^j(\omega) = \sum_{j\in J} e^j_\infty(\omega)$ for μ-a.e. $\omega \in \Omega$, (2) $U^j_\infty(g^j) > U^j_\infty(f^j_\infty)$ for all $j \in J$.

For supposition there were such J and $(g_j)_{j\in J}$. We will show that this contradicts the fact that $(f_k^i)_{I\in I}$ belongs to $C(\mathcal{E}_k)$ for every k. For every $j \in J$ and $k \in \mathbb{N}$ let g_k^j be an arbitrary version of the conditional expectation $E_\mu(g^j \mid \cap_{j\in J}\mathcal{F}^j_k)$. Then g_k^j belongs to $\mathcal{L}^1_{\Sigma^j_k}(\Omega, \mathcal{F}^j, \mu^j)$, by Definition 2.1(2). For every $k \in \mathbb{N}$ we have $\mathcal{G}_k \subset \mathcal{G}_{k+1}$ by Definition 2.1(1), where we denote $\mathcal{G}_k := \cap_{j\in J}\mathcal{F}^j_k$. Hence, $(g_k^j)_k$ is a martingale. So, by the same martingale convergence result as above, we obtain $g_k^j \to h^j := E_\mu(g^j \mid \mathcal{G})$ μ-a.e., with $\mathcal{G} := \vee_{k\in\mathbb{N}}\mathcal{G}_k$. By Lemma A.1(3) this gives

$$U^j_\infty(h^j) = \lim_k U^j_k(g^j_k) \text{ for every } j \in J.$$

We claim now that $U^j_\infty(h^j) \geq U^j_\infty(g^j)$ for every $j \in J$. Indeed, since h^j is $\mathcal{G}$-measurable we have

$$-U^j_\infty(h^j) = \sup_{p\in\mathcal{L}^\infty(\mathcal{G})} \left[\int_\Omega \langle h^j(\omega), p(\omega)\rangle \mu^j(\mathrm{d}\omega) - (-U^j_\infty)^*(p) \right], \tag{2.1}$$

by the Fenchel duality result in Castaing and Valadier (1977, VII-7). Here $(-U^j_\infty)^*(p) := \int_\Omega (-u^j_\infty)^*(\omega, p(\omega))\mu^j(\mathrm{d}\omega)$, the integrand being defined by pointwise Fenchel conjugation: $(-u^j_\infty)^*(\omega, p(\omega)) := \sup_{s\in S}[-\langle s, p(\omega)\rangle + u^j_\infty(\omega, s)]$. Now notice that $\int_\Omega \langle h^j, p\rangle \mathrm{d}\mu^j = \int_\Omega \langle g^j, p\rangle \mathrm{d}\mu^j$ for every $p \in \mathcal{L}^\infty(\mathcal{G})$, by the definition of h^j. Since also $\mathcal{G} \subset \mathcal{F}^j_\infty$, it follows from (2.1) that

$$-U^j_\infty(h^j) \leq \sup_{p\in\mathcal{L}^\infty(\mathcal{F}^j)} \left[\int_\Omega \langle g^j(\omega), p(\omega)\rangle \mu^j(\mathrm{d}\omega) - (-U^j_\infty)^*(p) \right],$$

where the right hand side equals $-U^j_\infty(g^j)$, again by Castaing and Valadier (1977, VII-7). This proves the claim. Combining the preceding inequalities, we obtain

$$\limsup_k U^j_k(f^j_k) \leq U^j_\infty(f^j) < U^j_\infty(g^j) \leq U^j_\infty(h^j) = \lim_k U^j_k(g^j_k) \text{ for every } j \in J,$$

in view of property (2) of $(g^j)_{j\in J}$. Hence, for large enough k we have $U^j_k(f^j_k) < U^j_k(g^j_k)$ for every $j \in J$. To obtain the desired contradiction it is now sufficient to prove that $(g^j_k)_{j\in J}$ satisfies $\sum_{j\in J} g^j_k = \sum_{j\in J} e^j_k$ a.e. By Definition 2.1(4) and property (1) of $(g^j)_{j\in J}$ we have

$$\sum_{j \in J} e_k^j = E_\mu \left(\sum_{j \in J} e_\infty^j \mid \mathcal{G}_k \right) = E_\mu \left(\sum_{j \in J} g^j \mid \mathcal{G}_k \right) = \sum_{j \in J} E_\mu(g^j \mid \mathcal{G}_k)$$
$$= \sum_{j \in J} g_k^j \ \mu\text{-a.e.}$$

and this finishes the proof. □

2.2 Related literature

After this paper had been written, we became aware of a related upper semicontinuity result of Einy et al. (2005). These can be seen as complementary results, for neither our theorem above nor theirs implies the other result. The main differences are as follows: (1) Our state space is a measure space, whereas they consider a topological state space. (2) They topologize the information sets, so that they can additionally consider convergence of information (in the Boylan metric); in contrast, our private information sub-σ-algebras must converge monotonically. (3) Our utility functions are allowed to depend measurably on the state variable and continuously on the consumption variable, whereas in Einy et al. (2005) the utility functions must be jointly continuous in these two variables and monotone in the consumption variable. (4) They use strong L^1-convergence of the allocations, whereas we only need weak L^1-convergence. (5) In terms of our paper, all their correspondences Σ_k^i are identically equal to the nonnegative orthant of a Euclidean space. These substantial differences already explain the complementarity of the results; moreover, the proofs are also different.

It should be noted that Einy et al. (2005) in their Example 1 show that upper semicontinuity can fail if a certain strong relationship between the information σ-algebras $\mathcal{F}_\infty^i$ for the limit economy and the approximating σ-algebras $\mathcal{F}_k^i$ is not maintained. In our paper such strong a relationship is fulfilled via monotonicity (see Definition 2.1(1) below), whereas Einy et al. (2005) allow for a more general setup in this regard. However, in some other respects our upper semicontinuity result, stated in Theorem 2.1, is more general, as was explained above. Also, our lower semicontinuity result (Theorem 3.1 in the next section) has no counterpart in Einy et al. (2005).

A version of Theorem 2.1 has been proved in Koutsoungeras and Yannelis (1999). However, the convergence is only in the allocation space and not all the characteristics of the economy are allowed to converge. In a way, the above theorem provides a solution to the question posed in footnote 12 of Koutsoungeras and Yannelis (1999).

It should be noticed that our positive results and those of Einy et al. (2005) are complemented by the discontinuity results of Krasa and Shafer (2001), who consider variations in the prior probability distributions of the agents.

Finally, Correia-da-Silva and Herves-Beloso (2004) show that the Boylan topology used in Einy et al. (2005) has the property that a small perturbation in the private information of an agent may render it incompatible with the private information of others. As a consequence of this one may face the state that from no trade one can move to trade. In that sense, small perturbations in the private information may result in significant changes in outcomes. To this end Correia-da-Silva and Herves-Beloso (2004) introduce a new topology on information which evaluates the similarity between private information fields, taking into account their compatibility (i.e., the events that are commonly observed). With this new topology they prove the upper semicontinuity of the private core and recast the results of Einy et al. (2005).

3 Lower semicontinuity of the private core

For the next result we introduce the following notion: a subset S' of S is said to be norm-increasing by addition (NIBA) if $\max(\|s\|, \|s'\|) \leq \|s + s'\|$ for every pair $s, s' \in S'$. For instance, any subset of the nonnegative orthant of a Euclidean space is NIBA for the Euclidean norm. We have the following result on approximate lower semicontinuity (observe that only for $\epsilon = 0$ this would yield classical lower semicontinuity).

Theorem 3.1 *Under the above conditions, suppose that $\mathcal{E}_k \Rightarrow \mathcal{E}_\infty$ and that $\Sigma_k^i(\omega)$ is NIBA for every $i \in I$, $k \in \mathbb{N}$ and $\omega \in \Omega$. Let $(f_\infty^i)_{i\in I} \in \Pi_{i\in I}\mathcal{L}^1_{\Sigma^i_\infty}(\Omega, \mathcal{F}^i_\infty, \mu^i)$ belong to the core $C(\mathcal{E}_\infty)$ of the economy $\mathcal{E}_\infty$. Then for every $\epsilon > 0$ there exists a sequence $(f_k^i)_{i\in I}$, $k \in \mathbb{N}$, that converges weakly in $\Pi_{i\in I}\mathcal{L}^1_S(\Omega, \mathcal{F}^i, \mu)$ to $(f_\infty^i)_{i\in I}$ and such that $(f_k^i)_{i\in I} \in C_\epsilon(\mathcal{E}_k)$ for every $k \in \mathbb{N}$.*

Proof Let $(f_\infty^i)_{i\in I}$ be in $\mathcal{C}(\mathcal{E}_\infty)$ and fix $\epsilon > 0$. Define $f_k^i := E_\mu(f_\infty^i \mid \cap_{j\in I}\mathcal{F}_k^j)$ for $j \in I$ and $k \in \mathbb{N}$; then by part (2) of Definition 2.1 it follows that $f_k^i \in \mathcal{L}^1_{\Sigma_k^i}$. Also, by Definition 2.1(4),

$$\sum_{i\in I} f_k^i = E_\mu\left(\sum_{i\in I} f_\infty^i \mid \cap_{j\in I}\mathcal{F}_k^j\right) = E_\mu\left(\sum_{i\in I} e_\infty^i \mid \cap_{j\in I}\mathcal{F}_k^j\right) = \sum_{i\in I} e_k^i$$

for every $k \in \mathbb{N}$. For every $i \in I$ it follows by Proposition A.1 that $(f_k^i)_k$ converges to $h^i := E_\mu(f_\infty^i \mid \vee_k \cap_{i\in I} \mathcal{F}_k^i)$, both μ-a.e. and in L^1-seminorm. By $f_k^i(\omega) \in \Sigma_k^i(\omega)$ μ-a.e., this implies $h^i(\omega) \in \Sigma_\infty(\omega)$ μ-a.e., in view of Definition 2.1(2). So $h^i \in \mathcal{L}^1_{\Sigma^i_\infty}(\Omega, \mathcal{F}^i_\infty, \mu)$ for every $i \in I$. Then by Lemma A.1(3) $\lim_k U_k^i(f_k^i) = U_\infty^i(h_\infty^i) \geq U_\infty^i(f_\infty^i)$ for every $i \in I$, where the inequality follows similar to what we did in the previous proof.

Further, by Definition 2.1(1) and (4) the martingale $(e_k^i)_k$ converges a.e. and in L^1-seminorm to e_∞^i (see footnote 6). We claim that there is a subsequence $(f_{k_p})_p$ of $(f_k)_k$ such that $f_{k_p} \in \mathcal{C}_\epsilon(\mathcal{E}_{k_p})$ for every p. For if this were not true, then there would exist $k_0 \in \mathbb{N}$ such that $f_k \notin \mathcal{C}(\mathcal{E}_k)$ for all $k \geq k_0$. But then to every $k \geq k_0$ there would correspond $J_k \subset I$ and an associated $(g_k^j)_{j\in J_k}$ with $g_k^j \in \mathcal{L}^1_{\Sigma_k^j}(\Omega, \mathcal{F}^j, \mu)$ and

$$\sum_{j\in J_k} g_k^j = \sum_{j\in J_k} e_k^j \text{ a.e. and } U_k^j(g_k^j) \geq U_k^j(f_k^j) + \epsilon \tag{3.2}$$

for all $j \in J_k$. Because there are at most 2^m different subsets of I, we may suppose without loss of generality that all these J_k are equal to one and the same subset

J of I. Below we shall demonstrate that there exist subsequences $(g^j_{k_p})_p$ such that $g^j_{k_p}$ weakly converges to some $\bar{g}^j$ in $\mathcal{L}^1_S(\Omega, \mathcal{F}^j, \mu)$ for every $j \in J$. Then $\bar{g}^j \in \mathcal{L}^1_{\Sigma^j_\infty}$ follows by Lemma A.1(2) and $\limsup_p U^j_{k_p}(g^j_{k_p}) \leq U^j_\infty(\bar{g}^j)$ for every $j \in J$. But by the inequality in (3.2) this gives $\limsup_p U^j_{k_p}(g^j_{k_p}) \geq \limsup_p U^j_{k_p}(f^j_{k_p}) + \epsilon$, with $\limsup_p U^j_{k_p}(f^j_{k_p}) = \lim_p U^j_{k_p}(f^j_{k_p}) \geq U^j_\infty(f^j_\infty)$ for every $j \in J$. So we conclude that $U^j_\infty(\bar{g}^j) \geq U^j_\infty(f^j_\infty) + \epsilon$ for every $j \in J$. The equality in (3.2) gives $\sum_{j \in J} \bar{g}^j = \sum_{j \in J} e^j_\infty$ a.e. in the limit, which gives the desired contradiction.

To wrap up the proof, it remains to demonstrate that there exist subsequences $(g^j_{k_p})_p$ such that $g^j_{k_p}$ weakly converges to some $\bar{g}^j$ in $\mathcal{L}^1_S(\Omega, \mathcal{F}^j, \mu)$ for every $j \in J$. Observe that, by the above, $(\sum_{j \in J} e^j_k)_k L^1$-converges to $\sum_{j \in J} e^j_\infty$, as also the corresponding norm functions (ϕ_k), with $\phi_k := \| \sum_{j \in J} e^j_k \|$, converge in $\mathcal{L}^1_{\mathbb{R}}$ to $\| \sum_{j \in J} e^j \|$. In particular, this implies that the sequence $(\phi_k)_k$ is uniformly integrable. So, by virtue of de la Vallée Poussin's theorem (Dellacherie and Meyer 1975, II.22), there exists a function $h' : \mathbb{R}_+ \to \mathbb{R}_+$, convex continuous nondecreasing with $\lim_{\xi \to \infty} h'(\xi)/\xi = +\infty$, such that $\sup_k \int_\Omega h'(\|\phi_k\|) \mathrm{d}\mu < +\infty$. It is enough to fix an arbitrary $j \in J$. By NIBA and monotonicity of h' it follows that $h'(\phi_k) \geq h'(\| f^j_k \|)$ on Ω, whence $\sup_k \int_\Omega h'(\| f^j_k \|) \mathrm{d}\mu < +\infty$. So by Theorem A.2 it follows that $(f^j_k)_k$ is relatively weakly sequentially compact. □

4 Concluding remarks

1. The technique introduced in this paper can be applied to prove the upper and lower semicontinuity of the private Shapley value (e.g., see Koutsoungeras and Yannelis 1999). Furthermore, one can prove similar results for other core concepts for a differential information economy. As we mentioned in the introduction, we focus on the on the private core, because it has been shown to have attractive properties.
2. One can introduce discounting as done by Serfes (2001) and establish the counterparts of Theorems 2.1 and 3.1.
3. It is an open question if one can provide the counterparts of Theorems 2.1 and 3.1 for a differential information economy with a continuum of agents. There seem to be serious technical difficulties.

A Appendix: Some useful results

As before, let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and let S be a separable reflexive Banach space. Recall from Hess (1991, p. 178) (see also Hiai and Umegaki 1977) that for every measurable multifunction $\Sigma : \Omega \to 2^S$ such that $\omega \mapsto \mathrm{dist}(\Sigma(\omega), 0)$ is μ-integrable and for every sub-σ-algebra $\mathcal{F}'$ of $\mathcal{F}$ there exists a Σ'-measurable multifunction $\Sigma' : \Omega \to 2^S$ such that $\omega \mapsto \mathrm{dist}(\Sigma'(\omega), 0)$ is μ-integrable and

$$\mathcal{L}^1_{\Sigma'}(\Omega, \mathcal{F}', \mu) = L^1\text{-cl}\,\{E_\mu(f \mid \mathcal{F}') : f \in \mathcal{L}^1_\Sigma(\Omega, \mathcal{F}, \mu)\}.$$

Such a multifunction is determined μ-a.e. by the above relationship; in this paper it is deoted by $E_\mu(\Sigma \mid \mathcal{F}')$.

Let D be a separable metric space. When S is finite-dimensional, the following result is classical (Cesari 1983; Ioffe 1977). In its present form it follows, for instance, from Theorem 5.2 in Balder (2000) or from the main result in Balder (1987).

Theorem A.1 *Let $d_0 : \Omega \to D$ be a measurable function and let $\ell : \Omega \times S \times D \to \mathbb{R}$ be integrably bounded from below, measurable with respect to $\mathcal{F} \times \mathcal{B}(S) \times \mathcal{B}(D)$ and such that for every $\omega \in \Omega$:* (1) *$\ell(\omega, \cdot, \cdot)$ is lower semicontinuous on $S \times D$ and* (2) *$\ell(\omega, \cdot, d_0(\omega))$ is convex on S for every $d \in D$. Then*

$$\liminf_n \int_\Omega \ell(\omega, f_n(\omega), d_n(\omega))\mu(\mathrm{d}\omega) \geq \int_\Omega \ell(\omega, f_0(\omega), d_0(\omega))\mu(\mathrm{d}\omega)$$

for every sequence (f_n) weakly converging to f_0 in $\mathcal{L}^1_S$ and every sequence (d_n) of measurable functions $d_n : \Omega \to D$ converging pointwise a.e. to d_0.

Lemma A.1 *In Theorem 2.1 the following hold for every $i \in I$:*

(1) $f^i_\infty \in \mathcal{L}^1_{\Sigma^i_\infty}$,

(2) $\limsup_k U^i_k(f^i_k) \leq U^i_\infty(f^i_\infty)$,

(3) $U^i_\infty(g^i) = \lim_k U^i_k(g^i_k)$ *for every $g^i \in \mathcal{L}^1_{\Sigma^i_\infty}$ and every sequence (g^i_k) converging μ^i-a.e. to g^i.*

Proof Fix $i \in I$. (1) By weak convergence of (f^i_k) to f^i_∞ we have $f^i_\infty(\omega) \in \mathrm{cl}\,\mathrm{co}\, w\text{-Ls}\, f^i_k(\omega)$ for μ^i-almost every ω, because S is reflexive (e.g., see Theorem 5.2 in Balder (2000) and Yannelis (1989)). Recall here that w-Ls $f^i_k(\omega)$ denotes the set of all limit points of the sequence $(f^i_k(\omega))$ in the weak topology on S. So by requirement (2) of Definition 2.1 $f^i_\infty(\omega)$ belongs to $\Sigma^i_\infty(\omega)$ for a.e. ω in Ω.

(2) Let $D := \mathbb{N} \cup \{\infty\}$ be the usual Alexandrov compactification of the set $\mathbb{N}$ of natural numbers. Define $\ell : \Omega \times D \times S \to \mathbb{R}$ as follows: (1) if $k \in \mathbb{N}$ then set $\ell(\omega, s, k) := -u^i_k(\omega, s)\phi^i(\omega)$ if $s \in \Sigma^i_k(\omega)$ and set $\ell(\omega, s, k) := +\infty$ otherwise; (2) if $k = \infty$ then set $\ell(\omega, s, \infty) := -u^i_\infty(\omega, s)\phi^i(\omega)$ if $s \in \Sigma^i_\infty(\omega)$ and set $\ell(\omega, s, \infty) := +\infty$ otherwise. Using Definition 2.1(2)–(3), it is easy to see that the function $(s, k) \mapsto \ell(\omega, s, k)$ is lower semicontinuous on $S \times D$ for μ-a.e. ω in Ω. Also, ℓ is bounded from below by the function $-\phi^i\psi^i$, and the latter is μ-integrable over Ω (recall that ψ^i was introduced following Definition 2.1). Therefore, by applying Theorem A.1 it follows that

$$\liminf_k \int_\Omega \ell(\omega, f^i_k(\omega), k)\mu(\mathrm{d}\omega) \geq \int_\Omega \ell(\omega, f^i_\infty(\omega), \infty)\mu(\mathrm{d}\omega).$$

Here one works with $d_k := k$ and $d_0 := \infty$. By definition of ℓ, the fact that f^i_k belongs to $\mathcal{L}^1_{\Sigma^i_k}$ for every $k \in \mathbb{N}$ and the fact, proven above, that y^i belongs to $\mathcal{L}^1_{\Sigma^i_\infty}$, we get the above inequality in the following form: $\limsup_k U^i_k(f^i_k) \leq U^i_\infty(f^i_\infty)$. Finally, in view of Definition 2.1(3) the result in (3) follows by a straightforward application of the Lebesgue dominated convergence theorem. □

The following result is Proposition V-2-6 of Neveu (1972), Neveu (1965)):

Proposition A.1 *For every $\mathcal{F}$-measurable function $f : \Omega \to 2^S$ and for every increasing sequence $(\mathcal{F}_n)_n$ of σ-algebras the martingale sequence $(f_n)_n$, given by $f_n := E_\mu(f \mid \mathcal{F}_n)$, satisfies $\lim_n |f_n(\omega) - g(\omega)| = 0$ for μ-a.e. ω in Ω, where $g := E_\mu(f \mid \vee_{n=1}^{\infty} \mathcal{F}_n)$. Moreover, if $f \in \mathcal{L}^1_S(\Omega, \mathcal{F}, \mu)$, then (f_n) also converges to f in L^1-seminorm.*

The following relative compactness result follows directly from (Brooks and Dinculeanu 1977, Theorem 1, Remarks 1,2); it uses the fact that S is reflexive.

Theorem A.2 *Suppose that $(f_k) \subset \mathcal{L}^1_S(\Omega, \mathcal{F}, \mu)$ is such that $\sup_k \int_\Omega h'(\| f_k(\omega) \|) \mu(\mathrm{d}\omega) < +\infty$ for some $h' : \mathbb{R}_+ \to \mathbb{R}_+$, convex continuous nondecreasing and such that $\lim_{\xi \to \infty} h'(\xi)/\xi = +\infty$. Then (f_k) has a subsequence that weakly converges to some function in $\mathcal{L}^1_S(\Omega, \mathcal{F}, \mu)$.*

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