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Cone Conditions in General Equilibrium Theory[1]

Charalambos D. Aliprantis[2]

Department of Economics, Purdue University, West Lafayette, Indiana 47907-1310
aliprantis@mgmt.purdue.edu

Rabee Tourky

Department of Economics, University of Melbourne, Parkville, Victoria 3052, Australia
r.tourky@ecomfac.unimelb.edu.au

and

Nicholas C. Yannelis

Department of Economics, University of Illinois at Urbana-Champaign, Champaign, Illinois 61820
nyanneli@uiuc.edu

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The modern convex-analytic rendition of the classical welfare theorems characterizes optimal allocations in terms of supporting properties of preferences by nonzero prices. While supporting convex sets in economies with finite dimensional commodity spaces is usually a straightforward application of the separation theorem, it is not that automatic in economies with infinite dimensional commodity spaces. In the last 30 years several characterizations of the supporting properties of convex sets by non-zero prices have been obtained by means of cone conditions. In this paper, we present a variety of cone conditions, study their interrelationships, and illustrate them with many examples. *Journal of Economic Literature* Classification Numbers: D46, D51. © 2000 Academic Press

1. INTRODUCTION

Of the many insights of the old neoclassical school of economics, the characterization of economic optimality in terms of the equality of marginal rates of substitution, has remained a most enduring (and endearing) result in economic theory. The modern convex-analytic version of this

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[2] Corresponding author.

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classical insight characterizes optimality in terms of the supporting of preferences by a continuous linear functional (a price). Such a characterization is possible in the presence of finitely many commodities since every closed convex set in a finite dimensional space can be supported by a non-zero linear functional at each of its boundary points. This remarkable property can be attributed to a strictly finite dimensional peculiarity which ensures that every finite dimensional convex set has a non-empty interior in the smallest affine subspace that contains it; see for instance [17, Theorem 11.2.7, p. 341].

However, this supporting property fails when there are infinitely many commodities and when the arena of discourse is an infinite dimensional commodity space. In such a setting, supporting optimal allocations by means of prices (the second welfare theorem) is far more onerous a problem. In fact, convex sets with empty interior arise naturally in many economic models with infinitely many commodities. For example, if the positive cone of the commodity space has an empty interior, then every lower bounded consumption set has an empty interior.

This difficulty has been a subject of investigation throughout the second half of the twentieth century. Infinite dimensional results that are related to the second welfare theorem appeared quite early in the literature; see the works of G. Debreu [24], E. Malinvaud [35], M. Majumdar [34], B. Peleg and M. E. Yaari [38], and R. Radner [41]. Furthermore, since the works of K. J. Arrow [15], G. Debreu [24], and T. F. Bewley [18], it has become apparent that one of the major differences between economic models with finite and infinite dimensional commodity spaces is that in the finite dimensional setting the positive cone of the commodity space has an interior point.[3] Therefore, the standard infinite dimensional setting is one where the well-known cheeper point problem cannot be readily assumed away—and one may even appreciate the problem elucidated here by considering curious finite dimensional examples that "mimic" this infinite dimensional difficulty.[4]

A solution to this problem was presented in a seminal paper by A. Mas-Colell [36]. His solution was based on the works by Aliprantis and

[3] Recall that the positive cone of an infinite dimensional commodity space has a non-empty interior basically only if it is a majorizing subspace of some $C(\Omega)$-space (see [2, Sect. 7.5])—which is not the most appropriate setting for many economic models.

[4] For example, suppose that there are three commodities, i.e., the commodity space is $\mathbb{R}^3$. Assume that consumption sets are the positive orthant of $\mathbb{R}^3$ and suppose that the price space is smaller than $\mathbb{R}^3$ and is in fact some *two* dimensional subspace E of $\mathbb{R}^3$. It is clear that if we require that all decentralizing prices be elements of E instead of $\mathbb{R}^3$, then one needs extra assumptions on preferences to guarantee the validity of this stronger version of the second welfare theorem. The problem here is that the positive orthant of $\mathbb{R}^3$ has no interior points in the weak topology $\sigma(\mathbb{R}^3, E)$.

Brown and Chichilnisky and Kalman. In particular, Mas-Colell replaces the requirement that the total endowment of resources be an interior point of the consumption sets with two assumptions. First, that the commodity space be a locally solid vector lattice and that the price space be its topological dual—a setting introduced into general equilibrium theory by C. D. Aliprantis and D. J. Brown [4]. Second, that preferences satisfy a cone condition that he termed *uniform properness*—an adaptation of a condition that was used in the work of G. Chichilnisky and P. J. Kalman [22]. In essence, this assumption guarantees that there exists a fixed open convex cone containing the total endowment of the resources such that preferences can be supported at *every* point by prices that are *positive in the direction of this open cone*. In this sense, Mas-Colell required that the marginal rates of substitution be bounded.

It is by now well accepted in economics to regard a preference relation supported by a price at a given point as being "proper" at that point. Furthermore, the properness concept has appeared in many forms and various contexts and applications including macroeconomic theory; see, for instance [1, 3, 5–8, 12, 14, 27, 28, 40, 42–47, 49]. However, the relationship between cone conditions and supporting properties of convex sets made its debut in mathematics at least since the 1948 work of V. L. Klee [30, Corollary 1, p. 769]; see also [26, Theorem V.9.10, p. 452; 32, p. 195; 31, p. 457]. Klee characterized the supporting property of convex sets in terms of cones and in a manner that is closely related to Mas-Colell's notion of properness—indeed, Mas-Colell's uniform properness condition is an adaptation of Klee's condition. It appears that in economic theory, G. Chichilnisky and P. J. Kalman [22, Theorem 2.1] were the first to use Klee's cone condition to support convex sets; see also [20].[5] Welfare properties of optimal allocations based on "pointwise" properness were obtained by A. Araujo and P. Monteiro [13] and K. Podczeck [40].

The objective of this paper is to discuss the connection between various cone conditions, paying special attention to cone conditions that have been used in general equilibrium theory. For a complete (almost) up to date discussion on the existence of supporting functionals the reader may want to consult the excellent monograph of R. R. Phelps [39].

We first turn our attention to the classical notions of properness. We compare the cone condition for unordered preferences introduced by Yannelis and Zame [49]—called *extreme desirability*—with the uniform properness condition of Mas-Colell. Yannelis and Zame's extreme desirability condition [49] appears to be the most widely used cone condition

[5] G. Chichilnisky and G. M. Heal also established results on the existence of the equilibrium and the second welfare theorem using variations on Klee's cone condition; these were later published in [21].

for proving existence of equilibrium with unordered preferences. Though it is well known that extreme desirability is the same as uniform properness when preferences are complete preorderings, the relationship between the two conditions when preferences are neither complete nor transitive has remained obscure. We establish several results, and show that when preferences are irreflexive and have open lower sections extreme desirability implies uniform properness. We also show that these two conditions are no longer comparable if this continuity assumption is dropped.

Moreover, we show that when there is a continuum of agents assuming that each agent has uniformly proper preferences, or that each satisfies the extreme desirability assumption, does not guarantee that the existence of equilibrium result of R. Aumann [11], the second theorem of welfare economics, and R. Aumann's core-equivalence theorem [10] hold. There is a sharp difference between models with a finite number and a continuum of consumers. In the later case one needs to impose stronger assumptions; see, for example, the positive results of A. Rustichini and N. C. Yannelis [43, 44].

We then turn our attention to an investigation of more recent notions of properness. There are several shortcomings of the uniform properness assumption and recent notions of properness have been introduced to alleviate these shortcomings. Uniform properness requires that the marginal rates of substitution be bounded. In particular, this assumption excludes preferences whose marginal rates of substitution display a "fanned" effect. For example, preferences that satisfy the Inada condition or preferences that display changing risk aversion and that have been noted in the literature on non-expected utility theory, see for instance Machina [33]. Furthermore, in most of the results that prove the existence of equilibrium in the literature using uniformly proper preferences the authors also assume that the strong assumption that consumption sets coincide with the positive cone of the commodity space. We shall, in this paper, establish that the notions of properness used in the recent results of [9, 46] are indeed more general than uniform properness, that they allow for "fanned" marginal rates of substitution, and that they allow for a richer class of consumption sets.

A final caveat is in order. We do not discuss the various cone conditions that have appeared in the literature on general equilibrium theory with non-convexities (see for instance [23]). Though, these cone conditions are closely related to Klee's cone condition and therefore to uniform properness, a review of this literature is beyond the scope of the present paper.

The paper is organized as follows. The Appendix (Section 4) lists some definitions and examines the relationship between cone conditions the separation of convex sets, and supporting properties. It also includes Klee's cone condition—the forerunner of properness. In Section 2 we turn our attention to uniform properness and extreme desirability. Section 3 discusses the more recent notions of properness.

2. UNIFORM PROPERNESS AND EXTREME DESIRABILITY

Many of the problems that arise in general equilibrium with infinite dimensional commodity spaces can be appreciated by studying appropriate finite dimensional examples. In fact, most of the examples in this paper are set in a two commodity general equilibrium model. Let us begin this section by considering an adaptation of K. J. Arrow's classical *exceptional case* [15, Fig. 3, p. 27]. In Fig. 1, there are two commodities and one consumer and utility function $u(x, y) = \sqrt{x} + \sqrt{y}$. The initial endowment of the consumer is $\omega = (0, 1)$, which is the only Pareto optimal allocation. However, since the marginal rate of substitution is infinite at ω, there is no positive price that supports the consumer's preferences at ω and gives a non-zero value to ω.[6] This is a well known failure of the second welfare theorem. Of course, here the problem can be assumed away by requiring that ω be an interior point of the positive orthant of $\mathbb{R}^2$. However, the standard infinite dimensional setting is one where the well-known cheeper point problem cannot be readily assumed away.

We shall fix some notation before proceeding any further. For the rest of this paper L will denote a Hausdorff locally convex ordered topological vector space with positive cone L_+ and topology τ. A correspondence $P\colon X \twoheadrightarrow X$ on a subset X of L is said to be:

(1) *irreflexive*, if $x \notin P(x)$ for all $x \in X$,

(2) *transitive*, if $x, y, z \in X$, $x \in P(y)$, and $y \in P(z)$ imply $x \in P(z)$,

(3) *convex-valued*, if $P(x)$ is a convex subset of X for all $x \in X$,

(4) *locally non-satiated*, if for each $x \in X$ and each neighborhood V of x we have $V \cap P(x) \neq \varnothing$ (or, equivalently, if $x \in \overline{P(x)}$ for each $x \in X$), and

(5) with *open lower sections*, if for each $y \in X$ the set $P^{-1}(y) = \{x \in X : y \in P(x)\}$ is a τ-open subset of X.

[6] In exchange economies with consumption sets that coincide with the positive cone of the commodity space, the notion of a quasi-equilibrium is vacuous without requiring that the equilibrium price gives a non-zero value to the total endowment of resources. Indeed, if the total endowment ω is chosen so that there exists a non-zero positive price p for which $p \cdot \omega = 0$, then every feasible allocation is a quasi-equilibrium with respect to the price p. Therefore, for the quasi-equilibrium notion to be useful, we must assume that the supporting price p satisfies $p \cdot \omega \neq 0$.

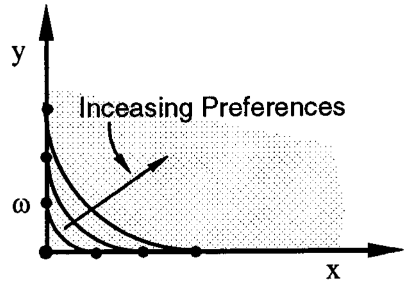


FIGURE 1

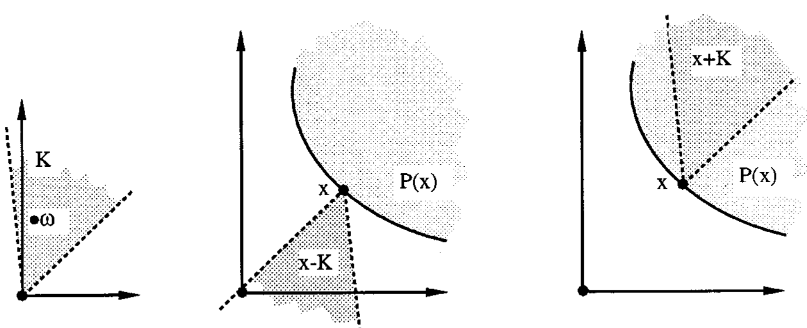
We now state the definitions of Mas-Colell's [36] uniform properness and the extreme desirability of Yannelis and Zame [49].

Definition 2.1. Let $P: X \twoheadrightarrow X$ be a correspondence on a subset X of L. Then:

(1) (Mas-Colell [36]) P is said to be *ω-uniformly proper* for some $\omega \in L$ if there exists a τ-open convex cone K with vertex zero such that $\omega \in K$ and $(x - K) \cap P(x) = \varnothing$ for every $x \in X$.

(2) (Yannelis and Zame [49]) A bundle $\omega \in X$ is called *extremely desirable* for the correspondence P if there exists a non-empty τ-open convex cone K with vertex zero such that $\omega \in K$ and $(x + K) \cap X \subseteq P(x)$ for every $x \in X$.

The geometric interpretation of these definitions (when $L = \mathbb{R}^2$ and $X = \mathbb{R}^2_+$) is illustrated in Fig. 2.



(a) The open cone containing ω

(b) The uniform propernes of P

(c) The extreme desirability of ω

FIGURE 2

We emphasize here that if the uniform properness assumption is to have its intended effect, then one must require that preferences be v-uniformly proper for some positive bundle v that is order dominated by the total endowment of resources ω, i.e., $0 \leqslant v \leqslant \omega$. Similarly, for the extreme desirability condition to be useful one needs to assume that there exists an extremely desirable bundle that is order dominated by the total endowment of resources. In fact, a quick glance at Arrow's exceptional case above shows that the preference relation is $(1, 1)$-uniformly proper—and that the bundle $(1, 1)$ is extremely desirable—but it is not ω-uniformly proper. It is also clear how ω-uniform properness excludes Arrow's exceptional case—it places a bound on the marginal rates of substitution at ω. However, it is not at all clear how the extreme desirability of ω precludes this exceptional case. The next results shed some light on this matter.

The next theorem is the major result in this section. It states that under very mild conditions, the existence of an extremely desirable commodity implies uniform properness.

THEOREM 2.2. *If a correspondence $P\colon L_+ \twoheadrightarrow L_+$ is irreflexive, convex-valued with open lower sections and $0 < \omega \in L$ is extremely desirable, then P is ω-uniformly proper.*

Proof. Let K be a τ-open convex cone satisfying $\omega \in K$ and $(x+K) \cap L_+ \subseteq P(x)$ for every $x \in L_+$. To finish the proof, it suffices to show that $(x-K) \cap P(x) = \emptyset$ for every $x \in X$. To this end, assume by way of contradiction that this is not the case.

This means that there exist vectors $x, z \in L_+$ such that $z \in (x-K) \cap P(x)$. Since $x \notin P(x)$, it follows that $x \neq z$. Clearly, $z \in P(x)$ is equivalent to $x \in P^{-1}(z)$. From

$$\lim_{\alpha \uparrow 1} [(1-\alpha)\, x + \alpha z] = z \qquad \text{and} \qquad \lim_{\alpha \uparrow 1} [\alpha x + (1-\alpha)\, z] = x,$$

$z \in x-K$, $x \in P^{-1}(z)$, the τ-openess of $x-K$ (in L), and the openess of $P^{-1}(z)$ (in L_+), it follows that there exists some $0 < \alpha_0 < 1$ such that

$$(1-\alpha)\, x + \alpha z \in x - K \qquad \text{and} \qquad \alpha x + (1-\alpha)\, z \in P^{-1}(z)$$

for all $\alpha_0 < \alpha < 1$. Now let $\gamma = (1+\alpha_0)/2$ and $x' = \gamma x + (1-\gamma)\, z \in L_+$. From $x' \in P^{-1}(z)$, we see that $z \in P(x')$. Moreover, from $(1-\gamma)\, x + \gamma z \in x - K$, there exists some $k \in K$ such that $(1-\gamma)\, x + \gamma z = x - k$, and so $z = \gamma x + (1-\gamma)\, z - k = x' - k \in x' - K$. Thus, $x' - z \in K$. Next consider the vector

$$y = \beta x' + (1-\beta)\, z = \beta\gamma x + (1-\beta\gamma)\, z,$$

and notice that for some $\beta > 1$ (close to 1) we must have $\alpha_0 < \beta\gamma < 1$. This implies $y \in L_+$. On the other hand, taking into account that $x' - z \in K$, we see that for this $\beta > 1$ we have

$$y = \beta x' + (1-\beta)\, z = x' + (\beta - 1)(x' - z) \in x' + K\,.$$

Therefore, $y \in (x' + K) \cap L_+ \subseteq P(x')$.

Now re-writing $y = \beta x' + (1-\beta)\, z$ in the form $x' = \frac{1}{\beta} y + (1 - \frac{1}{\beta})\, z$ and taking into account that $P(x')$ is a convex set (and that $y, z \in P(x')$), we see that $x' \in P(x')$, which contradicts the irreflexivity of P. Hence, $(x - K) \cap P(x) = \varnothing$ for every $x \in X$, and so P is ω-uniformly proper. ▮

The next example shows that one cannot dispense with the "openess of the lower sections" in Theorem 2.2.

EXAMPLE 2.3. Let $L = \mathbb{R}^2$ be equipped with the Euclidean topology and the pointwise ordering and let $\omega = (0, 1)$. Let $\geqslant_\ell$ denote the lexicographic ordering of $\mathbb{R}^2$; that is, $(x, y) \geqslant_\ell (x', y')$ if either $x > x'$ or else $x = x'$ and $y \geqslant y'$. Now define the correspondence $P\colon \mathbb{R}^2_+ \twoheadrightarrow \mathbb{R}^2_+$ by

$$P(x, y) = \begin{cases} \{(x', y') \in \mathbb{R}^2_+ : (x', y') \geqslant_\ell (0, 1)\} & \text{if} \quad (x, y) = (0, 1) \\ \{(x', y') \in \mathbb{R}^2_+ : y' > y\} & \text{otherwise.} \end{cases}$$

It is easy to see that the vector ω is extremely desirable for P with respect to the open convex cone K with vertex zero given by

$$K = \{(x, y) \in \mathbb{R}^2 : y > 0 \text{ and } y > -x\}.$$

However, if C is an arbitrary open convex cone with vertex zero containing ω, then it should be clear that $(\omega - C) \cap P(\omega) \neq \varnothing$ holds. This shows that P cannot be ω-uniformly proper. Finally, notice that from

$$P^{-1}(\omega) = \{(x, y) \in \mathbb{R}^2_+ : \omega \in P(x, y)\} = \{(x, y) \in \mathbb{R}^2_+ : y < 1\} \cup \{\omega\},$$

it easily follows that $P^{-1}(\omega)$ is not an open subset of $\mathbb{R}^2_+$ (since ω cannot be an interior point of $P^{-1}(\omega)$ in L_+), and so P does not have open lower sections. ▮

A converse of Theorem 2.2 is true for complete preferences.

THEOREM 2.4. *Let $\succcurlyeq$ be a complete preference relation on a subset X of L, and as usual define its* strict preference correspondence *$P\colon X \twoheadrightarrow X$ by*

$$P(x) = \{y \in X : y \succcurlyeq x \text{ and } x \not\succcurlyeq y\} = \{y \in X : y \succ x\}$$

for each $x \in X$. *If* $\{y \in X : y \succcurlyeq x\} \subseteq \overline{P(x)}$ *holds for every* $x \in X$ *and* P *is* ω*-uniformly proper, then* ω *is extremely desirable for* P.

Proof. Let K be a τ-open convex cone with vertex zero satisfying $(z - K) \cap P(z) = \emptyset$ for each $z \in X$ and $\omega \in K$. Assume that for a pair of bundles $x, y \in X$ we have $y \in x + K$, i.e., $x \in y - K$. It suffices to show that $y \in P(x)$.

To see this, assume by way of contradiction that $y \notin P(x)$. This implies (in view of the completeness of $\succcurlyeq$) that $x \succcurlyeq y$, so that $x \in \overline{P(y)}$. Now from $x \in y - K$ and the opennes of K, we see that there exists a τ-neighborhood V_x of x such that $V_x \subseteq y - K$. Also from $x \in \overline{P(y)}$, it follows that there exists some $v \in V_x$ such that $v \in P(y)$. This implies $v \in (y - K) \cap P(y) = \emptyset$, which is impossible. This shows that $y \in P(x)$, and so $(x + K) \cap X \subseteq P(x)$ for each $x \in X$, proving that ω is extremely desirable for P. ▮

The following theorem is related to Theorem 2.2.

THEOREM 2.5. *Let* $P : X \twoheadrightarrow X$ *be a correspondence on a subset of* L *such that whenever* $x, y \in X$ *satisfy* $x \in P(y)$, *then* $y \notin P(x)$. *If* $\omega \in L$ *is extremely desirable for* P, *then* P *is* ω*-uniformly proper.*

Proof. Let K be a non-empty open convex cone with vertex zero such that $\omega \in K$ and $(x + K) \cap X \subseteq P(x)$ for each $x \in X$. We claim that $(x - K) \cap P(x) = \emptyset$ holds for each $x \in X$. Indeed, if for some $x \in X$ we have $z \in P(x)$ and $z \in x - K$, then $x \in z + K$, which (in view of the extreme desirability of P) implies $x \in P(z)$, a contradiction. ▮

There are many examples of uniformly proper preferences without any extremely desirable commodity. For instance, the trivial case of satiated preferences where $P(x) = \emptyset$ for all x, has no extremely desirable commodity but it is uniformly proper. To continue our discussion, we need the notion of majorization between correspondences

DEFINITION 2.6. A correspondence $Q : X \twoheadrightarrow X$ *majorizes* (or *dominates*) another correspondence $P : X \twoheadrightarrow X$ if $P(x) \subseteq Q(x)$ holds for all $x \in X$.

Likewise, a correspondence $Q : X \twoheadrightarrow X$ on a topological space *weakly majorizes* (or *weakly dominates*) another correspondence $P : X \twoheadrightarrow X$ if $P(x) \subseteq \overline{Q(x)}$ holds for all $x \in X$.

In the next theorem we establish that certain uniformly proper preferences are majorized by preferences with extremely desirable bundles.

THEOREM 2.7. *Let* X *be a convex subset of* L *such that* $x + \alpha\omega \in X$ *for all* $x \in X$ *and* $\alpha \geqslant 0$, *and let* $P : X \twoheadrightarrow X$ *be a locally non-satiated, convex-valued,* *correspondence. Assume further that P has open lower sections in X and is ω-uniformly proper for some bundle $\omega \in X$. The following statements hold*:

(1) *There exists a correspondence that weakly majorizes P which is irreflexive, convex-valued, has an open graph in $X \times X$, and has ω as an extremely desirable bundle.*

(2) *If, in addition, P is irreflexive and has an open graph in $X \times X$, then there exists a correspondence that majorizes P which is irreflexive, convex-valued, has an open graph in $X \times X$, and has ω as an extremely desirable bundle.*

Proof. (1) Let K be a nonempty τ-open convex cone with vertex zero satisfying $\omega \in K$ and $(x - K) \cap P(x) = \emptyset$ for each $x \in X$. We claim that the correspondence $Q: X \twoheadrightarrow X$ defined by

$$Q(x) = [K + P(x)] \cap X$$

satisfies the desired properties.

Note first that Q is convex-valued. Moreover, if $y \in P(x)$, then from $\alpha\omega + y \in Q(x)$ for each $\alpha \geqslant 0$ and $\lim_{\alpha \downarrow 0}(\alpha\omega + y) = y$, it follows that $y \in \overline{Q(x)}$. Thus, $P(x) \subseteq \overline{Q(x)}$ for each $x \in X$, and so Q weakly dominates P.

We show next that the bundle ω is extremely desirable for Q. To see this, fix $x \in X$, take an arbitrary $y \in (x + K) \cap X$ and let $z = y - x \in K$. Since z is an interior point of K, there is some neighborhood V of zero such that $z + V \subseteq K$. Let $\{x_\alpha\}$ be a net in $P(x)$ that converges to x; such a net always exists since P is locally non-satiated. Clearly, $y_\alpha = z + x_\alpha \to y$. Noting that $-V$ is also a neighborhood of 0, we see that for some α large enough $y_\alpha \in y - V$, and so $y \in x_\alpha + z + V \subseteq x_\alpha + K$. Thus, $y \in Q(x)$ and $(x + K) \cap X \subseteq Q(x)$, proving that ω is extremely desirable for Q.

We show next that Q is irreflexive. To see this, suppose by way of contradiction that $x \in Q(x)$ for some $x \in X$. This implies that there is some $y \in P(x)$ such that $x \in y + K$. Therefore, $(x - K) \cap P(x) \neq \emptyset$, which is a contradiction.

Now we show that the graph of Q is open in $X \times X$. To see this, let $x, y \in X$ satisfy $(x, y) \in \text{Graph}(Q)$, i.e., $y \in Q(x)$. This means that there is some $z \in P(x)$ such that $y \in z + K$; clearly $z + K$ is a neighborhood of y. Since $x \in P^{-1}(z)$ and $P^{-1}(z)$ is an open subset of X, there exists a neighborhood V of zero such that $(x + V) \cap X \subseteq P^{-1}(z)$. To complete the proof, we shall show that the neighborhood $W = [(x + V) \times (z + K)] \cap (X \times X)$ of (x, y) in $X \times X$ satisfies $W \subseteq \text{Graph}(Q)$.

To this end, let $(x', y') \in W$. Then $x', y' \in X$, $x' = x + v$ for some $v \in V$ and $y' = z + k$ for some $k \in K$. It follows that $x' \in P^{-1}(z)$. This implies $z \in P(x')$, and so $y' = z + k \in [P(x') + K] \cap X = Q(x')$. Therefore, $(x', y') \in \text{Graph}(Q)$, and so $W \subseteq \text{Graph}(Q)$, proving that $\text{Graph}(Q)$ is open in $X \times X$.

(2) Let $C = K \cup \{0\}$ and note that C is a convex cone. We claim that the correspondence $R: X \twoheadrightarrow X$ defined by

$$R(x) = [C + P(x)] \cap X$$

satisfies the desired properties.

Note first that $R(x) = Q(x) \cup P(x)$ for all $x \in X$. Therefore, R dominates P and the bundle ω is also extremely desirable for R, since $Q(x) \subseteq R(x)$. Furthermore, R is convex-valued.

We show next that R is irreflexive. To see this, suppose by way of contradiction that $x \in R(x)$ for some $x \in X$. Since Q is irreflexive it must be that $x \notin Q(x)$. Therefore, $x \in P(x)$, which contradicts the assumption that P is irreflexive.

Now we show that the graph of R is open in $X \times X$. To see this, let $x, y \in X$ satisfy $(x, y) \in \text{Graph}(R)$, i.e., $y \in R(x)$. Now if $y \in Q(x)$, then there exists a neighborhood W of $X \times X$ such that $W \subseteq \text{Graph}(Q) \subseteq \text{Graph}(R)$. If on the other hand $y \in P(x)$, then there exists a neighborhood W of $X \times X$ such that $W \subseteq \text{Graph}(P) \subseteq \text{Graph}(R)$, proving that $\text{Graph}(R)$ is open in $X \times X$. ∎

We now turn our attention to economies with a continuum of agents. In their papers, A. Rustichini and N. C. Yannelis [43, 44] use notions of properness that are stronger than extreme desirability to extend R. Aumann's [10] important core-equivalence theorem to separable Banach lattices. We present an example that shows that ω-uniform properness not strong enough an assumption when there are infinitely many agents. We show that the existence of equilibrium result of R. Aumann [11], the second theorem of welfare economics, and Aumann's core-equivalence theorem fail when endowments of resources are not in the interior of the consumption sets. This is even though consumption sets coincide with the positive cone of $\mathbb{R}^2$ and preferences are strictly monotone and ω-uniformly proper.[7]

The following example can be seen as an extension of Arrow's exceptional case to economies with a continuum of agents. Notice that each agent has bounded marginal rates of substitution. However, the "mean" marginal rate of substitution at ω is infinite.

Example 2.8. The space of agents is a complete atomless measure space (Ω, Σ, μ); we assume that $\mu(\Omega) = 1$. The commodity space is $\mathbb{R}^2$ and

[7] This example complements the examples of R. Tourky and N. C. Yannelis [48], where it is shown that core-equivalence and the existence of equilibrium fail if the space of agents is atomless and the commodity space is non-separable or preferences do not satisfy Aumann's measurability assumption.

consumption sets are $\mathbb{R}^2_+$. For each consumer, $t \in \Omega$, the endowment of resources ω_t is the point $(0, 1)$. Let $\omega = \int_\Omega \omega_t \, d\mu(t) = (0, 1)$.

Now note that if $t \mapsto (x_t, y_t) \in \mathbb{R}^2_+$ is an allocation and $\int_S (x_t, y_t) \, d\mu(t) = \int_S \omega_t \, d\mu(t)$ for some measurable coalition S (i.e., $S \in \Sigma$), then

$$x = 0 \qquad \text{and} \qquad 0 \leqslant y_t \leqslant 1 \text{ for } \mu\text{-almost all } t. \tag{$*$}$$

In particular, any feasible allocation must satisfy property $(*)$.

Now let V_n be a disjoint sequence of non-negligible measurable subsets of Ω. Such a sequence exits since (Ω, Σ, μ) is atomless. For each $t \in V_n$ define the utility function $U_t \colon \mathbb{R}^2_+ \to \mathbb{R}_+$ by

$$U_t(x, y) = nx + y \, .$$

The functions U_t are linear and strictly monotone. Furthermore, each U_t is ω-uniformly proper. We need to show that Aumann's measurability assumption also holds. This assumption states that if $t \mapsto r(t)$ and $t \mapsto s(t)$ are two measurable allocations, then the set $S = \{t \in \Omega : U_t(r(t)) > U_t(s(t)\}$ is a measurable subset of Ω. Since each U_t is a linear function, it is clear that $S_n = \{t \in V_n : U_t(r(t)) > U_t(s(t)\}$ is a measurable set for each n. Hence, $S = \bigcup_{n=1}^\infty S_n$ is also measurable since it is the union of countable many measurable sets. Therefore, Aumann's measurability assumption holds.

Thus, with the exception of the strict positivity of ω, all of Aumann's assumptions are satisfied. We show next that the existence of equilibrium and Aumann's core-equivalence theorem fail in this economy.

Consider the allocation $t \mapsto \omega_t$. It is clear by $(*)$ and the monotonicity of preferences that no non-negligible coalition of agents can improve upon this allocation. Therefore, the allocation is in the core of the economy. Moreover, it is the only core allocation in this economy, since by $(*)$ it Pareto dominates all other feasible allocations.

Now suppose that there is some price system (p, q) that supports this allocation as a valuation equilibrium. That is, for any n,

$$(x, y) \in \mathbb{R}^2_+$$

and

$$U_t(x, y) = nx + y > 1 = U_t(\omega_t) \qquad \text{imply} \quad px + qy > p0 + q1 = q.$$

Take the point $(0, 2)$, it must be that $2q > q$ and $q > 0$. Taking the point $(\frac{1}{q}, 0)$ and $n > q$ we see that $\frac{p}{q} > q > 0$ and that $p > 0$. Finally, take the point $(\frac{q}{2p}, 0)$ and $n > \frac{2p}{q}$. We get $\frac{1}{2}q > q$, which is a contradiction.

Therefore, there is no price system that supports this core allocation as a valuation equilibrium—this also shows that the second welfare theorem is not valid. Also, since this is the only core allocation, the economy does not have a Walrasian equilibrium. ▌

The literature contains many instances where the authors assume that ω is extremely desirable in the Mackey topology of $\langle L, L' \rangle$ and that the preference correspondence is non-transitive and has *weakly* open lower sections. We close the section with an example of a non-transitive, non-empty, and convex-valued preference correspondence with a weakly open graph which is also irreflexive, strictly monotone and ω-uniformly proper—and also proper according to Definition 3.1 in the next section. Obviously, such a preference is not representable by a utility function.

EXAMPLE 2.9. The commodity space is the Hilbert space $L = \ell_2$ equipped with its norm topology τ. Let p_1 and p_2 be the two strictly positive prices of $L' = \ell_2$ given by

$$p_1 = (1, 1, \tfrac{1}{2}, \tfrac{1}{2^2}, \tfrac{1}{2^3}, ...) \qquad \text{and} \qquad p_2 = (1, 4, \tfrac{1}{2}, \tfrac{1}{2^2}, \tfrac{1}{2^3}, ...).$$

Next, we consider the correspondences $P_1, P_2 : \ell_2^+ \twoheadrightarrow \ell_2^+$ defined by

$$P_1(x) = \{ y \in \ell_2^+ : p_1 \cdot y > p_1 \cdot x \text{ and } p_2 \cdot y > p_2 \cdot x \},$$

and

$$P_2(x) = \{ y \in \ell_2^+ : p_1 \cdot y > p_1 \cdot x \}.$$

Now define the correspondence $P : \ell_2^+ \twoheadrightarrow \ell_2^+$ by

$$P(x) = \begin{cases} P_1(x) & \text{if} \quad p_1 \cdot x \leqslant 1 \\ P_2(x) & \text{if} \quad p_1 \cdot x > 1. \end{cases}$$

Then the correspondence P has the following properties:

(1) P is non-empty, convex.

(2) P is irreflexive.

(3) P is strictly monotone.

(4) P has open graph in the weak topology (hence also the norm topology) of $\ell_2^+ \times \ell_2^+$.

(5) P is v-proper for every vector $v > 0$, which is also extremely desirable (for the norm topology).

To see this consider the correspondence $\hat{P} : \ell_2^+ \twoheadrightarrow \ell_2$ defined by

$$\hat{P}(x) = \begin{cases} \hat{P}_1(x) & \text{if} \quad p_1 \cdot x \leqslant 1 \\ \hat{P}_2(x) & \text{if} \quad p_1 \cdot x > 1, \end{cases}$$

where

$$\hat{P}_1(x) = \{ y \in \ell_2 : p_1 \cdot y > p_1 \cdot x \text{ and } p_2 \cdot y > p_2 \cdot x \},$$

and

$$\hat{P}_2(x) = \{ y \in \ell_2 : p_1 \cdot y > p_1 \cdot x \}.$$

Then $\hat{P}(x) \cap \ell_2^+ = P(x)$ for all $x \in \ell_2^+$ and $x + v$ is in the *norm* interior of $\hat{P}(x)$ for all $x \in \ell_2^+$ and each non-zero positive vector $v \in \ell_2^+$.

(6) P is a non-transitive correspondence.
To see this, consider the tree vectors of ℓ_2^+:

$$x = (2, 0, 0, 0, ...), \qquad y = (\tfrac{1}{2}, 1, 0, 0, 0, ...), \qquad \text{and} \qquad z = (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, ...).$$

Then, an easy computation shows that $x \in P(y)$, $y \in P(z)$, and $x \notin P(z)$. ∎

3. PROPER CORRESPONDENCES

Let us return to the adaptation in the previous section of Arrow's exceptional case. It is easy to see that in that particular example one does not need the full strength of ω-uniform properness or extreme desirability. Indeed, a boundedness condition on the indifference curve at ω will suffice. Such an assumption was offered by R. Tourky [46] who—inspired by Mas-Colell's uniformly proper production sets [37]—termed it M-properness. He proved a limit theorem for the core of an economy using only this properness condition and without making any extra assumption on the consumption sets. Furthermore, the condition has recently been used by Aliprantis, Tourky, and Yannelis [9] to extend the literature on the existence of equilibrium and on the welfare theorems in infinite dimensional spaces to commodity spaces that are not lattice ordered. The condition allows for unbounded consumption sets and for preferences with unbounded marginal rates of substitution. In the next definition, we restrict our attention to the case of lower bounded consumption sets.

DEFINITION 3.1 (Tourky). Let X be a subset of L_+. A correspondence $P: X \twoheadrightarrow X$ is said to be *v-proper*, where v is a non-zero vector in L_+, if there exists another correspondence $\hat{P}: X \twoheadrightarrow L$ (which is convex-valued if P is also convex-valued) such that for each $x \in X$:

(i) the vector $x + v$ is a τ-interior point of $\hat{P}(x)$; and

(ii) $\hat{P}(x) \cap L_+ = P(x)$.

Now assume that $\succcurlyeq$ is a preference relation on a subset X of L_+. Then the preference $\succcurlyeq$ gives rise to two natural correspondences $P, Q\colon X \twoheadrightarrow X$ defined by

$$P(x) = \{ y \in X : y \succ x \} \qquad \text{and} \qquad Q(x) = \{ y \in X : y \succcurlyeq x \}.$$

The correspondence Q is called the *preference correspondence* of $\succcurlyeq$ and P is called (as we saw in Theorem 2.4) the *strict preference correspondence* of $\succcurlyeq$. We say that the preference relation $\succcurlyeq$ is *ω-uniformly proper* if its preference correspondence Q is ω-uniformly proper.

For complete preferences, it turns out that Tourky's properness condition is weaker than the Mas-Colell properness condition.

THEOREM 3.2. *Let $\succcurlyeq$ be a complete and transitive preference defined on the positive cone L_+, and let $\omega > 0$. If $\succcurlyeq$ is ω-uniformly proper, then its strict preference correspondence $P\colon L_+ \twoheadrightarrow L_+$, defined by*

$$P(x) = \{ y \in L_+ : y \succ x \},$$

is Tourky ω-proper according to Definition 3.1. *The converse is false even if $\succcurlyeq$ is continuous*, *strictly monotone*, *and convex.*

Proof. Assume that $\succcurlyeq$ satisfies the stated properties. Pick an open convex τ-neighborhood W of zero such that the open convex cone K with vertex zero generated by $\omega + W$ satisfies

$$(x - K) \cap Q(x) = (x - K) \cap \{ z \in L_+ : z \succcurlyeq x \} = \varnothing$$

for each $x \in L_+$. Notice that $x + \frac{1}{2}\omega \succ x$, i.e., $x + \frac{1}{2}\omega \in P(x)$ holds for each $x \in X$. Indeed, if this is not the case, then (by completeness) we must have $x \succcurlyeq x + \frac{1}{2}\omega$ for some $x \in X$, and so $(x + \frac{1}{2}\omega) - \frac{1}{2}\omega = x \succcurlyeq x + \frac{1}{2}\omega$. This implies $x \in (x + \frac{1}{2}\omega - K) \cap Q(x + \frac{1}{2}\omega) = \varnothing$, a contradiction. Next, define the correspondence $\hat{P}\colon L_+ \twoheadrightarrow L$ by

$$\hat{P}(x) = P(x) \cup [K + P(x)] = P(x) + [K \cup \{0\}].$$

Since $x + \omega + \frac{1}{2}W = (x + \frac{1}{2}\omega) + \frac{1}{2}(\omega + W) \subseteq \hat{P}(x)$, we see that $x + \omega$ is an interior point of $\hat{P}(x)$ and $\hat{P}$ is convex-valued if P is convex-valued. Moreover, we claim that $\hat{P}(x) \cap L_+ = P(x)$. To see this, note first that $P(x) \subseteq \hat{P}(x) \cap L_+$ is trivially true. Now let $y \in \hat{P}(x) \cap L_+$. Assuming $y \in P(x) + K$, this means that there exist $z \in P(x)$, $\lambda > 0$ and some $u \in W$ such that $y = z + \lambda(u + \omega)$. We claim that $y \succ x$. If this is not the case, then by completeness, we must have $x \succcurlyeq y$. Now notice that the vector $z \in L_+$ satisfies

$$z = y - \lambda(\omega + u) \succ x \succcurlyeq y,$$

which implies $z \in (y - K) \cap Q(y) = \varnothing$, which is a contradiction. Hence, $y \succ x$, and so $\hat{P}(x) \cap L_+ = P(x)$. Therefore, the correspondence P is Tourky ω-proper.

We now show by means of an example that the converse is not true. The example below is a modification of an example presented by R. Tourky in [46]. Let $L = \mathbb{R}^2$ and $X = \mathbb{R}^2_+$. Also, let $\omega = (0, 1)$. Consider the preference $\succcurlyeq$ represented by the continuous, quasi-concave and strictly monotone utility function $U: \mathbb{R}^2_+ \to \mathbb{R}$ defined by

$$U(x, y) = \tfrac{1}{2}(x + \sqrt{x^2 + 4y}).$$

Since for $c \geqslant 0$ we have $\frac{1}{2}(x + \sqrt{x^2 + 4y}) = c$ if and only if $y + cx = c^2$, we see that the indifference curves of the utility function U are straight lines, some of which are shown in Fig. 3(a).

Next, we claim that the preference $\succcurlyeq$ is not ω-uniformly proper. Indeed, if a cone K contains ω as an interior point, then it is easy to see that at some point $x = (x_0, y_0) \in \mathbb{R}^2_+$ we must have $(x - K) \cap \{(x, y) \in \mathbb{R}^2_+ : (x, y) \succcurlyeq (x_0, y_0)\} \neq \varnothing$, which shows that $\succcurlyeq$ is not ω-uniformly proper; see Fig. 3(b).

Now consider the strict preference correspondence $P: \mathbb{R}^2_+ \twoheadrightarrow \mathbb{R}^2_+$ of U, defined by

$$P(x, y) = \{(s, t) \in \mathbb{R}^2_+ : U(s, t) > U(x, y)\}.$$

We claim that the correspondence P is Tourky ω-proper according to Definition 3.1. To see this, fix $(x_1, y_1) \in \mathbb{R}^2_+$, and notice that

$$\begin{aligned} P(x_1, y_1) &= \{(x, y) \in \mathbb{R}^2_+ : U(x, y) > U(x_1, y_1)\} \\ &= \{(x, y) \in \mathbb{R}^2_+ : y + cx > c^2\}, \end{aligned}$$

where $c = \frac{1}{2}(x_1 + \sqrt{x_1^2 + 4y_1})$. Now if we define $\hat{P}: \mathbb{R}^2_+ \twoheadrightarrow \mathbb{R}^2$ by

$$\hat{P}(x_1, y_1) = \{(x, y) \in \mathbb{R}^2 : y + cx > c^2\},$$

then $\hat{P}(x_1, y_1)$ is an open convex subset of $\mathbb{R}^2$ such that $\hat{P}(x_1, y_1) \cap \mathbb{R}^2_+ = P(x_1, y_1)$ and the vector $(x_1, y_1) + \omega = (x_1, 1 + y_1)$ is an interior point of $\hat{P}(x_1, y_1)$. ∎

The example in the above proof shows how uniform properness does not allow for the unbounded "fanning" of the the marginal rates of substitution.

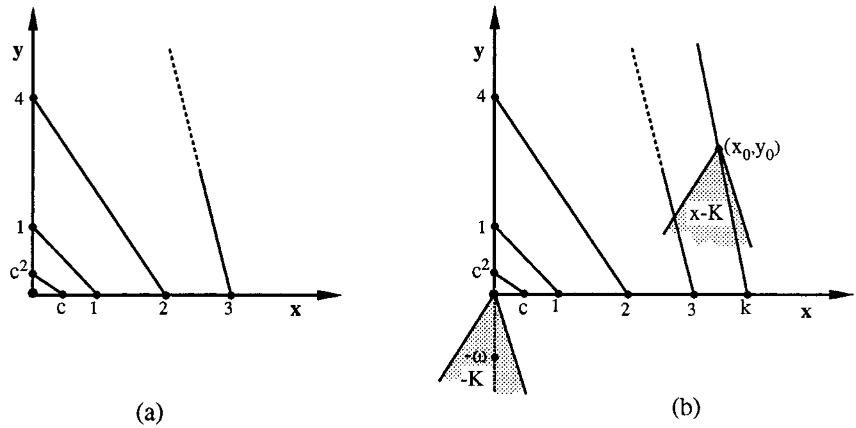


FIGURE 3

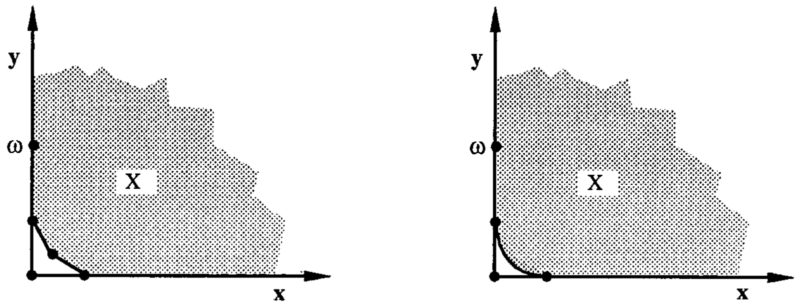
Preferences that display such behavior include those that satisfy the Inada condition. Furthermore, the literature on non-expected utility is permeated with examples of "fanned" indifference curves. Indeed, Machina [33] showed that "fanning" can explain several empirical anomalies in the theory of choice under uncertainty. Tourky properness allows for unbounded fanning of indifference curves, however, it precludes marginal rates of substitution that are infinite. That is, it precludes the preferences in Arrow's counter-example. Notice that the preferences in Fig. 1 are not ω-uniformly proper and thus are not Tourky proper, since the marginal rate of substitution is infinite at ω.

Tourky properness also allows for consumptions sets that are different from the positive cone of the commodity space. In most of the results that prove the existence of equilibrium in the literature using uniformly proper preferences it is also assumed that the consumption sets coincide with the positive cone of the commodity space. However, K. Back [16] produced examples which show that these results cannot be extended to more general convex lower bounded sets without making additional assumptions on the consumption sets. To compensate for this, K. Back introduced the following notion of proper consumption sets.

Definition 3.3 (Back [16]). A subset X of L_+ is said to be *Back ω-uniformly proper*, where ω is a non-zero bundle, if there exists an open cone K with vertex zero such that $\omega \in K$ and $(K + X) \cap L_+ \subseteq X$.

See Fig. 4 for Back and non-Back uniformly proper sets.

With this concept of properness Back [16] proved the existence of equilibrium in exchange economies. He also showed that the Back ω-uniformly



(a) A Back uniformly proper set (b) A non-Back uniformly proper set

FIGURE 4

proper consumption sets are extendible to convex sets containing ω as an interior point—a result which was also obtained earlier in [42] for uniformly proper preferences.

The next results presents a connection between Back and Tourky properness. They show that Tourky properness allows for more general consumption sets than Back's proper consumption sets.

THEOREM 3.4. *Let X be a subset of L_+ and let $P\colon X \twoheadrightarrow X$ be a transitive correspondence with an extremely desirable bundle $\omega \in L_+$. If the set X is also Back ω-uniformly proper, then P is also a Tourky ω-proper correspondence.*

Proof. It is easy to see that our assumptions guarantee the existence of an open convex cone K with vertex zero such that:

(1) $\omega \in K$,

(2) $(X+K) \cap L_+ \subseteq X$, and

(3) $(x+K) \cap L_+ \subseteq P(x)$ for each $x \in X$.

Now define the correspondence $\hat{P}\colon X \twoheadrightarrow L$ via the formula

$$\hat{P}(x) = K \cup \{0\} + P(x).$$

We claim that $\hat{P}$ satisfies the desired properties. Notice first that $\hat{P}(x)$ is a convex set if $P(x)$ is also convex.

We claim that $\hat{P}(x) \cap L_+ = P(x)$ holds for each $x \in X$. To see this fix $x \in X$ and note that $P(x) \subseteq \hat{P}(x) \cap L_+$ is trivially true. For the reverse inclusion, let $y \in \hat{P}(x) \cap L_+$ and assume that y is of the form $y = k + z$ with $k \in K$ and $z \in P(x)$. Then, $y = z + k \in (z+K) \cap L_+ \subseteq P(z)$, and so $y \in P(z)$. Now using that $z \in P(x)$, it follows from the transitivity of P that $y \in P(x)$. Thus, $\hat{P}(x) \cap L_+ \subseteq P(x)$, and therefore $\hat{P}(x) \cap L_+ = P(x)$.

To finish the proof, we must verify that $x+\omega$ is an interior point of $\hat{P}(x)$ for each $x \in X$. So, fix $x \in X$. Since ω, $\frac{\omega}{2} \in K$ and K is an open set, there exists some neighborhood V of zero such that $\frac{\omega}{2}+V \subseteq K$ and $\omega+V \subseteq K$. We claim that $\omega+x \in P(x)$ and that $\omega+x+V \subseteq \hat{P}(x)$. To see this, notice first that $\frac{\omega}{2}+x \in L_+$ and $\frac{\omega}{2}+x \in x+K$ imply $\frac{\omega}{2}+x \in (x+K) \cap L_+$, which (in view of (3)) yields $\frac{\omega}{2}+x \in X$. Then, $\frac{\omega}{2}+x \in (x+K) \cap X$, and so, from (2), $\frac{\omega}{2}+x \in P(x)$. (A similar argument shows that $\omega+x \in P(x)$.) Now note

$$x+\omega+V=\frac{\omega}{2}+\left(\frac{\omega}{2}+x+V\right) \subseteq K+P(x) \subseteq \hat{P}(x).$$

Therefore, $\omega+x$ is an interior point of $\hat{P}(x)$, and so (according to Definition 3.1) P is a ω-proper correspondence.

In the next example we construct a consumption set with a lower bound and a smooth "substance" frontier. This consumption set does not satisfy Back's ω-uniform properness condition. We then construct a transitive continuous preference ordering on that set which gives rise to a Tourky ω-proper correspondence.

EXAMPLE 3.5. Consider the Euclidean space $\mathbb{R}^2$ under its canonical ordering and let $\omega=(0,1)$. Also let B be the closed unit ball in $\mathbb{R}^2$ centered at the point $(1,1)$. Now consider the consumption set

$$X=B \cup \{(x, y) \in \mathbb{R}^2_+ : x+y \geqslant 1\}.$$

Clearly, X is closed, convex, and bounded from below; see Fig. 5. However, X is not Back ω-uniformly proper. This follows easily by observing that if K is an open convex cone with vertex zero containing ω, then $(K+X) \cap \mathbb{R}^2_+ \nsubseteq \mathbb{R}^2_+$.

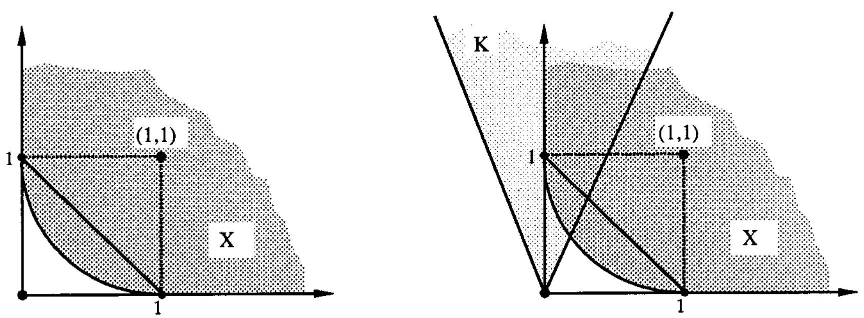


FIGURE 5

We now define a correspondence $P: X \twoheadrightarrow X$ as follows. For any $(x, y) \in X$ satisfying $x + y \geqslant 1$, we let

$$P(x, y) = \{(x', y') \in \mathbb{R}^2_+ : x' + y' > x + y\}.$$

For any other $(x, y) \in X$ let H be the straight line passing through (x, y) and $(0, 1)$ and let $Q(x, y)$ be the open half-plane determined by H that contains $(1, 1)$. Clearly, $Q(x, y)$ is an open convex set that contains $(x, y) + \omega$. For these points $(x, y) \in X$, we let

$$P(x, y) = Q(x, y) \cap X.$$

An easy verification shows that P is transitive, convex, monotone, and has an open graph in $X \times X$. It is also easy to check that $P(x, y)$ is Tourky ω-proper. (In addition, it can be seen that P is not ω-uniformly proper and that ω is not extremely desirable.) In fact, for each $x \in X$ let

$$\hat{P}(x) = P(x) \cup [Q(x) \cap \{(x, y) : x \leqslant 0\}]$$

and note that $\hat{P}$ satisfies the desired properties. ∎

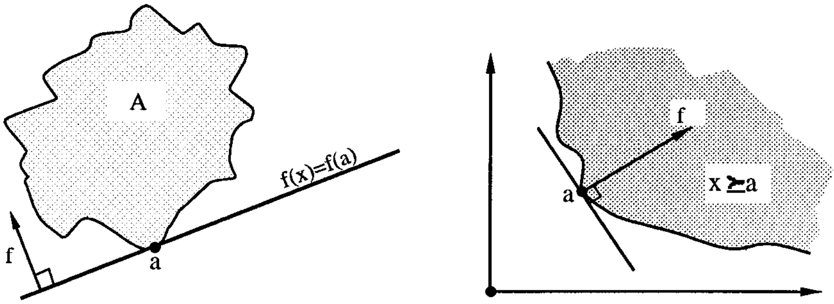
4. APPENDIX: MATHEMATICAL BACKGROUND

Let A be a non-empty subset of a topological vector space. A point $a \in A$ is said to be a *support point* of A if there exists a non-zero continuous linear functional f (called a *supporting linear functional* of A at a) satisfying $f(a) \leqslant f(x)$ for all $x \in A$.[8] In other words, a non-zero continuous linear functional f supports a set A at some point $a \in A$ if and only if f attains its minimum value over the set A at the point a. We also say that A has the *support property* at a if A is supported at a. It should be clear that only boundary points of a set can be support points.

A preference relation $\succcurlyeq$ defined on a subset S of a topological vector space is said to be *supported* at some point $a \in S$ by a non-zero continuous linear functional f if $x \in S$ and $x \succcurlyeq a$ imply $f(x) \geqslant f(a)$, i.e., if the better-than-a set of $\succcurlyeq$ is supported by f at a. The geometric interpretation of the support property is shown in Fig. 6.

The supporting property at a point is closely related with the notion of a cone. We shall discuss this relationship next. But first, let us start with the definition of a cone.

[8] Any non-zero continuous linear functional f satisfying $f(x) \geqslant f(a)$ for each $x \in A$ is called a *tangent functional* by Dunford and Schwartz [26, Definition V.9.4, p. 447].



(a) Supporting the set A at a by f (b) Supporting a preference at a by f

FIGURE 6

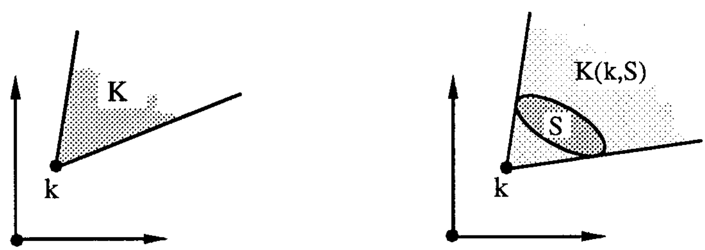
Definition 4.1. A non-empty subset K of a vector space is called a *cone* with vertex k, if for each $x \in K$ we have $k+\alpha(x-k)=\alpha x+(1-\alpha)k \in K$ for any $\alpha \geqslant 0$; see Figure 7(a). If a cone is also a convex set, then it is called a *convex cone*.

In a topological vector space, an *open cone* K with vertex k is a non-empty open set such that for each $x \in K$ we have $k+\alpha(x-k) \in K$ for all $\alpha > 0$.

Every subspace is a cone with vertex zero. Notice that if K is a cone with vertex k, then $c+K$ is a cone with vertex $c+k$. In a vector space, the intersection of all cones with vertex k that contain a given non-empty subset S is called the *cone generated* by S with vertex k and is denoted $K(k, S)$. Clearly,

$$K(k, S)=\{k+\alpha(x-k): \alpha \geqslant 0 \text{ and } x \in S\}.$$

Similarly, the *convex cone with vertex k* generated by S is the smallest convex cone with vertex k and coincides with the convex hull of $K(k, S)$; see



(a) A cone with vertex k (b) The cone generated by S

FIGURE 7

Fig. 7(b). A straightforward verification shows that if S is a convex set, then $K(k, S)$ is a convex cone for each k. In a topological vector space, the *closed cone generated by a subset* with vertex k is the intersection of all closed cones with vertex k that contain S. Clearly, the closed cone generated by S with vertex k is $\overline{K(k, S)}$.

The next result describes a basic supporting property of a cone. (This result was established in [30, 31], see also the first part of [26, Theorem V.9.10, p. 452].)

LEMMA 4.2 (Klee). *In a locally convex space the vertex of a convex cone is a point of support of the cone if and only if the cone is not dense.*

The following is a simple theorem that characterizes the support property of a convex set at a given vector—it is the basic principle behind all cone conditions. Note that the equivalence between (1) and (2) was first established by V. L. Klee [30, Corollary 1, p. 769].

THEOREM 4.3. *Let C be a convex subset in a locally convex space and let c be a boundary point of C. If $c \in C$, then the following statements are equivalent.*

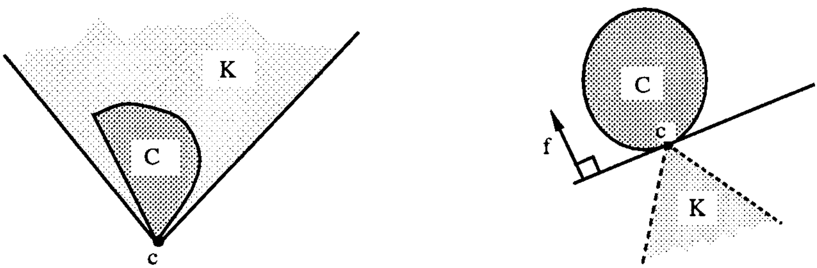
(1) *The vector c is a point of support of C.*

(2) *There is a non-dense convex cone K with vertex c that includes C, i.e., $C \subseteq K$; or, equivalently, the convex cone $K(c, C)$ is not dense.*

(3) *There exists an open convex cone K with vertex c such that $K \cap C = \varnothing$; or, equivalently, there exists an open convex cone with vertex zero such that $(c + K) \cap C = \varnothing$.*

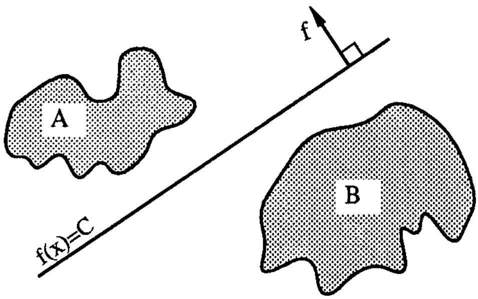
(4) *There exist a non-zero vector v and a neighborhood V of zero such that $c - \alpha v + z \in C$ with $\alpha > 0$ implies $z \notin \alpha V$.*

The geometry of the situation in Theorem 4.3 is shown in Fig. 8.



(a) Condition (2) (b) Condition (3)

FIGURE 8



The separation of the sets A and B by f

FIGURE 9

Recall that two subsets A and B of a topological vector space can be *separated* if there exist a non-zero continuous linear functional f (called a *separating linear functional*) and some constant C such that $f(b) \leqslant C \leqslant f(a)$ holds for all $a \in A$ and all $b \in B$ (or equivalently, if $f(b) \leqslant f(a)$ holds for all $a \in A$ and all $b \in B$.) The geometrical meaning of the separation of two sets is shown in Fig. 9.

LEMMA 4.4. *Two non-empty subsets A and B in a locally convex space can be separated if and only if the convex cone with vertex zero generated by the set $A - B$ is not dense.*

LEMMA 4.5. *In a finite dimensional Hausdorff topological vector space, if zero does not belong to a non-empty convex set C, then the convex cone with vertex zero generated by C is not dense.*

In finite dimensional vector spaces two non-empty disjoint convex sets can always be separated.

THEOREM 4.6. *In a finite dimensional Hausdorff topological vector space any two non-empty disjoint convex sets can be separated.*

As mentioned before, only boundary points of a set can be support points. Unfortunately, not every boundary point of a non-empty closed convex set is a support point. The next result presents two cases where the support points of a closed convex subset are precisely its boundary points. In most of the results that prove the existence of equilibrium in the literature using uniformly proper preferences they also assume that the consumption sets coincide with the positive cone of the commodity space.

LEMMA 4.7. *Let C be a non-empty convex subset of a topological vector space (X, τ). If either,*

(1) *(X, τ) is finite dimensional and τ is Hausdorff, or*

(2) *the convex set C has a τ-interior point,*

then every boundary point of C that lies in C is a support point.

In spite of the fact that in infinite dimensional spaces not every boundary point of a non-empty closed convex set is a support point, we have the following remarkable result due to E. Bishop and R. R. Phelps [19]. There are many proofs of this result which are variations of the original proof; see, for instance, [39, Theorem 3.18, p. 48], [29, Theorem 3.8.14, p. 127], and [2, Theorem 8.60, p. 329].

THEOREM 4.8 (Bishop–Phelps). *If C is a non-empty closed convex subset of a Banach space, then the set of support points of C is dense in the boundary of C.*

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