Collusion and distribution of profits under differential information*

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Abstract. We examine a Cournot game with differential private information. We study collusion under different information rules, i.e., when firms pool their private information, use their common knowledge information, or decide not to share their private information at all. We put the industry profits under the three different information schemes in a hierarchy. In addition, we look at the incentive compatibility problem and we show that only collusion under common knowledge information is incentive compatible. Finally, we deal with the issue of how the industry profits are distributed among the firms, in a way that asymmetries are captured. We propose the *Shapley value* as a proper way to distribute the industry profits among the firms. We also point out that the α-core associated with the Cournot game with differential information is non-empty.

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1 Introduction

We study the collusion of firms with differential information. A game with differential information consists of a finite number of firms, where each firm is characterized by its strategy set, its payoff function, its private information (which is a partition of an exogeneously given probability measure space) and a prior. When firms collude, they choose an output level that maximizes joint expected profits. The information firms can use in the collusive agreement varies. Firms may pool their information, may use their private information, or they may choose to use their common knowledge information. Each type of information sharing yields different profits and most importantly creates different incentives to the individual firms for

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misreporting their true information.

The main focus of our paper is to address the following questions: i) How should firms share their private information in collusive agreement such that industry profits are the highest? ii) How are the collusive profits under different types of information sharing compared to profits from non-cooperative production? iii) How should colluding firms share their private information in a coalitional incentive compatible way? iv) How should industry profits be distributed among firms in a way which captures the contribution or the "worth" of each firm to total profits?

Other work on the subject [e.g., Donsimoni et al. (1986) and Crampton and Palfrey (1990)] follow an approach similar to ours by assuming that firms abide by the cartel agreement. The problem of explicit collusion in an industry with heterogeneous firms and private information was first considered formally by Roberts (1983). He derives properties of the incentive compatibility constraints associated with a revelation game. He found that without side payments, if firms are sufficiently similar, then monopoly collusion cannot be achieved, but if side payments are allowed such collusion is possible with a dominant strategy mechanism essentially equivalent to the Vickrey (second-price) auction. Rotemberg and Saloner (1990) investigate a price leadership scheme in a differentiated products duopoly in which the firms are asymmetrically informed. Crampton and Palfrey (1990) study the issue of cartel enforcement when the cost of each firm is private information. An enforceable cartel is one which is feasible, incentive compatible and individually rational. They show that if defection results in either Cournot or Bertrand competition, the incentive problem in large cartels is severe enough to prevent the cartel from achieving the monopoly outcome. Laffont and Martimort (1997) study collusion of agents whose objectives are not aligned with that of their organization under asymmetric information.

Our model is different from the above ones. In particular, we have a general model and we address the issue of collusion in a differential information game for the first time. We show that collusion under the pooled information yields the highest industry profits. However, this type of information sharing is not coalitional incentive compatible.[1] We present examples with two firms where one firm can distinguish between two states of nature and the other cannot and the firm with the "superior" information finds it profitable to misreport the true state of nature to the other firm. Only collusion under the common knowledge information is coalitional incentive compatible. It is important to emphasize here that we look at the coalitional incentive compatibility and not at the individual incentive compatibility as, for example, Crampton and Palfrey (1990). An individual incentive compatible

[1] A collusive agreement is coalitional incentive compatible when there does not exist a coalition of firms that can misreport the true state of nature and benefit its members. For a precise definition see definition 7.1.

outcome may not be coalitional incentive compatible which in turn means that coalitions of firms, rather than individual firms, may have an incentive not to report truthfully the realized state of nature. We also propose that a sensible rule for allocating production and distributing the profits among the firms is according to the Shapley value of each firm. The Shapley value rule yields individually rational and Pareto optimal outcomes, captures the informational asymmetries between the firms as well as the contribution of each firm to the total profits. We also point out that the α-core of the differential information game is non-empty. We provide several examples that illustrate and clarify our results.

The rest of the paper is organized as follows. In Section 2, we provide the notation and definitions. In Section 3, we present the model and Section 4 contains existence results. In Section 5, we outline three information rules and in Section 6 we rank the industry profits under the different information rules. Section 7 addresses the incentive compatibility issue. In Section 8, the issue of profit distribution is addressed. Finally, Section 9 contains two illustrative examples.

2 Notation and definitions

2.1 Notation

$\mathbb{R}^l$ denotes the l-fold Cartesian product of the set of real numbers.
$\mathbb{R}^l_+$ denotes the positive cone of $\mathbb{R}^l$.
$\mathbb{R}^l_{++}$ denotes the strictly positive elements.
2^A denotes the set of all non-empty subsets of the set A.
$\emptyset$ denotes the empty set.
$\setminus$ denotes set theoretic subtraction.

2.2 Definitions

If X and Y are sets, the *graph* of the set-valued function (or correspondence), $\phi : X \to 2^Y$ is denoted by

$$G_\phi = \{(x, y) \in X \times Y : y \in \phi(x)\}.$$

Let $(\Omega, \mathcal{F}, \mu)$ be a complete, finite measure space, and X be a separable Banach space. The set-valued function $\phi : \Omega \to 2^X$ is said to have a *measurable graph* if $G_\phi \otimes \beta(X)$, where $\beta(X)$ denotes the Borel σ-algebra on X and $\otimes$ denotes the product σ-algebra. The set-valued function $\phi : \Omega \to 2^X$ is said to be *lower measurable* or just *measurable* if for every open subset V of X, the set

$$\{\omega \in \Omega : \phi(\omega) \cap V \neq \emptyset\}$$

is an element of $\mathcal{F}$. It is well known that if $\phi : \Omega \to 2^X$ has a measurable graph, then ϕ is lower measurable. Furthermore, if $\phi(\cdot)$ is closed valued and lower measurable then $\phi : \Omega \to 2^X$ has a measurable graph. A theorem of Aumann tells us that if $(\Omega, \mathcal{F}, \mu)$ is a complete finite measure space, X is a separable metric space and $\phi : \Omega \to 2^X$ is a non-empty valued correspondence having a measurable graph, then $\phi(\cdot)$ admits a *measurable selection*, i.e., there exists a measurable function $f : \Omega \to X$ such that $f(\omega) \in \phi(\omega), \mu - a.e.$

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and X be a Banach space. Following Diestel-Uhl (1977), the function $f : \Omega \to X$ is called *simple* if there exist $x_1, x_2, ..., x_n$ in X and $\alpha_1, \alpha_2, ..., \alpha_n$ in $\mathcal{F}$ such that $\sum_{i=1}^{n} x_i \chi_{\alpha_i}$ where $\chi_{\alpha_i}(\omega) = 1$ if $\omega \in \alpha_i$ and $\chi_{\alpha_i}(\omega) = 0$ if $\omega \notin \alpha_i$. A function $f : \Omega \to X$ is said to be *μ-measurable* if there exists a sequence of simple function $f_n : \Omega \to X$ such that $\lim_{n\to\infty} \|f_n(\omega) - f(\omega)\| = 0$ for almost all $\omega \in \Omega$. A *μ-measurable* function $f : \Omega \to X$ is said to be *Bochner integrable* if there exists a sequence of simple functions $\{f_n : n = 1, 2, ...\}$ such that

$$\lim_{n\to\infty} \int_{\Omega} \|f_n(\omega) - f(\omega)\| d\mu(\omega) = 0.$$

In this case we define for each $E \in \mathcal{F}$ the integral to be

$$\int_E f(\omega) d\mu(\omega) = \lim_{n\to\infty} \int_E f_n(\omega) d\mu(\omega).$$

It can be shown [see Diestel-Uhl (1977), Theorem 2, p.45] that if $f : \Omega \to X$ is a μ- measurable function then f is Bochner integrable if and only if $\int_\Omega \|f(\omega)\| d\mu(\omega) < \infty$.

For $1 \leq p < \infty$, we denote by $L_p(\mu, X)$ the space of equivalence classes of X-valued Bochner integrable functions $x : \Omega \to X$ normed by

$$\|x\|_p = \left(\int_\Omega \|x(\omega)\|^p d\mu(\omega)\right)^{\frac{1}{p}}.$$

It is a standard result that normed by the functional $\|\cdot\|_p$ above, $L_p(\mu, X)$ becomes a Banach space [see Diestel-Uhl (1977), p.50].

3 The Cournot game with differential information

We assume that there are n firms, $\{i = 1, \ldots, n\}$, that produce an output $q = \{q_1, \ldots, q_n\}$. The subscript $-i$ will be used to denote all firms other than firm i. Let $(\Omega, \mathcal{F}, \mu)$ be a complete probability measure space. We interpret Ω as the states of nature of the world and assume that it is large enough to include all events that we consider to be interesting. As usual $\mathcal{F}$ denotes the σ-algebra of events and μ is a common probability measure. Let Y be a separable Banach space denoting the production space.

Definition 3.1: A *Cournot game with differential information* is a set $\mathcal{C} = \{(Q_i, \pi_i, \mathcal{F}_i, \mu) : i = 1, \ldots, n\}$, where

i) $Q_i : \Omega \to 2^Y$ is the *random production set* of firm i;

ii) $\pi_i : Q(\omega) \to \mathbb{R}$ is the *random profit function*[2] of firm i, (where $Q(\omega) = Q_1(\omega) \times \cdots \times Q_n(\omega)$);

iii) $\mathcal{F}_i$ is a sub σ-algebra of $\mathcal{F}$, which denotes the *private information* of firm i;

iv) μ is a probability measure on Ω denoting the *common prior.*

Let L_{Q_i} denote the set of all Bochner integrable and $\mathcal{F}_i$-measurable selections from the production set Q_i of firm i, i.e.,

$$L_{Q_i} = \{q_i \in L_1(\mu, Y) : q_i : \Omega \to Y \text{ is } \mathcal{F}_i - \text{measurable and } q_i(\omega) \in Q_i(\omega) \text{ and } \mu - a.e.\}.$$

Let $L_Q = L_{Q_1} \times \cdots \times L_{Q_n}$. Given a Cournot game, a production plan for firm i is an element $q_i \in L_{Q_i}$.

The *ex-ante expected profit function*[3] of firm i, $\Pi_i : L_Q \to \mathbb{R}$ is defined as[4]

$$\Pi_i(q_i, q_{-i}) = \int_{\omega\in\Omega} \pi_i(q_i(\omega), q_{-i}(\omega))d\mu(\omega).$$

A *Cournot-Nash equilibrium* for $C = \{(Q_i, \pi_i, \mathcal{F}_i, \mu) : i = 1, \ldots, n\}$ is an element $q^* \in L_Q$ such that for all i,

$$\Pi_i(q^*) = \max_{y_i \in L_{Q_i}} \Pi_i(q^*_{-i}, y_i).$$

We can now state the assumptions needed to prove the existence of a Cournot-Nash equilibrium.

(A.1)

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[2] *If $p(\omega) : Q(\omega) \to R$ is the inverse demand function and $C_i : Q_i(\omega) \to R$ is the cost function of firm i, then $\pi_i(q(\omega)) = p(q(\omega))q_i(\omega) - C_i(q_i(\omega))$. We could have allowed the payoff function π to depend also on the state of nature ω. The results of the paper remain valid.*

[3] The entire analysis would go through if instead of the ex-ante profit function we used the interim one. That is, the *conditional (interim) expected profit function* of firm i $\Pi_i(\cdot,\cdot) : L_{Q_i} \times Q_{-i}(\omega) \to R$ is defined as

$$\Pi_i(q_i, \bar{q}_{-i}) = \int_{\omega' \in E_i(\omega)} \pi_i(\omega', q_i, \bar{q}_{-i}(\omega'))k_i(\omega'|E_i(\omega))d\mu(\omega'),$$

where

$$k_i(\omega'|E_i(\omega)) = \begin{cases} 0 & \text{if } \omega' \notin E_i(\omega) \\ \frac{q_i(\omega')}{\int_{\tilde{\omega}\in E_i(\omega)} q_i(\tilde{\omega})d\mu(\tilde{\omega})} & \text{if } \omega' \in E_i(\omega) \end{cases}$$

is the prior of agent i, (where k_i is a Radon-Nikodym derivative such that $\int k_i(\omega)d\mu(\omega) = 1$ and $E_i(\omega)$ denotes the event in firm i's partition which contains the realized state of nature).

[4] For simplicity we assume that the profit function does not depend on Ω. As we mentioned above all the results of the paper remain valid.

$Q_i : \Omega \to 2^Y$, is a non-empty, convex, weakly compact-valued and integrably bounded correspondence having an $\mathcal{F}_i$-measurable graph, i.e., $G_{Q_i} \in \mathcal{F}_i \otimes B(Y)$.

(A.2)

i) For each i, $\pi_i(\cdot,\cdot) : Q(\omega) \to \mathbb{R}$, is weakly continuous.

ii) The function π_i is concave in the i-th coordinate for all i.

iii) π_i is integrably bounded.

4 Existence of a Cournot-Nash equilibrium

We can now state the first existence result. We assume that there exists a finite or countable partition $\Lambda_i, (i = 1, \ldots, n)$ of Ω, and the σ-algebra $\mathcal{F}_i$ is generated by Λ_i.

Theorem 4.1: *Let $\mathcal{C} = \{(Q_i, \pi_i, \mathcal{F}_i, \mu) : i = 1, \ldots, n\}$ be a Cournot game satisfying (A.1) - (A.2). Then, there exists a Cournot-Nash equilibrium.*

Proof: For each i, define the correspondence $\varphi_i : L_{Q_{-i}} \to 2^{L_{Q_i}}$ by

$$\varphi_i(q^*_{-i}) = \{y_i \in L_{Q_i} : \Pi_i(q^*) = \max_{y_i \in L_{Q_i}} \Pi_i(q^*_{-i}, y_i)\}.$$

Also define the correspondence $F : L_Q \to 2^{L_Q}$ by

$$F(q) = \Pi^n_{i=1} \varphi_i(q^*_{-i}).$$

As in Yannelis and Rustichini (1991), we will show that the correspondence F satisfies all the hypotheses of the Fan-Glicksberg Fixed Point Theorem. It can then be easily checked that a fixed point of the correspondence F is by construction a Cournot-Nash equilibrium for $\mathcal{C}$. We will complete the proof in three steps.

I. *L_Q is non-empty, convex, weakly compact and metrizable.*

The non-emptiness of L_Q follows from the Aumann measurable selection theorem. Also, since each Q_i is non-empty, convex and weakly compact, it follows from Diestel's Theorem that each L_{Q_i} is a weakly compact subset of $L_1(\mu, Y)$. Obviously, each L_{Q_i} is convex. Furthermore, since each L_{Q_i} is a weakly compact subset of a separable Banach space $L_1(\mu, Y)$, it is also metrizable [for more details see Yannelis and Rustichini (1991), Theorem 5.1].

II. *The function Π_i is weakly continuous for each i.*

Since, by assumption, π_i is concave, weakly continuous and π_i is integrably bounded, the result follows by an application of Theorem 2.8 in Balder and Yannelis (1993).

III. *Each correspondence $\varphi_i : L_{Q_{-i}} \to 2^{L_{Q_i}}$, is non-empty, convex valued and weakly u.s.c.*

Since $\Pi(q)$ is a concave function of q_i on L_{Q_i}, it follows that φ_i is convex valued. By virtue of Berge's Maximum Theorem, it follows that φ_i is weakly u.s.c. Finally, an appeal to Weierstrass' Theorem it is guaranteed that φ_i is also a non-empty valued correspondence.

Now since each φ_i is non-empty, closed, convex valued and weakly u.s.c., it follows that likewise is $F : L_Q \to 2^{L_Q}$. Thus, the correspondence F satisfies all the conditions of the Fan-Glicksberg Fixed Point Theorem. Consequently, there exists some $q^* \in L_Q$ such that $q^* \in F(q^*)$. □

5 Collusion under different information rules

It is a well known result that a Cournot-Nash equilibrium may not be Pareto optimal. In other words, there is a surplus that has not been extracted by the firms. If the firms collude and play a cooperative game, then a Pareto optimal outcome will be reached. This problem has been examined when firms have symmetric information. However, in the presence of differential information there may be different ways for the firms to collude, depending on how they want to share their private information. Before we proceed, let's define the three different information rules that we will consider in the sequel.

Definition 5.1: A *Pooled information rule* is the one where firms share their information, i.e., the information they use is, $\mathcal{F}'_j = \vee_{i=1}^n \mathcal{F}_i, j = 1, \ldots, n$, where $\vee$ denotes the *join.*[5]

Definition 5.2: A *Private information rule* is the one where firms use their own private information, i.e., $\mathcal{F}_i, i = 1, \ldots, n$.

Definition 5.3: A *Common knowledge information rule* is the one where firms use only the information that is common to them, i.e., $\mathcal{F}'_j = \wedge_{i=1}^n \mathcal{F}_i, j = 1, \ldots, n$, where $\wedge$ denotes the *meet.*[6]

Let $L^p_{Q_i}$ denote the set of all Bochner integrable and $\vee_{i=1}^n \mathcal{F}_i$- measurable selections from the production set Q_i of firm i, i.e.,

$$\begin{aligned} L^p_{Q_i} &= \{q_i \in L_1(\mu, Y) : q_i : \Omega \to Y \text{ is } \vee_{i=1}^n \mathcal{F}_i - \text{measurable and} \\ q_i(\omega) &\in Q_i(\omega),\ \mu - a.e.\}. \end{aligned}$$

Also let $L^p_Q = L^p_{Q_1} \times \cdots \times L^p_{Q_n}$.

Let $L^c_{Q_i}$ denote the set of all Bochner integrable and $\wedge_{i=1}^n \mathcal{F}_i-$ measurable selections from the production set Q_i of firm i, i.e.,

$$\begin{aligned} L^c_{Q_i} &= \{q_i \in L_1(\mu, Y) : q_i : \Omega \to Y \text{ is } \wedge_{i=1}^n \mathcal{F}_i - \text{measurable and} \\ q_i(\omega) &\in Q_i(\omega), \mu - a.e.\}. \end{aligned}$$

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[5] That is the smallest σ-algebra containing all of the sub σ-algebras $\mathcal{F}_i, i = 1, \ldots, n$.

[6] That is the largest σ-algebra contained in all of the sub σ-algebras $\mathcal{F}_i, i = 1, \ldots, n$.

Also let $L_Q^c = L_{Q_1}^c \times \cdots \times L_{Q_n}^c$.

A *collusion equilibrium under the pooled information rule* for $C = \{(Q_i, \pi_i, \mathcal{F}_i, \mu) : 1, \ldots, n\}$ is an element $q^* \in L_Q^p$ such that

$$\Pi^p(q^*) = \max_{q \in L_Q^p} \sum_{i=1}^{n} \Pi_i(q).$$

A *collusion equilibrium under the private information rule* for $C = \{(Q_i, \pi_i, \mathcal{F}_i, \mu) : i = 1, \ldots, n\}$ is an element $q^* \in L_Q$ such that

$$\bar{\Pi}(q^*) = \max_{q \in L_Q} \sum_{i=1}^{n} \Pi_i(q).$$

A *collusion equilibrium under the common knowledge information rule* for $C = \{(Q_i, \pi_i, \mathcal{F}_i, \mu) : i = 1, \ldots, n\}$ is an element $q^* \in L_Q^c$ such that

$$\Pi^c(q^*) = \max_{q \in L_Q^c} \sum_{i=1}^{n} \Pi_i(q).$$

Next we present the second existence result of this paper.

Theorem 5.1: *Under assumptions (A.1)-(A.2), a collusion equilibrium exists for all the information rules.*

Proof: Notice that the objective function is weakly continuous and L_Q^p, L_Q and L_Q^c are non-empty and weakly compact. Therefore the maximum is attained and the argmax is the set of all equilibrium points. □

6 Comparison of profits under the three information rules

In this section, we will put the industry profits under the three different information rules in a hierarchy. It is known that under symmetric information the industry profits when firms collude are greater than or equal to the industry profits derived from the Cournot-Nash game. But what happens under differential information?

Let $\Pi^p(q^*), \bar{\Pi}(q^*)$ and $\Pi^c(q^*)$ be the value functions under the three information rules, as defined in the previous section.

Proposition 6.1: $\Pi^p(q^*) \geq \bar{\Pi}(q^*) \geq \Pi^c(q^*)$.

Proof: First observe that $\wedge_{i=1}^n \mathcal{F}_i \subseteq \mathcal{F}_i, i = 1, \ldots, n, \subseteq \vee_{i=1}^n \mathcal{F}_i$. This implies that $L_Q^c \subseteq L_Q \subseteq L_Q^p$. Since the objective functions are the same, the desired result follows. □

Let $\Pi^N(q^*)$ denote the industry profits derived from the Nash game.

Proposition 6.2: $\Pi^p(q^*) \geq \bar{\Pi}(q^*) \geq \tilde{\Pi}^N(q^*)$.

Proof: Obvious. □

However, as the following proposition indicates, we cannot compare the industry profits derived under the common knowledge rule with those derived from the Cournot-Nash game. The reason is that when firms collude using the common information, essentially they throw away some valuable information, which means lower profit. On the other hand, the joint maximization alone, gives them higher profits. In this general setting we cannot tell which effect outweighs the other.

Proposition 6.3: *The industry profits derived from the collusion under the common knowledge information rule and the ones derived from the Cournot-Nash game are not comparable.*

Proof: Consider a Cournot game with two firms $\{1,2\}$, three states of nature, i.e., $\Omega = \{a, b, c\}$ and one homogeneous output q. Each state occurs with the same probability. Each firm's private information is given by the following partition of the state space,

$$\mathcal{F}_1 = \{a, b, c\}, \mathcal{F}_2 = \{\{a\}, \{b\}, \{c\}\}.$$

The inverse demand function is, $p(\omega) = 5 - 1.5(q_1(\omega) + q_2(\omega))$. The cost function which is measurable with respect to each firm's private information is: For firm 1, $C_1(\omega, q_1(\omega)) = .4q_1^2$ for all $\omega \in \Omega$ and for firm 2,

$$C_2(\omega, q_2(\omega)) = \begin{cases} q_2^2 & \text{if } \omega = a \\ .4q_2^2 & \text{if } \omega = b \\ .8q_2^2 & \text{if } \omega = c. \end{cases}$$

Notice that firm 1 has trivial information, while firm 2 has complete information. The following production plan is a Cournot-Nash equilibrium,

$$q_1(\omega) = 1.00275, \text{ for all } \omega \in \Omega;$$

$$q_2(\omega) = \begin{cases} .699 & \text{if } \omega = a \\ .919 & \text{if } \omega = b \\ .759 & \text{if } \omega = c. \end{cases}$$

Observe that the production plan is also measurable with respect to each firm's private information. The expected industry profits from the Cournot-Nash game are 3.296.

Now assume that firms collude using the common knowledge information rule. Since the information that is common to both of them is the trivial information, the production plan must be constant in all states. This is,

$$q_1(\omega) = .9194, q_2(\omega) = .502, \text{ for all } \omega \in \Omega.$$

The expected industry profits are 3.5537.

Thus, in this example the profits from collusion with common knowledge information are higher than the profits from the Cournot-Nash game. Next we present an example where the profits from the Cournot - Nash game are higher than the profits from collusion with common knowledge information.

Consider now a Cournot game with two firms $\{1, 2\}$ that produce a homogeneous output q. Uncertainty is generated by the marginal cost functions $c_i : \Omega \to \mathbb{R}_+, i = 1, 2$. Firm 1 has trivial information and its cost function is: $C_1 = c_1 q_1^2$, where $c_1 = 5$ for all $\omega \in \Omega$. Firm 2 has complete information and its cost function is: $C_2 = c_2 q_2^2$. We assume that c_2 is distributed uniformly on $[0, 10]$. The above information about the Cournot game is common knowledge. Moreover, the inverse demand function is $p(\omega) = 50 - 1.5(q_1(\omega) + q_2(\omega))$. Since firm 1 has trivial information, its production plan will be constant across all states, while firm's 2 production plan will be contingent on each realization of the random variable c_2. Therefore, the Cournot-Nash equilibrium is

$$q_1(\omega) = 3.317, \text{ for all } \omega \in \Omega \text{ and } q_2(\omega) = \frac{22.5122}{1.5 + c_2(\omega)}.$$

The expected industry profits from the Cournot-Nash game are 174.747.

Now assume that firms collude using the common knowledge information rule. This now implies that production must be constant across all states. The production plan is

$$q_1(\omega) = q_2(\omega) = 3.125, \text{ for all } \omega \in \Omega.$$

The expected industry profits now are 156.25. □

7 Coalitional incentive compatibility

One of the basic questions that one may ask is whether the different equilibrium notions we defined previously are coalitional incentive compatible. That is, whether a coalition of firms has an incentive to misreport the true state of nature and benefit its members. This is an important question especially for the collusion equilibrium. If a collusion equilibrium is not coalitional incentive compatible, then it is not sustainable. We define rigorously below the notion of coalitional incentive compatibility which is related to the one in Krasa-Yannelis (1994).

Definition 7.1: An output function $q \in L_Q$ is said to be *coalitional incentive compatible* if and only if the following does not hold: There exist a coalition of firms[7] $S \subset I$ and two states a, b that members of $I \setminus S$ cannot distinguish (i.e., a and b are in the same partition for the firms in $I \setminus S$)

[7] I is the set of all firms.

and such that members of S are better off by announcing b whenever a has actually occurred. Formally, $q \in L_Q$ is said to be coalitional incentive compatible for C if it is not true that there exist coalition S, and states a, b with[8] $a \in \cap_{i \notin S} E_i(b)$, such that[9] $\pi_i(q^S(b), q^{I \setminus S}(b)) > \pi_i(q^S(a), q^{I \setminus S}(b))$, for all $i \in S$, that is, each firm in coalition S is strictly better off announcing that state b occurred rather than the true state a and firms not in S are unable to distinguish between state a and b.

It turns out that a Cournot-Nash equilibrium is incentive compatible. Also a collusion equilibrium under the common knowledge information rule is coalitional incentive compatible. However, a collusion equilibrium under the pooled information rule and under the private information rule may not be coalitional incentive compatible.

Proposition 7.1: *A Cournot-Nash equilibrium for $C = \{(Q_i, \pi_i, \mathcal{F}_i, \mu) : i = 1, \dots, n\}$ is incentive compatible.*

Proof: Since we are dealing with a non-cooperative concept it is appropriate to reduce the coalition S to the singleton coalition, i.e., $S = \{i\}$. Then, $-i$ denotes all the firms but i. Suppose that $q^* \in L_Q$ is a Cournot-Nash equilibrium and there exist a, b, where $a \in E_{-i}(b)$, such that

$$\pi_i(q_i^*(b), q_{-i}^*(b)) > \pi_i(q_i^*(a), q_{-i}^*(b)).$$

First, since q_{-i}^* is $\mathcal{F}_{-i}$-measurable, it is implied that $q_{-i}^*(a) = q_{-i}^*(b)$. Thus, for all $\omega \in E_i(a) \cap E_{-i}(a)$ and $t \in E_i(b) \cap E_{-i}(a)$,

$$\pi_i(q_i^*(t), q_{-i}^*(t)) > \pi_i(q_i^*(\omega), q_{-i}^*(\omega)).$$

Now consider the following production plan for firm i,

$$\tilde{q}_i(\omega) = \begin{cases} q_i^*(\omega) = q_i^*(t) & \text{if } \omega \in E_i(a) \cap E_{-i}(a) \\ q_i^*(\omega) & \text{otherwise.} \end{cases}$$

It follows that

$$\int \pi_i(\tilde{q}_i(\omega), q_{-i}^*(\omega)) d\mu > \int \pi_i(q_i^*(\omega), q_{-i}^*(\omega)) d\mu.$$

This contradicts the fact that q^* is a Nash equilibrium. □

Proposition 7.2: *A collusion equilibrium under the private information rule for $C = \{(Q_i, \pi_i, \mathcal{F}_i, \mu) : i = 1, \dots, n\}$ may not be coalitional incentive compatible.*

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[8] $E_i(b)$, is the event in firms' information partition that contains the realized state b.

[9] q^S and $q^{I \setminus S}$ are vectors of outputs for firms in coalition S and $I \setminus S$ respectively.

Proof: Consider two firms $\{1,2\}$ that produce a homogeneous output. The state space is $\Omega = \{a,b,c\}$ where each state occurs with probability $\frac{1}{3}$ and the private information of each firm is: $\mathcal{F}_1 = \{\{a,b\},\{c\}\}$ and $\mathcal{F}_2 = \{\{a,c\},\{b\}\}$. We denote by $q_1(\omega), q_2(\omega)$ the production in state ω of firm 1 and firm 2 respectively. The inverse demand that firms face is $p = (5 - 1.5(q_1(\omega) + q_2(\omega)))$. The marginal cost function of each firm, which is measurable with respect to each firm's private information is

$$c_1(\omega) = \begin{cases} .25 & \text{if } \omega = a, b \\ 1 & \text{if } \omega = c \end{cases};$$

$$c_2(\omega) = \begin{cases} .25 & \text{if } \omega = a, c \\ 1 & \text{if } \omega = b. \end{cases}$$

A collusion equilibrium under the private information rule is

$$q_1^*(a) = q_1^*(b) = .916, q_1^*(c) = .416,$$

$$q_2^*(a) = q_2^*(c) = .916, q_2^*(b) = .416,$$

and the profit of each firm in each state is

$$\pi_1(a) = 1.833, \pi_1(b) = 2.52, \pi_1(c) = .832,$$

$$\pi_2(a) = 1.833, \pi_2(b) = .832, \pi_2(c) = 2.52.$$

The ex-ante expected profit for the *industry* is $\Pi_1 + \Pi_2 = 1.725 + 1.725 = 3.45$.

Suppose that state b occurs. Firm 2's profit is .832. However, if firm 2 reports that state a occurred and produce as if a had actually occurred, its profit is 1.146. Since $1.146 > .832$ the collusion equilibrium with differential information is not incentive compatible. □

Remark: It follows from the above proposition that a collusion equilibrium under the pooled information rule may not be incentive compatible as well. The reason is that although information is now symmetric, still firms cannot distinguish between the states that could not distinguish before the pooling took place. Hence, the above proposition is applicable here as well.

Proposition 7.3: *A collusion equilibrium under the common knowledge information rule is incentive compatible.*

Proof: It is rather obvious, since now there do not exist states a and b such that one firm can distinguish between the two and the others cannot. □

8 Distribution of profits

So far, we showed that when firms collude (under the pooled and the private information rules), industry profits are higher than the profits obtained by the Cournot-Nash game. Moreover, profits may be higher when firms collude under the common knowledge information rule. The question that remains to be answered is how are these extra profits being distributed among the firms in a way that captures the contribution of each firm to the total profits.

8.1 The private value production plan

We propose that each firm should be rewarded according to its contribution to the total profits. One way to do this is to reward each firm according to its *Shapley value*. Then a production plan will be determined, so as each firm will get its Shapley value. This production plan will be the *value production plan*.

As in the definition of the standard value allocation concept, we must first derive a transferable profit game (TP) in which each firm's profits are weighted by a factor λ_i, $(i = 1, \dots, n)$, which allows for profits comparisons. In the value allocation itself no side payments are necessary.[10] A game with side payments is then defined as follows:

Definition 8.1.1: A game with side payments $\Gamma = (I, V)$ consists of a finite set of agents (firms) $I = \{1, \dots, n\}$ and a superadditive, real valued function V defined on 2^I such that $V(\emptyset) = 0$. Each $S \subset I$ is called a coalition and $V(S)$ is the "worth" of the coalition S.

The Shapley value of the game Γ (Shapley 1953) is a rule that assigns to each firm i a "payoff", Sh_i, given by the formula,[11]

$$Sh_i(V) = \sum_{\substack{S \subset I \\ S \supset \{i\}}} \frac{(|S| - 1)!(|I| - |S|)!}{|I|!} [V(S) - V(S \setminus \{i\})].$$

The Shapley value has the property that $\sum_{i \in I} Sh_i(V) = V(I)$, i.e., the Shapley value is Pareto efficient. Moreover, it is individually rational, i.e., $Sh_i \geq V(\{i\}), \forall i$.

We now define for each Cournot game with differential information, $\mathcal{C}$, and for each set of weights, $\{\lambda_i : i = 1, \dots, n\}$, the associated game with side payments (I, V_λ^p) (we also refer to this as a "transferable profits" (TP) game) as follows:

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[10] See Emmons and Scafuri (1985, p.60) for further discussion.

[11] The Shapley value measure is the sum of the expected marginal contributions a firm can make to all the coalitions of which it is a member.

For each coalition $S \subset I$, let

$$V_\lambda^p(S) = \max_q \sum_{i \in S} \lambda_i \int \pi_i(q^S(\omega), q^{I \setminus S}(\omega)) d\mu(\omega) = \max_{q \in L_Q} \sum_{i \in S} \lambda_i \Pi_i(q). \quad (8.1)$$

We are now ready to define the private value production plan.

Definition 8.1.2: An output plan $q \in L_Q$ is said to be a *private value production plan* of the Cournot game with differential information, C, if there exist $\lambda_i \geq 0$ ($i = 1, \ldots, n$, which are not all equal to zero), with

$$\lambda_i \Pi_i(q) = Sh_i(V_\lambda^p), \forall i,$$

where $Sh_i(V_\lambda^p)$ is the Shapley value of firm i derived from the game (I, V_λ^p), defined in (8.1).

The above definition says that the expected profits of each firm multiplied by its weight λ_i must be equal to its Shapley value derived from the (TP) game (I, V_λ^p).

An immediate consequence of Definition 8.1.2 is that the private value production plan is individually rational (profits for firm i are greater than of equal from the ones derived from the Cournot-Nash game). This follows immediately from the fact that the game (I, V_λ^p) is superadditive for all weights. In addition, it is Pareto efficient.[12]

We are now ready to state the first existence result of this section.

Theorem 8.1.1: *Let* $C = \{(Q_i, \pi_i, \mathcal{F}_i, \mu) : i = 1, \ldots, n\}$ *be a Cournot game as defined in Section 3, satisfying assumptions (A.1)-(A.2).*

Then, a private value production plan exists in C.

Proof: This result can be proved along the lines of Krasa and Yannelis (1996), Theorem 1. □

Remark: One can easily show that a pooled information (where the information that is being used is the pooled information) value production plan[13] exists as well.

8.2 The common knowledge value production plan

We now introduce another notion of a value production plan for the Cournot game with differential information. The difference stems from the measurability restriction on the type of production plans. It is an analog of the coarse core of Yannelis (1991). We call it a common knowledge value production plan, since the information that is being used is the common knowledge information. As we saw in the previous section a common

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[12] For more details see Krasa and Yannelis (1996), p.169.

[13] Since it is not incentive compatible we will not examine it thoroughly.

knowledge production plan is coalitional incentive compatible. Therefore, it is of great importance if we know that there is also a way to distribute the surplus (if there is any) in a manner that the contribution of each firm is rewarded.

We now define for each Cournot game with differential information, $\mathcal{C}$, and for each set of weights, $\{\lambda_i : i = 1, \dots, n\}$, the associated game with side payments (I, V^c_λ) (we also refer to this as a "transferrable profits" (TP) game) as follows:

For each coalition $S \subset I$, let

$$V^c_\lambda(S) = \max_q \sum_{i \in S} \lambda_i \int \pi_i(q^S(\omega), q^{I \setminus S}(\omega)) d\mu(\omega) \tag{8.2}$$

subject to

i) for each i, q_i is $\wedge_{i=1}^n \mathcal{F}_j$-measurable.

The *common knowledge value production plan* can now be defined as in definition (8.1.2), except that we replace (8.1) by (8.2) and also replace V^p by V^c.

Thus, in contrast to the private value production plan, we now require the production plan within a coalition to be based on the common knowledge information. Notice that the common knowledge value production plan is coalitional incentive compatible. However, we cannot prove a general existence theorem. In fact, if, for example, one firm has "trivial" information and the other has "full" information, the common knowledge information implies that the trivial information must be used and therefore the superadditivity condition of the function $V^c_\lambda(\cdot)$ may be violated, i.e., there can exist coalitions S, T with $S \cap T = \emptyset$ and $V^c_\lambda(S) + V^c_\lambda(T) > V^c_\lambda(S \cup T)$.[14] In the proof of proposition 6.3, we present an example with two firms where the Cournot-Nash equilibrium yields higher profits than collusion under common knowledge information, which destroys the superadditivity condition. This causes problems with the existence of a common knowledge value production plan. Therefore, we cannot prove a general existence theorem of a common knowledge value production plan.

9 Examples

Below we give examples with two firms, with differential information, that collude using the common knowledge information rule and the distribution of profits is determined by a common knowledge value production plan. These examples illustrate how the common knowledge production plan is determined and also show that firms with superior information, while keeping the other characteristics of the firms (i.e., marginal cost) fixed, have higher Shapley value and higher share of the industry profits. The profits

[14] See Krasa and Yannelis (1996), p.177, for more details.

from collusion are higher than the Cournot-Nash profits and the Shapley value of each firm captures its contribution to the total industry profits. It is important to note here that in these examples the value allocation is in the core and therefore the cartel can be viewed as *stable*, in the sense that no coalition of firms can deviate from the cartel agreement and become strictly better off.

Example 9.1

Consider two firms $\{1,2\}$ that produce a homogeneous product. The state space is $\Omega = \{a,b\}$ where each state occurs with probability $\frac{1}{2}$ and the private information of each firm is: $\mathcal{F}_1 = \{\{a\},\{b\}\}$ and $\mathcal{F}_2 = \{\{a,b\}\}$. We denote by $q_1(\omega), q_2(\omega)$ the production in state ω of firm 1 and firm 2 respectively. The inverse demand that firms face is: $p = (5 - \beta(\omega)(q_1(\omega) + q_2(\omega)))$. The marginal cost is zero for both firms. The slope β takes on the following values:

$$\beta(\omega) = \begin{cases} .8 & \text{if } \omega = a \\ 1.2 & \text{if } \omega = b \end{cases}$$

A Cournot-Nash equilibrium is

$$q_1^*(a) = 2.2916, q_1^*(b) = 1.25,$$

$$q_2^*(a) = q_2^*(b) = 1.666.$$

Notice that the production is measurable with respect to each firm's private information. The ex-ante expected profit for the *industry* is $\Pi_1 + \Pi_2 = 3.03819 + 2.77778 = 5.81597$.

Now suppose that the two firms collude under the common knowledge information rule. The information they use now is the trivial information and the optimum total production is 2.5. The expected industry profits are: 6.25. The problem that arises is how this surplus will be distributed among the two firms. Or put it in different words, what is the production that will be assigned to each firm? Without taking the information superiority of the first firm into account, both firms are identical. Hence, one solution would be just to split the profits. However, this is not a "fair solution" since firm 1 contributes more to the coalition than firm 2 does. The value production plan allocation we discussed above provides a more sensible outcome.

The Shapley value of the two firms (with $\lambda_1 = \lambda_2 = 1$) is

$$Sh_1 = \frac{1}{2}[6.25 - 2.77778] + \frac{1}{2}[3.03819] = 3.25521$$

$$Sh_2 = \frac{1}{2}[6.25 - 3.03819] + \frac{1}{2}[2.77778] = 2.99479.$$

Then, a value production will be a solution to the following problem:

$$(5 - (q_1 + q_2))q_1 = 3.25521$$

$$(5-(q_1+q_2))q_2 = 2.99479.$$

Hence, a value production plan is $q_1 = 1.30208$ and $q_2 = 1.19792$. To conclude, the firm with the superior information gets rewarded in the value production plan, by being assigned a higher level of production and thus higher profits ($\Pi_1 = 3.25521$ and $\Pi_2 = 2.99479$).

In the next example, the asymmetry of information comes from the cost side.

Example 9.2

Consider two firms $\{1, 2\}$ that produce a homogeneous product. The state space is $\Omega = \{a, b\}$ where each state occurs with probability $\frac{1}{2}$ and the private information of each firm is: $\mathcal{F}_2 = \{\{a\}, \{b\}\}$ and $\mathcal{F}_1 = \{\{a, b\}\}$. We denote by $q_1(\omega), q_2(\omega)$ the production in state ω of firm 1 and firm 2, respectively. The inverse demand that firms face is $p = (5 - 1.5(q_1(\omega) + q_2(\omega)))$. The marginal cost of each firm, which is measurable with respect to each firm's private information, is

$$c_2(\omega) = \begin{cases} .8 & \text{if } \omega = a \\ 1.2 & \text{if } \omega = b \end{cases},$$

$$c_1(\omega) = \begin{cases} 1 & \text{if } \omega = a \\ 1 & \text{if } \omega = b. \end{cases}.$$

A Cournot-Nash equilibrium is

$$q_1^*(a) = q_1^*(b) = .888889,$$

$$q_2^*(a) = .955556, q_2^*(b) = .822222.$$

The ex-ante expected profit for the *industry* is $\Pi_1 + \Pi_2 = 1.18519 + 1.19185 = 2.37704$.

Now suppose that the two firms collude under the common knowledge information rule. The information they use now is the trivial information and the optimum total production is 1.33333. The expected industry profits are 2.66667.

The Shapley value of the two firms (with $\lambda_1 = \lambda_2 = 1$) is

$$Sh_1 = \frac{1}{2}[2.66667 - 1.19185] + \frac{1}{2}[1.18519] = 1.33,$$

$$Sh_2 = \frac{1}{2}[2.66667 - 1.18519] + \frac{1}{2}[1.19185] = 1.33667.$$

Then, a value production will be a solution to the following problem:

$$(5 - 1.5(q_1 + q_2))q_1 - q_1 = 1.33,$$

$$(5 - 1.5(q_1 + q_2))q_2 - q_2 = 1.33667.$$

Hence, a value production plan is $q_1 = .665$ and $q_2 = .668333$. To conclude, as in the above example, the firm with the superior information gets rewarded in the value production plan, by being assigned a higher level of production and thus higher profits.

10 Concluding remarks

Remark 1: Alternatively, one could have used the notion of the $\alpha-$core which is defined as follows: We say that $q \in L_Q$ is an α-core of the game C if

> it is not true that there exist $S \subset I$ and $(y_i)_{i\in S} \in \Pi_{i\in S} L_{Q_i}$ such that for any $z^{I\backslash S} \in \Pi_{i\notin S} L_{Q_i}, \Pi_i(y^S, z^{I\backslash S}) > \Pi_i(q)$ for all $i \in S$.

It follows that under our assumptions in Section 3 the α-core is nonempty [see Yannelis (1991)]. A collusive agreement that is an element of the α-core is individually rational, Pareto optimal and coalitional stable. Although these are clearly desirable properties, we do not have a straightforward way of selecting an element from the core that would capture the "worth" of each firm. To this end, the Shapley value provides a relatively easy way of figuring out the contribution of each firm to the total profits and how to distribute them among the firms.

Remark 2: In the two firms case, the Shapley value is in the core and therefore in this case the duopoly with differential information can be viewed as stable. This is not the case for more than two firms unless the corresponding TU game is convex. Zhao (1998) provides necessary and sufficient conditions for the deterministic TU game to be convex. In a subsequent paper we intend to examine the conditions which guarantee the convexity of the side-payments game defined in (8.1) or (8.2).

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