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Coalitional Bayesian Nash implementation in differential information economies*

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Summary. A mechanism *coalitionally implements* a social choice set if any outcome of the social choice set can be achieved as a coalitional Bayesian Nash equilibrium of a mechanism and vice versa. We say that a social choice set is *coalitionally implementable* if there is a mechanism which coalitionally implements it. Our main theorem proves that a social choice set is coalitionally implementable if and only if it is interim individually rational, interim efficient, coalitional Bayesian incentive compatible, and satisfies a coalitional Bayesian monotonicity condition as well as a closure condition. As an application of our main result, we show that the private core and the private Shapley value of an economy with differential information are coalitionally implementable.

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JEL Classification Numbers: C71, D51, D78, D82.

1 Introduction

An economy with differential information consists of a finite set of agents, each of whom is characterized by a random utility function, a random initial endowment, a private information set, and a prior (a precise definition can be found in Section 2.1).

The traditional notion which has been adopted in the literature to analyze trade in a differential information economy is the (Walrasian) rational expectations equilibrium. One of the criticisms of the above notion is that it does not

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provide a mechanism which describes how the equilibrium prices reflect information asymmetries in the economy. To this end we adopt the private core (Yannelis, 1991) and the private value (Krasa and Yannelis, 1994) in order to analyze the trading procedure in a differential information economy. The private core and the private value are not fully cooperative in a differential information economy framework, because within a coalition agents make redistributions of their initial endowments based on their own private information (without necessarily sharing it). Hence, despite the fact that coalitions of agents get together and make redistributions (the cooperative aspect of the concepts), there is a noncooperative element in that agents in the coalition bargain using their differential information. This noncooperative feature of the private core and the private value results in allocations which are always coalitionally incentive compatible.[1] Moreover, these concepts provide sensible and reasonable outcomes in situations where the traditional rational expectations equilibrium fails to do so [for examples of this effect, see Koutsougeras and Yannelis (1993, pp. 206-207) and Krasa and Yannelis (1994, pp. 890-892) as well as the Example 3.1 in Section 3 of this paper].[2]

The outcomes generated by the private core or value are of interest because they resemble contracts, and contracts are a common means by which agents execute trade. In particular, in a contract it is common for agents to make an agreement ex ante (or interim), which is executed ex-post (for example insurance contracts). The allocation rules we consider, the private core and the private value, have the following properties that we believe are desirable. First, information asymmetries matter and agents benefit from superior information. Second, optimal contracts (i. e., private core or private value allocations) always exist, which is not the case for the rational expectations equilibrium. This matches the observation that contracts are more common than competitive markets in situations where differential information makes trade difficult.[3] In view of these attractive features that the private core and the private value possess, it is important to know whether or not they are implementable, *i.e.*, can a game be constructed whose equilibrium outcomes coincide with the private core or the private value? This knowledge will enable us not only to understand better the outcomes that these allocation rules generate but also to distinguish and compare them from the traditional (Walrasian) rational expectations equilibrium.

Our implementation results indicate that indeed information asymmetries matter and the stringent informational conditions needed for the Bayesian Nash implementation of the Walrasian expectations equilibrium (see for example Blume and Easley, 1990; Palfrey and Srivastava, 1987; Postlewaite and Schmeidler, 1986) are not needed. In particular, Palfrey and Srivastava (1987) have shown that the core (their core notion is different than the one adopted in this paper) of an economy with differential information may not be implementable as a Bayesian

[1] See Section 2.4 for a precise definition.

[2] See also Ichiishi and Radner (1999) and Ichiishi and Sertel (1998) for related core notions.

[3] These points are made formally in Example 3.1 of Section 3 where we refer the reader for further discussion.

Nash equilibrium.[4] However, despite the negative result of Palfrey and Srivastava (1987), we demonstrate that indeed our private core notion is implementable as a coalitional (strong) Bayesian Nash equilibrium, *i.e.*, we can construct a game (mechanism) whose coalitional Bayesian Nash equilibrium outcomes coincide with the private core. By focusing on the coalitional implementation of a social choice set, we reconsider the problem of implementation in differential information economies studied in a series of papers by Blume and Easley (1987), Jackson (1991), Palfrey and Srivastava (1986, 1987, 1989), and Postlewaite and Schmeidler (1986, 1987). To be precise, we say that a mechanism *coalitionally implements* a social choice set if any outcome of the social choice set can be achieved as a coalitional Bayesian Nash equilibrium of a mechanism, and vice versa. We say that a social choice set is *coalitionally implementable* if there is a mechanism which coalitionally implements it.

The main purpose of this paper is to show that a social choice set is coalitionally implementable if and only if it is interim individually rational, interim efficient, coalitional Bayesian incentive compatible, and satisfies a coalitional Bayesian monotonicity condition as well as a closure condition. As an application of this result, we show that the private core and the private Shapley value of an economy with differential information are coalitionally implementable. In doing so, we build on the incomplete information monotonicity condition of Jackson (1991), Palfrey and Srivastava (1987, 1989), and Postlewaite and Schmeidler (1986), and introduce new concepts. We define a coalitional form of monotonicity which is appropriate for our model.

Finally, it should be mentioned that we not only examine the problem of coalitional Bayesian implementation for the first time and provide characterization results, but we also make several other advances. First, we are able to address the incentive compatibility issue in a coalitional way. This is of great importance because individually incentive compatible contracts may not be necessarily coalitional incentive compatible Hence, if one considers multilateral contracts, the individually incentive compatibility may not be sufficient to guarantee that the contract may be viable. Secondly, we implement the Shapley value without restricting trade to be bilateral (e. g., Gul, 1989) or to the transferable utility case (e. g., Winter, 1994). Hence, we also contribute to the literature of finding ways to rationalize the Shapley value. Thirdly, we offer a new construction of a mechanism which takes into account coalitional deviations, is not wasteful, and is feasible.

The paper is organized as follows: In Section 2, we describe the model and characterize the coalitional implementation. In Sections 3 and 4, we show that the private core and the private value are coalitionally implementable. Concrete examples are presented in Section 5. Finally, some concluding remarks are collected in Section 6.

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[4] That is, one cannot construct a game whose set of Bayesian Nash equilibria coincides with their core notion.

2 Coalitional implementation

We begin with some notation and definitions. $|A|$ denotes the number of elements in the set A. If A is a set, we denote by χ_A the characteristic function having the property that $\chi_A(\omega)$ is one if $\omega \in A$ and it is zero otherwise. $\setminus$ denotes the set theoretic subtraction.

2.1 Differential information economies

Below we define the notion of an economy with differential information. Let $(\Omega, \mathscr{F}, \mu)$ be a probability measure space denoting the states of the world, $\boldsymbol{R}^\ell$ be an Euclidean space denoting the commodity space and $I = \{1, 2, \ldots, N\}$ be a finite set of agents. For simplicity, we assume that $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ is a finite set of states.[5] An **economy with differential information** is described by $\mathscr{E} = \{(X_i, u_i, \mathscr{F}_i, \mu, e_i) : i \in I\}$, where

(1) $X_i = \boldsymbol{R}^\ell_+$ is the **consumption set** of agent $i \in I$,
(2) $u_i : \Omega \times \boldsymbol{R}^\ell_+ \to \boldsymbol{R}$ is the **state-dependent utility function** of agent $i \in I$,
(3) $\mathscr{F}_i$ is a finite measurable partition of Ω denoting the **private information** of agent $i \in I$,
(4) μ is a probability measure on Ω denoting the **common prior** of each agent,
(5) $e_i : \Omega \to \boldsymbol{R}^\ell_{++}$ is an $\mathscr{F}_i$-measurable function[6] denoting the **state-dependent initial endowment** of agent $i \in I$.

We assume that the structure of the differential information economy is common knowledge among all agents. We call a set of states an event. An event E_i, which is an element of the partition $\mathscr{F}_i$, is the largest set of states that agent i cannot distinguish. Let $E_i(\omega)$ denote the event of $\mathscr{F}_i$ which contains $\omega \in \Omega$. This means that when the true state ω occurs, agent i knows only that the event $E_i(\omega)$ occurs. Assume that $\mu(\omega) > 0$ for every $\omega \in \Omega$.

Let $L_{\mathscr{F}}$ be the set of $\mathscr{F}$-measurable functions which maps Ω to $\boldsymbol{R}^\ell_+$, L_{X_i} the set of $\mathscr{F}_i$-measurable functions which maps Ω to $\boldsymbol{R}^\ell_+$, and L_i the set of $\mathscr{F}_i$-measurable functions which maps Ω to $\boldsymbol{R}^\ell$. The **conditional expected utility function** of agent i is a function $V_i : \Omega \times L_{\mathscr{F}} \to \boldsymbol{R}$ defined by[7]

$$V_i(\omega, x_i) = \frac{1}{\mu(E_i(\omega))} \sum_{\omega' \in E_i(\omega)} u_i(\omega', x_i(\omega'))\mu(\omega').$$

An element $x = (x_i)_{i \in I} \in L_X := \prod_{i \in I} L_{X_i}$ is called an **allocation**. The **set of feasible allocations** is given by $\boldsymbol{A} = \{x \in L_X : \sum_{i \in I} x_i = \sum_{i \in I} e_i\}$. For each i, an element $z_i \in L_i$ with $z_i = x_i - e_i$ is a **net trade** of agent i. The **set of feasible net trades** is given by $\boldsymbol{Z} = \{z \in L : \sum_{i \in I} z_i = 0\}$ where $L = \prod_{i \in I} L_i$. Let $\hat{Z} = \{\hat{z} \in \prod_{i \in I} Y_i : \sum_{i \in I} \hat{z}_i = 0\}$, where $Y_i = \boldsymbol{R}^\ell$ for every $i \in I$. Notice that the initial endowment vector denoted by $e = (e_i)_{i \in I}$ is an element of L_X.

[5] One would allow for infinitely many states and infinitely many commodities. We refer the reader to Hahn and Yannelis (1995) for the details.
[6] A function $f : \Omega \to \boldsymbol{R}$ is *$\mathscr{F}_i$-measurable* if $f(\omega) = f(\omega')$ for every $\omega, \omega' \in E_i \in \mathscr{F}_i$.
[7] One could allow agents to have different priors.

2.2 Coalitional implementation

A **social choice set** Γ is a subset of $\boldsymbol{A}$. A **mechanism** for an economy with differential information $\mathscr{E}$ is a pair (M, f) where $M = \prod_{i \in I} M_i$ is the set of messages and $f : M \to \hat{Z}$ is an outcome function. If $M = F$ with $F = \prod_{i \in I} \mathscr{F}_i$, the mechanism (F, f) is a **direct revelation mechanism**. A **strategy** for agent i is a function $\sigma_i : \mathscr{F}_i \to M_i$. We use the following notation: $\sigma = (\sigma_i)_{i \in I}$, $\sigma(E(\omega)) = (\sigma_i(E_i(\omega)))_{i \in I}$, $f(\sigma)(\omega) = f(\sigma(E(\omega)))$, $E = (E_i)_{i \in I}$. For $S \subset I$, $\sigma_S = (\sigma_i)_{i \in S}$, $\sigma_{-S} = (\sigma_i)_{i \notin S}$, $\sigma_S(E_S(\omega)) = (\sigma_i(E_i(\omega)))_{i \in S}$, $\sigma_{-S}(E_{-S}(\omega)) = (\sigma_i(E_i(\omega)))_{i \notin S}$, $E_S = (E_i)_{i \in S}$.

Definition 2.2.1. *A strategy vector σ is a* **Bayesian Nash equilibrium (BNE)** *for the mechanism (M, f) if for every $i \in I$, for every $\omega \in \Omega$, and for every strategy $\sigma_i' : \mathscr{F}_i \to M_i$,*

$$V_i(\omega, e_i + f_i(\sigma)) \geq V_i(\omega, e_i + f_i(\sigma_i', \sigma_{-i})).$$

When agents are allowed to form coalitions, one may define a stronger equilibrium concept.

Definition 2.2.2. *A strategy vector σ is a* **coalitional Bayesian Nash equilibrium (CBNE)** *for the mechanism (M, f) if it is not true that there exists a state $\omega \in \Omega$, a coalition $S \subset I$, and a strategy $\sigma_S' : \prod_{i \in S} \mathscr{F}_i \to \prod_{i \in S} M_i$ such that*

$$V_i(\omega, e_i + f_i(\sigma_S', \sigma_{-S})) > V_i(\omega, e_i + f_i(\sigma)), \ \forall i \in S.$$

In this paper, we consider full implementation which requires that the set of equilibrium outcomes of the mechanism exactly coincide with the given social choice set. This does not allow the existence of any undesirable equilibrium outcome in the mechanism.

Definition 2.2.3. *A mechanism (M, f)* **coalitionally implements (c-implements)** *a social choice set Γ if*

(1) For any $x \in \Gamma$, there exists a coalitional Bayesian Nash equilibrium σ for (M, f) such that $e + f(\sigma) = x$,
(2) If σ is a coalitional Bayesian Nash equilibrium for (M, f), then $e + f(\sigma) \in \Gamma$.

A social choice set Γ is **coalitionally implementable (c-implementable)** if there is a mechanism (M, f) which c-implements Γ. Given a mechanism (M, f), we assume that for every $i \in I$ and every strategy vector σ, there is a strategy σ_i' for agent i such that $f_i(\sigma_i', \sigma_{-i}) = 0$. That is, we restrict attention to such mechanisms.

2.3 Interim efficiency and interim individual rationality

A social choice set Γ is **interim efficient (IE)** if for every $x \in \Gamma$, there is no $x' \in \boldsymbol{A}$ such that for some $\omega \in \Omega$, $V_i(\omega, x_i') > V_i(\omega, x_i)$ for every $i \in I$[8] Since for each $\omega \in \Omega$, $u_i(\omega, \cdot)$ is monotone and continuous, so is $V_i(\omega, \cdot)$. Therefore, one can easily show that the above definition of interim efficiency coincides with that of a stronger interim efficiency, *i.e.*, a social choice set Γ is interim efficient if for every $x \in \Gamma$, there is no $x' \in \boldsymbol{A}$ such that for some $\omega \in \Omega$, $V_i(\omega, x_i') \geq V_i(\omega, x_i)$ for every $i \in I$ with strict inequality for some $i \in I$.

Proposition 2.3.1. *If a social choice set Γ is c-implementable by a mechanism (M, f) and $f : M \to \hat{Z}$ is onto, then it is IE.*

Proof. Suppose, by way of contradiction, that (M, f) c-implements Γ but Γ is not IE. Then, there exists an allocation $x = e + z \in \Gamma$ such that for some state $\omega \in \Omega$ and for some allocation $x' = e + z' \in \boldsymbol{A}$,

$$V_i(\omega, e_i + z_i') > V_i(\omega, e_i + z_i), \forall i \in I.$$

Since Γ is c-implementable, we have a CBNE σ such that $f(\sigma) = z$. Because f is onto, there is a strategy profile σ' such that $f(\sigma'(\omega)) = z'(\omega) \in \hat{Z}$ for every $\omega \in \Omega$. Hence, for every $i \in I$,

$$V_i(\omega, e_i + f_i(\sigma')) > V_i(\omega, e_i + f_i(\sigma)),$$

a contradiction to the fact that σ is a CBNE. □

A social choice set Γ is **interim individually rational (IIR)** if for every $x \in \Gamma$ and for every $\omega \in \Omega$, $V_i(\omega, x_i) \geq V_i(\omega, e_i)$ holds for every $i \in I$. One can easily show that interim individual rationality is a necessary condition for coalitional implementation. The following result is the counterpart of that by Hurwicz et al. (1984, Proposition, p. 14).

Proposition 2.3.2. *If a social choice set Γ is c-implementable, then it is IIR.*

Proof. Suppose (M, f) c-implements Γ but Γ is not IR. Then there is $x = e+z \in \Gamma$ such that there exists $\omega \in \Omega$ and $i \in I$ such that $V_i(\omega, x_i) < V_i(\omega, e_i)$. Since Γ is c-implementable, we have a CBNE σ such that $f(\sigma) = z$. Since we assume that for every i and every σ, there exists σ_i' such that $f(\sigma_i', \sigma_{-i}) = 0$, we have $V_i(\omega, e_i + f(\sigma_i', \sigma_{-i})) > V_i(\omega, e_i + f(\sigma))$, a contradiction. □

[8] Note that our interim efficiency notion is different than usual one (e. g., Holmström-Myerson, 1983) in that we use a weaker quantifier "for some $\omega \in \Omega$." Due to the private information measurability, our notion is not so strong as it seems and cannot be directly compared with the usual one. For comparisons of different efficiency notions in differential information economies, see Hahn and Yannelis (1997).

2.4 Coalitional Bayesian incentive compatibility

When agents have differential information, arbitrary allocations are not generally viable. In particular, arbitrary allocations might not be incentive compatible in the sense that groups of agents may misreport their information without other agents noticing it, and hence achieve different payoffs. We will show that a social choice set must satisfy an incentive compatibility criterion in order to be coalitionally implementable.

An allocation $x = e + z \in \boldsymbol{A}$ is *coalitionally Bayesian incentive compatible* if it is not true that there exists a coalition S and states ω^*, ω' ($\omega^* \neq \omega'$) with $\omega' \in \bigcap_{i \notin S} E_i(\omega^*)$ such that

$$\frac{1}{\mu(E_i(\omega^*))} \sum_{\omega \in E_i(\omega^*)} u_i(\omega, e_i(\omega) + z_i(\omega'))\mu(\omega) > \frac{1}{\mu(E_i(\omega^*))} \sum_{\omega \in E_i(\omega^*)} u_i(\omega, e_i(\omega) + z_i(\omega))\mu(\omega)$$

for every $i \in S$. In essence, this concept assures that no coalition S can make redistributions among themselves in states that the complementary coalition cannot distinguish, and become better off. In other words, if state ω^* occurs and the agents in the coalition $I \setminus S$ cannot distinguish between the state ω^* and ω', it must be the case that the agents of coalition S cannot become better off by announcing ω' instead of the actually occurred ω^*. The measurability implies that $\omega' \notin E_i(\omega^*)$ for every agent i in the coalition S.

As in Palfrey and Srivastava (1989), a **deception** for agent i is a function $\alpha_i : \mathscr{F}_i \to \mathscr{F}_i$. Let $\alpha_i^* : \mathscr{F}_i \to \mathscr{F}_i$ be the truth-telling strategy for agent i. A deception vector $\alpha = (\alpha_i)_{i \in I}$ is **compatible** with F if $\alpha(\omega) := \bigcap_{i \in I} \alpha_i(E_i(\omega)) \neq \emptyset$ for every $\omega \in \Omega$. In a direct revelation mechanism, a deception is a strategy such that for every $\omega \in \Omega$, agent i reports $\alpha_i(E_i(\omega))$ instead of $E_i(\omega)$. Notice that when $\sigma_i : \mathscr{F}_i \to M_i$ is a strategy and α_i is a deception of agent i, their composition $\sigma_i \circ \alpha_i : \mathscr{F}_i \to M_i$ is also a strategy of agent i.

We use the following notation: $E^S(\omega) = \bigcap_{i \in S} E_i(\omega)$, $E^{-S}(\omega) = \bigcap_{i \notin S} E_i(\omega)$, $\alpha_S(\omega) = E_\alpha^S(\omega) = \bigcap_{i \in S} \alpha_i(E_i(\omega))$, $\alpha_{-S}(\omega) = E_\alpha^{-S}(\omega) = \bigcap_{i \notin S} \alpha_i(E_i(\omega))$, $(\sigma \circ \alpha)_S = (\sigma_i \circ \alpha_i)_{i \in S}$.[9] Let $z \in \boldsymbol{Z}$ be a feasible net trade. If $\alpha(\omega) \neq \emptyset$, let $z \circ \alpha(\omega) = z(\alpha(\omega)) = z(\omega')$ for all $\omega' \in \alpha(\omega)$, otherwise let $(z \circ \alpha)(\omega) = 0$. Note that $(z \circ \alpha)_i = z_i \circ \alpha$ and $(z \circ \alpha^*)(\omega) = z(\omega)$. Recall from Lemma 1 of Palfrey and Srivastava (1989, p. 120) that for every $i \in I$, if $\omega' \in E_i(\omega)$, then $\alpha(\omega') \subset E_i(\alpha(\omega))$ for every $i \in I$, where $E_i(\alpha(\omega))$ is the event that contains $\alpha(\omega)$. In

[9] For example, consider the following information structure:

$$\mathscr{F}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}, \mathscr{F}_2 = \{\{\omega_1, \omega_3\}, \{\omega_2\}\}, \mathscr{F}_3 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}$$

Let us define a deception α as follows: for every ω, $\alpha_i(E_i(\omega)) = E_i(\omega_1), \forall i = 1, 2$ and $\alpha_3(E_3(\omega)) = E_3(\omega)$. Then for the coalition $S = \{1, 3\}$, $\alpha_S^*(\omega_3) = E^S(\omega_3) = \{\omega_3\}$, $\alpha_{-S}^*(\omega_3) = E^{-S}(\omega_3) = \{\omega_1, \omega_3\}$, $\alpha_S(\omega_3) = E_\alpha^S(\omega_3) = \{\omega_1\}$.

view of this Lemma, we immediately conclude that if $z \in \boldsymbol{Z}$, then $z \circ \alpha \in \boldsymbol{Z}$ for every deception α.

Using the notion of deception, we can define coalitional Bayesian incentive compatibility as follows.

Definition 2.4.1. *A social choice set Γ is said to be* **coalitionally Bayesian incentive compatible (CBIC)** *if for every $x = e + z \in \Gamma$, it is not true that there exists a state $\omega \in \Omega$, a coalition $S \subset I$, and a deception $\alpha_S : \prod_{i \in S} \mathscr{F}_i \to \prod_{i \in S} \mathscr{F}_i$ such that for every $i \in S$,*

$$V_i(\omega, e_i + [z \circ (\alpha_S, \alpha^*_{-S})]_i) > V_i(\omega, x_i),$$

where $e + z \circ (\alpha_S, \alpha^*_{-S}) \in \boldsymbol{A}$.

This notion of incentive compatibility states that it is not possible for any coalition S to become better off by announcing a false event. Observe that if S is a singleton, then the CBIC condition is reduced to standard Bayesian incentive compatibility. It is straightforward to show (see Theorem 2.4.1 below) that the coalitional Bayesian incentive compatibility is a necessary condition for coalitional implementation, *i.e.*, if a social choice set is implementable as a coalitional Bayesian Nash equilibrium, then it is coalitionally Bayesian incentive compatible. Note that this is the counterpart of the standard Bayesian Nash implementation results (see, for example, Jackson, 1991; Palfrey and Srivastava, 1989; Postlewaite and Schmeidler, 1986), *i.e.*, if a social choice set is implementable as a Bayesian Nash equilibrium, then it is Bayesian incentive compatible.

Theorem 2.4.1. *If a social choice set Γ is c-implementable, then it is CBIC.*

Proof. Let (M, f) c-implement Γ and $x = e + z \in \Gamma$. Then there is a CBNE σ^* with $f(\sigma^*) = z$. Now suppose that x is not CBIC, then there exists a state $\omega \in \Omega$, a coalition $S \subset I$, and a deception $\alpha_S : \prod_{i \in S} \mathscr{F}_i \to \prod_{i \in S} \mathscr{F}_i$ such that for every $i \in S$,

$$V_i(\omega, e_i + [z \circ (\alpha_S, \alpha^*_{-S})]_i) > V_i(\omega, x_i),$$

with $e + [z \circ (\alpha_S, \alpha^*_{-S})] \in \boldsymbol{A}$, which is equivalent to

$$V_i(\omega, e_i + f_i((\sigma^* \circ \alpha)_S, \sigma^*_{-S})) > V_i(\omega, e_i + f_i(\sigma^*)),$$

a contradiction to the fact that σ^* is a CBNE for the mechanism (M, f). Hence, Γ is CBIC. □

2.5 Coalitional Bayesian monotonicity

In the literature of Nash implementation with complete information, Maskin (1977) first recognized that a monotonicity condition is necessary. The Maskin-type monotonicity condition states the following: Denote the ex post preference of agent i at the state ω by $\succeq_i(\omega)$. If the outcome x is in a social choice set $\Gamma(\omega)$ and $x \notin \Gamma(\omega')$ where $\omega' \neq \omega$, then there exist an agent i and an outcome

x' such that $x \succeq_i (\omega) x'$ but $x' \succ_i (\omega') x$ (see also Saijo, 1988; Williams, 1986). In an incomplete information setting, Palfrey and Srivastava (1987, 1989) and Postlewaite and Schmeidler (1986) introduced a Bayesian monotonicity condition, which is an extension of that of Maskin (1977). Below we introduce a coalitional form of Bayesian monotonicity.

Definition 2.5.1. *A social choice set Γ satisfies* **Bayesian monotonicity (BM)** *if for every $x = e + z \in \Gamma$, whenever $e + z \circ \alpha \in \boldsymbol{A} \setminus \Gamma$ for α compatible with F, there exists a state $\omega \in \Omega$, an agent $i \in I$, and a net trade $z' \in \boldsymbol{Z}$ such that $e + z' \circ \alpha \in \boldsymbol{A}$, $e + z' \circ (\alpha_i, \alpha^*_{-i}) \in \boldsymbol{A}$,*

(1) $V_i(\omega, e_i + (z' \circ \alpha)_i) > V_i(\omega, e_i + (z \circ \alpha)_i)$, *and*
(2) $V_i(\omega', e_i + z_i) \geq V_i(\omega', e_i + [z' \circ (\alpha_i, \alpha^*_{-i})]_i), \ \forall \omega' \in \Omega.$

In our context, Palfrey and Srivastava (1989) require instead of (2) above that:

$$(2') \ V_i(\omega', e_i + z_i) \geq V_i(\omega', e_i + [z' \circ (\alpha_i^{\omega}, \alpha^*_{-i})]_i), \ \forall \omega' \in E_{\alpha}^{-i}(\omega),$$

where $\alpha_i^{\omega}(E_i) = \alpha_i(E_i(\omega))$ for every $E_i \in \mathscr{F}_i$. Our definition of Bayesian monotonicity is not directly comparable with those of Palfrey and Srivastava (1989) and Jackson (1991) because of the private information measurability. But if we impose the private information measurability on their notions, our definition is the same with that of Jackson (1991) since his conditions must hold for all deceptions (including incompatible ones) but incompatible deceptions here make the first condition violated. However, our definition is stronger than that of Palfrey and Srivastava (1989) since they require the second condition to hold only for the states which the other agents collectively report and for the restricted deceptions. Below we introduce a coalitional form of the above definition.

Definition 2.5.2. *A social choice set Γ satisfies* **coalitional Bayesian monotonicity (CBM)** *if for every $x = e + z \in \Gamma$, whenever $e + z \circ \alpha \in \boldsymbol{A} \setminus \Gamma$ for α compatible with F, there exists a state $\omega \in \Omega$, a coalition $S \subset I$, and a net trade $z' \in \boldsymbol{Z}$ such that $e + z' \circ \alpha \in \boldsymbol{A}$, $e + z' \circ (\alpha_S, \alpha^*_{-S}) \in \boldsymbol{A}$,*

(1) $\forall i \in S, \ V_i(\omega, e_i + (z' \circ \alpha)_i) > V_i(\omega, e_i + (z \circ \alpha)_i)$, *and*
(2) $\exists i \in S, \ V_i(\omega', e_i + z_i) \geq V_i(\omega', e_i + [z' \circ (\alpha_S, \alpha^*_{-S})]_i), \ \forall \omega' \in \Omega.$

Note that if S is a singleton, the coalitional Bayesian monotonicity is equivalent to the Bayesian monotonicity. Since $\{i\}$ is a coalition, the Bayesian monotonicity implies the coalitional Bayesian monotonicity but not vice versa. This means that if an allocation is eliminated by the Bayesian Nash equilibrium criterion, then it must be excluded by the coalitional Bayesian Nash equilibrium criterion. The following theorem and its proof shed light on the implications of coalitional Bayesian monotonicity to the coalitional implementation. It is the coalitional counterpart of a result in Palfrey and Srivastava (1989, Theorem 2, p. 124).

Theorem 2.5.1. *If a social choice set Γ is c-implementable, then it satisfies the CBM condition.*

Proof. Let (M, f) c-implement Γ and $x = e + z \in \Gamma$. Then there exists a CBNE σ of (M, f) with $f(\sigma) = z$. Assume that for some α compatible with F, $e + z \circ \alpha \in \boldsymbol{A} \setminus \Gamma$. Note that $f(\sigma \circ \alpha) = z \circ \alpha$. Since Γ is c-implementable and $e + f(\sigma \circ \alpha) = e + z \circ \alpha \in \boldsymbol{A} \setminus \Gamma$, the strategy vector $\sigma \circ \alpha$ is not a CBNE. Therefore, there exists a state $\omega \in \Omega$, a coalition $S \subset I$, and a strategy vector $\sigma'_S \in \prod_{i \in S} \mathscr{F}_i \to \prod_{I \in S} M_i$ such that for every $i \in S$, $V_i(\omega, e_i + f_i(\sigma'_S, (\sigma \circ \alpha)_{-S})) > V_i(\omega, e_i + f_i(\sigma \circ \alpha))$.[10] Now for every $i \in S$, define $\overline{\sigma}_i$ by $\overline{\sigma}_i(E_i) = \sigma'_i(E_i(\omega))$ for every $E_i \in \mathscr{F}_i$ and let $z' = f(\overline{\sigma}_S, \sigma_{-S})$. Then since $z' \circ \alpha = f(\sigma'_S, (\sigma \circ \alpha)_{-S})$, it follows that $V_i(\omega, e_i + (z' \circ \alpha)_i) > V_i(\omega, e_i + (z \circ \alpha)_i)$ for every $i \in S$. Note that $z' \circ (\alpha_S, \alpha^*_{-S}) = f(\sigma'_S, \sigma_{-S})$. Since σ is a CBNE, the coalition S with the strategy σ'_S cannot improve upon σ. That is, there exists some $i \in S$ such that $V_i(\omega', e_i + z_i) = V_i(\omega', e_i + f_i(\sigma)) \geq V_i(\omega', e_i + f_i(\sigma'_S, \sigma_{-S})) = V_i(\omega', e_i + [z' \circ (\alpha_S, \alpha^*_{-S})]_i)$ for every $\omega' \in \Omega$. Hence Γ satisfies the CBM condition. □

2.6 Closure

Denote by $\bigwedge_{i \in I} \mathscr{F}_i$ the *finest common coarsening* of $\{\mathscr{F}_i : i \in I\}$, *i.e.*, the finest partition of Ω which is coarser than $\mathscr{F}_i$ for every $i \in I$. An event E is said to be *common knowledge at* ω if $(\bigwedge_{i \in I} \mathscr{F}_i)(\omega) \subset E$ where $(\bigwedge_{i \in I} \mathscr{F}_i)(\omega)$ is the event of $\bigwedge_{i \in I} \mathscr{F}_i$ containing ω. Notice that $(\bigwedge_{i \in I} \mathscr{F}_i)(\omega)$ itself is common knowledge at ω. We also call $\bigwedge_{i \in I} \mathscr{F}_i$ the *common knowledge partition* of Ω.

Following Postlewaite and Schmeidler (1986), we define z^* to be the **common knowledge concatenation** of $\{z^k \in L : k = 1, \ldots, m\}$ if $z^*(\omega) = \sum_{k=1}^m z^k(\omega) \chi_{E^k}(\omega)$ where $\{E^k : k = 1, \ldots, m\}$ is the common knowledge partition of Ω. Let $\{z^k \in L : k = 1, \ldots, m\}$ be a collection of net trades such that $e + z^k \in \Gamma$. If the common knowledge concatenation z^* of $\{z^k : k = 1, \ldots, m\}$ has the property that $e + z^* \in \Gamma$, then Γ is said to satisfy **closure (C)**. It turns out that a c-implementable social choice set Γ satisfies the closure condition as Lemma below indicates.

Lemma 2.6.1. *If a social choice set Γ is c-implementable, then it satisfies the condition C.*

Proof. Suppose that (M, f) c-implements Γ. Let $\{E^k : k = 1, \ldots, m\}$ be the common knowledge partition and $e + z^k \in \Gamma$ for $k = 1, \ldots, m$. Define $z^* = \sum_{k=1}^m z^k \cdot \chi_{E^k}$. We must show that $e + z^* \in \Gamma$. Let σ^k be a CBNE such that $f(\sigma^k) = z^k$. Then the strategy vector σ defined by $\sigma(E(\omega)) = \sum_{k=1}^m \sigma^k(E(\omega)) \chi_{E^k}(\omega)$ is also a CBNE. For otherwise there exists a state $\omega \in \Omega$, a coalition $S \subset I$, and $\sigma'_S : \prod_{i \in S} \mathscr{F}_i \to \prod_{i \in S} M_i$ such that for every $i \in S$,

$$V_i(\omega, e_i + f_i(\sigma'_S, \sigma_{-S})) > V_i(\omega, e_i + f_i(\sigma)),$$

which is equivalent to

[10] Since it does not matter which message agent $i \in S$ sends at $\omega' \notin E_i(\omega)$, without loss of generality, we can choose σ'_S such that $\sigma'_i(E_i(\omega')) = \sigma'_i(E_i(\omega))$ at every $\omega' \in \Omega$ for all $i \in S$.

$$V_i(\omega, e_i + f_i(\sigma'_S, \sigma^k_{-S})) > V_i(\omega, e_i + f_i(\sigma^k)),$$

where $\omega \in E^k$ for some k. Then σ^k is not a CBNE, a contradiction. Furthermore, $f(\sigma) = z^*$. Since Γ is c-implementable, $e + z^* \in \Gamma$. □

2.7 Sufficient conditions for coalitional implementation

In this section, we will show that interim individual rationality, interim efficiency, coalitional Bayesian incentive compatibility, coalitional Bayesian monotonicity, and closure are sufficient conditions for coalitional implementation. As in the previous literature, the proof is constructive. It is an extension of the constructions in Postlewaite and Schmeidler (1986) and Palfrey and Srivastava (1989), which allows us to consider deviations by coalitions. It should be noted that, as in Hurwicz et al. (1984), our mechanism is not wasteful and also maintains the feasibility of the outcomes out of equilibrium.

Before stating the main theorem, it is worth mentioning the case where there is only one good in the economy. If there is only one good, the measurability of allocations implies that the set of interim efficient allocations is equivalent to the set of feasible allocations. In this case, the initial endowment is the unique interim efficient and interim individually rational allocation (see footnote 15) and it is clearly c-implementable. It is enough to consider the mechanism (M, f) where $f(\sigma) = 0$ for every strategy profile σ. Hence, in the theorem below it is assumed that there is more than one good.

Theorem 2.7.1. *Assume that $N \geq 3$. If a social choice set Γ is IIR, IE, CBIC, and satisfies CBM and C, then it is c-implementable.*

Proof. Consider the message space of agent i, $M_i = \{m_i = (E_i, z^i, n_i) \in \mathscr{F}_i \times \boldsymbol{Z} \times \boldsymbol{N}_0 : e + z^i \in \Gamma\}$ for every i, where $\boldsymbol{N}_0 = \{0, 1, 2, 3, \ldots\}$. Thus every agent i reports his/her private information event E_i, net trade profile $z^i = (z^i_j)_{j \in I}$ of the economy, and a nonnegative integer n_i. In principle, we can divide the message space M into two main groups. One is a region where the reported private information events have nonempty intersection. In this region, the mechanism designer cannot tell whether someone is lying about his/her private information event. This region consists of M^0, M^1, M^4, and M^5 (see below for the definitions of the regions and outcome function). In the region M^6 where the reported information events have empty intersection, some agent reports a non-zero integer. In M^6, the mechanism designer knows that some one is lying about his/her private information event. The outcome function in this region assigns no trade. The remaining regions are M^2 and M^3. When all agents report the integer zero and every agent except agent 1 reports the same net trade configuration, the message belongs to the region M^2. The mechanism makes agent 1 give away his/her reported endowments to the other agents who will equally share them with each other. In the region M^3, where all agents report the integer 0 but the message does not belong to either M^0 or M^2, agent 1 takes the reported endowments of all the other agents.

For the former regions (*i.e.*, M^0, M^1, M^4, M^5), more explanation is needed. First of all, in the region M^0, every agent agrees on the net trade profile of the economy and the integer zero, but reports his/her own private information event. In this case, the outcome function assigns the agreed net trade at the consented states. In the region $M^1(S)$, agents in the coalition S unanimously report the net trade profile of the economy and a nonzero integer, but they report their own private information events. However, the agents in the complementary coalition use strategies in the same fashion as in M^0. The outcome function assigns the net trade suggested by the coalition S at the agreed states in $M^{11}(S)$ where some agent in the coalition S does not prefer his/her proposed net trade to the net trade proposed by the complementary coalition at their agreed states. The outcome function assigns no trade in $M^{12}(S)$. In this case, every agent in the coalition S prefers the net trade of the coalition at some agreed state of the complementary coalition. In the region $M^4(S)$, agents in the coalition S use strategies without unanimity. In the complementary coalition, agents send messages in the same fashion as in M^0. In this region, the outcome is determined by the "integer game", *i.e.*, the agent who has the lowest index among the agents reporting the highest integer receives the reported endowments of all the other agents. Finally, the region M^5 collects all the messages which are not in M^0, M^1, M^2, M^3, M^4 and M^6. In particular, there is no agent who reports the integer zero. As in M^4, the outcome is determined by the integer game, but there is no tie-breaker of choosing the agent with the lowest index so that the winners (the agents reporting the highest number) evenly share the sum of the endowments of the losers.

We now formalize the above discussion. Let S be a nonempty proper coalition of I and let us write $m_i = (m_i^1, m_i^2, m_i^3)$ for each $i \in I$. For every $i \in I$, define $z_i[E_S](\omega) = z_i(\omega')$ where $\omega' \in E^S \cap E^{-S}(\omega)$. Define the sets:

$$\begin{aligned}
M^0 &= \{m \in M : m_i = (E_i, z, 0), \forall i \in I; \bigcap_{i\in I} E_i \neq \emptyset\},\\
M^1(S) &= \{m \in M \setminus M^0 : m_i = (E_i, z', n), n \neq 0, \forall i \in S; m_i = (E_i, z, 0),\\
&\quad \forall i \notin S; \bigcap_{i\in I} E_i \neq \emptyset\},\\
M^{11}(S) &= \{m \in M^1(S) : \exists i \in S, V_i(\omega, e_i + z_i) \geq V_i(\omega, e_i + z_i'[E_S]),\\
&\quad \forall \omega \in E^{-S}\},\\
M^{12}(S) &= M^1(S) \setminus M^{11}(S),\\
M^1 &= \bigcup_S M^1(S),\\
M^2 &= \{m \in M : m_1 = (E_1, z', 0); m_i = (E_i, z, 0), \forall i \neq 1\},\\
M^3 &= \{m \in M \setminus (M^0 \cup M^2) : m_i^3 = 0, \forall i \in I\},\\
M^4(S) &= \{m \in M \setminus \bigcup_{k=0}^{3} M^k : m_i = (E_i, z, 0), \forall i \notin S; \bigcap_{i\in I} E_i \neq \emptyset\},
\end{aligned}$$

$$
\begin{aligned}
M^4 &= \bigcup_S M^4(S),\\
M^5 &= \{m \in M \setminus \bigcup_{k=0}^{4} M^k : \bigcap_{i\in I} m_i^1 \neq \emptyset\},\\
M^6 &= \{m \in M \setminus M^3 : \bigcap_{i\in I} m_i^1 = \emptyset\}.
\end{aligned}
$$

Define the outcome function $f : M \to \hat{Z}$ as follows: For every $i \in I$,

$$
f_i(m) = \begin{cases}
z_i(\omega), & \omega \in \bigcap_{i\in I} m_i^1 \quad \text{if } m \in M^0,\\
z_i'(\omega), & \omega \in \bigcap_{i\in I} m_i^1 \quad \text{if } m \in M^{11}(S) \text{ for some } S,\\
0 & \text{if } m \in M^{12}(S) \text{ for some } S,\\
e_1(\omega)/(N-1), & \omega \in m_1^1 \quad \text{if } m \in M^2,\ i \neq 1,\\
-e_1(\omega), & \omega \in m_1^1 \quad \text{if } m \in M^2,\ i = 1,\\
\sum_{j\neq i} e_j(\omega_j), & \omega_j \in m_j^1 \quad \text{if } m \in M^3,\ i = 1,\\
-e_i(\omega_i), & \omega_i \in m_i^1 \quad \text{if } m \in M^3,\ i \neq 1,\\
\sum_{j\neq i} e_j(\omega), & \omega \in \bigcap_{i\in I} m_i^1 \quad \text{if } m \in M^4,\ i = \min\{k : k \in K\},\\
-e_i(\omega), & \omega \in \bigcap_{i\in I} m_i^1 \quad \text{if } m \in M^4,\ i \neq \min\{k : k \in K\},\\
\frac{1}{|K|}\sum_{j\notin K} e_j(\omega), & \omega \in \bigcap_{i\in I} m_i^1 \quad \text{if } m \in M^5,\ i \in K,\\
-e_i(\omega), & \omega \in \bigcap_{i\in I} m_i^1 \quad \text{if } m \in M^5,\ i \notin K,\\
0 & \text{if } m \in M^6,
\end{cases}
$$

where $K = \{k \in I : m_k^3 = \max_{i\in I} m_i^3\}$.

Since the mechanism is common knowledge, every agent knows that his/her (implicitly) reported endowment can be confiscated. Therefore, it is must be the case that when each agent reports his/her private information (and implicitly reports his/her initial endowment), he/she cannot overreport his/her initial endowment. That is, for every $i \in I$ and every α_i, $e_i(\omega') \leq e_i(\omega)$ for every $\omega' \in m_i^1 = \alpha_i(E_i(\omega))$ when ω occurs. Since $e_i(\omega) + z_i^j(\omega') \geq e_i(\omega') + z_i^j(\omega') \geq 0$ for $\omega' \in \bigcap_{i\in I} m_i^1$ with $m \in M^0 \cup M^1(S)$ for some S and for every $i, j \in I$ when state ω occurs, it follows that the allocations induced by the mechanism are always positive, *i.e.*, $e_i(\omega) + f_i(m) \geq 0$ for every i, every ω and every m. Furthermore, $\sum_{i\in i} f_i(m) = 0$ for every $m \in M$. Hence the mechanism is feasible.

By Lemma 2.7.2 below, for every $x = e + z \in \Gamma$, we have a CBNE σ for (M, f) such that $f(\sigma) = z$. By Lemma 3.7.4 below, we conclude that for every CBNE strategy σ for (M, f), $e + f(\sigma) \in \Gamma$. Hence Γ is c-implementable. □

Lemma 2.7.2 below establishes that the mechanism of Theorem 2.7.1 satisfies the first requirement for coalitional implementation (condition (1) of Definition 2.2.3). Lemma 2.7.4 below shows that the mechanism of Theorem 2.7.1 satisfies the second requirement for coalitional implementation (condition (2) of Definition 2.2.3).

Lemma 2.7.2. *For every $x = e + z \in \Gamma$, let σ be such that $\sigma_i(E_i(\omega)) = (E_i(\omega), z, 0)$ for all i and for all ω. Then σ is a CBNE for the mechanism (M, f) and $f(\sigma) = z$.*

Proof. See Appendix. □

For the proof of Lemma 2.7.4 we need a result (Lemma 2.7.3) which guarantees that no CBNE message lies outside of the region M^0. Indeed, if a message lies inside the region M^0, there is no profitable coalitional deviation. Moreover, if a message lies outside of the region M^0, there is always a profitable coalitional deviation.

Lemma 2.7.3. *If σ is a CBNE for (M, f), then $\sigma(E(\omega)) \in M^0$ for all $\omega \in \Omega$.*

Proof. See Appendix. □

Although all agents do not truthfully report their private information events, the equilibrium still belongs to the social choice set as long as they agree on the net trade configuration and the integer zero.

Lemma 2.7.4. *If σ is a CBNE for (M, f), then $e + f(\sigma) \in \Gamma$.*

Proof. See Appendix. □

Remark 2.7.1. If there is only one agent in the economy, the initial endowment is the unique feasible allocation, which is trivially c-implementable. Assume that $N = 2$ and that the initial endowment is not interim efficient. If a social choice set Γ is IR, IE, CBIC, and satisfies CBM and C, then it is c-implementable. The proof of Theorem 2.7.1 can be modified as follows. The mechanism f assigns the same net trade as before except on M^2, where f assigns no trade. Thus, there is no profitable deviation from M^0 to M^2 and there is a profitable deviation from M^2 to M^0, since the initial endowment is not Pareto optimal. Note that $M^3 = \emptyset$. The other arguments continue to hold.

Remark 2.7.2. It should be noted that for the complete information case Dutta and Sen (1991) provided a strong Nash implementation theorem, which is different than ours. In particular, they do not require individual rationality. In economic environments, they identified only the sufficiency conditions for strong Nash implementation. One can substitute the strong Maskin monotonicity condition with the individual rationality condition and the coalitional monotonicity condition (which is weaker than the strong Maskin monotonicity) to get a full charaterization result for strong Nash implementation.

3 Coalitional implementation of the private core

The core notion defined below (see also Yannelis, 1991) does not necessarily allow agents in a coalition to share their private information. In fact, allowing agents in a coalition to use either their common knowledge information or their pooled information, one may face serious problems as Example 3.1 will indicate (see also Koutsougeras and Yannelis, 1993, Section 5). More importantly, however, we will show in Section 5 that a core notion which allows for pooling of information may not be c-implementable.

Definition 3.1. *An allocation* $x \in \boldsymbol{A}$ *is an* **(ex ante) private core allocation** *of the economy with differential information* $\mathscr{E}$ *if it is not true that there exists a coalition* $S \subset I$ *and* $(y_i)_{i\in S} \in \prod_{i\in S} L_{X_i}$ *such that*

(1) $\sum_{i\in S} y_i = \sum_{i\in S} e_i$, *and*
(2) *for every* $i \in S$,

$$\sum_{\omega\in\Omega} u_i(\omega, y_i(\omega))\mu(\omega) > \sum_{\omega\in\Omega} u_i(\omega, x_i(\omega))\mu(\omega).$$

The **(ex ante) private core** *is the set of all ex ante private core allocations for* $\mathscr{E}$.

Definition 3.2. *An allocation* $x \in \boldsymbol{A}$ *is an* **(interim) private core allocation** *of the economy with differential information* $\mathscr{E}$ *if it is not true that there exists a state* $\omega \in \Omega$, *a coalition* $S \subset I$, *and* $(y_i)_{i\in S} \in \prod_{i\in S} L_{X_i}$ *such that*

(1) $\sum_{i\in S} y_i = \sum_{i\in S} e_i$, *and*
(2) *for every* $i \in S$, $V_i(\omega, y_i) > V_i(\omega, x_i)$.

The **(interim) private core** *is the set of all interim private core allocations for* $\mathscr{E}$ *and it is denoted by* $\boldsymbol{C}(\mathscr{E})$.

The only difference between the two above concepts is that agents in the ex ante private core use their ex ante expected utility functions and in the interim private core, their interim expected utility functions. The example below will illustrate that despite the fact that they have the same properties, i. e., they are coalitional incentive compatible and they take into account the informational superiority of an individual, they may be different.

Example 3.1. Consider an economy with differential information with three agents, one good, and three states (*i.e.*, $\Omega = \{\omega_1, \omega_2, \omega_3\}$) with equal probability (*i.e.*, $\mu(\{\omega\}) = 1/3$ for every $\omega \in \Omega$), where utility functions, initial endowment, and private information sets are given as follows:

$$\begin{array}{lll} u_1(\omega, x) = \sqrt{x}, & e_1 = (10, 10, 0), & \mathscr{F}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}, \\ u_2(\omega, x) = \sqrt{x}, & e_2 = (10, 0, 10), & \mathscr{F}_2 = \{\{\omega_1, \omega_3\}, \{\omega_2\}\}, \\ u_3(\omega, x) = \sqrt{x}, & e_3 = (0, 0, 0), & \mathscr{F}_3 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}. \end{array}$$

It can be shown that the allocation $x^* = (x_1, x_2, x_3)$ is in the ex ante private core where

$$x_1^* = (8, 8, 2),\ x_2^* = (8, 2, 8),\ x_3^* = (4, 0, 0). \tag{3.1}$$

In the above example, agents 1 and 2 cannot undertake any risk sharing among themselves (the trades between agents 1 and 2 are state independent and these trades do not make them better off) without agent 3. Since agent 3 has superior information, she acts as an intermediary who executes the correct trades (makes a Pareto improvement) and as a consequence gets rewarded for this service.

It should be noted that the allocation (3.1) in the ex ante private core is entirely different than that of any traditional rational expectations equilibrium

(REE). Indeed, in any REE, agent 3 gets zero because his/her budget set is zero in each state. However, in any ex ante private core allocation, agent 3 gets positive consumption[11] in state ω_1. It follows that agent 3 plays the role of an intermediary who makes a Pareto improvement for the economy as a whole and he/she gets rewarded for this. It is important to note that if the private information of agent 3 is $\mathscr{F}_3' = \{\{\omega_1, \omega_2, \omega_3\}\}$, then agent 3 gets $(0, 0, 0)$ and in this case the initial endowment is the unique ex ante private core allocation. Hence, contrary to the REE,[12] changes in the private information of an agent have effects on the resulting ex ante private core.[13]

Suppose now that agents 1 and 2 pool their information to obtain the allocation:

$$x_1^* = x_2^* = (10, 5, 5),\ x_3^* = (0, 0, 0).$$

However, such a contract may not be viable because the above allocation is not incentive compatible. Simply notice that agent 1 becomes better off by reporting state ω_3 if state ω_1 occurs and agent 2 cannot distinguish ω_1 from ω_3. Using the same reasoning, one can easily see that agent 2 has an incentive to report ω_2 whenever state ω_1 occurs and agent 1 cannot detect his because he/she is not able to distinguish ω_1 from ω_2. Hence, pooling of information violates coalitional incentive compatibility (see also Example 5.1 in Section 5).

Finally, notice that the initial endowment is the *unique* interim private core allocation[14] and it is not in the ex ante private core since the initial endowment is blocked by the grand coalition with the allocation x^* given by (3.1).

In the next example, we have an interim private core allocation which is not the initial endowment. Hence, the example below indicates that an interim private core allocation exists.

Example 3.2. Consider an economy with differential information with two agents, two goods (*i.e.*, x^1, x^2), and three equally probable states, where utility functions,

---

[11] This can be proved as follows: Suppose not and let x be an ex ante private core allocation such that agent 3's consumption at state ω_1 is zero. Since $x \in \boldsymbol{A}$, $x_3 = (0, 0, 0)$ and $x_1 - e_1 = -(x_2 - e_2)$. Since $x_i - e_i$ is $\mathscr{F}_i$-measurable for $i = 1, 2$, $x_i - e_i$ is $(\mathscr{F}_1 \wedge \mathscr{F}_2)$-measurable for $i = 1, 2$. Note that $\mathscr{F}_1 \wedge \mathscr{F}_2 = \{\{\omega_1, \omega_2, \omega_3\}\}$. Therefore, $x_1 - e_1 = (c, c, c)$ and $x_2 - e_2 = (-c, -c, -c)$ for some $c \in \boldsymbol{R}$. If $c < 0$, agent 1 blocks x since $x_1 < e_1$. If $c > 0$, agent 2 blocks x for the same reason. Thus $x = e$. However, the grand coalition with the allocation given in the above Example blocks $x = e$, a contradiction.

[12] Notice that by changing the private information of agent 3 from $\mathscr{F}_3 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}$ to $\mathscr{F}_3' = \{\{\omega_1, \omega_2, \omega_3\}\}$, the REE does not affect the consumption of agent 3, *i.e.*, he/she always gets zero since his/her budget set is zero in every state.

[13] Similar examples can be constructed for the interim private core, the ex ante private value, and the interim private value (see also Krasa and Yannelis, 1994, Section 4).

[14] In an economy with one good per state, *i.e.*, the interim private core is the initial endowment. First, notice that the initial endowment is in the interim private core. Otherwise, there exists a state ω, a coalition S, and $(y_i)_{i \in S}$ such that $\sum_{i \in S} y_i = \sum_{i \in S} e_i$ and $V_i(\omega, y_i) > V_i(\omega, e_i)$ for every $i \in S$. Since there is only one good, by monotonicity and measurability, we have $y_i(\omega) > e_i(\omega)$ for every $i \in S$, a contradiction. If there is another interim private core allocation $x \neq e$, the feasibility implies that there is an agent $i \in I$ such that $e_i(\omega) > x_i(\omega)$ for some $\omega \in \Omega$. Since there is only one good, $V_i(\omega, e_i) > V_i(\omega, x_i)$ by measurability and monotonicity. This implies that this agent is a blocking coalition against x at ω, a contradiction.

random initial endowments, and private information structures are given as follows:

$$u_1(\omega, x^1, x^2) = \sqrt{x^1 x^2},\ \forall\omega \quad e_1 = ((3,1),(3,1),(5,3)),\ \mathscr{F}_1 = \{\{\omega_1,\omega_2\},\{\omega_3\}\},$$
$$u_2(\omega, x^1, x^2) = \sqrt{x^1 x^2},\ \forall\omega, \quad e_2 = ((1,3),(3,5),(3,5)),\ \mathscr{F}_2 = \{\{\omega_1\},\{\omega_2,\omega_3\}\}.$$

The allocation

$$x^* = (x_1^*, x_2^*) = (((2,2),(2,2),(4,4)),((2,2),(4,4),(4,4)))$$

is the unique interim private core allocation which is different from the initial endowment.

In order to show that the interim private core is c-implementable, we will need some Lemmata.

Lemma 3.1. *The interim private core $\boldsymbol{C}(\mathscr{E})$ is IIR and IE.*

Proof. It is immediate from the definition. □

Lemma 3.2. *The interim private core $\boldsymbol{C}(\mathscr{E})$ satisfies the CBM condition.*

Proof. Let $x = e + z \in \boldsymbol{C}(\mathscr{E})$ and $e + z \circ \alpha \in \boldsymbol{A} \setminus \boldsymbol{C}(\mathscr{E})$. We must show that there exists a state $\omega^* \in \Omega$, a coalition $S \subset I$, and a net trade $z' \in \boldsymbol{Z}$ such that $e + z' \circ \alpha \in \boldsymbol{A}$, $e + z' \circ (\alpha_S, \alpha_{-S}^*) \in \boldsymbol{A}$,

(a) for every $i \in S$, $V_i(\omega^*, e_i + (z' \circ \alpha)_i) > V_i(\omega^*, e_i + (z \circ \alpha)_i)$, and
(b) for some $i \in S$, $V_i(\omega', e_i + z_i) \geq V_i(\omega', e_i + [z' \circ (\alpha_S, \alpha_{-S}^*)]_i)$ for all $\omega' \in \Omega$.

Since $e + z \circ \alpha \notin \boldsymbol{C}(\mathscr{E})$, there exists a state $\omega^* \in \Omega$, a coalition $S \subset I$, and $z^* \in \boldsymbol{Z}$ such that $\sum_{i \in S} z_i^* = 0$ and for every $i \in S$,

$$V_i(\omega^*, e_i + z_i^*) > V_i(\omega^*, e_i + (z \circ \alpha)_i). \tag{3.2}$$

Now define $z' = (z_i')_{i \in I} \in \boldsymbol{Z}$ by

$$z_i'(\omega') = \begin{cases} z_i^*(\omega^*) & \text{if } \omega' \in (\bigwedge_{i \in I} \mathscr{F}_i)(\alpha(\omega^*)), \\ 0 & \text{otherwise.} \end{cases}$$

Then $e + z' \circ \alpha \in \boldsymbol{A}$ and $(z' \circ \alpha)_i(\omega^*) = z_i^*(\omega^*)$ for every $i \in S$. Thus, it follows from (3.2) that for every $i \in S$,

$$V_i(\omega^*, e_i + (z' \circ \alpha)_i) > V_i(\omega^*, e_i + (z \circ \alpha)_i).$$

Thus, condition (a) holds.

Also note that for every $\omega' \in \Omega$,

$$[z' \circ (\alpha_S, \alpha_{-S}^*)](\omega') = \begin{cases} z^*(\omega^*), & \text{if } E_\alpha^S(\omega') \subset (\bigwedge_{i \in I} \mathscr{F}_i)(\alpha(\omega^*)), \\ 0, & \text{otherwise,} \end{cases}$$

which implies that $e + z' \circ (\alpha_S, \alpha_{-S}^*) \in \boldsymbol{A}$. Since $e + z \in \boldsymbol{C}(\mathscr{E})$, it must be the case that for some $i \in S$,

$$V_i(\omega', e_i + z_i) \geq V_i(\omega', e_i + [z' \circ (\alpha_S, \alpha^*_{-S})]_i), \ \forall \omega' \in \Omega.^{15} \tag{3.3}$$

Hence condition (b) holds, and this completes the proof of the Lemma. □

Lemma 3.3. *The interim private core $\boldsymbol{C}(\mathscr{E})$ is CBIC.*

Proof. Let $x = e + z \in \boldsymbol{C}(\mathscr{E})$ and suppose that x is not CBIC. Then there exists a state $\omega \in \Omega$, a coalition S, and a deception $\alpha_S : \prod_{i \in S} \mathscr{F}_i \to \prod_{i \in S} \mathscr{F}_i$ such that $(\alpha_S, \alpha^*_{-S})$ is compatible with F and for every $i \in S$,

$$V_i(\omega, e_i + [z \circ (\alpha_S, \alpha^*_{-S})]_i) > V_i(\omega, x_i),$$

where $e + z \circ (\alpha_S, \alpha^*_{-S}) \in \boldsymbol{A}$. Since for every $\omega' \in E^S_\alpha(\omega) \bigcap E^{-S}(\omega)$ it holds that $z_i(\omega') = z_i(\omega)$, *i.e.*, $[(z \circ (\alpha_S, \alpha^*_{-S})]_i(\omega) = z_i(\omega)$ for every $i \notin S$, it must be the case that for every $i \notin S$,

$$V_i(\omega, e_i + [z \circ (\alpha_S, \alpha^*_{-S})]_i) = V_i(\omega, x_i). \tag{3.4}$$

Since $V_i(\omega, \cdot)$ is continuous for every $i \in I$, there exists an $\varepsilon > 0$ such that for every $i \in S$,

$$V_i(\omega, e_i + [z \circ (\alpha_S, \alpha^*_{-S})]_i - \varepsilon \cdot \mathbf{1}) > V_i(\omega, x_i). \tag{3.5}$$

Now define $x' = (x'_i)_{i \in I}$ by

$$x'_i = \begin{cases} e_i + [z \circ (\alpha_S, \alpha^*_{-S})]_i - \varepsilon \cdot \mathbf{1} & \text{if } i \in S, \\ e_i + [z \circ (\alpha_S, \alpha^*_{-S})]_i + \frac{|S|}{|I \setminus S|} \varepsilon \cdot \mathbf{1} & \text{if } i \notin S. \end{cases}$$

Note that x'_i is $\mathscr{F}_i$-measurable and x' is a feasible allocation since $e + z \circ (\alpha_S, \alpha^*_{-S}) \in \boldsymbol{A}$. However, (3.5) implies that $V_i(\omega, x'_i) > V_i(\omega, x_i)$ for every $i \in S$. Because $V_i(\omega, \cdot)$ is monotone for every $i \in I$, (3.4) implies that $V_i(\omega, x'_i) > V_i(\omega, x_i)$ for every $i \notin S$, a contradiction to the fact that $x \in \boldsymbol{C}(\mathscr{E})$. □

Lemma 3.4. *The interim private core $\boldsymbol{C}(\mathscr{E})$ satisfies the condition C.*

Proof. Let $\{E^k : k = 1, \ldots, m\}$ be the common knowledge partition and $e + z^k \in \boldsymbol{C}(\mathscr{E})$ for $k = 1, \ldots, m$. Define $z^* = \sum_{i=1}^m z^k \cdot \chi_{E^k}$. Suppose, by way of contradiction, that $e + z^* \notin \boldsymbol{C}(\mathscr{E})$. Then there exists a state ω, a coalition $S \subset I$, and $x_S \in \prod_{i \in S} L_{X_i}$ such that $\sum_{i \in S} x_i = \sum_{i \in S} e_i$ and for every $i \in S$,

$$V_i(\omega, x_i) > V_i(\omega, e_i + z^*_i),$$

which is equivalent to

$$V_i(\omega, x_i) > V_i(\omega, e_i + z^k_i),$$

where $\omega \in E^k$ for some k. Then $e + z^k$ is not an interim private core allocation, a contradiction. □

[15] For if (3.3) does not hold, then there is a state $\omega \in \Omega$ such that $V_i(\omega, e_i + [z' \circ (\alpha_S, \alpha^*_{-S})]_i) > V_i(\omega, e_i + z_i)$ for every $i \in I$, a contradiction to the fact that $e + z \in \boldsymbol{C}(\mathscr{E})$.

Theorem 3.5. *If $N \geq 3$, the interim private core $\boldsymbol{C}(\mathscr{E})$ is c-implementable.*

Proof. By Lemmata 3.1 - 3.4, the interim private core $\boldsymbol{C}(\mathscr{E})$ is IIR, IE, CBIC, and satisfies CBM and C. Thus, by virtue of Theorem 2.7.1, we can conclude that the interim private core is c-implementable. □

Note that when there are two agents in the economy and the initial endowment is not interim efficient, the interim private core $\boldsymbol{C}(\mathscr{E})$ is implementable.

4 Coalitional implementation of the private value

As is the case with the private core notions defined above, the private value (see also Krasa and Yannelis, 1994) does not necessarily allow agents to share their private information. The problems that arise whenever coalitions of agents either pool their information or use their common knowledge information are discussed in Krasa and Yannelis (1994, 1996).

We introduce an interim version of a private value allocation. For each economy with differential information $\mathscr{E}$, for each state $\omega \in \Omega$, and for each set of weights $\{\lambda_i(\omega) : i \in I\}$, we can now associate a TU-game $G = (I, W)$ according to the following rule: For each $\omega \in \Omega$ and each $S \subset I$, let

$$W(\omega, S) = \max\left\{\sum_{i\in S}\lambda_i(\omega)V_i(\omega, x_i) : \sum_{i\in S}x_i = \sum_{i\in S}e_i; x_i \in L_{X_i}\right\}. \tag{4.1}$$

The **interim Shapley value** of the TU-game $G = (I, W)$ is a rule which assigns to each agent i a payoff $\Psi_i(\omega, W)$ at each state ω, which is given by:

$$\Psi_i(\omega, W) = \sum_{S\subset I, S\ni i}\frac{(|S|-1)!(N-|S|)!}{N!}[W(\omega, S) - W(\omega, S\setminus\{i\})].$$

Note that the interim Shapley value is individually rational and Pareto optimal, *i.e.*, $\Psi(\omega, W) \geq W(\omega, \{i\})$ for every $\omega \in \Omega$ and for every $i \in I$, and $\sum_{i\in I}\Psi(\omega, W) = W(\omega, I)$ for every ω.

Definition 4.1. *An allocation $x \in \boldsymbol{A}$ is an* **(interim) private value allocation** *of the economy with differential information $\mathscr{E}$ if for every $\omega \in \Omega$, there exist $\lambda(\omega) = (\lambda_i(\omega))_{i\in I} \in \boldsymbol{R}_+^N \setminus \{0\}$ such that for each $i \in I$,*

$$\lambda_i(\omega)V_i(\omega, x_i) = \Psi_i(\omega, W),$$

where $\Psi_i(\omega, W)$ is the interim Shapley value derived from the TU-game $G = (I, W)$ defined by (4.1). The **interim private value** *is the set of all interim private value allocations for $\mathscr{E}$ and it is denoted by $\boldsymbol{V}(\mathscr{E})$.*

Theorem 4.1. *If $N \geq 3$ and $\lambda \gg 0$, then the interim value $\boldsymbol{V}(\mathscr{E})$ is c-implementable.*

Proof. Since the interim private value $\boldsymbol{V}(\mathscr{E})$ is IIR, IE, CBIC, and satisfies CBM and C (see Hahn and Yannelis, 1995, for the details), Theorem 2.7.1 implies that the interim private value is c-implementable. □

Similarly with the interim private core, if there are two agents in the economy and the initial endowment is not interim efficient, the interim value $\boldsymbol{V}(\mathscr{E})$ with $\lambda \gg 0$ is c-implementable.

5 Examples of non-c-implementation

According to Palfreya and Srivastava (1987), the rational expectations equilibrium (REE) social choice set is Bayesian Nash implementable but neither the interim efficient social choice set nor the interim core is Bayesian Nash implementable because neither one satisfies the Bayesian monotonicity condition. Note that Palfrey and Srivastava define an interim efficiency notion without information sharing at all and the initial endowment of each agent e_i is not $\mathscr{F}_i$-measurable. Hence their core notion is quite different than ours and one can easily show that it is not c-implementable. Also, one can construct examples to show that the interim private core is not Bayesian Nash implementable. We show below that the interim fine core (which allows for information pooling within a coalition) is not c-implementable.

Definition 5.1. *A feasible allocation x with x_i being $\bigvee_{i\in I}\mathscr{F}_i$-measurable for every $i \in I$ is an* **interim fine core allocation** *of the economy with differential information $\mathscr{E}$ if it is not true that there exist a state $\omega \in \Omega$, a coalition $S \subset I$, and $(y_i)_{i\in S}$ such that y_i is $\bigvee_{i\in S}\mathscr{F}_i$-measurable for every $i \in S$, $\sum_{i\in S} y_i = \sum_{i\in S} e_i$, and $V_i(\omega, y_i) > V_i(\omega, x_i)$ for every $i \in S$.*

The above core concept is different from the (interim) private core in that agents in a coalition now are allowed to pool their information instead of making redistributions based on their individual private information only (as the private core requires). This notion is analogous to fine core notion of Wilson (1978) (see also Srivastava, 1984a, b; Yannelis, 1991). The interim fine core and the interim fine value need not be c-implementable because they violate the CBIC condition as the following example indicates.

Example 5.1. Consider an economy with differential information with three agents, two goods (*i.e.*, x^1, x^2), and three equally probable states, where utility functions, random initial endowments, and private information sets are given as follows:

$$
\begin{aligned}
u_1(\omega, x^1, x^2) &= \sqrt{x^1 x^2}, & e_1 &= ((7,1),(7,1),(4,1)), & \mathscr{F}_1 &= \{\{\omega_1,\omega_2\},\{\omega_3\}\},\\
u_2(\omega, x^1, x^2) &= \sqrt{x^1 x^2}, & e_2 &= ((7,1),(7,1),(4,1)), & \mathscr{F}_2 &= \{\{\omega_1,\omega_2\},\{\omega_3\}\},\\
u_3(\omega, x^1, x^2) &= \sqrt{x^1 x^2}, & e_3 &= ((1,10),(1,7),(1,7)), & \mathscr{F}_3 &= \{\{\omega_1\},\{\omega_2,\omega_3\}\}.
\end{aligned}
$$

The allocation $x^* = (x_1^*, x_2^*, x_3^*)$ with

$$x_1^* = ((33/8,\ 33/10), (13/3,\ 13/5), (5/2,\ 5/2))$$
$$x_2^* = ((33/8,\ 33/10), (13/3,\ 13/5), (5/2,\ 5/2))$$
$$x_3^* = ((54/8,\ 54/10), (19/3,\ 19/5), (4, 4))$$

is an interim fine core allocation. But it is not CBIC. To see this, suppose that ω_2 is realized and let $z^* = x^* - e$. Consider the coalition $S = \{1, 2\}$ and the deception $\alpha_i(E_i(\omega)) = \{\omega_3\}$ for every $\omega \in \Omega$ and for $i \in S$. Since

$$V_i(\omega_2, e_i + [z^* \circ (\alpha_S, \alpha^*_{-S})]_i) > V_i(\omega_2, x_i^*)$$

for $i \in S$, it follows that x^* is not CBIC. Therefore, the interim fine core is not c-implementable.[16]

We will show that the fine core allocation in Example 5.1 is also a fully revealing REE allocation[17] which in turn violates the CBIC condition and therefore it is not c-implementable: Consider the same economy and the same allocation x^* as in Example 5.1. The price-allocation pair $(p^*, x^*) = (p^*, x_1^*, x_2^*, x_3^*)$ with $p^* = ((4/5, 1), (3/5, 1), (1, 1))$ constitutes a fully revealing rational expectations equilibrium. However, since the allocation x^* is not CBIC as it is shown in Example 5.1, we can conclude that the set of REE allocations is not c-implementable. It should be noted that one cannot c-implement the ex ante private core because such allocation rules are not necessarily interim individually rational.

Finally, one may wonder as to whether or not an extension of the coalition-proof Nash equilibrium concept to differential information economies can be adopted here instead of the coalitional Bayesian Nash equilibrium concept. The answer is no because such a concept will yield outcomes which are not necessarily interim efficient.

6 Conclusions

We introduced the idea of coalitional Bayesian implementation with the main objective to examine solution concepts that the standard Bayesian Nash implementation literature does not cover. In particular, we presented necessary and sufficient conditions for the coalitional implementation of a social choice set, and as a consequence of this, we showed that the private core and private value are indeed coalitionally implementable.

It is important to note that our c-implementation results do not rule out information asymmetries, *i.e.*, an agent who has superior information that is useful to the rest of the economy will be rewarded. This is in sharp contrast with the

[16] A similar example can be constructed for the interim private value allocation (see Hahn and Yannelis (1995) for the details).

[17] When prices and allocations are $(\bigvee_{i \in I} \mathscr{F}_i)$-measurable, one can define the notion of a $(\bigvee_{i \in I} \mathscr{F}_i)$-revealing REE and it can be easily checked (the proof is similar to that of Debreu-Scarf) that the set of $(\bigvee_{i \in I} \mathscr{F}_i)$-revealing REE allocations is contained in the fine core. A related result has been proved by Srivastava (1984b).

Bayesian Nash implementation results of the rational expectations equilibrium, where the stringent informational assumptions rule out information asymmetries.

It should be emphasized that the idea of coalitional Bayesian implementation is quite natural for resource allocation concepts because the outcomes that the game generates are always interim efficient, contrary to the standard Bayesian Nash implementation. Moreover, the assumptions needed for the c-implementation of our solution concepts are quite attractive from a normative viewpoint.

Finally our results support the conjectures and findings of Wilson (1978), Srivastava (1984a, b), Yannelis (1991), Krasa and Yannelis (1994, 1996), and Koutsougeras and Yannelis (1993). Specifically, these authors note that some information sharing (*e.g.*, pooling of information) may not be incentive compatible and may also rule out the information superiority of an individual. Our examples in Section 6 indicate that indeed such core and value notions which allow for pooling information need not be c-implementable (as is the case for the REE). In that sense, our main c-implementation theorem is especially useful, because it not only delineates a set of reasonable and mild conditions which are necessary and sufficient for c-implementation, but it also enables us to conclude that the private core and private value do provide a successful alternative to the (Walrasian) rational expectation equilibrium.

Appendix

Proof of Lemma 2.7.2. First notice that $\sigma(E(\omega)) \in M^0$ for every $\omega \in \Omega$. Observe that $f(\sigma) = z$ by the definition of f. Consider an arbitrary state $\omega \in \Omega$. Let S be a coalition deviating from the strategy vector σ and denote the deviating strategy of S by σ'_S with $\sigma'_i(E_i(\omega)) \neq \sigma_i(E_i(\omega))$ and $\sigma'_i(E_i(\omega')) = \sigma_i(E_i(\omega')), \forall \omega' \notin E_i(\omega)$ for every $i \in S$. Let $\sigma' = (\sigma'_S, \sigma_{-S})$.

First, consider a proper coalition S. Then $\sigma'(E(\omega)) \in [\bigcup_{k=0}^{4} M^k] \cup M^6$. Notice that it is impossible that $\sigma'(E(\omega)) \in M^5$ because no agent reports the integer zero in M^5. If $\sigma'(E(\omega)) \in M^0$, CBIC of z implies that the coalition S cannot misreport to become better off. If $\sigma'(E(\omega)) \in M^{11}(S)$, the definition of f and the property of V_i on this region implies that there exists at least one agent in the coalition S who cannot become better off by deviating. If $\sigma'(E(\omega)) \in M^{12}(S)$, or $\sigma'(E(\omega)) \in M^6$, the new outcome is no trade and IIR of $e + z$ implies that no agent in the coalition S can become better off. If $\sigma'(E(\omega)) \in M^2$, where only agent 1 deviates, it is clear that agent 1 becomes worse off. Suppose that $\sigma'(E(\omega)) \in M^3$, then by the definition of f, at least one agent is worse off since every agent (except agent 1) transfers his/her reported endowment to agent 1. If $\sigma'(E(\omega)) \in M^4(S)$ with $|S| \geq 2$, then IIR of z and the monotonicity of preferences would imply that an agent in the coalition S who is not the winner of the "integer game" would be worse off.

Let S be a grand coalition, *i.e.*, $S = I$, $\sigma'(E(\omega)) \in \bigcup_{k=0}^{6} M^k$. The interim efficiency of $e+z$ implies that every agent in the grand coalition cannot become better off.

Since no coalitional deviation from σ is profitable, we conclude that σ is a CBNE and $f(\sigma) = z$. □

In the argument below, we set $\sigma_i(E_i(\omega)) = (\sigma_i^1(E_i(\omega)), \sigma_i^2(E_i(\omega)), \sigma_i^3(E_i(\omega)))$.

Proof of Lemma 2.7.3. Suppose, by way of contradiction, that $\sigma(E(\omega)) \notin M^0$ for some ω. Let us define $\tilde{\sigma}_S$ to be a deviation from σ_S by the coalition S as follows: For every $i \in S$,

$$\begin{aligned}\tilde{\sigma}_i(E_i(\omega)) &= (\sigma_i^1(E_i(\omega)), \sigma_i^2(E_i(\omega)), n^*),\\ \tilde{\sigma}_i(E_i(\omega')) &= \sigma_i(E_i(\omega')), \forall \omega' \notin E_i(\omega),\end{aligned}$$

where $n^* = 1 + \max\{\sigma_i^3(E_i(\omega)) : i \in I\}$. Let $\tilde{\sigma} = (\tilde{\sigma}_S, \sigma_{-S})$. Then there are the following cases to consider:

(1) Suppose that $\sigma(E(\omega)) \in M^{11}(S)$ for some S. Let $\sigma_i(E_i(\omega)) = (E_i, z', n)$ for every $i \in S$. Observe that $0 \leq e_i + z_i' \leq \sum_{i \in I} e_i$ for every $i \in I$. Consider any $\omega' \in \bigcap_{i \in I} \sigma_i^1(E_i(\omega))$. Then $z_i'(\omega') < \sum_{j \neq i} e_j(\omega')$ for every $i \in I$. For, otherwise there is an agent k in I such that $z_k'(\omega') = \sum_{i \neq k} e_i(\omega')$. By feasibility, $\sum_{i \neq k} z_i'(\omega') = -\sum_{i \neq k} e_i(\omega)$. Since $z_i'(\omega') + e_i(\omega') \geq 0$ for every $i \neq k$, it follows that $z_i'(\omega') = -e_i(\omega') \ll 0$ for every $i \neq k$, a contradiction to the IIR of z'. Hence, some agent i in $I \setminus S$ will deviate to $M^4 \cup M^5$ using the strategy $\tilde{\sigma}_i$ to win the integer game and become better off.

(2) Suppose that $\sigma(E(\omega)) \in M^{12}(S)$ for some S. Note that $\sum_{j \neq i} e_j(\omega') \gg 0$ for every $i \in I$ and every $\omega' \in \bigcap_{j \in I} \sigma_j^1(E_j(\omega))$. Some agent i in $I \setminus S$ will deviate to $M^4 \cup M^5$ by using strategy $\tilde{\sigma}_i$. For he/she who gets $-e_i(\omega')$ at σ would become the winner of the integer game, obtain $\sum_{j \neq i} e_j(\omega')$, and become better off, since the new message $\tilde{\sigma}(E(\omega))$ would belong to $M^4 \cup M^5$.

(3) Suppose that $\sigma(E(\omega)) \in M^2$. If $\bigcap_{i \in I} E_i \neq \emptyset$, then an agent $i \in I \setminus \{1\}$ will deviate using the strategy $\tilde{\sigma}_i$. Since the new message $\tilde{\sigma}(E(\omega))$ belongs to M^4, he/she wins the integer game and becomes better off. If $\bigcap_{i \in I} E_i = \emptyset$, agent 1 will deviate to M^6 using the strategy $\tilde{\sigma}_1$ and get no trade.

(4) Suppose that $\sigma(E(\omega)) = (E_i, z^i, 0)_{i \in I} \in M^3$. We first consider the case where $\bigcap_{i \in I} E_i \neq \emptyset$. An agent $i \in I \setminus \{1\}$ will deviate using the strategy $\tilde{\sigma}_i$ to become better off. Since the new message $\tilde{\sigma}(E(\omega))$ lies in M^4 and he/she wins the integer game, he/she becomes better off. If $\bigcap_{i \in I} E_i = \emptyset$, an agent $i \in I \setminus \{1\}$ will deviate using the strategy $\tilde{\sigma}_i$ to become better off. Since the new message $\tilde{\sigma}(E(\omega))$ lies in M^6 and he/she obtains no trade, he/she becomes better off.

(5) If $\sigma(E(\omega)) \in M^4 \cup M^5$, an agent i who is one of the losers in the integer game will use the strategy $\tilde{\sigma}_i$ and become better off. Since the new message $\tilde{\sigma}(E(\omega))$ lies in $M^4 \cup M^5$, agent i becomes the winner of the integer game and gets $\sum_{j \neq i} e_j(\omega') \gg -e_i(\omega')$ for every $\omega' \in \bigcap_{j \in I} \sigma_j^1(E_j(\omega))$.

(6) Suppose that $\sigma(E(\omega)) = (E_i, z^i, n_i)_{i\in I} \in M^6$. Fix any agent k such that $\sigma_k(E_k(\omega)) = (E_k, z^k, n_k)$ with $n_k \neq 0$. Then the coalition $S = I \setminus \{k\}$ will deviate by using the strategy $\bar{\sigma}_S$ such that $\bar{\sigma}_i(E_i(\omega)) = (\bar{E}_i, \bar{z}^i, n_k + 1)$ and $\bar{\sigma}_i(E_i(\omega')) = \sigma_i(E_i(\omega')), \forall \omega' \notin E_i(\omega)$ for every $i \in S$, where $E_k \cap [\bigcap_{i\neq k} \bar{E}_i] \neq \emptyset$ and $\bar{z}^i \neq \bar{z}^j$ for some $i, j \in I \setminus \{k\}$. Since the new message $\bar{\sigma}(E(\omega))$ with $\bar{\sigma} = (\bar{\sigma}_S, \sigma_{-S})$ lies in M^5, the agent k, who is not the winner of the integer game, gives to the coalition S his/her reported endowment $e_k(\omega') \gg 0$ with $\omega' \in E_k \cap [\bigcap_{i\neq k} \bar{E}_i]$, which all agents in the coalition S evenly share. Therefore, every agent in the coalition S becomes better off.

From (1) through (6), it follows that σ is not a CBNE for (M, f), a contradiction. □

Proof of Lemma 2.7.4. By Lemma 2.7.3, $\sigma(E(\omega)) \in M^0$ for all ω. Since Γ satisfies C, we get $\sigma_i^2(E_i(\omega)) = z^*$ with $e + z^* \in \Gamma$ for every $i \in I$ and for every $\omega \in \Omega$. Define $\alpha_i(E_i(\omega)) = \sigma_i^1(E_i(\omega))$ for every $i \in I$. Then it follows from the definition of the mechanism that $f(\sigma) = z^* \circ \alpha$. We have to show that $e + f(\sigma) = e + z^* \circ \alpha \in \Gamma$. Suppose, by way of contradiction, that $e + z^* \circ \alpha \in \boldsymbol{A} \setminus \Gamma$. By CBM, there exists a state $\omega^* \in \Omega$, a coalition $S \subset I$, and $z' \in \boldsymbol{Z}$ such that $e + z' \circ \alpha \in \boldsymbol{A}$, $e + z' \circ (\alpha_S, \alpha^*_{-S}) \in \boldsymbol{A}$,

(1) $\forall i \in S,\ V_i(\omega^*, e_i + (z' \circ \alpha)_i) > V_i(\omega^*, e_i + (z^* \circ \alpha)_i)$, and
(2) $\exists i \in S,\ V_i(\omega', e_i + z_i^*) \geq V_i(\omega', e_i + [z' \circ (\alpha_S, \alpha^*_{-S})]_i),\ \forall\, \omega' \in \Omega$.

If they use the strategy σ'_S with $\sigma'_i(E_i(\omega^*)) = (\alpha_i(E_i(\omega^*)), z', 1)$ and $\sigma'_i(E_i(\omega)) = \sigma_i(E_i(\omega)), \forall \omega \notin E_i(\omega^*)$ for every $i \in S$, by (2) they move from M^0 to $M^{11}(S)$. Furthermore, since $f(\sigma'_S, \sigma_{-S}) = z' \circ \alpha$, (1) implies that every agent $i \in S$ becomes better off at ω^*, *i.e.*, $V_i(\omega^*, e_i + f_i(\sigma'_S, \sigma_{-S})) > V_i(\omega^*, e_i + f_i(\sigma))$, a contradiction to the fact that σ is a CBNE for (M, f). □

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