JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **122**, 393–407 (1987)

Caratheodory-Type Selections and Random Fixed Point Theorems

TAESUNG KIM

Department of Economics, University of Minnesota, Minneapolis, Minnesota 55455

KAREL PRIKRY*

Department of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

AND

NICHOLAS C. YANNELIS

Department of Economics, University of Minnesota, Minneapolis, Minnesota 55455, and Department of Economics, Tulane University, New Orleans, Louisiana 70118

Submitted by G.-C. Rota

Received September 10, 1985

We provide some new Caratheodory-type selection theorems, i.e., selections for correspondences of two variables which are continuous with respect to one variable and measurable with respect to the other. These results generalize simultaneously Michael's [21] continuous selection theorem for lower-semicontinuous correspondences as well as a Caratheodory-type selection theorem of Fryszkowski [10]. Random fixed point theorems (which generalize ordinary fixed point theorems, e.g., Browder's [6]) follow as easy corollaries of our results. © 1987 Academic Press, Inc.

1. INTRODUCTION

The two major types of selection theorems are continuous selection results of Michael-type [21], and measurable selection results of von Neumann–Aumann-type [1, 8, 19]. Both types of selection theorems have found important applications in general equilibrium theory as well as in

* Research partially supported by an NSF grant.

0022-247X/87 $3.00

Copyright © 1987 by Academic Press, Inc.
All rights of reproduction in any form reserved.

other areas of analysis [11, 15, 17, 23, 29, 30, 31]. The present paper is concerned with so-called Caratheodory-type selections which combine both measurable and continuous selections via the setting of a product space. Thus Caratheodory-type selections are selections for correspondences of two variables which are continuous with respect to one of the variables and measurable with respect to the other. The main reference for this type of result appears to be Fryszkowski's paper [10], which in turn generalizes results of Cellina and Castaing [7]. We ourselves have encountered need for this type of result in [17, 31] while studying equilibria in economies with a measure space of agents.

Our main result (Theorem 3.1) is a simultaneous generalization of Michael's continuous selection theorem for lower-semicontinuous correspondences (Theorem 3.1‴, [21, p. 368]) as well as of Fryszkowski's Theorem 1 [10, p. 44]. As an application of Theorem 3.1 we obtain a random fixed point theorem, Theorem 3.3, which generalizes the fixed point theorem of Bohnenblust–Karlin [4] as well as a result of Itoh [16]. The random fixed point theorems, in turn, are generalizations of ordinary fixed point theorems, e.g., Browder's [6], but only when the underlying space is separable.

Now, we would like to comment on the relationship between our results and those of Fryszkowski. We consider the following setting. Let T be a measure space, and X and Y be topological spaces. Let $\phi\colon T\times X\to 2^Y$ be a (possibly empty-valued) correspondence and

$$U=\{(t,x)\in T\times X\colon \phi(t,x)\neq\varnothing\}.$$

A function $f\colon U\to Y$ such that $f(t,x)\in\phi(t,x)$ for all $(t,x)\in U$, $f(\cdot,x)$ is measurable for each x and $f(t,\cdot)$ is continuous for each t is said to be a *Caratheodory-type selection* for the correspondence ϕ. In the actual results, ours as well as Fryszkowski's, it is assumed that T is a complete measure space, X is complete separable metric, and Y is a Banach space. In addition, Fryszkowski assumes that X is locally compact and $U=T\times X$.

In our view, Fryszkowski's arguments are somewhat ad hoc. In effect, by an application of Michael's continuous selection theorem, he reduces the problem to an application of the measurable selection theorem of [19] to an auxiliary correspondence into a function space. This type of argument cannot be readily adapted to the setting when U is an arbitrary subset of $T\times X$. Likewise, we do not need the local compactness. By comparison, our arguments are more direct. To a large extent we simply just mimic Michael's proof. That is, we carry out a "parametrized" version of his proof, where the parameter t ranges over the measurable space T. We think that the details here are far from routine. We need a number of results scattered throughout the literature on measurable selections (Castaing–

Valadier [8] is especially helpful) as well as some simple ideas of "descriptive set-theoretic" character.

Finally, we suggest that allowing U to be an arbitrary subset of $T \times X$, rather than $T \times X$ itself, is a significant generalization. We needed this type of results in [17] and [31] to extend the theory of Nash equilibria, developed in [9, 11, 13, 14, 20, 22, 24, 26, 29, 30] to the setting of an arbitrary measure space of agents and an infinite dimensional strategy space. Moreover, random fixed point theorems also follow as easy corollaries of this more general version.

In [18] we have proved the Caratheodory-type selection theorem which we needed in [17]. This result did not require the full strength of Michael's methods but only a rather direct argument involving partitions of unity. The present paper deals primarily with the kinds of situations arising in applications of the theory of measurable selections in analysis (lower-semicontinuous closed-valued mappings), see [1, 2, 8]. By contrast, [18] is focused on a different and narrower situation which arose in [17].

The paper is organized as follows. Section 2 contains notation and definitions. Section 3 contains the statements of the main results of the paper. Several technical lemmata needed for the proof of our main results are given in Section 4. Finally, Section 5 contains proofs of the main results.

2. Notation and Definitions

2.1. *Notation*

2^A	the set of all subsets of the set A,
cl A	the closure of the set A,
$\setminus$	the set theoretic subtraction,

If $\phi: X \to 2^Y$ is a correspondence then $\phi|_U: U \to 2^Y$ denotes the restriction of ϕ to U,

$B(x, \varepsilon)$	the open ball centered at x of radius ε,
diam	diameter,
dist	distance,
proj	projection.

2.2. *Definitions*

Let X and Y be sets. The graph G_ϕ of a correspondence $\phi: X \to 2^Y$ is the set $G_\phi = \{(x, y) \in X \times Y: y \in \phi(x)\}$. If X and Y are topological spaces, $\phi: X \to 2^Y$ is said to be *lower semicontinuous* (l.s.c.) if the set $\{x \in X: \phi(x) \cap V \neq \emptyset\}$ is open in X for every open subset V of Y.

If $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are measurable spaces and $\phi: X \to 2^Y$ is a correspondence, ϕ is said to have a *measurable graph* if G_ϕ belongs to the product σ-algebra $\mathcal{A} \otimes \mathcal{B}$. We are often interested in the situation where $(X, \mathcal{A})$ is a measurable space, Y is a topological space and $\mathcal{B} = \mathcal{B}(Y)$ is the Borel σ-algebra of Y. For a correspondence ϕ from a measurable space into a topological space, if we say that ϕ has a measurable graph, it is understood that the topological space is endowed with its Borel σ-algebra (unless specified otherwise). In the same setting as above, i.e., $(X, \mathcal{A})$ a measurable space and Y a topological space, ϕ is said to be *lower measurable* if

$$\{x: \phi(x) \cap V \neq \emptyset\} \in \mathcal{A} \qquad \text{for every } V \text{ open in } Y.$$

A *continuous selection* for a correspondence ϕ between topological spaces is a function f defined on the set $\tilde{X} = \{x \in X: \phi(x) \neq \emptyset\}$, continuous on this set and satisfying $f(x) \in \phi(x)$ for $x \in \tilde{X}$.

A *measurable selection* is defined analogously: if $\tilde{X}$ is as above, we require f to be measurable on $\tilde{X}$ with respect to the σ-algebra of relatively measurable subsets of $\tilde{X}$. That is, if $(X, \mathcal{A})$ is a measurable space, the relatively measurable subsets of $\tilde{X}$ are sets of the form $\tilde{X} \cap A$, where $A \in \mathcal{A}$. Of main interest is the case when $\tilde{X}$ belongs to $\mathcal{A}$ and then, of course, the relatively measurable subsets of $\tilde{X}$ are just the (ordinary) measurable subsets of $\tilde{X}$.

Let $(X, \mathcal{A})$ be a measurable space and Y and Z be topological spaces. Let $\phi: X \times Z \to Y$ be a (possibly empty-valued) correspondence. Let $U = \{(x, z) \in X \times Z: \phi(x, z) \neq \emptyset\}$. A *Caratheodory selection for* ϕ is a function $f: U \to Y$ such that $f(x, z) \in \phi(x, z)$; for each $x \in X$, $f(x, \cdot)$ is continuous on $U^x = \{z \in Z: (x, z) \in U\}$, and for each $z \in Z$, $f(\cdot, z)$ is measurable on $U_z = \{x \in X: (x, z) \in U\}$.

If $(X, \mathcal{A})$, $(Y, \mathcal{B})$ and $(Z, \mathcal{C})$ are measurable spaces, $U \subseteq X \times Z$ and $f: U \to Y$, we call f *jointly measurable* if for every $B \in \mathcal{B}$, $f^{-1}(B) = U \cap A$ for some $A \in \mathcal{A} \otimes \mathcal{C}$. It is a standard result that if Z is a separable metric space, Y is metric and $f: X \times Z \to Y$ is such that for each fixed $x \in X$, $f(x, \cdot)$ is continuous and for each fixed $z \in Z$, $f(\cdot, z)$ is measurable, then f is jointly measurable (where $\mathcal{B} = \mathcal{B}(Y)$, $\mathcal{C} = \mathcal{B}(Z)$) However, when U is a proper subset of $X \times Z$, the situation is more delicate. The appropriate result is stated as Lemma 4.12.

Recall that an open cover $\mathcal{U}$ of a topological space X is *locally finite* if every $x \in X$ has a neighborhood intersecting only finitely many sets in $\mathcal{U}$. With all these preliminaries out of the way, we can now turn to our main results.

3. Main Theorems

The theorem below generalizes the Caratheodory-type selection theorem of Fryszkowski [10].

THEOREM 3.1. *Let (T, τ, μ) be a complete finite measure space, Y be a separable Banach space and Z be a complete, separable metric space. Let $\phi: T \times Z \to 2^Y$ be a convex, closed (possibly empty-) valued correspondence such that:*

(i) *$\phi(\cdot, \cdot)$ is lower measurable with respect to the σ-algebra $\tau \otimes \mathscr{B}(Z)$ and*

(ii) *for each $t \in T$, $\phi(t, \cdot)$ is* l.s.c.

Then there exists a Caratheodory-type selection for ϕ. Moreover this selection is jointly measurable.

THEOREM 3.2. *The statement of Theorem* 3.1 *remains true without closed valueness of $\phi: T \times Z \to 2^Y$ if either*

(i) *Y is finite dimensional or*

(ii) *$\phi(t, x)$ has a nonempty interior for all $(t, x) \in U$.*

Let T be any measure space and X be a nonempty subset of any linear topological space. Let ϕ be a correspondence from $T \times X$ into 2^X. The correspondence ϕ is said to have a *random fixed point* if there exists a measurable function $x: T \to X$ such that $x(t) \in \phi(t, x(t))$ for almost all t in T. Below we provide a random fixed point theorem. This result generalizes a theorem of Bohnenblust and Karlin [4]. For other random fixed point results see [16, 23].

THEOREM 3.3. *Let (T, τ, μ) be a complete finite measure space, and X be a nonempty compact convex subset of a separable Banach space Y. Let $\phi: T \times X \to 2^X$ be a nonempty, convex, closed valued correspondence such that:*

(i) *$\phi(\cdot, \cdot)$ is lower measurable and*

(ii) *for each $t \in T$, $\phi(t, \cdot)$ is* l.s.c.

Then ϕ has a random fixed point.

THEOREM 3.4. *The statement of Theorem* 3.3 *remains true without closed valueness of $\phi: T \times X \to 2^X$ if either*

(i) *Y is finite dimensional or*

(ii) *$\phi(t, x)$ has a nonempty interior for all $(t, x) \in T \times X$.*

Remark 3.1. We wish to comment further on Theorem 3.1. Let $U = \{(t, x) \in T \times Z: \phi(t, x) \neq \varnothing\}$. It follows at once from (i) that $U \in \tau \otimes \mathscr{B}(Z)$. Hence clearly $U_x \in \tau$ for every $x \in Z$, and by (ii) U^t is open for every $t \in T$.

Assumption (i) can be relaxed to supposing the lower measurability with respect to the σ-algebra of $\tau \otimes \mathscr{B}(Z)$-relatively measurable subsets of U together with the hypothesis $U_x \in \tau$ for every $x \in Z$, and Theorem 3.1 still holds.

Remark 3.2. Theorem 3.4(i) can be seen as a generalization of a version of a theorem of Gale–Mas-Colell [11] given in Yannelis–Prabhakar [29, Theorem 3.4].

4. LEMMATA

LEMMA 4.1 (Aumann). *If (T, τ, μ) is a complete finite measure space, Y is a complete, separable metric space, and $F: T \to 2^Y$ is a correspondence with measurable graph, i.e., $G_F \in \tau \otimes \mathscr{B}(Y)$, then there is a measurable function $f: \operatorname{proj}_T(G_F) \to Y$ such that $f(t) \in F(t)$ for all $t \in \operatorname{proj}_T(G_F)$.*

Proof. See Aumann [1] or Castaing–Valadier [3].

LEMMA 4.2 (Projection theorem). *Let (T, τ, μ) be a complete finite measure space and Y be a complete, separable metric space. If G belongs to $\tau \otimes \mathscr{B}(Y)$, its projection $\operatorname{proj}_T(G)$ belongs to τ.*

Proof. See Theorem III.23 in Castaing–Valadier [3].

LEMMA 4.3. *Let (T, τ) be a measurable space, Z be an arbitrary topological space, and W_n, $n = 1, 2, \ldots$, be correspondences from T into Z with measurable graphs. Then the correspondences $\bigcup_n W_n(\cdot)$, $\bigcap_n W_n(\cdot)$ and $Z \setminus W_n(\cdot)$ have measurable graphs.*

The proof is obvious.

LEMMA 4.4. *Let (T, τ, μ) be a complete finite measure space, Z be a complete separable metric space and $W: T \to 2^Z$ be a correspondence with measurable graph. Then for every $x \in Z$, $\operatorname{dist}(x, W(\cdot))$ is a measurable function, where $\operatorname{dist}(x, \varnothing) = +\infty$.*

Proof. First, observe that $S = \{t \in T: W(t) \neq \varnothing\}$ belongs to τ by Lemma 4.2. Now $\{s \in S: \operatorname{dist}(x, W(s)) < \lambda\} = \{s \in S: W(s) \cap B(x, \lambda) \neq \varnothing\} = \operatorname{proj}_T[G_W \cap (T \times B(x, \lambda))]$. Another application of Lemma 4.2 concludes the proof.

LEMMA 4.5. *Let (T, τ, μ) be a complete finite measure space, Z be a complete separable metric space, and $W: T \to 2^Z$ be a correspondence with measurable graph. Then the correspondence $V: T \to 2^Z$ defined by*

$$V(t) = \{x \in Z: \operatorname{dist}(x, W(t)) > \lambda\} \qquad (\textit{where } \lambda \textit{ is any real number})$$

has a measurable graph. The same holds for the correspondence $V(t) = \{x \in Z\colon \operatorname{dist}(x, W(t)) \leqslant \lambda\}$.

Proof. Consider the function $g\colon T \times Z \to [0, +\infty]$ given by $g(t, x) = \operatorname{dist}(x, W(t))$. By Lemma 4.4, $g(\cdot, x)$ is measurable for each x, and obviously $g(t, \cdot)$ is continuous for each t. It is well known that g is therefore jointly measurable, i.e., measurable with respect to the product σ-algebra $\tau \otimes \mathscr{B}(Z)$. For this result see, e.g., Lemma III.14, [3]. Finally, $G_V = g^{-1}([\lambda, +\infty])$, hence $G_V \in \tau \otimes \mathscr{B}(Z)$.

LEMMA 4.6. *Let* (T, τ) *be a measurable space and* Z *be a separable metric space. If* $F\colon T \to 2^Z$ *is a lower measurable correspondence, then the correspondence* $\psi\colon T \to 2^Z$ *defined by* $\psi(t) = \operatorname{cl} F(t)$ *has a measurable graph.*

Proof. See Himmelberg [12, Theorem 3.3].

LEMMA 4.7. *Let* $(S, \mathcal{A})$ *be a measurable space,* X *a separable metric space and* $W\colon S \to 2^X$ *a lower measurable correspondence. Then the correspondence* $V\colon S \to 2^X$ *defined by*

$$V(s) = \{x \in X\colon \operatorname{dist}(x, W(s)) < \lambda\},$$

where λ *is any real number, has a measurable graph with respect to the* σ*-algebra* $\mathcal{A} \otimes \mathscr{B}(X)$.

Proof. We consider the function

$$g(s, x) = \operatorname{dist}(x, W(s)),$$

where $g(s, x) = +\infty$ if $W(s) = \varnothing$. Since $W(\cdot)$ is lower measurable, it follows at once that $g(\cdot, x)$ is measurable for every fixed x, for

$$\{s \in S\colon \operatorname{dist}(x, W(s)) < \lambda\} = \{s \in S\colon W(s) \cap B(x, \lambda) \neq \varnothing\}$$

and the latter set belongs to $\mathcal{A}$ by the assumption of lower measurability. (In particular, $\{s \in S\colon W(s) \neq \varnothing\} \in \mathcal{A}$.)

Obviously, for each fixed $s \in S$, $g(s, \cdot)$ is continuous. Hence by Lemma III.14, [8] (or see Lemma 4.12), g is measurable with respect to the product σ-algebra $\mathcal{A} \otimes \mathscr{B}(X)$. Finally

$\{(s, x)\colon x \in V(s)\} = g^{-1}((-\infty, \lambda)) \in \mathcal{A} \otimes \mathscr{B}(X)$, which is the desired result.

LEMMA 4.8. *Let* $(S_i, \mathcal{A}_i)$ *for* $i = 1, 2$, *be measurable spaces.* $S \subseteq S_1$, $S \in \mathcal{A}_1$, $h\colon S \to S_2$ *be a measurable function and* $A \in \mathcal{A}_1 \otimes \mathcal{A}_2$. *Then*

$$\operatorname{proj}_{S_1}(G_h \cap A) \in \mathcal{A}_1.$$

Proof. See [18, Lemma 4.4].

LEMMA 4.9. *Let* (T_i, τ_i) *for* $i = 1, 2, 3$, *be measurable spaces*, $y\colon T_1 \to T_3$ *be a measurable function and* $\phi\colon T_1 \times T_2 \to 2^{T_3}$ *be a correspondence with measurable graph, i.e.*, $G_\phi \in \tau_1 \otimes \tau_2 \otimes \tau_3$. *Let* $W\colon T_1 \to 2^{T_2}$ *be defined by*

$$W(t) = \{x \in T_2\colon y(t) \in \phi(t, x)\}.$$

Then W *has a measurable graph, i.e.*, $G_W \in \tau_1 \otimes \tau_2$.

Proof. See [18, Lemma 4.5].

LEMMA 4.10. *Let* (T, τ) *be a measurable space*, $S \subseteq T$, $S \in \tau$ *and* Y *be a complete, separable metric space. Let* $\phi\colon T \to 2^Y$ *be a lower measurable correspondence and* $f\colon S \to Y$ *be a measurable function. Then the correspondence* $\psi\colon T \to 2^Y$ *defined by*

$$\psi(t) = \phi(t) \cap (f(t) + B(0, \varepsilon))$$

is lower measurable. Here we understand that $f(t) + B(0, \varepsilon) = \varnothing$ *if* $t \notin S$.

Proof. We must show that $\{t \in T\colon \psi(t) \cap U \neq \varnothing\} \in \tau$ for every open subset U of Y. Set $\theta(t) = (\phi(t) \cap U) + B(0, \varepsilon)$. Observe that

$$\begin{aligned}\{t \in T\colon \psi(t) \cap U \neq \varnothing\} &= \{t \in T\colon (\phi(t) \cap U) \cap (f(t) + B(0, \varepsilon)) \neq \varnothing\} \\ &= \{t \in S\colon f(t) \in \theta(t)\} = \operatorname{proj}_T(G_f \cap G_\theta).\end{aligned}$$

Since U is open, $\phi(t) \cap U$ is lower measurable, and since $\theta(t) = \{y \in Y\colon \operatorname{dist}(y, \phi(t) \cap U) < \varepsilon\}$, $\theta(\cdot)$ has a measurable graph by Lemma 4.7. Therefore by Lemma 4.8, $\operatorname{proj}_T(G_f \cap G_\theta) \in \tau$. Therefore $\{t \in T\colon \psi(t) \cap U \neq \varnothing\} \in \tau$. This completes the proof of the lemma.

LEMMA 4.11. *Let* X, Y *be topological spaces and* $\phi\colon X \to 2^Y$, $\psi\colon X \to 2^Y$ *be nonempty valued* l.s.c. *correspondences. Let* V *be an open entourage for some uniform structure on* Y. *Suppose that* $\phi(x) \cap V(\psi(x)) \neq \varnothing$ *for all* $x \in X$. *Then the correspondence* $\theta\colon X \to 2^Y$ *defined by* $\theta(x) = \phi(x) \cap V(\psi(x))$ *is* l.s.c.

Proof. See Michael [21, Proposition 2.5].

LEMMA 4.12. *Let* (T, τ) *be a measurable space*, Z *be a separable metric space*, Y *be a metric space and* $U \subseteq T \times Z$ *be such that*:

(i) *for each* $t \in T$ *the set* $U^t = \{x \in Z\colon (t, x) \in U\}$ *is open in* Z *and*

(ii) *for each* $x \in Z$ *the set* $U_x = \{t \in T\colon (t, x) \in U\}$ *belongs to* τ.

Moreover, let $f\colon U \to Y$ be such that for each $t \in T$, $f(t, \cdot)$ is continuous on U^t and for each $x \in Z$, $f(\cdot, x)$ is measurable on U_x.

Then f is relatively jointly measurable with respect to the σ-algebra $\tau \otimes \mathscr{B}(Z)$, i.e., for every V open in Y,

$$\{(t, x) \in U: f(t, x) \in V\} = U \cap A$$

for some $A \in \tau \otimes \mathscr{B}(Z)$.

Finally, condition (ii) is implied by either one of the following two conditions:

(ii′) $U \in \tau \otimes \mathscr{B}(Z)$;

(ii″) $\tilde{Z}$ is complete separable metric; for some $U' \subseteq T \times Z \times \tilde{Z}$ such that $U' \in \tau \otimes \mathscr{B}(Z) \otimes \mathscr{B}(\tilde{Z})$, $U = \mathrm{proj}_{T \times Z}(U')$ and (T, τ) is the underlying measure space of some complete finite measure space (T, τ, μ).

Remark. When $U = T \times Z$ the above lemma is standard; see, e.g., Lemma III.14, [8]. The proof from [8] gives even the more general version of Lemma 4.12 after only minor modifications. We give the details for reader's convenience.

Proof. Let x_n $(n = 0, 1, 2,...)$ be dense in Z. For $p \geqslant 1$ put $f_p(t, x) = f(t, x_n)$, for $(t, x) \in U$, if n is the smallest integer such that $x \in B(x_n, 1/p)$ and $(t, x_n) \in U$. It is easy to see that $f_p(t, x) = f(t, x_n)$ if (t, x) belongs to the set

$$\left[U_{x_n} \times \left(B\left(x_n, \frac{1}{p}\right) \Big\backslash \bigcup_{m<n} B\left(x_m, \frac{1}{p}\right)\right)\right] \cap U.$$

Note that by the hypothesis (ii), $U_{x_n} \in \tau$.

Now, we observe that f_p is defined everywhere on U. To see this, let $(t, x) \in U$. By (i), U' is open. Hence let $\varepsilon > 0$ be such that $B(x, \varepsilon) \subseteq U'$. Since x_n $(n = 0, 1, 2,...)$ are dense in Z, there is some n such that $x_n \in B(x, \min(\varepsilon, 1/p))$. Thus $x_n \in U'$. Hence $x \in B(x_n, 1/p)$ and $(t, x_n) \in U$. Therefore, $f_p(t, x)$ is defined.

We shall now show that f_p is relatively jointly measurable. For this purpose let V be open in Y and set

$$S_n = \{t \in U_{x_n}: f(t, x_n) \in V\}.$$

Since $U_{x_n} \in \tau$ and $f(\cdot, x_n)$ is measurable on U_{x_n}, it follows that $S_n \in \tau$. It is now easy to see that

$$f_p^{-1}(V) = \bigcup_{n=0}^{\infty} \left[S_n \times \left(B\left(x_n, \frac{1}{p}\right) \Big\backslash \bigcup_{m<n} B\left(x_m, \frac{1}{p}\right)\right)\right] \cap U.$$

Thus f_p is relatively jointly measurable.

Since $f(t, \cdot)$ is continuous on U', we obtain at once that $f_p(t, x)$ converges to $f(t, x)$ when p approaches infinity. Thus $f(t, x)$ is relatively jointly measurable. That a limit of relatively jointly measurable functions is relatively jointly measurable is clear, since relative joint measurability is just the ordinary measurability with respect to an appropriate σ-algebra; in our case, with respect to the σ-algebra of subsets of U which are of the form $U \cap A$ where $A \in \tau \otimes \mathscr{B}(Z)$.

That (ii′) implies (ii) is well known and easy to see. Finally let U and U' be as in (ii″) and let $x \in Z$. Set

$$U'' = \{(t, y): (t, x, y) \in U'\}.$$

Again clearly $U'' \in \tau \otimes \mathscr{B}(\tilde{Z})$ and it is easy to see that $\text{proj}_T(U'') = U_x$. Hence $U_x \in \tau$ by Lemma 4.2.

5. Proof of the Main Theorems

We begin with an approximate Caratheodory-type selection result which is needed for the proof of Theorem 3.1.

Main Lemma 5.1 (Approximate Caratheodory-Type Selection). *Let (T, τ, μ) be a complete measure space, Y be a separable Banach space and Z be a complete, separable, metric space. Let ϕ: $T \times Z \to 2^Y$ be a convex valued correspondence (possibly empty) such that*

(i) *$\phi(\cdot, \cdot)$ is lower measurable and*

(ii) *for each $t \in T$, $\phi(t, \cdot)$ is* l.s.c.

Let $U = \{(t, x) \in T \times Z$: $\phi(t, x) \neq \varnothing\}$, and for each $x \in X$, let $U_x = \{t \in T$: $(t, x) \in U\}$ and for each $t \in T$, let $U^t = \{x \in Z$: $(t, x) \in U\}$. Then there exists an approximate or ε-Caratheodory-type Selection from $\phi \mid U$, i.e., given $\varepsilon > 0$, there exists a function f^ε: $U \to Y$ such that $f^\varepsilon(t, x) \in \phi(t, x) + B(0, \varepsilon)$, and for each $x \in Z$, $f^\varepsilon(\cdot, x)$ is measurable on U_x and for each $t \in T$, $f^\varepsilon(t, \cdot)$ is continuous on U^t.

Proof. Since Y is separable we may choose $\{y_n: n = 1, 2, \ldots\}$ to be a countable dense subset of Y. For each $t \in T$ and $\varepsilon > 0$, let $W_n^\varepsilon(t) = \{x \in Z$: $y_n \in [\phi(t, x) + B(0, \varepsilon)]\}$. It follows from (ii) that for each $t \in T$ and $n = 1, 2, \ldots$, $W_n^\varepsilon(t)$ is open in Z. Since for each $(t, x) \in U$, $\phi(t, x) \neq \varnothing$, the set $\{W_n^\varepsilon(t)$: $n = 1, 2, \ldots\}$ is an open cover of U^t. Note that $\phi(t, x) + B(0, \varepsilon) = \{y \in Y$: $\text{dist}(y, \phi(t, x)) < \varepsilon\}$. Setting $S = T \times Z$, $X = Y$, $\mathscr{A} = \tau \otimes \mathscr{B}(Z)$ and $W(s) = \phi(t, x)$ for $s = (t, x) \in S$ in Lemma 4.7, we conclude that $\phi(\cdot, \cdot) + B(0, \varepsilon)$ has a measurable graph. By Lemma 4.9, $W_n^\varepsilon(\cdot)$ has a measurable graph. This is also easy to see directly. For each $m=1, 2,\dots,$ define the operator $(\ )_m$ on subsets of Z by

$$(W)_m=\left\{w\in W\colon \operatorname{dist}(w, Z\backslash W)\geqslant\frac{1}{2^m}\right\}.$$

For $n=1, 2,\dots,$ let $V_n^\varepsilon(t)=W_n^\varepsilon(t)\backslash\bigcup_{k=1}^{n-1}(W_k^\varepsilon(t))_n$. It can be easily checked that $\{V_n^\varepsilon(t)\colon n=1, 2,\dots\}$ is a locally finite open cover of the set U'. Since $W_n^\varepsilon(\cdot)$ has a measurable graph, by Lemmata 4.3 and 4.5, $V_n^\varepsilon(\cdot)$ has a measurable graph. Let $\{g_n^\varepsilon(t, x)\colon n=1, 2,\dots\}$ be a partition of unity subordinated to the open cover $\{V_n^\varepsilon(t)\colon n=1, 2,\dots\}$, for instance, for each $n=1, 2,\dots,$ let

$$g_n^\varepsilon(t, x)=\frac{\operatorname{dist}(x, Z\backslash V_n^\varepsilon(t))}{\sum_{k=1}^\infty \operatorname{dist}(x, Z\backslash V_k^\varepsilon(t))}.$$

Then $\{g_n^\varepsilon(t, \cdot)\colon n=1, 2,\dots\}$ is a family of continuous functions $g_n^\varepsilon(t, \cdot)\colon U'\to [0, 1]$ such that $g_n^\varepsilon(t, x)=0$ for $x\notin V_n^\varepsilon(t)$ and $\sum_{n=1}^\infty g_n^\varepsilon(t, x)=1$ for all $(t, x)\in U$. Define $f^\varepsilon\colon U\to Y$ by $f^\varepsilon(t, x)=\sum_{n=1}^\infty g_n^\varepsilon(t, x)\, y_n$. Since $\{V_n^\varepsilon(t)\colon n=1, 2,\dots\}$ is locally finite, each x has a neighborhood N_x which intersects only finitely many $V_n^\varepsilon(t)$. Hence, $f^\varepsilon(t, \cdot)$ is a finite sum of continuous functions on N_x and it is therefore continuous on N_x. Consequently, $f^\varepsilon(t, \cdot)$ is a continuous function on U'. Moreover, for any n such that $g_n^\varepsilon(t, x)>0$, $x\in V_n^\varepsilon(t)\subset W_n^\varepsilon(t)=\{z\in Z\colon\ y_n\in[\phi(t, z)+B(0, \varepsilon)]\}$, i.e., $y_n\in\phi(t, x)+B(0, \varepsilon)$. So $f^\varepsilon(t, x)$ is a convex combination of elements from the convex set $\phi(t, x)+B(0, \varepsilon)$. Therefore, $f^\varepsilon(t, x)\in\phi(t, x)+B(0, \varepsilon)$ for all $(t, x)\in U$. Since $V_n^\varepsilon(\cdot)$ has a measurable graph, by Lemmata 4.3 and 4.4, $\operatorname{dist}(x, Z\backslash V_n^\varepsilon(\cdot))$ is a measurable function for every $x\in Z$. Hence, for each n, x, $g_n^\varepsilon(\cdot, x)$ is a measurable function. Consequently, $f^\varepsilon(\cdot, x)$ is measurable for each x. Therefore f^ε is an approximate or ε-Caratheodory-type selection from $\phi\,|\,U$. This completes the proof of the Main Lemma.

Proof of Theorem 3.1. Now, we construct inductively, functions $f_l\colon U\to Y$, $l=1, 2,\dots,$ such that

(a) $f_l(t, \cdot)$ is continuous on U' and $f_l(\cdot, x)$ is measurable on U_x,

(b) $f_l(t, x)\in\phi(t, x)+B(0, 1/2^l)$, $l=1, 2,\dots,$

(c) $f_l(t, x)\in f_{l-1}(t, x)+2B(0, 1/2^{l-1})$, $l=2, 3,\dots$.

The existence of f_1 satisfying (a) and (b) for $l=1$, is guaranteed by the Main Lemma 5.1. Suppose that we have $f_1,\dots, f_k$ satisfying (a), (b), and (c) for $l=1, 2,\dots, k$. We must find $f_{k+1}\colon U\to Y$ which satisfies (a), (b), and (c) for $l=k+1$. Now define $\phi_{k+1}(t, x)=\phi(t, x)\cap(f_k(t, x)+B(0, 1/2^k))$. Then $\phi_{k+1}(t, x)$ is nonempty, by the induction hypothesis, and $\phi_{k+1}(t, \cdot)$ is l.s.c.

by Lemma 4.11. It follows from Lemma 4.10 that $\phi_{k+1}(\cdot,\cdot)$ is lower measurable. By the Main Lemma 5.1 there exists $f_{k+1}: U \to Y$ such that $f_{k+1}(t,x) \in \phi_{k+1}(t,x) + B(0, 1/2^{k+1})$. But then $f_{k+1}(t,x) \in (f_k(t,x) + B(0, 1/2^k)) + B(0, 1/2^{k+1}) \subset (f_k(t,x) + 2B(0, 1/2^k))$ which is (c) and $f_{k+1}(t,x) \in \phi(t,x) + B(0, 1/2^{k+1})$ which is (b). By (c), $\{f_l: l = 1, 2, ...\}$ is uniformly Cauchy, and therefore converges uniformly to $f: U \to Y$. Since ϕ is closed valued $f(t,x) \in \phi(t,x)$ for all $(t,x) \in U$. Furthermore, $f(t,\cdot)$ is continuous in U' and $f(\cdot, x)$ is measurable on U_x. By Lemma 4.12, $f(\cdot,\cdot)$ is jointly measurable. This completes the proof of the theorem.

MAIN LEMMA 5.2. *Under the conditions of Theorem 3.1 there exists a countable collection $\mathscr{F}$ of Caratheodory-type selections from $\phi | U$ such that for every $(t,x) \in U$, $\{f(t,x): f \in \mathscr{F}\}$ is dense in $\phi(t,x)$.*

Proof. Let $\{E^n: n = 1, 2, ...\}$ be a convex open basis of Y. For each $n = 1, 2, ...,$ $U^n = \{(t,x) \in T \times Z: \phi(t,x) \cap E^n \neq \varnothing\} \in \tau \otimes \mathscr{B}(Z)$. For each $t \in T$, define $U^n(t) = \{x \in Z: (t,x) \in U^n\}$. Note that for each $t \in T$, $U^n(t)$ is open in Z. Moreover, $U^n(\cdot)$ has a measurable graph. For each $k = 1, 2, ...,$ and $t \in T$, let $A^n_k(t) = \{x \in Z: \operatorname{dist}(x, Z \backslash U_n(t)) \geq 1/2^k\}$. By Lemma 4.5, $A^n_k(\cdot)$ has a measurable graph. Note that $\bigcup_{k=1}^{\infty} A^n_k(t) = U^n(t)$, and for each $t \in T$, A^n_k is closed in Z. Define $\phi^n_k: T \times Z \to 2^Y$ by

$$\phi^n_k(t,x) = \begin{cases} \operatorname{cl}(\phi(t,x) \cap E^n) & \text{if } x \in A^n_k(t) \\ \phi(t,x) & \text{if } x \notin A^n_k(t). \end{cases}$$

Since for each $t \in T$, $A^n_k(t)$ is closed in Z, $\phi^n_k(t,\cdot)$ is l.s.c. Moreover, since for every open subset V of Y, $\{(t,x): \phi^n_k(t,x) \cap V \neq \varnothing\} = \{(t,x): \operatorname{cl}(\phi(t,x) \cap E^n) \cap V \neq \varnothing, x \in A^n_k(t)\} \cup \{(t,x): \phi(t,x) \cap V \neq \varnothing, x \notin A^n_k(t)\} \in \tau \otimes \mathscr{B}(Z)$, $\phi^n_k(\cdot,\cdot)$ is lower measurable. By Theorem 3.1 there exist Caratheodory-type selection $f^n_k(\cdot,\cdot)$ from $\phi^n_k(\cdot,\cdot)$. Let $\mathscr{F}$ be the collection of all f^n_k, $n, k = 1, 2, ...$. Then $\mathscr{F}$ is a countable collection of Caratheodory-type selections from $\phi | U$, and it can be easily seen that $\{f(t,x): f \in \mathscr{F}\}$ is dense in $\phi(t,x)$ for all $(t,x) \in U$. This completes the proof of the lemma.

We will need the following notions. If K is a closed, convex subset of a normed linear space, then a *supporting set* of K is a closed convex subset S of K, $S \neq K$, such that if an interior point of a segment in K is in S, then the whole segment is in S. The set of all elements of K which are not in any supporting set of K will be denoted by $I(K)$. The following facts below are due to Michael [21, p. 372].

FACT 5.1. If any convex subset K of Y is either closed or has an interior point or is finite dimensional, then $I(\operatorname{cl} K) \subset K$.

FACT 5.2. Let K be a nonempty, closed, convex separable subset of a Banach space Y, and $\{y_i: i=1, 2, ...\}$ be a dense subset of K. If

$$z_i = y_i + \frac{(y_i - y_1)}{\max(1, \|y_i - y_1\|)} \qquad \text{for all } i \text{ and } z = \sum_{i=1}^{\infty} (\tfrac{1}{2})^i z_i,$$

then $z \in I(K)$.

Proof. See Michael [17, Lemma 5.1, p. 372].

Proof of Theorem 3.2. Define $\psi: T \times Z \to 2^Y$ by $\psi(t, x) = \text{cl}\, \phi(t, x)$. Since $\phi(t, \cdot)$ is l.s.c. so is $\psi(t, \cdot)$. Moreover, ψ is lower measurable. By the Main Lemma 5.2 there exist Caratheodory-type selections $\{g_k(t, x): k = 1, 2, ...\}$ dense in $\psi(t, x)$ for all $(t, x) \in U$. Let for each $k = 1, 2, ...,$

$$f_k(t, x) = g_1(t, x) + \frac{g_k(t, x) - g_1(t, x)}{\max(1, \|g_k(t, x) - g_1(t, x)\|)},$$

$$f(t, x) = \sum_{k=1}^{\infty} \frac{1}{2^k} f_k(t, x).$$

By Fact 5.2, $f(t, x) \in I(\psi(t, x))$ for all $(t, x) \in U$. Since the series defining f converges uniformly, it follows that for each $t \in T$, $f(t, \cdot)$ is continuous and for each $x \in X$, $f(\cdot, x)$ is measurable. By Fact 5.1, $f(t, x) \in I(\psi(t, x)) \subset \varphi(t, x)$ if either (i) or (ii) of Theorem 3.2 are satisfied. This completes the proof of the theorem.

Proof of Theorem 3.3. It follows from Theorem 3.1 that there exists a function $f: T \times X \to X$ such that $f(t, x) \in \phi(t, x)$ for all $(t, x) \in T \times X$, and for each $x \in X$, $f(\cdot, x)$ is measurable and for each $t \in T$, $f(t, \cdot)$ is continuous. Moreover, $f(\cdot, \cdot)$ is jointly measurable.

For each $t \in T$, let $F(t) = \{x \in X: g(t, x) = 0\}$, where $g(t, x) = f(t, x) - x$. It follows from the Tychonoff fixed point theorem that the function $f(t, \cdot)$: $X \to X$ has a fixed point. Therefore, for each $t \in T$, $F(t) \neq \emptyset$. Since g is jointly measurable, F has a measurable graph. Hence by Aumann's measurable selection theorem (Lemma 4.1), there exists a measurable function $x^*: T \to X$ such that for almost all t in T, $x^*(t) \in F(t)$, i.e., $x^*(t) = f(t, x^*(t)) \in \phi(t, x^*(t))$. This completes the proof of the theorem.

Proof of Theorem 3.4. The argument is similar to that adopted in the proof of Theorem 3.3 except that one must use now Theorem 3.2 instead of Theorem 3.1.

References

1. J. P. Aubin and A. Cellina, "Differential Inclusions," Springer-Verlag, New York, 1984.
2. J. P. Aubin and I. Ekeland, "Applied Nonlinear Analysis," Wiley–Interscience, New York, 1984.
3. R. J. Aumann, Measurable utility and the measurable choice theorem, *in* "La Decision," pp. 15–26, C.N.R.S., Aix-en-Provence, 1967.
4. H. Bohnenblust and S. Karlin, On a theorem of Ville, *in* "Contributions to the Theory of Games I" (H. Kuhn and A. Tucker, Eds.), Princeton Univ. Press, Princeton, N.J., 1950.
5. P. A. Borglin and H. Keiding, Existence of equilibrium actions and equilibrium: A note on the "new" existence theorems, *J. Math. Econ.* **3** (1976), 313–316.
6. F. Browder, The fixed point theory of multivalued mappings in topological vector spaces, *Math. Ann.* **177** (1968), 283–301.
7. C. Castaing, Sur l'existence des sections séparément mesurable et séparément continues d'une multi-application, *in* "Travaux du Seminaire d'analyse Convexe," Univ. des. Sci. et Techniques du Languedoc, No. 5, p. 14, 1975.
8. C. Castaing and M. Valadier, "Convex Analysis and Measurable Multifunctions," Lecture Notes in Mathematics, No. 580, Springer-Verlag, New York, 1977.
9. G. Debreu, A social equilibrium existence theorem, *Proc. Nat. Acad. Sci. U.S.A.* **38** (1952), 886–893.
10. A. Fryszkowski, Caratheodory-type selectors of set-valued maps of two variables, *Bull. Acad. Polon. Sci.* **25** (1977), 41–46.
11. D. Gale and A. Mas-Colell, An equilibrium existence theorem for a general model without ordered preferences, *J. Math. Econ.* **2** (1975), 9–15.
12. C. J. Himmelberg, Measurable relations, *Fund. Math.* **87** (1975), 53–72.
13. M. A. Khan, "Equilibrium points of nonatomic games over a non-reflexive Banach space," *J. Approx. Theory* **43** (1985), 370–376.
14. M. A. Khan, On extensions of the Cournot–Nash theorem, *in* "Advances in Equilibrium Theory" (C. D. Aliprantis *et al.*, Eds.), Springer-Verlag, Berlin, 1985.
15. M. A. Khan and N. S. Papageorgiou, "On Cournot–Nash Equilibria in Generalized Quantitative Games with a Continuum of Players," Univ. of Illinois, Urbana, 1985.
16. S. Itoh, Random fixed point theorems with applications to random differential equations in Banach spaces, *J. Math. Anal. Appl.* **67** (1979), 261–273.
17. T. Kim, K. Prikry, and N. C. Yannelis, "Equilibria in Abstract Economies with a Measure Space of Agents and with an Infinite Dimensional Strategy Space," Univ. of Minnesota, Minneapolis, 1985.
18. T. Kim, K. Prikry, and N. C. Yannelis, "On a Caratheodory-Type Selection Theorem," Univ. of Minnesota, Minneapolis, 1985.
19. K. Kuratowski and C. Ryll-Nardzewski, A general theorem on selectors, *Bull. Acad. Polon. Sci. Ser. Sci. Marsh. Astronom. Phys.* **13** (1965), 397–403.
20. A. Mas-Colell, On a theorem of Schmeidler, *J. Math. Econ.* **13** (1984), 201–206.
21. E. Michael, Continuous selections I, *Ann. Math.* **63** (1956), 363–382.
22. J. F. Nash, Noncooperative games, *Ann. Math.* **54** (1951), 286–295.
23. N. S. Papageorgiou, Random fixed point theorems for measurable multifunctions in Banach spaces, *Proc. Amer. Math. Soc.*, in press.
24. D. Schmeidler, Equilibrium points of nonatomic games, *J. Statist. Phys.* **7** (1973), 295–300.
25. W. Shafer and H. Sonnenschein, Equilibrium in abstract economies without ordered preferences, *J. Math. Econ.* **2** (1975), 345–348.
26. S. Toussaint, On the existence of equilibrium with infinitely many commodities and without ordered preferences, *J. Econ. Theory* **33** (1984), 98–115.

27. E. Wesley, Borel preference orders in markets with a continuum of traders, *J. Math. Econ.* **3** (1976), 155–165.
28. A. Wieczorek, On the measurable utility theorem, *J. Math. Econ.* **7** (1980), 165–173.
29. N. C. Yannelis and N. D. Prabhakar, Existence of maximal elements and equilibria in linear topological spaces, *J. Math. Econ.* **12** (1983), 233–245.
30. N. C. Yannelis and N. D. Prabhakar, "Equilibrium in Abstract Economies with an Infinite Number of Agents, an Infinite Number of Commodities and without Ordered Preferences," Univ. of Minnesota, Minneapolis, 1983.
31. N. C. Yannelis, Equilibria in non-cooperative models of competition, *J. Econ. Theory*, in press.