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RESEARCH ARTICLE

# Capital market equilibrium without riskless assets: heterogeneous expectations

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**Abstract** The existence theorem of Allingham (Econometrica 59:1169–1174, 1991) for the capital asset pricing model (CAPM) is generalized to the case where agents have heterogeneous expectations on the return distribution and the mean-variance utility functions are quasiconcave. This result is built upon new conditions which are distinct from and weaker than the conditions imposed on the CAPM in the literature.

**Keywords** Asset market equilibrium · Satiation · CAPM · Heterogeneous expectations

**JEL Classification Numbers** G12 · G11 · D51 · D52

## 1 Introduction

The notion of arbitrage has been considered as an important conceptual framework for studying the existence of asset market equilibrium since the seminal work Hart (1974). The arbitrage-based literature investigates the existence of equilibrium in asset

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markets by presupposing that all the preferred sets of investors are unbounded.[1] This presupposition may conflict with satiated preferences because the set of satiations is bounded in general. As discussed below, the mean-variance utility functions may reach satiation in asset markets without riskless assets if they reveal variance aversion. Thus, the existence theorems of the arbitrage-based literature do not apply to the capital asset pricing model (CAPM) without riskless assets. The traditional approaches to equilibrium theory of commodity markets are not applicable to the CAPM with satiation portfolios either, because they exclude the case that satiation occurs only inside the set of feasible and individually rational allocations.

Nielsen (1990) and Allingham (1991) investigate the existence of equilibrium in the classical capital asset pricing model without riskless assets. They assume that agents have homogeneous expectations on the return distribution. Nielsen (1990) focuses on special cases where either risk aversion of agents is constant or satiation portfolios are expressed as multiples of the total endowments. Allingham (1991) introduces a technical lemma to show the existence of equilibrium in the case where the mean-variance utility function is strictly concave. The result of Allingham (1991) subsumes as a special case the existence theorem of Nielsen (1990). It does not extend beyond the prototype of the CAPM, however, because the technical lemma of Allingham (1991) does not work any more in the case where investors have heterogeneous expectations on the return distribution.

The purpose of this paper is to show the existence of equilibrium asset prices in the CAPM without riskless assets where agents have heterogeneous expectations on the return distribution and the mean-variance utility functions are quasiconcave.[2] To do this, we present a new sufficient condition for equilibrium to exist which is much weaker than the literature. The mean-variance utility function has distinct features in the absence of riskless assets. More specifically, as shown later, the set of portfolios which are preferred to any given portfolio is bounded under the convexity condition on the preferences. If the utility function is continuous, the preferred set is closed and therefore, compact. This implies that the mean-variance utility function reaches satiation. Equilibrium asset prices may not be positive in the presence of satiation portfolios in general. We present a necessary and sufficient condition under which equilibrium asset prices are strictly positive. Consequently, the result of the paper generalizes the existence proof of Allingham (1991) to the case with heterogeneous expectations on the return distribution and quasiconcave mean-variance utility functions.

Sun and Yang (2003) attempt to extend the results of Allingham (1991) to the case with heterogeneous expectations by generalizing the assumptions and the technical lemma of Allingham (1991). Won and Chay (2006) examine the existence issue in the CAPM with heterogeneous expectations and concave mean-variance utility functions in the framework of Won and Yannelis (2006). Won and Chay (2006), however, fail to fully characterize the conditions for the existence of equilibrium and the positivity of equilibrium prices in the CAPM context. This paper has several merits over the

[1] For an extensive review of the arbitrage-based approaches, see Dana et al. (1999), Page et al. (2000), and Allouch (2002) and the references therein.

[2] For CAPM implications of heterogeneous expectations, see Jarrow (1980) which discusses the effect of heterogeneous expectations on asset prices in the framework of the CAPM.

literature. First, the conditions for the existence of equilibrium and the positivity of equilibrium prices are "CAPM-context-free" so that they can work beyond the classical framework of the CAPM.[3] This is not the case with the aforementioned literature which is built on the specific structure of the mean-variance economy. Second, the covariance matrice of the return distribution is not required to have full rank. This is a great merit in analyzing the effect of portfolio constraints like short-selling restrictions on equilibrium prices of redundant assets such as options and futures. In contrast, the existing literature requires that the covariance matrices of asset returns have full rank and therefore, is not applicable to the constrained markets with redundant assets. Third, we introduce new conditions for the existence of equilibrium and the positivity of equilibrium prices. These conditions are much weaker than in the literature. Finally, the mean-variance utility functions are here assumed to be quasiconcave. This assumption subsumes as a special case the literature with the concave mean-variance utility functions.

There exist interesting attempts to address the existence issue with satiable preferences in a general framework by using a weaker notion of equilibrium. For e.g., Mas-Colell (1992) introduces "equilibrium with slack" in which allows agents to keep some positive income unused. The result of Mas-Colell (1992) does not apply, however, to asset pricing models because he fails to characterize conditions under which "weak equilibrium" coincides with competitive equilibrium. Allouch and Le Van (2007) extend Mas-Colell (1992) and provide a condition under which the weak equilibrium coincides with Walrasian equilibrium. The consequence of Allouch and Le Van (2007) does not apply to the case where satiation occurs only inside the set of feasible and individually rational allocations.

This paper is organized as follows. In the next section, we introduce an asset market economy and illustrate that equilibrium fails to exist in the presence of satiation portfolios. In Sect. 3, we show that the mean-variance utility function reaches satiation in the absence of riskless portfolios. The main consequence of the paper is provided in Sect. 4 followed by concluding remarks.

## 2 Asset market equilibrium

There are $\ell$ assets, indexed by $j = 1, \ldots, \ell$. A *portfolio* of assets can be represented by a $\ell$-dimensional vector $x \in \mathbb{R}^\ell$, where $j$th coordinate indicates the number of shares of the $j$th asset included in the portfolio. There are $m$ investors, indexed by $i = 1, \ldots, m$. Let $I = \{1, \ldots, m\}$ denote the set of investors. It is assumed that every investor is allowed to take unlimited short sales, i.e., the set of feasible portfolios is $\mathbb{R}^\ell$ for all $i \in I$. Each investor $i$ is endowed with an initial portfolio $e_i \in \mathbb{R}^\ell$ of assets.

For each $j = 1, \ldots, \ell$, let $\tilde{r}^j$ denote the return of asset $j$ and $\tilde{r}$ the random vector with $\tilde{r}^j$ as the $j$th component. Investors are allowed to have heterogeneous expectations about asset returns. Let $\mathbb{E}_i$ denote the expectation operator for investor $i$. We set $r_i = \mathbb{E}_i[\tilde{r}]$ and $\Omega_i = \mathbb{E}_i[(\tilde{r} - r_i)(\tilde{r} - r_i)^\top]$ where $\top$ denotes the transpose of a matrix.

[3] Examples include the asset pricing models with higher moments of the return distribution such as skewness and kurtosis.

Then $r_i$ and $\Omega_i$ indicate the mean vector and the covariance matrix of the returns on assets perceived by investor $i \in I$. For a portfolio $x_i \in \mathbb{R}^\ell$, let $\mu_i(x_i)$ and $\sigma_i(x_i)$ denote its mean return and standard deviation, respectively. Clearly, $\mu_i(x_i) = x_i \cdot r_i$ and $\sigma_i(x_i)^2 = x_i^\top \Omega_i x_i$ for all $i \in I$.

We assume that for each $i \in I$, the preferences of investor $i$ over portfolios are represented by a mean-variance function $v_i$. Specifically, the utility of investor $i$ who holds a portfolio $x_i \in \mathbb{R}^\ell$ is expressed as

$$v_i(\mu_i(x_i), \sigma_i(x_i)^2) = v_i(x_i \cdot r_i,\ x_i^\top \Omega_i x_i). \tag{2.1}$$

For each $x_i \in \mathbb{R}^\ell$, we set $u_i(x_i) = v_i(\mu_i(x_i), \sigma_i(x_i)^2)$. For a portfolio $x_i \in \mathbb{R}^\ell$, the preferred set $R_i(x_i)$ and the strictly preferred set $P_i(x_i)$ are defined as $R_i(x_i) = \{x_i' \in \mathbb{R}^\ell : u_i(x_i') \geq u_i(x_i)\}$ and $P_i(x_i) = \{x_i' \in \mathbb{R}^\ell : u_i(x_i') > u_i(x_i)\}$, respectively. The asset market economy is denoted by $\mathcal{E} = (\mathbb{R}^\ell, u_i, e_i)_{i \in I}$.

A collection $x = (x_1, \ldots, x_m)$ of portfolios with $x_i \in \mathbb{R}^\ell$ for each $i \in I$ is called an allocation of the economy $\mathcal{E}$. The initial allocation of portfolios is given by $e = (e_1, \ldots, e_m)$ and the *market portfolio* by $\sum_{i \in I} e_i$. An allocation $x \in \mathbb{R}^{\ell m}$ is *attainable* if it satisfies $\sum_{i \in I}(x_i - e_i) = 0$. As usual, the budget set of agent $i$ is defined as $\mathcal{B}_i(p) = \{x_i \in \mathbb{R}^\ell : p \cdot x_i \leq p \cdot e_i\}$. An *asset market equilibrium* for the asset market economy $\mathcal{E}$ is a pair $(p, x) \in (\mathbb{R}^\ell \setminus \{0\}) \times \mathbb{R}^{\ell m}$ such that (i) $x_i \in \mathcal{B}_i(p)$ for all $i \in I$, (ii) $P_i(x_i) \cap \mathcal{B}_i(p) = \emptyset$ for all $i \in I$, (iii) $\sum_{i \in I}(x_i - e_i) = 0$. An allocation is said to be *individually rational* if $x_i \in R_i(e_i)$ for all $i \in I$. We define the set $A = \left\{x \in \mathbb{R}^{\ell m} : \sum_{i \in I}(x_i - e_i) = 0 \quad \text{and} \quad x_i \in R_i(e_i) \quad \text{for all } i \in I\right\}$. Then the set $A$ is a collection of attainable and individually rational allocations of $\mathcal{E}$. Let $A_i$ be the projection of $A$ onto $\mathbb{R}^\ell$.

To handle the difficulty with satiation, for each $x$ we divide the set $I$ of investors into the sets $I(x) = \{i \in I : P_i(x_i) \neq \emptyset\}$ and $I^s(x) = I \setminus I(x)$. Investor $i \in I(x)$ does not reach satiation and $i \in I^s(x)$ reaches satiation at the allocation $x$. For each $i \in I$, let $S_i = \{x_i \in \mathbb{R}^\ell : P_i(x_i) = \emptyset\}$ denote the set of satiation portfolios. We illustrate that the presence of satiation portfolios leads to the non-existence of asset market equilibrium.

*Example 2.1* We take a specific example of the capital asset pricing model with two investors and two assets. Investors are endowed with the initial asset holdings $e_1 = (1/2, 1)$ and $e_2 = (3/2, 1)$, respectively. We assume that the two investors have the same expectation about the mean returns but different beliefs about the covariance matrix of the asset returns as following.

$$r_1 = r_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \Omega_1 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.2}$$

Their preferences are represented by the mean-variance utility function.

$$v_i(\mu, \sigma) = a_i \mu - \frac{1}{2} b_i \sigma^2. \tag{2.3}$$

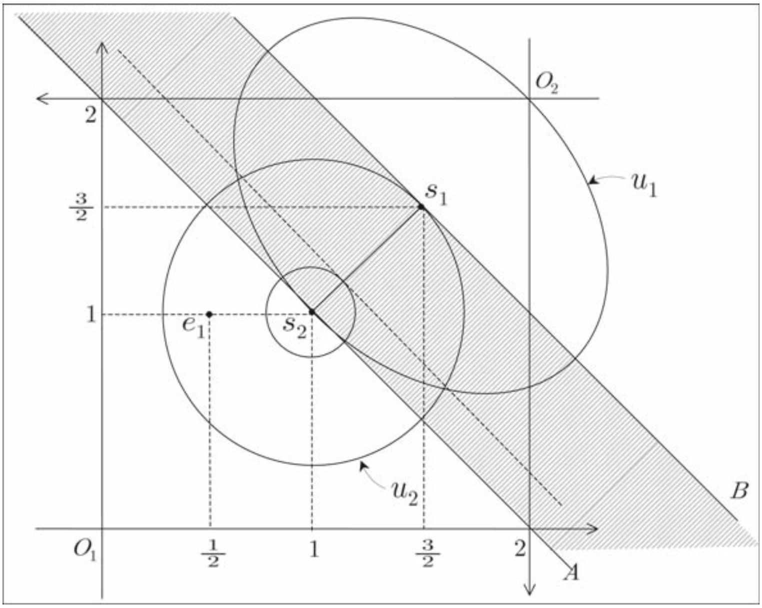


**Fig. 1** The non-existence of equilibrium

We assume that $(a_1, b_1) = (3, 1/2)$ and $(a_2, b_2) = (1, 1)$. Then the utility of a portfolio $x \in \mathbb{R}^2$ for each $i = 1, 2$ has the form

$$u_1(x) = 3x \cdot r_1 - \frac{1}{4}x^\top \Omega_1 x = 3(x_1 + x_2) - \frac{1}{4}(3x_1^2 + 2x_1x_2 + 3x_2^2),$$
$$u_2(x) = x \cdot r_2 - \frac{1}{2}x^\top \Omega_2 x = (x_1 + x_2) - \frac{1}{2}(x_1^2 + x_2^2).$$

It is easy to check that investor 1 is satiated at $s_1 = (3/2, 3/2)$ while investor 2 is satiated at $s_2 = (1, 1)$. As depicted in Fig. 1, indifference curves for each $i = 1, 2$ form an ellipse and a circle centered at the satiation point, respectively. We set $t_1 = (1, 1)$ and $t_2 = (1/2, 1/2)$. Clearly, $(s_1, t_2)$ and $(t_1, s_2)$ are a feasible allocation.

Now we show that the economy has no equilibrium. Suppose to the contrary that $(p^*, x^*, y^*)$ is an equilibrium. Then it falls into one of the three cases: (i) $x^* = s_1$, (ii) $y^* = s_2$, and (iii) $x^* \neq s_1$ and $y^* \neq s_2$. If $x^* = s_1$, then $y^* = t_2$ and therefore $p^* = (1, 1)$.[4] In this case, $p^* \cdot s_1 = 3 > 3/2 = p^* \cdot e_1$, which is impossible. If $y^* = s_2$, then $x^* = t_1$ and therefore $p^* = (1, 1)$ by normalization. In this case, $p^* \cdot t_1 = 2 > 3/2 = p \cdot e_1$, which is impossible. Now consider the case (iii). Since $x^* \neq s_1$ and $y^* \neq s_2$, we have $Du_1(x^*) \neq 0$ and $Du_2(y^*) \neq 0$. By the first order condition for utility maximization, there exist $\lambda_1 > 0$ and $\lambda_2 > 0$ such that $p^* = \lambda_1[3(1, 1) - (1/2)(3x_1^* + x_2^*, x_1^* + 3x_2^*)] = \lambda_2[(1, 1) - y^*]$. Since $(x^*, y^*) \in A$, it follows that

[4] For simplicity, we can take $\lambda = 1$ by normalization.

$$x^*+y^* = \frac{5}{2}(1,1)-\left(\frac{1}{4\lambda_1}(3p_1^*+p_2^*)+\frac{1}{\lambda_2}p_1^*,\ \frac{1}{4\lambda_1}(p_1^*+3p_2^*)+\frac{1}{\lambda_2}p_2^*\right)=(2,2).$$

By the symmetry of the above equations, one can show that $p_1^* = p_2^*$ and therefore, $x_1^* = x_2^*$ and $y_1^* = y_2^*$. Putting them in the budget constraints, we have $x^* = (3/4, 3/4)$ and $y^* = (5/4, 5/4)$. But this is contradictory because $p^* = \lambda_1[3(1,1)-(1/2)(3x_1^*+x_2^*, x_1^*+3x_2^*)] = \lambda_1(3/2, 3/2)$, while $p^* = \lambda_2[(1,1)-y^*] = -\lambda_2(1/4, 1/4)$. Hence the economy fails to have equilibrium.

The non-existence of equilibrium can be checked diagrammatically in Fig. 1. The line segment between $s_1$ and $s_2$ corresponds to the set of Pareto optimal allocations. Thus, the price which supports an optimal allocation must be a vector in the line through $s_1$ and $s_2$. Let $A$ and $B$ denote a line orthogonal to the supporting price at an allocation where one of the agents is satiated. Notice that the initial allocation $e = (e_1, e_2)$ is outside the closed band between the lines $A$ and $B$. Thus, any budget line which passes through the initial endowment and an optimal allocation fails to be orthogonal to the line through $s_1$ and $s_2$. □

## 3 Characterizations of the CAPM

We characterize the properties of asset markets where agents are allowed to have heterogeneous expectations on the return distribution and the mean-variance utility functions are quasiconcave. To do this, for each $i \in I$, we make the following assumptions:

**Assumption C1.** $v_i$ is differentiable and strictly quasiconcave.[5]
**Assumption C2.** For each point $(\mu, \sigma^2) \in \mathbb{R}^2$ and any positive number $a$,

$$v_i(\mu,\sigma^2) \le v_i(\mu+a,\sigma^2), \quad v_i(\mu,\sigma^2+a) < v_i(\mu,\sigma^2).$$

**Assumption C3.** Each $\Omega_i$ is positive definite.

Assumption **C1** subsumes the conditions of Allingham (1991) on preferences as a special case because Allingham (1991) assumes that $v_i$ is strictly concave. Assumption **C2** states that $v_i$ is increasing in expected return and strictly decreasing in variance. Assumption **C3** implies the absence of non-trivial riskless portfolios. The following lemma shows that $u_i$ is differentiable and quasiconcave.

**Lemma 3.1** *The utility function $u_i$ is differentiable and strictly quasiconcave under Assumptions* **C1**–**C3**.

*Proof* By Assumption **C1**, $v_i$ is differentiable. Since $\mu_i$ and $\sigma_i$ are a differentiable function from $\mathbb{R}^\ell$ to $\mathbb{R}$, so is $u_i$ as a composite of differentiable functions. By Assumption **C3**, the quadratic form $x^\top \Omega_i x$ is a convex function of $x$. Let $x_i$ and $z_i$ be a point in $\mathbb{R}^\ell$. Then it follows that for all $\alpha$ in $[0, 1]$,

[5] A function $f : \mathbb{R}^\ell \to \mathbb{R}$ is strictly quasiconcave if for all $\lambda \in (0, 1)$ and $x, x'$ in $\mathbb{R}^\ell$, $f(x') > f(x)$ implies $f(\lambda x + (1-\lambda)x') > f(x)$.

$$(\alpha x_i + (1-\alpha)z_i)^\top \Omega_i (\alpha x_i + (1-\alpha)z_i) \le \alpha x_i^\top \Omega_i x_i + (1-\alpha) z_i^\top \Omega_i z_i.$$

Since $v_i$ is strictly quasiconcave and non-increasing in the second argument, we see that for all $\alpha$ in [0, 1],

$$\begin{aligned}
& u_i\,(x_i + (1-\alpha)z_i) \\
&\quad = v_i\left(\alpha x_i \cdot r_i + (1-\alpha)z_i \cdot r_i,\ (\alpha x_i + (1-\alpha)z_i)^\top \Omega_i\,(\alpha x_i + (1-\alpha)z_i)\right) \\
&\quad \ge v_i\left(\alpha x_i \cdot r_i + (1-\alpha)z_i \cdot r_i,\ \alpha x_i^\top \Omega_i x_i + (1-\alpha) z_i^\top \Omega_i z_i\right) \\
&\quad > \min\{v_i(x_i \cdot r_i,\ x_i^\top \Omega_i x_i),\ v_i(z_i \cdot r_i,\ z_i^\top \Omega_i z_i)\} \\
&\quad = \min\{u_i(x_i), u_i(z_i)\}.
\end{aligned}$$

□

By Lemma 3.1, $P_i(x_i)$ is a convex set for all $x_i \in \mathbb{R}^\ell$. The following lemma shows that the set $R_i(x_i)$ is compact for all $x_i \in \mathbb{R}^\ell$.

**Lemma 3.2** *Suppose that Assumptions* **C1–C3** *hold. Then for each $x_i \in \mathbb{R}^\ell$, $R_i(x_i)$ is compact.*

*Proof* By Assumption **C1** and Lemma 3.1, $R_i(x_i)$ is closed and convex for any $x_i \in \mathbb{R}^\ell$. Suppose that $R_i(x_i)$ is not bounded. Then there exists a non-zero $y \in \mathbb{R}^\ell$ which is a direction of recession of $R_i(x_i)$.[6] For any $\lambda \ge 0$, therefore, we have $u_i(x_i + \lambda y) \ge u_i(x_i)$. It follows from the quasiconcavity of $v_i$ that for any $\alpha \in [0, 1]$,

$$\begin{aligned}
& v_i\left((1-\alpha)x_i \cdot r_i + \alpha\lambda(x_i/\lambda + y)\cdot r_i,\ (1-\alpha)x_i^\top \Omega_i x_i + \alpha\lambda^2 (x_i/\lambda + y)^\top \Omega_i (x_i/\lambda + y)\right) \\
&\quad = v_i\left((1-\alpha)x_i \cdot r_i + \alpha(x_i + \lambda y)\cdot r_i,\ (1-\alpha)x_i^\top \Omega_i x_i + \alpha(x_i + \lambda y)^\top \Omega_i (x_i + \lambda y)\right) \\
&\quad > \min\{v_i\left(x_i \cdot r_i,\ x_i^\top \Omega_i x_i\right),\ v_i\left((x_i + \lambda y)\cdot r_i,\ (x_i + \lambda y)^\top \Omega_i (x_i + \lambda y)\right)\} \\
&\quad = \min\{u_i(x_i),\ u_i(x_i + \lambda y)\} \\
&\quad = u_i(x_i).
\end{aligned}$$

By setting $\alpha = 1/\lambda^2$ and $\lambda \to \infty$ in the previous relations, we obtain

$$v_i\left(x_i \cdot r_i,\ x_i^\top \Omega_i x_i + y^\top \Omega_i y\right) \ge v_i\left(x_i \cdot r_i,\ x_i^\top \Omega_i x_i\right).$$

Since Assumption **C3** implies $y^\top \Omega_i y > 0$, the above inequality contradicts Assumption **C2**. □

For each $\gamma \in \mathbb{R}$ and each $i \in I$, we set $x_i(\gamma) = \gamma \Omega_i^{-1} r_i$ and $\theta_i = r_i^\top \Omega_i^{-1} r_i$. The results of Lemma 3.2 leads to the following proposition.

**Proposition 3.1** *Suppose that Assumptions* **C1–C3** *is satisfied. Then for each $i \in I$, the following hold:*

(i) *The set $R_i(e_i)$ is compact.*

[6] A vector $v$ is said to be a direction of recession of a set $S$ in $\mathbb{R}^\ell$ if there exists a point $x \in S$ such that $x + \lambda v \in S$ for every $\lambda \in \mathbb{R}_+$.

(ii) *There exist satiation portfolios in* $R_i(e_i)$*, i.e.,* $S_i \neq \emptyset$.

(iii) *Suppose that each* $u_i$ *is differentiable and* $r_i \neq 0$ *for all* $i \in I$*. Then for some* $\gamma_i \in \mathbb{R}$*,* $x_i(\gamma_i) = \gamma_i \Omega_i^{-1} r_i$ *is a satiation portfolio if and only if* $\gamma_i$ *solves the following equation*

$$D_1 v_i \left(\gamma_i \theta_i,\ \gamma_i^2 \theta_i\right) + 2\gamma_i D_2 v_i \left(\gamma_i \theta_i,\ \gamma_i^2 \theta_i\right) = 0, \tag{3.1}$$

*where, for each* $j = 1, 2$*,* $D_j$ *denotes the partial differentiation with respect to the* $j$*-th argument of* $v_i$.

*Proof* (i) By Lemma 3.2, $R_i(e_i)$ is compact.

(ii) Since $u_i$ is continuous and $R_i(e_i)$ is compact, there must exist a portfolio in $R_i(e_i)$ which maximizes $u_i$ over $\mathbb{R}^\ell$.

(iii) Let $x_i$ denote a satiation portfolio for each $i \in I$. Then the first-order condition for utility maximization yields

$$D_1 v_i \left(x_i \cdot r_i,\ x_i^\top \Omega_i x_i\right) r_i + 2 D_2 v_i \left(x_i \cdot r_i,\ x_i^\top \Omega_i x_i\right) \Omega_i x_i = 0. \tag{3.2}$$

We set

$$\gamma_i = -\frac{D_1 v_i \left(x_i \cdot r_i,\ x_i^\top \Omega_i x_i\right)}{2 D_2 v_i \left(x_i \cdot r_i,\ x_i^\top \Omega_i x_i\right)}.$$

Then (3.2) gives $x_i = \gamma_i \Omega_i^{-1} r_i$. By putting $x_i$ into (3.2), we have

$$\left[D_1 v_i(\gamma_i \theta_i,\ \gamma_i^2 \theta_i) + 2\gamma_i D_2 v_i(\gamma_i \theta_i,\ \gamma_i^2 \theta_i)\right] r_i = 0. \tag{3.3}$$

Since $r_i \neq 0$, it follows that

$$D_1 v_i(\gamma_i \theta_i,\ \gamma_i^2 \theta_i) + 2\gamma_i D_2 v_i(\gamma_i \theta_i,\ \gamma_i^2 \theta_i) = 0. \tag{3.4}$$

Conversely, suppose that there exists $\gamma_i$ which solves (3.4). We set $x_i = \gamma_i \Omega_i^{-1} r_i$. By putting it into (3.3), we obtain the same relation as in (3.2). This implies that $x_i$ is a satiation point of $u_i$. □

## 4 Existence of positive equilibrium prices

We present a new set of conditions for the existence of equilibrium and the positivity of equilibrium prices in the CAPM. For each $x \in \mathbb{R}^{\ell m}$ with $I(x) \neq \emptyset$, we define the set

$$H(x) = \sum_{i \in I(x)} con\,[P_i(x_i) - \{x_i\}] \tag{4.1}$$

where, for a set $S \subset \mathbb{R}^\ell$, $con(S) = \bigcup_{\lambda > 0} \lambda S$, i.e., $con(S)$ is the cone generated by the set $S$. Since $P_i(x_i)$ is open and convex, $H(x)$ is an open, convex cone. For each $i \in I$, we make the following assumptions.

**Assumption D1.** Let $x$ be an allocation in $A$ with $I^s(x) \neq \emptyset$. Then for each $k \in I^s(x)$, there exists $i \in I(x)$ such that

$$\nabla u_i(x_i) \cdot (x_k - e_k) \geq 0.$$

**Assumption D2.** For all $x \in A$, $\mathbb{R}^\ell_+ \setminus \{0\} \subset H(x)$.

Assumption **D1** requires that for all $x \in A$ with $I^s(x) \neq \emptyset$, there be an agent $i \in I(x)$ for each $k \in I^s(x)$ which evaluates the satiation portfolio $x_k$ of agent $k$ more than the initial endowment $e_k$ with the marginal valuation at his own choice $x_i$. Assumption **D2** ensures the strict positivity of equilibrium prices. As shown below, this condition is necessary and sufficient for equilibrium prices to be strictly positive.

To exploit the special property of mean-variance utility functions, we define the function $b_i : \mathbb{R}^\ell \to \mathbb{R}$ such that for all $x_i \in \mathbb{R}^\ell$,

$$b_i(x_i) = -\frac{D_2 v_i(x_i \cdot r_i,\ x_i^\top \Omega_i x_i)}{D_1 v_i(x_i \cdot r_i,\ x_i^\top \Omega_i x_i)}. \tag{4.2}$$

The function $b_i$ measures the degree of risk aversion in the CAPM.[7] Then Assumption **D1** is written in an alternative way:

**D1a.** Let $x$ be an allocation in $A$ with $I^s(x) \neq \emptyset$. Then for each $k \in I^s(x)$, there exists $i \in I(x)$ such that

$$\left(\frac{r_i}{2b_i(x_i)} - \Omega_i x_i\right) \cdot (x_k - e_k) \geq 0.$$

Assumption **D1** alone is required for the existence of equilibrium. The following example shows that Assumption **D1** is necessary and sufficient for equilibrium to exist in the economy of Example 2.1.

*Example 4.1* In the economy of Example 2.1, $t_1 = s_2 = (1, 1)$, $x = (t_1, s_2)$ is a feasible allocation, and $I(x) = \{1\}$. Recall that for some $\lambda > 0$, $\nabla u_1(t_1) = \lambda(1, 1)$. Thus, we have $(1, 1) \cdot (s_2 - e_2) = -1/2 < 0$ and therefore, $\nabla u_1(t_1) \cdot (s_2 - e_2) < 0$. Thus, Assumption **D1** is violated in the economy of Example 2.1.

In fact, we can show that Assumption **D1** is necessary and sufficient for equilibrium to exist in Example 2.1 for all the initial allocations $(e_1, e_2)$ where $e_1 = (a, b)$ and $e_2 = (2-a, 2-b)$ for some $(a, b) \in \mathbb{R}^2$. To check Assumption **D1** in the economy with $(e_1, e_2)$, we consider the inequalities $(1, 1)\cdot(s_1 - e_1) = (1, 1)\cdot[(3/2, 3/2) - (a, b)] \geq 0$ and $(1, 1) \cdot (s_2 - e_2) = (1, 1) \cdot [(1, 1) - (2 - a, 2 - b)] \geq 0$. Both inequalities yield $2 \leq a + b \leq 3$. It is easy to check in Fig. 1 that equilibrium exists in the modified economy of Example 2.1 if and only if $2 \leq a + b \leq 3$. The region of the initial allocations which satisfy Assumption **D1** is depicted by the shaded area between the lines A and B in Fig. 1.

Two remarks on Assumptions **D1** and **D2** are in order.

[7] For details, see Allingham (1991).

*Remark 4.1* Assumptions **D1** and **D2** are quite different from those of Sun and Yang (2003) and moreover, strictly weaker than them, as shown below. For the purpose of comparison, we reproduce “Assumption 1” of Sun and Yang (2003) which is made to build their main existence theorem:

**SY.** For every non-zero $\alpha \in \mathbb{R}^m_+$, every $i \in I$, and every $\beta \in V$, it holds that

$$\text{(i)} \sum_{i\in I}\left(\frac{\Omega_i^{-1} r_i}{2\beta_i} - e_i\right) \neq 0 \quad \text{and}$$

$$\text{(ii)} \left(\frac{\Omega_i^{-1} r_i}{2\beta_i} - e_i\right)^{\top}\left(\sum_{i\in I}\alpha_i \Omega_i^{-1}\right)^{-1}\left[\sum_{i\in I}\left(\frac{\Omega_i^{-1} r_i}{2\beta_i} - e_i\right)\right] \geq 0,$$

where $V = \{(\beta_1, \ldots, \beta_m) \in \mathbb{R}^m : \beta_i = b_i(x_i), i \in I$, for some $x_i$ with $u_i(x_i) \geq u_i(e_i)\}$.

*Claim* The condition **SY** implies Assumption **D1**.

*Proof* For each $i \in I$, let $\bar{x}_i$ be a satiation portfolio, i.e., $\bar{x}_i \in S_i$. By (iii) of Proposition 3.1, we have $\bar{x}_i = \Omega_i^{-1} r_i / 2b_i(\bar{x}_i)$ for each $i \in I$. Suppose that the economy satisfies the condition **SY**. In particular, (i) of the condition **SY** implies that $\sum_{i\in I}(\bar{x}_i - e_i) \neq 0$ or $I(x) \neq \emptyset$ for all $x \in A$. Let $x$ be an allocation in $A$. Recalling that $\Omega_i^{-1} r_i / 2b_i(x_i) - x_i = 0$ for each $i \in I^s(x)$, it follows from (ii) of the condition **SY** that for each nonzero $\alpha \in \mathbb{R}^m_+$, each $x \in X$, and each $k \in I^s(x)$,

$$\begin{aligned}
0 &\leq \left(\frac{\Omega_k^{-1} r_k}{2b_k(x_k)} - e_k\right)^{\top}\left(\sum_{i\in I}\alpha_i \Omega_i^{-1}\right)^{-1}\left[\sum_{i\in I}\left(\frac{\Omega_i^{-1} r_i}{2b_i(x_i)} - e_i\right)\right] \\
&= (x_k - e_k)^{\top}\left(\sum_{i\in I}\alpha_i \Omega_i^{-1}\right)^{-1}\left[\sum_{i\in I}\left(\frac{\Omega_i^{-1} r_i}{2b_i(x_i)} - x_i\right)\right] \\
&= (x_k - e_k)^{\top}\left(\sum_{i\in I}\alpha_i \Omega_i^{-1}\right)^{-1}\left[\sum_{i\in I(x)}\left(\frac{\Omega_i^{-1} r_i}{2b_i(x_i)} - x_i\right)\right].
\end{aligned}$$

Thus, for each $k \in I^s(x)$, there exists $i \in I(x)$ such that

$$(x_k - e_k)^{\top}\left(\sum_{i\in I}\alpha_i \Omega_i^{-1}\right)^{-1}\left(\frac{\Omega_i^{-1} r_i}{2b_i(x_i)} - x_i\right) \geq 0. \tag{4.3}$$

Now we choose $\alpha$ in $\mathbb{R}^m_+$ such that $\alpha_i = 1$ and $\alpha_j = 0$ for all $j \neq i$. Then it follows from (4.3) that

$$(x_k - e_k) \cdot \left(\frac{r_i}{2b_i(x_i)} - \Omega_i x_i\right) \geq 0.$$

Thus, we conclude that the condition **SY** implies Assumption D1. □

Readers will recognize that Assumption **D1** is strictly weaker (in fact, much weaker) than the condition **SY** when they read through the proof of CLAIM. Their difference is also revealed in the markets where preferences are not satiated. In this case, Assumption **D1** trivially holds but the condition **SY** still imposes certain restrictions on preferences.

*Remark 4.2* Assumption **D1** has an additional advantage over the conditions of Sun and Yang (2003) and Allingham (1991). They require that the covariance matrix $\Omega_i$ have full rank for each $i \in I$ by allowing no redundant assets like options and futures in markets. This assumption is not valid any more in asset markets where portfolio constraints such as short-selling restrictions enables redundant assets to create new opportunities for income transfers. In contrast, this paper does not require the covariance matrix to fulfill the full rank condition in obtaining the main consequences of the paper. Thus, the current approach can be extended to constrained asset markets with redundant assets.

The following result shows that Assumption **D2** is necessary and sufficient for equilibrium prices to be strictly positive.

**Proposition 4.1** *Suppose that $u_i$ is differentiable for all $i \in I$. Let $(p, x)$ be an equilibrium with $I(x) \neq \emptyset$. Then Assumption* **D2** *holds if and if $p \gg 0$.*

*Proof* Since $(p, x)$ be an equilibrium, for each $i \in I(x)$, $y_i \in P_i(x_i)$ implies that $p \cdot x_i < p \cdot y_i$ and therefore, $p \cdot z_i > 0$ for all $z_i \in con[P_i(x_i) - \{x_i\}]$. Since $u_i$ is differentiable, $p$ is the unique vector which supports $con[P_i(x_i) - \{x_i\}]$ at zero and therefore, $con[P_i(x_i) - \{x_i\}]$ equals the open half space $\{z \in \mathbb{R}^\ell : p \cdot z > 0\}$. Thus we have $H(x) = \{z \in \mathbb{R}^\ell : p \cdot z > 0\}$.

Suppose that $p \gg 0$. Then $p \cdot z > 0$ for all $z \in \mathbb{R}^\ell_+ \backslash \{0\}$. This implies $\mathbb{R}^\ell_+ \backslash \{0\} \subset H(x)$. Conversely, suppose that $\mathbb{R}^\ell_+ \setminus \{0\} \subset H(x)$. For each $j \in \{1, \ldots, \ell\}$, let $\mathbf{1}_j$ denote the vector in $\mathbb{R}^\ell$ which has 1 in its $j$th coordinate and zero in the other coordinates. Clearly, $\mathbf{1}_j \in \mathbb{R}^\ell_+ \setminus \{0\}$. Since $\mathbb{R}^\ell_+ \setminus \{0\} \subset H(x)$, we must have $p \cdot \mathbf{1}_j = p_j > 0$ for all $j \in \{1, \ldots, \ell\}$ where $p_j$ denotes the $j$th coordinate of $p$. Therefore, we conclude that $p \gg 0$. □

We provide the existence of equilibrium in the CAPM with satiable preferences and heterogeneous expectations on the mean and variance of the return distribution. This result extends the equilibrium existence theorem of Allingham's (1991) homogeneous expectation model to the case with heterogeneous expectation and moreover, generalizes the existence theorem of Sun and Yang (2003).

**Theorem 4.1** *Suppose that $\sum_{i \in I} e_i \notin \sum_{i \in I} S_i$.*[8] *Then under Assumptions* **C1**–**C3** *and* **D1**–**D2***, the CAPM has an equilibrium $(p, x)$ with $p \gg 0$.*

*Proof* Let $\Delta$ be the unit closed ball in $\mathbb{R}^\ell$. According to (i) of Proposition 3.1, $A$ is bounded, so that one can find a compact convex cube $K$ with center 0 such that

8 The supposition excludes the unrealistic case that every agent reaches satiation in equilibrium. For details on the existence of equilibrium in this case, see Won and Yannelis (2006).

$A_i \cup \Delta \subset intK$ for each $i \in I$. For each $i \in I$, we set $X_i = K$. Then it is obvious that $X_i$ is compact and convex and $A_i \subset X_i$ for every $i \in I$. We set $X = \prod_{i \in I} X_i$ and define the correspondences $\hat{\mathcal{B}}_i : \Delta \to 2^{X_i}$ and $\hat{\mathcal{B}}_i^\circ : \Delta \to 2^{X_i}$ such that for each $p \in \Delta$,

$$\hat{\mathcal{B}}_i(p) = \{x_i \in X_i : p \cdot x_i \le p \cdot e_i + (1 - \|p\|)\},$$
$$\hat{\mathcal{B}}_i^\circ(p) = \{x_i \in X_i : p \cdot x_i < p \cdot e_i + (1 - \|p\|)\}.$$

Notice that $e_i \in A_i \subset intK = intX_i$ and therefore, $\hat{\mathcal{B}}_i^\circ(p) \neq \emptyset$ for all $p \in \Delta$.

We set $X_0 = \Delta$ and $I_0 = I \cup \{0\}$, and consider the abstract economy $\Gamma = (X_i, \mathcal{A}_i, \mathcal{P}_i)_{i \in I_0}$ where

(1) the correspondence $\mathcal{A}_i : \Delta \times X \to 2^{X_i}$ for each $i \in I_0$ is defined such that $\mathcal{A}_0(p, x) = \Delta$ and $\mathcal{A}_i(p, x) = \hat{\mathcal{B}}_i^\circ(p)$ for each $i \in I$, and
(2) the correspondence $\mathcal{P}_i : \Delta \times X \to 2^{X_i}$ for each $i \in I_0$ is defined such that

$$\mathcal{P}_0(p, x) = \left\{q \in \Delta : q \cdot \textstyle\sum_{i \in I}(x_i - e_i) > p \cdot \sum_{i \in I}(x_i - e_i)\right\},$$
$$\mathcal{P}_i(p, x) = P_i(x_i) \cap K := \hat{P}_i(x_i), \ \forall i \in I.$$

Since $\mathcal{E}$ satisfies Assumptions **C1**–**C3** for all $i \in I$, the abstract economy $\Gamma$ satisfies the conditions of Theorem 6.1 of Yannelis and Prabhakar (1983). Thus, there must exist a pair $(p^*, x^*) \in \Delta \times X$ which satisfies (a) $x_i^* \in \hat{\mathcal{B}}_i(p^*)$, $\forall i \in I$, (b) $\hat{P}_i(x_i^*) \cap \hat{\mathcal{B}}_i^\circ(p^*) = \emptyset$, $\forall i \in I$, and (c) $p^* \cdot z^* \ge p \cdot z^*$, $\forall p \in \Delta$, where $z_i^* = x_i^* - e_i$ and $z^* = \sum_{i \in I}(x_i^* - e_i)$.

To see that $z^* = 0$, suppose to the contrary that $z^* \neq 0$. Then it follows from (c) that $p^* \cdot z^* > 0$ and $\|p^*\| = 1$. Then by (a), we obtain $p^* \cdot z^* \le 0$, which leads to a contradiction. Hence, we have $z^* = 0$. Since $\sum_{i \in I} e_i \notin \sum_{i \in I} S_i$, we have $(S_1 \times \cdots \times S_n) \cap A = \emptyset$ and therefore, $I(x) \neq \emptyset$ for all $x \in A$. Thus, we obtain $I(x^*) \neq \emptyset$.

We claim that $p^* \cdot x_i^* = p^* \cdot e_i + (1 - \|p^*\|)$, $\forall i \in I(x^*)$. Since $x_i^* \in \hat{\mathcal{B}}_i(p^*)$, we have $p^* \cdot x_i^* \le p^* \cdot e_i + (1 - \|p^*\|)$. We need to show $p^* \cdot x_i^* \ge p^* \cdot e_i + (1 - \|p^*\|)$. Since $i \in I(x^*)$ and $x_i^* \in intK$, $\hat{P}_i(x_i^*) \neq \emptyset$. Thus, we can choose $x_i \in \hat{P}_i(x_i^*)$. Then it follows from (b) that $\alpha x_i + (1 - \alpha)x_i^* \in \hat{P}_i(x_i^*)$ for any $\alpha \in (0, 1]$ and $p^* \cdot \left[\alpha x_i + (1 - \alpha)x_i^*\right] \ge p^* \cdot e_i + (1 - \|p^*\|)$. As $\alpha \to 0$, we have $p^* \cdot x_i^* \ge p^* \cdot e_i + (1 - \|p^*\|)$. This proves that $p^* \cdot x_i^* = p^* \cdot e_i + (1 - \|p^*\|)$, $\forall i \in I(x^*)$. Observe that for each $i \in I(x^*)$, $x_i' \in P_i(x_i^*)$ has $p^* \cdot x_i' \ge p^* \cdot e_i + (1 - \|p\|) = p \cdot x_i^*$. That is, $p^*$ supports the convex set $P_i(x_i^*)$ at $x_i^*$. Since $u_i$ is differentiable, this implies that there exists $\lambda_i > 0$ for each $i \in I(x^*)$ such that $\lambda_i p^* = \nabla u_i(x_i^*)$.

We show that $\|p^*\| = 1$ and $p^* \cdot x_i^* = p^* \cdot e_i$, $\forall i \in I$. Suppose that $\|p^*\| < 1$. Then $p^* \cdot x_i^* = p^* \cdot e_i + (1 - \|p^*\|) > p^* \cdot e_i$ for every $i \in I(x^*)$. On the other hand, by Assumption **D1**, there exists $i \in I(x^*)$ for each $k \in I^s(x^*)$ such that $\nabla u_i(x_i^*) \cdot (x_k^* - e_k) \ge 0$. Thus, we have $p^* \cdot (x_k^* - e_k) \ge 0$ for all $k \in I^s(x^*)$. Summing up these inequalities over $I$, we obtain $p^* \cdot z^* > 0$, which contradicts $z^* = 0$. Thus, it must be the case that $\|p^*\| = 1$ and $p^* \cdot x_i^* = p^* \cdot e_i$, $\forall i \in I$.

Consequently, it holds that $x_i^* \in \mathcal{B}_i(p^*)$ and $\hat{P}_i(x^*) \cap \{x_i \in X_i : p^* \cdot x_i < p^* \cdot e_i\} = \emptyset$, $\forall i \in I$. Since $x_i^*$ is in the interior of $K$ for all $i \in I$, it is easy to see that $P_i(x^*) \cap \mathcal{B}_i^\circ(p^*) = \emptyset$ where $\mathcal{B}_i^\circ(p^*) = \{x_i \in \mathbb{R}^\ell : p^* \cdot x_i < p^* \cdot e_i\}$, $\forall i \in I$. Since both $P_i(x^*)$ and $\mathcal{B}_i^\circ(p^*)$ are open and $\mathcal{B}_i^\circ(p^*) \neq \emptyset$ for every $i \in I$, this implies that $P_i(x^*) \cap \mathcal{B}_i(p^*) = \emptyset$, $\forall i \in I$. Hence, $(p^*, x^*)$ constitutes an equilibrium of the asset market economy $\mathcal{E}$. It follows by Proposition 4.1 that $p^* \gg 0$. $\square$

## 5 Conclusion

This paper shows the existence of equilibrium and the positivity of equilibrium prices in the mean-variance economy with heterogeneous expectations on the return distribution. These consequences are based upon new conditions on the utility functions and the initial allocation of portfolios.

In contrast to the literature, our approach to the existence of equilibrium prices in the CAPM does not rely on the structure of the mean-variance economy. This is a definite advantage over the literature such as Allingham (1991) and Sun and Yang (2003) because the current approach can work beyond the framework of the traditional CAPM. Specifically, the consequences of the paper can be generalized to the case where higher-moments of the return distribution like skewness and kurtosis have impact on asset pricing. Another conceivable extension is to investigate the existence of equilibrium asset prices with redundant assets such as options and futures in asset markets which are subject to portfolio constraints.

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