Equilibria in Abstract Economies with a Continuum of Agents with Discontinuous and Non-Ordered Preferences

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Abstract

This paper focus on the problem of the existence of an equilibrium in abstract economies and exchange economies. Spanning over the literature we have managed to extend and generalize some previous results. In particular, we generalize the main theorem of Yannelis (1987) on the existence of an equilibrium in an abstract economy with a continuum of agents, by allowing for discontinuous preferences. As a corollary of this result, we extend the finite agent Cournot-Nash equilibrium existence theorems with discontinuous preferences (e.g., Reny (1999), Bareli-Meneghel (2013), He-Yannelis (2016), among others), to a continuum of agents. We also obtain an existence theorem for an abstract economy which allows for a convexifying effect on aggregation and nonconvex strategy and constraint sets. Furthermore, our new main theorem is used to prove the existence of a Walrasian equilibrium with a continuum of agents with discontinuous, non-ordered, interdependent and price-dependent preferences and thus extending the results of Aumman (1966) and Schmeidler (1969).

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1 Introduction

It is our pleasure to contribute to the celebration of the 50 years anniversary of the Journal of Mathematical Economics. One of the deepest problems in general equilibrium theory is the existence of an equilibrium. The JME was the leader in that field and in 1970's and 1980's seminal papers published by Mas-Colell (1974), Gale-Mas-Colell (1975, 1979), Shafer-Sonneschein (1975), Shafer (1976), Borglin-Keiding (1976). This work went beyond the traditional wisdom of the Arrow-Debreu-McKenzie existence results as the transitivity and completeness of preferences were dropped. Further contributions to the existence of an equilibrium with infinitely many commodities were made by Mas-Colell (1975), Bewley (1982), Mas-Colell (1984), Khan-Vohra (1984), Yannelis-Phrabakar (1983), Yannelis-Zame (1986) and Anderson et al. (2022), among others.

The Yannelis-Phrabakar (1983) existence theorem was extended to a continuum of agents in Yannnelis (1987). One of the main purposes of this paper is to generalize the Yannelis (1987) theorem on the existence of Cournot-Nash equilibrium (CNE) to discontinuous preferences. As a consequence of this, we extend all the recent results on the existence of CNE with a finite number of agents, e.g., Dasgupta-Maskin (1986), Lebrun (1996), Reny (1999, 2016a), Bagh-Jofre (2006), Bareli–Meneghel (2013), Mclennan-Monteiro-Tourky (2011), Carmona (2011), Carmona-Podczeck (2016), Prokopovich (2011, 2016), Prokopovych-Yannelis (2017), He-Yannelis (2016, 2017), Scalzo (2015), among others, to a continuum of agents.[1] Also, we apply our new result to a concrete exchange economy in order to obtain an extension of the classical Walrasian equilibrium theorems of Aumann (1966) and Schmeidler (1969) to an economy with a continuum of agents with discontinuous, non-ordered, interdependent and price dependent preferences. Thus, spanning over the literature with a continuum of agents we generalize and extend previous results, and consequently, we bring the existence of equilibrium work to the "state of the art".[2]

A short historical introduction is outlined below: Debreu (1952) generalized the non-cooperative existence of an equilibrium theorem of Nash (1951) by introducing the concept of a social system or abstract economy, which is a generalization of the Counot-Nash game. The theorem of Debreu was applied in Arrow-Debreu (1954) to prove the existence of a Walrasian equilibrium (competitive equilibrium). Debreu's theorem was obtained for an abstract economy with a finite number of agents whose preferences were

[1] see Reny (2020) for an excellent survey on the work of discontinuous noncooperative games.

[2] Recently and independently from our work using the excess demand approach related results have been obtained by Otsuka (2024). See Remark 6.5.

representable by continuous utility functions, i.e., preferences were transitive, complete, reflexive, and continuous. Motivated by the pioneering works of Shafer (1974) and Mas-Collel (1974), Shafer-Sonnenschein (1975) generalized the Debreu existence theorem to non-ordered preferences and subsequently, Shafer (1976) applied this result to an exchange economy (in a similar fashion with that of Arrow-Debreu), in order to prove the existence of a Walrasian equilibrium with non-ordered, interdependent and price dependent agent preferences. The Shafer-Sonneschein (1975) theorem was obtained for a finite number of agents and with a finite-dimensional strategy/commodity space. It was pointed out in Yannelis-Prabhakar (1983)[3] that the Shafer-Sonneschein as well as the Borglin-Keiding (1976) theorems fail if the commodity space or the set of agents is infinite. Yannelis-Prabhakar generalized the Shafer-Sonnenschein theorem to an infinite dimensional commodity space and to an infinite number of agents. However, no measure-theoretic structure of the set of agents (i.e., no continuum of agents) was modeled in that paper, this was done in Yannelis (1987)[4].

Recently the Shafer-Sonneschein (1975) and Yannelis-Prabhakar (1983) theorems, were generalized to discontinuous preferences, (see for example Carmona-Podczeck (2016), He-Yannelis (2016) and Reny (2016b), among others). The main question addressed in this paper is whether or not one can model the continuum of agents and generalize Yannelis' (1987) theorem to allow for discontinuous preferences. We provide an affirmative answer to this question. Our new result allows us to extend all the previous work on discontinuous games and economies to a continuum of agents as it was already pointed out above.

The paper proceeds as follows. Section 2 contains all notations and definitions. Section 3 is attributed to describing the economic model and assumptions. The main existence theorem is presented in Section 4. Section 5 is devoted to the existence of an equilibrium of an abstract economy with a convexifying effect. Section 6 ideals with the existence of a Walrasian equilibrium with a measure space of agents and non-ordered, interdependent, price-dependent, and discontinuous preferences. Section 7 is devoted to a technical proof. Several technical lemmata needed for the proof of our main theorem are concentrated in Section 8.

[3]See also Khan and Uyanik (2021) for a recent extension of Yannelis-Prabhakar (1983).

[4]We would like to mention that early contributions to games with a continuum of agents were made by Schmeidler(1973), Mas-Collel (1974), Khan (1986), Khan-Vohra (1984), and Khan-Papageorgiou (1987). However, these results do not extend the Yannelis-Prabhakar existence theorem.

2 Preliminaries

2.1 Notation

- 2^A denotes the set of all subsets of the set A,
- $\mathbb{R}$ denotes the set of real numbers,
- $\mathbb{R}^\ell$ denotes the ℓ-fold product of $\mathbb{R}$,
- $\mathrm{cl}A$ denotes the closure of the set A,
- $\mathrm{con}A$ denotes the convex hull of the set A,
- $\overline{\mathrm{con}}A$ denotes the closed convex hull of the set A,
- $\mathrm{bd}A$ denotes the boundary of the set A,
- $\setminus$ denotes the set-theoretic subtraction,
- If $\Phi : X \rightrightarrows Y$ is a correspondence then $\Phi_{|U} : U \rightrightarrows Y$ denotes the restriction of Φ to U,
- proj denotes projection.

2.2 Definitions

Let X, Y be two topological spaces. A correspondence $\Phi : X \rightrightarrows Y$ is said to be **upper-semicontinuous** (**u.s.c.**) if the set $\{x \in X : \Phi(x) \subseteq V\}$ is open in X for every open subset V of Y. The **graph** of the correspondence $\Phi : X \rightrightarrows Y$ is defined as

$$G_\Phi = \{(x, y) \in X \times Y : y \in \Phi(x)\}.$$

The correspondence $\Phi : X \rightrightarrows Y$ is said to have a **closed graph** if the set G_Φ is closed in $X \times Y$. A correspondence $\Phi : X \rightrightarrows Y$ is said to be **lower-semicontinuous** (**l.s.c.**) if the set $\{x \in X : \Phi(x) \cap V \neq \emptyset\}$ is open in X for every open subset V of Y. A correspondence $\Phi : X \rightrightarrows Y$ is said to have **open lower sections** if for each $y \in Y$, the set $\Phi^{-1}(y) = \{x \in X : y \in \Phi(x)\}$ is open in X. If $\Phi(x)$ is open in Y for each $x \in X$, Φ is said to have **open upper sections**.

Let $(T, \mathscr{T}, \mu)$ be a complete finite measure space, i.e., μ is a real-valued, non-negative, countable additive measure defined in a complete σ-field $\mathscr{T}$ of subsets of T such that $\mu(T) < \infty$. We denote by $L_1(\mu, \mathbb{R}^\ell)$ the space of equivalence classes of $\mathbb{R}^\ell$-valued Bochner integrable functions $f : T \to \mathbb{R}^\ell$ normed by $\|f\| = \int_T \|f(t)\| d\mu(t)$.

A correspondence $\Phi : T \rightrightarrows \mathbb{R}^\ell$ is said to be **integrably bounded** if there exists an element $g \in L_1(\mu, \mathbb{R})$ such that

$$\sup \{\|x\| : x \in \Phi(t)\} \leq g(t) \, \mu\text{-a.e.}$$

A correspondence $\Phi : T \rightrightarrows \mathbb{R}^\ell$ is said to have a **measurable graph** if $G_\Phi \in \mathscr{T} \otimes \mathscr{B}(\mathbb{R}^\ell)$, where $\mathscr{B}(\mathbb{R}^\ell)$ denotes the Borel σ-algebra on $\mathbb{R}^\ell$ and $\otimes$ denotes σ-product field. A correspondence $\Phi : T \rightrightarrows \mathbb{R}^\ell$ is said to be **lower measurable** if the set $\{t \in T : \Phi(t) \cap V \neq \emptyset\} \in \mathscr{T}$ for every open subset V of $\mathbb{R}^\ell$. It is worth pointing out that if $\mathscr{T}$ is complete and the correspondence $\Phi : T \rightrightarrows \mathbb{R}^\ell$ has a measurable graph, then Φ is lower measurable. Moreover, if Φ is closed valued and lower measurable, then Φ has a measurable graph. For an extensive survey, see Yannelis (1991a).

Let now, X be a topological space. Let $\Phi : X \rightrightarrows \mathbb{R}^\ell$ be a nonempty valued correspondence. A function $f : X \to \mathbb{R}^\ell$ is said to be a **continuous selection** from Φ if $f(x) \in \Phi(x)$ for all $x \in X$, and f is continuous. Let $(T, \mathscr{T}, \mu)$ be an arbitrary measure space. Let $\Psi : T \rightrightarrows \mathbb{R}^\ell$ be a non-empty valued correspondence. A function $f : T \to \mathbb{R}^\ell$ is said to be a **measurable selection** of Ψ if $f(t) \in \Psi(t)$ for all $t \in T$, and f is measurable. We denote by $\mathcal{S}^1_\Psi$ the set of integrable selections of Ψ, i.e.,

$$\mathcal{S}^1_\Psi := \left\{\psi \in L_1(\mu, \mathbb{R}^\ell) : \psi \text{ is a measurable selection of } \Psi\right\}.$$

For any correpondence $\Psi : T \rightrightarrows Y$, the **integral** of Ψ is defined by

$$\int_T \Psi \, d\mu = \left\{ \int_T \psi \, d\mu : \psi \in S^1_\Psi \right\}.$$

If the correspondence Ψ is integrably bounded and has measuranble graph or it is closed valued and lower measurable then by virtue of the Aumann or Kuratowski–Ryll-Nardzewski measurable selection theorem, the integral is nonempty.

We now define the concept of a **Caratheodory-type selection** which roughly speaking combines the notions of continuous selection and measurable selection. Let Z be a topological space and $\Phi : T \times Z \rightrightarrows \mathbb{R}^\ell$ be a nonempty valued correspondence. A function $f : T \times Z \to \mathbb{R}^\ell$ is said to be a **Caratheodory-type selection** from Φ if $f(t,z) \in \Phi(t,z)$ for all $(t,z) \in T \times Z$, $f(\cdot, z)$ is measurable for all $z \in Z$, and $f(t, \cdot)$ is continuous for all $t \in T$. For any correspondence $F : T \times Z \rightrightarrows \mathbb{R}^\ell$, define

$$U_F = \{(t,x) \in T \times Z : F(t,x) \neq \emptyset\}.$$

For any $t \in T$, let $U^t_F = \{x \in Z : (t,x) \in U_F\}$, and for any $x \in Z$, let $U^x_F = \{t \in T : (t,x) \in U_F\}$.

3 Abstract Economies and Assumptions

Let $(T, \mathscr{T}, \mu)$ be a complete, finite, and positive measure space of agents. For any correspondence $X : T \rightrightarrows \mathbb{R}^\ell$, we define

$$L_X = \left\{x \in L_1(\mu, \mathbb{R}^\ell) : x(t) \in X(t)\, \mu\text{-a.e.}\right\}.$$

An **abstract economy** Γ is a quadruple $\langle (T, \mathscr{T}, \mu), X, P, A\rangle$, where

(i) $(T, \mathscr{T}, \mu)$ is a **measure space** of agents;

(ii) $X : T \rightrightarrows \mathbb{R}^\ell$ is a **strategy** correspondence;

(iii) $P : T \times L_X \rightrightarrows \mathbb{R}^\ell$ is a **preference** correspondence such that $P(t, x) \subseteq X(t)$ for all $(t, x) \in T \times L_X$; and

(iv) $A : T \times L_X \rightrightarrows \mathbb{R}^\ell$ is a **constraint** correspondence such that $A(t, x) \subseteq X(t)$ for all $(t, x) \in T \times L_X$.

By definition, the preference correspondence P captures the idea of interdependence. The interpretation of these preference correspondences is that $y \in P(t, x)$ means that agent t strictly prefers y to $x(t)$ if the given strategies of other agents are fixed. Note that L_X is the set of all joint strategies. As in Khan-Vohra (1984), Schmeidler (1973) and Yannelis (1987), we endow L_X throughout the paper with the weak topology. This signifies a natural form of myopic behaviour on the part of the agents. In particular, an agent has to arrive at his/her decisions on the basis of knowledge of only finitely many (average) numerical characteristics of the joint strategies.

We now define the concept of an equilibrium in an abstract economy.

Definition 3.1. An **equilibrium** for an abstract economy Γ is an element $x^* \in L_X$ such that for μ-a.e. on T, we have $x^*(t) \in A(t, x^*)$ and $P(t, x^*) \cap A(t, x^*) = \emptyset$.

We introduce the following notion of "continuous inclusion property", which is a generalization of that in He and Yannelis (2017) to a large economy.

Definition 3.2. A correspondence $G : T \times L_X \rightrightarrows \mathbb{R}^\ell$ is said to have the **continuous inclusion property** if for each $y \in L_X$, there exists a correspondence $F_y : T \times L_X \rightrightarrows \mathbb{R}^\ell$ satisfying the following:

(i) If $U_G^y \neq \emptyset$ then there exists a collection $\{O_y^t : t \in U_G^y\}$ of weakly open neighbourhoods of y in L_X such that $F_y(t, x) \neq \emptyset$ and $F_y(t, x) \subseteq G(t, x)$ for all $x \in O_y^t$ and all $t \in U_G^y$;

(ii) $F_y(t,\cdot) : O_y^t \rightrightarrows \mathbb{R}^\ell$ is lower-semicontinuous for all $t \in U_G^y$ and $F_y(t,\cdot) : L_X \rightrightarrows \mathbb{R}^\ell$ is lower-semicontinuous for all $t \notin U_G^y$;[5] and

(iii) $F_y : T \times L_X \rightrightarrows \mathbb{R}^\ell$ is jointly lower measurable.

Definition 3.3. A correspondence $G : T \times L_X \rightrightarrows \mathbb{R}^\ell$ is said to have the **strong continuous inclusion property** if it satisfies the continuous inclusion property and the following measurability condition.

Measurability Condition: If $F_y \neq F_z$ for some $y, z \in L_X$, then any one of the following is true:

(a) The collection $\{F_y : y \in L_X\}$ and $\{O_y^t : (t,y) \in U_G\}$ are countable, and the set $\{(t,x) : x \in O_y^t\}$ is $\mathscr{T} \otimes \mathscr{B}_w(L_X)$-measurable for each $y \in L_X$, where $\mathscr{B}_w(L_X)$ is the Borel σ-algebra for the weak topology on L_X.[6]

(b) $U_G \in \mathscr{T} \otimes \mathscr{B}_w(L_X)$. For each $y \in L_X$, the correpondence $\mathbb{I} : T \times L_X \rightrightarrows L_X$, defined by $\mathbb{I}(t,x) = \{y \in U_G^t : x \in O_y^t\}$, is jointly lower measurable and is finite valued.[7] Furthermore, for each fixed $(t,x) \in T \times L_X$, the correpondence $H : L_X \rightrightarrows \mathbb{R}^\ell$, defined by $H(y) = F_y(t,x)$, is continuous and is contained in $X(t)$ for all $y \in L_X$.

The definition of continuous inclusion property is similar to those in He-Yannelis (2016) except for the closed graph condition of con$F_y(t,\cdot)$ in He-Yannelis (2016) is replaced with the lower-semicontinuity of $F_y(t,\cdot)$. In the next section, we use this condition to show the equilibrium existence theorem when the measure space is purely atomic. However, to deal with more general case, i.e, an economy with an arbitrary complete finite positive separable measure space of agents, we require the strong continuous inclusion property, which satisfies some measurability condition along with the continuous inclusion property.

[5]Note that the lower-semicontinuity of $F_y(t,\cdot) : L_X \rightrightarrows \mathbb{R}^\ell$ also implies that of $F_y(t,\cdot) : O_y^t \rightrightarrows \mathbb{R}^\ell$.

[6]Under our assumptions for the main result, we can show that L_X is a non-empty, closed, weakly compact and metrizable space, and hence it is also a separable metrizable space. Let $D = \{y_1, y_2, \cdots\}$ be a dense subset of L_X. We denote by $\mathcal{O}$ the collection of neighbourhood base at each point of D, i.e.,

$$\mathcal{O} = \left\{ B\left(y_k, \frac{1}{m}\right); k, m \in \mathbb{N} \right\}.$$

Therefore for each O_y^t, there exists some $V_y^t \in \mathcal{O}$ such that $y \in V_y^t \subseteq O_y^t$. Note that the collection $\{V_y^t : t \in U_G^y\}$ satisfies the condition (i) and (ii), and it is countable. However, the measurability of $\{(t,x) : x \in O_y^t\}$ may not imply that of $\{(t,x) : x \in V_y^t\}$. Hence, we assume that $\{O_y^t : (t,y) \in U_G\}$ is countable.

[7]The finiteness assumption is also follows from the assumption that $\{O_y^t : x \in L_X\}$ is locally finite.

Remark 3.4. It should be noted that if a correpondence $G : T \times L_X \rightrightarrows \mathbb{R}^\ell$ is jointly lower measurable and the correspondence $G(t, \cdot) : L_X \rightrightarrows \mathbb{R}^\ell$ is lower-semicontinuous for all $t \in T$ then G satisfies the strong continuous inclusion property. Indeed, we can take $O_y^t = U_G^t$ for all $t \in U_G^y$ if $U_G^y \neq \emptyset$ and $F_y = G$ for all $y \in L_X$. Since $F_y = F_z$ for all $y, z \in L_X$, the measurability condition is vacuously satisfied. The lower measurability of $F_y(t, \cdot) : O_y^t \rightrightarrows \mathbb{R}^\ell$ for all $t \in U_G^y$ follows from the fact that $G(t, \cdot) : L_X \rightrightarrows \mathbb{R}^\ell$ is lower-semicontinuous for all $t \in T$ and $U_G^t = \{x \in L_X : G(t, x) \cap \mathbb{R}^\ell \neq \emptyset\}$ is weakly open in L_X. The rest of the conditions are also immediate.

To define a set of assumptions for our main result, we let $\psi(t, x) := A(t, x) \cap \text{con}P(t, x)$ for all $(t, x) \in T \times L_X$.

Assumptions: Below, we state a set of assumptions that will be used in the main result of the paper:

(A.1) $X : T \rightrightarrows \mathbb{R}^\ell$ is a correspondence such that:

- (a) it is integrably bounded and for all $t \in T$, $X(t)$ is a non-empty, convex, closed subset of $\mathbb{R}^\ell$;
- (b) for every open subset V of $\mathbb{R}^\ell$, $\{t \in T : X(t) \cap V \neq \emptyset\} \in \mathscr{T}$.

(A.2) $A : T \times L_X \rightrightarrows \mathbb{R}^\ell$ is a correspondence such that:

- (a) for each $t \in T$, $A(t, \cdot) : L_X \rightrightarrows \mathbb{R}^\ell$ is upper-semicontinuous;
- (b) for all $(t, x) \in T \times L_X$, $A(t, x)$ is a non-empty, convex, closed subset of $\mathbb{R}^\ell$;
- (c) for each fixed $x \in L_X$, $A(\cdot, x)$ is lower measurable.

(A.3) $P : T \times L_X \rightrightarrows \mathbb{R}^\ell$ has the property that $x(t) \notin \text{con}P(t, x)$ for all $x \in L_X$ and almost all $t \in T$.

(A.4) $\psi : T \times L_X \rightrightarrows \mathbb{R}^\ell$ satisfies the continuous inclusion property if the economy Γ is purely atomic[8] and the strong continuous inclusion property, otherwise.

[8] The economy Γ is said to be *purely atomic* if $\mu(\{t\}) > 0$ for all $t \in T$. Note that He-Yannelis (2016) considers countably many agents in their existence proof, but their approach was not measure-theoretic as in this paper.

4 The main result

In this section, we study our main result, namely the existence of an equilibrium in an abstract economy with discontinuous preferences. We also show that our main result generalizes the main result of Yannelis (1987) and gives the existence theorem of a Nash equilibrium as a simple corollary.

4.1 The existence of an equilibrium in an abstract economy with discontinuous preferences

Before we state our main result we should mention that existence theorems in infinite dimensional commodity/strategy spaces for abstract economies without ordered preferences were based on extensions of the finite dimensional Michael continuous selection theorem (that was used in Gale and Mas-Collel (1975)), see for example, Yannelis-Prabhakar (1983), Wu-Shen (1996), Kim-Prikry-Yannelis (1987). Our proof follows this type of argument, i.e., a Carathéodory-type selection theorem which generalizes the continuous and measurable selections theorems simultaneously. However, due to discontinuity of preferences, our proof becomes significantly more difficult and challenging.

Theorem 4.1. *Under the assumptions* (A.1)-(A.4)*, there exists an equilibrium for the abstract economy* Γ*.*

Proof. Letting

$$\mathbb{J}_\psi = \left\{ x \in L_X : U_\psi^x \text{ is not null} \right\},$$

we consider the following two cases.

Case 1. $\mathbb{J}_\psi = \emptyset$. In this case, we have $\mu(U_\psi^x) = 0$, which implies $P(t,x) \cap A(t,x) = \emptyset$ for all $x \in L_X$ and almost all $t \in T$. Since A is closed-valued and $A(\cdot, x)$ is lower measurable, we conclude that $A(\cdot, x)$ has a measurable graph, for all $x \in L_X$. Define $\Psi : L_X \rightrightarrows L_X$ by

$$\Psi(x) = \{ y \in L_X : y(t) \in A(t,x) \, \mu\text{-a.e.} \}.$$

In view of Lemma 8.6, Ψ is non-empty valued and weakly upper-semicontinuous. Since A is convex valued, so is Ψ. Furthermore, Lemma 8.5 guarantees that L_X is non-empty, convex, and weakly compact. Hence by Fan-Glicksberg's fixed point theorem, there exists $x^* \in L_X$ such that $x^* \in \Psi(x^*)$, which means $x^*(t) \in A(t, x^*)$ μ-a.e. Since $P(t,x^*) \cap A(t,x^*) = \emptyset$ μ-a.e., it follows that x^* is an equilibrium for the abstract economy Γ.

Case 2. $\mathbb{J}_\psi \neq \emptyset$. By Theorem 7.1, we can find a correspondence $\Phi : T \times L_X \rightrightarrows \mathbb{R}^\ell$ satisfying the following:

(A) $\Phi(t, x) \subseteq \psi(t, x)$ for all $(t, x) \in U_\psi$;

(B) $U_\psi = U_\Phi$;

(C) $\Phi(t, \cdot) : L_X \rightrightarrows \mathbb{R}^\ell$ is lower-semicontinuous for all $t \in T$;

(D) Φ is jointly lower measurable; and

(E) There exists a Carathéodory-type selection $f : U_\Phi \to \mathbb{R}^\ell$ of $\Phi|_{U_\Phi}$.

Given an $x \in L_X$, the set U_Φ^x can be expressed as

$$U_\Phi^x = \mathrm{proj}_T \left(\left\{ (t, x) \in T \times L_X : \Phi(t, x) \cap \mathbb{R}^\ell \neq \emptyset \right\} \cap (T \times \{x\}) \right).$$

Since $\Phi(\cdot, \cdot)$ is jointly lower measurable, by the projection theorem (see Yannelis (1991b)), we have that U_Φ^x is measurable. Let $\Lambda : T \times L_X \rightrightarrows \mathbb{R}^\ell$ be a correspondence such that

$$\Lambda(t, x) = \begin{cases} \{f(t, x)\}, & \text{if } (t, x) \in U_\Phi; \\ A(t, x), & \text{otherwise.} \end{cases}$$

Clearly, Λ is non-empty and convex valued. In view of the lower-semicontinuity of $\Phi(t, \cdot)$, we conclude that

$$U_\Phi^t = \left\{ x \in L_X : \Phi(t, x) \cap \mathbb{R}^\ell \neq \emptyset \right\}$$

is a weakly open subset of L_X. Hence, by Lemma 8.12, we have that $\Lambda(t, \cdot)$ is upper-semicontinuous in the sense that $\{x \in L_X : \Lambda(t, x) \subset V\}$ is a weakly open subset of L_X for every open subset V of $\mathbb{R}^\ell$. As in Case 1, we conclude that $A(\cdot, x)$ has a measurable graph. It can be easily seen that for each $x \in L_X$, $\Lambda(\cdot, x)$ has a measurable graph. In fact, for all $x \in L_X$,

$$G_{\Lambda(\cdot, x)} = \left\{ (t, y) \in T \times \mathbb{R}^\ell : y \in \Lambda(t, x) \right\} = \mathbb{B} \cup \mathbb{C},$$

where

$$\mathbb{B} = \left\{ (t, y) \in T \times \mathbb{R}^\ell : y = f(t, x) \text{ and } t \in U_\Phi^x \right\}$$

and

$$\mathbb{C} = \left\{ (t, y) \in T \times \mathbb{R}^\ell : y \in A(t, x) \text{ and } t \notin U_\Phi^x \right\}.$$

As $\mathbb{B}, \mathbb{C}$ belong to $\mathscr{T} \otimes \mathscr{B}(\mathbb{R}^\ell)$, we infer that $G_{\Lambda(\cdot,x)} = \mathbb{B} \cup \mathbb{C}$ is in $\mathscr{T} \otimes \mathscr{B}(\mathbb{R}^\ell)$. Define $\Psi : L_X \rightrightarrows L_X$ by

$$\Psi(x) = \{y \in L_X : y(t) \in \Lambda(t, x)\, \mu\text{-a.e.}\}.$$

Again, by repeating the argument of Case 1, we can find an $x^* \in L_X$ such that $x^* \in \Psi(x^*)$, which means $x^*(t) \in \Lambda(t, x^*)$ μ-a.e. We claim that $\mu(U_\Phi^{x^*}) = 0$. This follows from the fact that for $t \in U_\Phi^{x^*}$, we have $x^*(t) = f(t, x^*) \in \Phi(t, x^*) \subseteq \psi(t, x^*) \subseteq \text{con}P(t, x^*)$,[9] a contradiction to Assumption (A.3) if $\mu(U_\Phi^{x^*}) > 0$. Therefore, μ-a.e. on T, $x^*(t) \in A(t, x^*)$ and $\psi(t, x^*) = \emptyset$, which implies that $P(t, x^*) \cap A(t, x^*) = \emptyset$ μ-a.e., i.e. x^* is an equilibrium for Γ. This completes the proof. □

Remark 4.2. As it was mentioned previously the definition of continuous inclusion property is similar to the one in He-Yannelis (2016) except for the closed graph condition of $\text{con}F_y(t, \cdot)$ in He-Yannelis (2016) is replaced with the lower-semicontinuity. This is due to the fact that our approch is measure-theoretic whereas the approach in He-Yannelis (2016) is purely non measure-theoretic. More precisely, the lower-semicnontinuity condition in our measure-theoretic setup guarantees the existence of a Carathéodory-type selection, which plays a pivotal role in our proof.

4.2 Corollaries of the main existence theorem

In this subsection, we show that our main result is a generalization of that in Yannelis (1987). We further show that, as an immediate corollary of our main result, one can establish the existence of a Nash equilibrium.

The followings assumptions for Γ were made in Yannelis (1987):

(B.1) (T, τ, μ) is a finite, positive, complete, separable measure space.

(B.2) $X : T \rightrightarrows \mathbb{R}^\ell$ is a correspondence such that:

- (a) it is integrably bounded and for all $t \in T$, $X(t)$ is a non-empty, convex, closed subset of $\mathbb{R}^\ell$;
- (b) for every open subset V of $\mathbb{R}^\ell$, $\{t \in T : X(t) \cap V \neq \emptyset\} \in \mathscr{T}$.

(B.3) $A : T \times L_X \rightrightarrows \mathbb{R}^\ell$ is a correspondence such that:

- (a) for each $t \in T$, $A(t, \cdot) : L_X \rightrightarrows \mathbb{R}^\ell$ is continuous;

[9] Note that, by (A), $\Phi(t, x) \subseteq \psi(t, x)$ for all $(t, x) \in U_\psi$.

(b) for all $(t,x) \in T \times L_X$, $A(t,x)$ is convex, closed, and nonempty;

(c) for each fixed $x \in L_X$, $A(\cdot,x)$ is lower measurable.

(B.4) $P : T \times L_X \rightrightarrows \mathbb{R}^\ell$ is a correspondence such that:

(a) for each $t \in T$, $P(t,\cdot) : L_X \rightrightarrows \mathbb{R}^\ell$ has an open graph in $L_X \times \mathbb{R}^\ell$;

(b) $x(t) \notin conP(t,x)$ for all $x \in L_X$ for almost all t in T;

(c) $\{(t,x) \in T \times L_X : A(t,x) \cap conP(t,x) \cap V \neq \emptyset\} \in \mathscr{T} \otimes \mathscr{B}_w(L_X)$ for every open subset V of $\mathbb{R}^\ell$.

Corollary 4.3. (Yannelis (1987)) *Under the assumptions* (B.1)-(B.4)*, there exists an equilibrium for the abstract economy* Γ.

Proof: Define $\psi : T \times L_X \rightrightarrows \mathbb{R}^\ell$ by letting $\psi(t,x) = A(t,x) \cap \text{con}P(t,x)$ for all $(t,x) \in T \times L_X$. Since $A(t,\cdot)$ is continuous and $P(t,\cdot)$ has open graph in $L_X \times \mathbb{R}^\ell$, it follows from Lemma 8.7 and Lemma 8.8 that $\psi(t,\cdot) : L_X \rightrightarrows \mathbb{R}^\ell$ is lower-semicontinuous for all $t \in T$. Clearly, ψ is jointly lower measurable. Since any jointly lower measurable correpondence ψ with $\psi(t,\cdot) : L_X \rightrightarrows \mathbb{R}^\ell$ lower-semicontinuous, always satisfies the strong continuous inclusion property, the assumptions (A.1)-(A.4) are satisfied. Hence, this corollary follows from Theorem 4.1.

A **game** G is a quadruple $\langle (T,\mathscr{T},\mu), X, P\rangle$, where

(i) $(T,\mathscr{T},\mu)$ is a **measure space** of agents;

(ii) $X : T \rightrightarrows \mathbb{R}^\ell$ is a **strategy** correspondence; and

(iii) $P : T \times L_X \rightrightarrows \mathbb{R}^\ell$ is a **preference** correspondence such that $P(t,x) \subseteq X(t)$ for all $(t,x) \in T \times L_X$.

We now define the concept of a Nash equilibrium in our game G.

Definition 4.4. A **Nash equilibrium** of G is an element $x^* \in L_X$ such that for μ-a.e. on T, we have $x^*(t) \in X(t)$ and $P(t,x^*) = \emptyset$.

Note that if $A(t,x) = X(t)$ for all $x \in L_X$, in our abstract economy Γ, that is, $A(t,\cdot)$ is a constant correspondence, we can assume that P has the continuous inclusion property, and the existence of Nash equilibrium follows as a corollary. To this end, we state the relevant assumptions.

(C.1) $X : T \rightrightarrows \mathbb{R}^\ell$ is a correspondence such that:

(a) it is integrably bounded and for all $t \in T$, $X(t)$ is a non-empty, convex, closed subset of $\mathbb{R}^\ell$;

(b) for every open subset V of $\mathbb{R}^\ell$, $\{t \in T : X(t) \cap V \neq \emptyset\} \in \mathscr{T}$.

(C.2) $P : T \times L_X \rightrightarrows \mathbb{R}^\ell$ has the property that $x(t) \notin \text{con}P(t, x)$ for all $x \in L_X$ and almost all $t \in T$.

(C.3) $P : T \times L_X \rightrightarrows \mathbb{R}^\ell$ has the continuous inclusion property if the economy is purely atomic and the strong continuous inclusion property, otherwise.

This corollary extends the finite player theorems on Nash equilibrium with discontinous games to infinitely many agents without ordered preferences.

Corollary 4.5. *Under the assumptions* (C.1)-(C.3)*, there exists a Nash equilibrium for the game G.*

Proof: Define the constrained correspondence $A : T \times L_X \rightrightarrows \mathbb{R}^\ell$ by letting $A(t, x) = X(t)$ for $(t, x) \in T \times L_X$. Note that the correspondece A satisfies the assumption $(A.2)$. Moreover, the correspondence $\psi : T \times L_X \rightrightarrows \mathbb{R}^\ell$, defined by $\psi(t, x) = A(t, x) \cap \text{con}P(t, x)$ for all $(t, x) \in T \times L_X$, satisfies the condition (C.3) because $\psi(t, x) = \text{con}P(t, x)$ for all $(t, x) \in T \times L_X$ and the condition (C.3) is satsified by $\text{con}P$ as it is satisfied by P. Thus, the assumptions (A.1)-(A.4) are verified and hence, this corollary follows from Theorem 4.1.

5 Convexifying effect

In order to obtain a convexifying effect, we will define the preference and constraint correspondences to depend on the average (integral) strategies of all all other agents. As in Section 4, we again assume that $(T, \mathscr{T}, \mu)$ is a complete finite positive separable measure space of agents.

An **abstract economy with a convexifying effect** $\widetilde{\Gamma}$ is a quadruple $\langle (T, \mathscr{T}, \mu), X, P, A \rangle$, where

(i) $(T, \mathscr{T}, \mu)$ is a **measure space** of agents;

(ii) $X : T \rightrightarrows \mathbb{R}^\ell$ is a **strategy** correspondence;

(iii) $\widetilde{P} : T \times \int_T X \, d\mu \rightrightarrows \mathbb{R}^\ell$ is a **preference** correspondence such that $\widetilde{P}(t, \widetilde{x}) \subseteq X(t)$ for all $(t, \widetilde{x}) \in T \times \int X \, d\mu$; and

(iv) $\widetilde{A} : T \times \int_T X \, d\mu \rightrightarrows \mathbb{R}^\ell$ is a **constraint** correspondence such that $\widetilde{A}(t, \widetilde{x}) \subseteq X(t)$ for all $(t, \widetilde{x}) \in T \times \int_T X \, d\mu$.

We now define the concept of an equilibrium in an abstract economy with a convexifying effect.

Definition 5.1. An **equilibrium** for an abstract economy with a convexifying effect $\widetilde{\Gamma}$ is an element $\widetilde{x}^* \in \int_T X \, d\mu$, i.e., there exists $x \in L_X$ with $\int_T x \, d\mu = \widetilde{x}^*$, such that for μ-a.e. on T, we have $x(t) \in \widetilde{A}(t, \widetilde{x}^*)$ and $\widetilde{P}(t, \widetilde{x}^*) \cap \widetilde{A}(t, \widetilde{x}^*) = \emptyset$.

Below we modify the notion of "(strong) continuous inclusion property", which is analogous to the one stated in Section 3 and is compatible with a convexifying effect. However, the lower measurability is replaced with the graph measurability.

Definition 5.2. A correspondence $G : T \times \int_T X \, d\mu \rightrightarrows \mathbb{R}^\ell$ is said to have the **continuous inclusion property** if for each $\widetilde{y} \in \int_T X \, d\mu$, there exists a correspondence $F_{\widetilde{y}} : T \times \int_T X \, d\mu \rightrightarrows \mathbb{R}^\ell$ such that

(i) If $U_G^{\widetilde{y}} \neq \emptyset$ then there exists a collection $\{O_{\widetilde{y}}^t : t \in U_G^{\widetilde{y}}\}$ of open neighbourhoods of $\widetilde{y}$ in $\int_T X \, d\mu$ such that $F_{\widetilde{y}}(t, \widetilde{x})$ is a non-empty, closed set with $F_{\widetilde{y}}(t, \widetilde{x}) \subseteq G(t, \widetilde{x})$ for all $\widetilde{x} \in O_{\widetilde{y}}^t$ and all $t \in U_G^{\widetilde{y}}$;

(ii) $F_{\widetilde{y}}(t, \cdot) : O_{\widetilde{y}}^t \rightrightarrows \mathbb{R}^\ell$ is lower-semicontinuous for all $t \in U_G^{\widetilde{y}}$ and $F_{\widetilde{y}}(t, \cdot) : \int_T X \, d\mu \rightrightarrows \mathbb{R}^\ell$ is lower-semicontinuous for all $t \notin U_G^{\widetilde{y}}$;

(iii) $F_{\widetilde{y}} : T \times \int_T X \, d\mu \rightrightarrows \mathbb{R}^\ell$ is jointly lower measurable.

Definition 5.3. A correspondence $G : T \times \int_T X \, d\mu \rightrightarrows \mathbb{R}^\ell$ is said to have the **strong continuous inclusion property** if it satisfies the continuous inclusion property and the following measurability condition.

Measurability Condition: If $F_{\widetilde{y}} \neq F_{\widetilde{z}}$ for some $\widetilde{y}, \widetilde{z} \in \int_T X \, d\mu$, then any one of the following is true:

(a) The collection $\{F_{\widetilde{y}} : \widetilde{y} \in \int_T X \, d\mu\}$ and $\{O_{\widetilde{y}}^t : (t, \widetilde{y}) \in U_G\}$ are finite,[10] and the set $\{(t, \widetilde{x}) : \widetilde{x} \in O_{\widetilde{y}}^t\}$ is $\mathscr{T} \otimes \mathscr{B}(\int_T X \, d\mu)$-measurable for each $\widetilde{y} \in \int_T X \, d\mu$.

[10]Countable collection condition in Definition 3.3 is replace with finite collection in this definition to guarantee that the function Φ constructed in the proof of Theorem 5.4 is closed valued.

(b) $U_G \in \mathscr{T} \otimes \mathscr{B}(\int_T X\, d\mu)$. For each $\widetilde{y} \in \int_T X\, d\mu$, the correpondence $\mathbb{I} : T \times \int_T X\, d\mu \rightrightarrows \int_T X\, d\mu$, defined by $\mathbb{I}(t, \widetilde{x}) = \{\widetilde{y} \in U_G^t : \widetilde{x} \in O_{\widetilde{y}}^t\}$, is jointly lower measurable and is finite valued. Furthermore, for each fixed $(t, \widetilde{x}) \in T \times \int_T X\, d\mu$, the correpondence $H : \int_T X\, d\mu \rightrightarrows \mathbb{R}^\ell$, defined by $H(\widetilde{y}) = \mathrm{con}F_{\widetilde{y}}(t, \widetilde{x})$, is continuous and is contained in $X(t)$ for all $\widetilde{y} \in \int_T X\, d\mu$.

To define a set of assumptions for our next result, we let $\widetilde{\psi}(t, \widetilde{x}) := \widetilde{A}(t, \widetilde{x}) \cap \widetilde{P}(t, \widetilde{x})$ for all $(t, \widetilde{x}) \in T \times \int_T X\, d\mu$.

Assumptions: As preference and constraint correpondences depend on the average of strategies of all agents rather than the set L_X, i.e., the set of joint strategies, we now state a set of assumptions (analogous to those in Section 3) that will be used in the next result. These assumptions are different from those in Section 3 in the sense that the convexity assumptions of the strategy and constraint correpondences are dropped. Further, (A.3′) below is weaker than (A.3).

(A.1′) $X : T \rightrightarrows \mathbb{R}^\ell$ is a correspondence such that:

- (a) it is integrably bounded and for all $t \in T$, $X(t)$ is a non-empty, closed subset of $\mathbb{R}^\ell$ containing 0;
- (b) for every open subset V of $\mathbb{R}^\ell$, $\{t \in T : X(t) \cap V \neq \emptyset\} \in \mathscr{T}$.

(A.2′) $\widetilde{A} : T \times \int_T X\, d\mu \rightrightarrows \mathbb{R}^\ell$ is a correspondence such that:

- (a) for each $t \in T$, $\widetilde{A}(t, \cdot) : \int_T X\, d\mu \rightrightarrows \mathbb{R}^\ell$ is upper-semicontinuous;
- (b) for all $(t, \widetilde{x}) \in T \times \int_T X\, d\mu$, $\widetilde{A}(t, \widetilde{x})$ is a non-empty, closed subset of $\mathbb{R}^\ell$;
- (c) for each fixed $x \in \int_T X\, d\mu$, $\widetilde{A}(\cdot, \widetilde{x})$ is lower measurable.

(A.3′) $\widetilde{P} : T \times \int_T X\, d\mu \rightrightarrows \mathbb{R}^\ell$ has the property that for almost all $t \in T$, $x(t) \notin \widetilde{P}(t, \widetilde{x})$ for all $\widetilde{x} \in \int_T X\, d\mu$ and all $x(t) \in X(t)$ with $\int_T x\, d\mu = \widetilde{x}$.

(A.4′) $\widetilde{\psi} : T \times \int_T X\, d\mu \rightrightarrows \mathbb{R}^\ell$ satisfies the strong continuous inclusion property.

Theorem 5.4. *Assume that the measure space $(T, \mathscr{T}, \mu)$ is atomless and the assumptions* (A.1′)-(A.4′) *are satisfied. Then, there exists an equilibrium for the abstract economy with a convexifying effect* $\widetilde{\Gamma}$.

Proof. Let

$$\widetilde{\mathbb{J}}_{\widetilde{\psi}} = \left\{ \widetilde{x} \in \int_T X \, d\mu : U_{\widetilde{\psi}}^{\widetilde{x}} \text{ is not null} \right\}.$$

The proof is splitted into two cases.

Case 1. $\widetilde{\mathbb{J}}_{\widetilde{\psi}} = \emptyset$. In this case, $\mu(U_{\widetilde{\psi}}^{\widetilde{x}}) = 0$. This implies that $U_{\widetilde{\psi}}^{\widetilde{x}}$ is measurable and $\widetilde{P}(t, \widetilde{x}) \cap \widetilde{A}(t, \widetilde{x}) = \emptyset$ for all $\widetilde{x} \in \int_T X \, d\mu$ and almost all $t \in T$. Since $\widetilde{A}$ is closed-valued and $\widetilde{A}(\cdot, \widetilde{x})$ is lower measurable, we conclude that $\widetilde{A}(\cdot, \widetilde{x})$ has a measurable graph for all $\widetilde{x} \in \int_T X \, d\mu$. Define $\Psi : \int_T X \, d\mu \rightrightarrows \int_T X \, d\mu$ by

$$\Psi(\widetilde{x}) = \int_T \widetilde{A}(\cdot, \widetilde{x}) \, d\mu.$$

Given that $\widetilde{A}(t, \widetilde{x}) \subseteq X(t)$ for all $(t, \widetilde{x}) \in T \times \int_T X \, d\mu$, we conclude that $\widetilde{A}(\cdot, \widetilde{x})$ is integrably bounded for all $\widetilde{x} \in \int_T X \, d\mu$. By the Aumann measurable selection theorem, we have $\int_T \widetilde{A}(\cdot, \widetilde{x}) \, d\mu$ is non-empty for all $\widetilde{x} \in \int_T X \, d\mu$ and hence, Ψ is non-empty valued. Applying Lemma 8.2 and Lemma 8.3, we have Ψ is compact and convex valued. By (A.1′)(a) and (A.2′)(b), we have that $\widetilde{A}$ is compact valued. Thus, by applying Lemma 8.9, we conclude that Ψ is upper-semicontinuous. A similar argument also guarantees that $\int_T X \, d\mu$ is non-empty, convex, and compact. Hence, by Katatuni's fixed point theorem, there exists some $\widetilde{x}^* \in \int_T X d\mu$ such that $\widetilde{x}^* \in \Psi(\widetilde{x}^*)$. Thus, there exists some $x \in L_X$ such that $\int_T x \, d\mu = \widetilde{x}^*$ and $x(t) \in A(t, \widetilde{x}^*)$ μ-a.e. Since $\widetilde{A}(t, \widetilde{x}^*) \cap \widetilde{P}(t, \widetilde{x}^*) = \emptyset$ μ-a.e., it follows that $\widetilde{x}^*$ is an equilibrium for the abstract economy with convexifying effect $\widetilde{\Gamma}$.

Case 2. $\widetilde{\mathbb{J}}_{\widetilde{\psi}} \neq \emptyset$. Similar to Theorem 4.1, we can find a correspondence $\Phi : T \times \int_T X \, d\mu \rightrightarrows \mathbb{R}^\ell$ satisfying the following:

(i) Φ is closed valued and $\Phi(t, \widetilde{x}) \subseteq \psi(t, \widetilde{x})$ *for all* $(t, \widetilde{x}) \in U_{\widetilde{\psi}}$;

(ii) $U_{\widetilde{\psi}} = U_\Phi$;

(iii) $\Phi(t, \cdot) : \int_T X \, d\mu \rightrightarrows \mathbb{R}^\ell$ *is lower-semicontinuous for all* $t \in T$; *and*

(iv) Φ is jointly lower measurable.

Let

$$\mathbb{K} = \left\{ \widetilde{x} \in \int_T X \, d\mu : \mu(U_\Phi^{\widetilde{x}}) > 0 \right\}.$$

By (ii), we have $\mathbb{K} = \widetilde{\mathbb{J}}_{\widetilde{\psi}}$.

Claim 1: $\mathbb{K}$ *is an open subset of* $\int_T X d\mu$. Suppose, by way of contradiction, that $\mathbb{K}$ is not an open subset of $\int_T X d\mu$. Then $\mathbb{K}$ is non-empty and there must exist a point $\widetilde{x}$ of $\mathbb{K}$ which is not an interior point of $\mathbb{K}$ in $\int_T X d\mu$. Then there is a sequence $\{\widetilde{x_n} : n \geq 1\}$ of points in $\int_T X d\mu$ converging to $\widetilde{x}$ in the Euclidean norm and $\widetilde{x_n} \notin \mathbb{K}$ for all $n \geq 1$. Define

$$B_n = \left\{t \in U_\Phi^{\widetilde{x}} : \Phi(t, \widetilde{x_n}) \neq \emptyset\right\}$$

for all $n \geq 1$. Since $\widetilde{x_n} \notin \mathbb{K}$, we have

$$\mu\left(\{t \in T : \Phi(t, \widetilde{x_n}) \neq \emptyset\}\right) = 0.$$

This implies that $\mu(B_n) = 0$ for all $n \geq 1$. Letting $B = \bigcup\{B_n : n \geq 1\}$, we see that $\mu(B) = 0$. Let $t_0 \in U_\Phi^{\widetilde{x}} \setminus B$. It follows that $\Phi(t_0, \widetilde{x}) \neq \emptyset$ and $\Phi(t_0, \widetilde{x_n}) = \emptyset$ for all $n \geq 1$. Since

$$U_\Phi^{t_0} = \{\widetilde{z} \in U_\Phi^{t_0} : \Phi(t_0, \widetilde{z}) \cap \mathbb{R}^\ell \neq \emptyset\},$$

by the lower-semicontinuity of $\Phi(t_0, \cdot)$, we conclude that $U_\Phi^{t_0}$ is open in $\int_T X \, d\mu$. Given that $\widetilde{x} \in U_\Phi^{t_0}$, we infer that $\widetilde{x}_n \in U_\Phi^{t_0}$ for all large n. This contradicts with the fact that $\Phi(t_0, \widetilde{x_n}) = \emptyset$ for all $n \geq 1$, whch verifes our claim.

Define $G : \mathbb{K} \rightrightarrows \mathbb{R}^\ell$ by letting

$$G(\widetilde{x}) = \int_T \widehat{\Phi}(\cdot, \widetilde{x}) \, d\mu,$$

where the correspondence $\widehat{\Phi} : T \times \mathbb{K} \rightrightarrows \mathbb{R}^\ell$ is defined by

$$\widehat{\Phi}(t, \widetilde{x}) = \begin{cases} \Phi(t, \widetilde{x}), & \text{if } \widetilde{x} \in \mathbb{K} \text{ and } t \in U_\Phi^{\widetilde{x}}; \\ \{0\}, & \text{otherwise.} \end{cases}$$

Since X is integrably bounded, so is $\widehat{\Phi}(\cdot, \widetilde{x})$ for any $\widetilde{x} \in \mathbb{K}$. Since Φ is closed valued, $\widehat{\Phi}$ is also closed valued. By the lower measurability of $\Phi(\cdot, \widetilde{x})$, one can verify that $\widehat{\Phi}(\cdot, \widetilde{x})$ is lower measurable and thus, it has a measurable graph for all $\widetilde{x} \in \mathbb{K}$. Hence, by the Aumann measurable selection theorem, we conclude that G is non-empty valued. Since $\widehat{\Phi}$ is closed valued, so is G. By virtue of Lemma 8.3, G is convex valued. Since $\Phi(t, \cdot)$ is lower-semicontinuous for all $t \in T$, by Lemma 8.10, we conclude that G is lower-semicontinuous. Thus, in view of the Michael continuous selection theorem, there is a continuous selection f of G. Let $\Psi : \int_T X d\mu \rightrightarrows \int_T X \, d\mu$ be a correspondence such that

$$\Psi(\widetilde{x}) = \begin{cases} \{f(\widetilde{x})\}, & \text{if } \widetilde{x} \in \mathbb{K}; \\ \int_T \widetilde{A}(\cdot, \widetilde{x}) \, d\mu, & \text{otherwise.} \end{cases}$$

As in Case 1, we can show that Ψ is non-empty, compact and convex valued, and $\int_T \widetilde{A}(\cdot,\cdot)\,d\mu : \int_T X\,d\mu \rightrightarrows \int_T X\,d\mu$ is upper-semicontinuous. Since $\mathbb{K}$ is an open subset of $\int_T X d\mu$, Lemma 8.12 implies that Ψ is upper-semicontinuous. Hence, by Kakatuni's fixed point theorem, there exists some $\widetilde{x}^* \in \int_T X d\mu$ such that $\widetilde{x}^* \in \widetilde{\Psi}(\widetilde{x}^*)$. If $\widetilde{x}^* \in \mathbb{K}$ then[11]

$$\widetilde{x}^* = f(\widetilde{x}^*) \in G(\widetilde{x^*}) = \int_{U_\Phi^{\widetilde{x}^*}} \Phi(\cdot,\widetilde{x}^*)\,d\mu \subseteq \int_{U_\Phi^{\widetilde{x}^*}} \widetilde{\psi}(\cdot,\widetilde{x}^*)\,d\mu \subseteq \int_{U_\Phi^{\widetilde{x}^*}} \widetilde{P}(\cdot,\widetilde{x}^*)\,d\mu.$$

Therefore, there exists a function $y : U_\Phi^{\widetilde{x}^*} \to \mathbb{R}^\ell$ such that $y(t) \in \widetilde{P}(t,\widetilde{x}^*)$ for all $t \in U_\Phi^{\widetilde{x}^*}$ and $\int_{U_\Phi^{\widetilde{x}^*}} y\,d\mu = \widetilde{x}^*$. Define $z : T \to \mathbb{R}^\ell$ such that $z(t) = y(t)$, if $t \in U_\Phi^{\widetilde{x}^*}$; and $z(t) = 0$, otherwise. It follows that $\int_T z\,d\mu = \widetilde{x}^*$ and $z(t) \in \widetilde{P}(t,\widetilde{x}^*)$ for all $t \in U_\Phi^{\widetilde{x}^*}$. This contradicts (A.3′). Therefore, $\widetilde{x}^* \notin \mathbb{K}$, which means $\widetilde{\psi}(t,\widetilde{x}^*) = \emptyset$ μ-a.e.[12] Furthermore, there exists some $x \in L_X$ such that $\int_T x\,d\mu = \widetilde{x}^*$ and $x(t) \in \widetilde{A}(t,\widetilde{x}^*)$ μ-a.e. Thus, $\widetilde{x}^*$ is an equilibrium for $\widetilde{\Gamma}$. This completes the proof. □

Remark 5.5. Analogous to Corollary 4.5, we can also establish the existence of Nash equilibrium in a game with a convexifying effect.

6 Existence of Walrasian equilibria

In this section, we consider an exchange economy and estabilsh the existence of a Walrasian equilibrium for a measure space of agents as an application of our equilibrium existence theorem of an abstract economy.

6.1 The existence of Walrasian equilibria with free disposal

We consider a standard exchange economy with a complete finite positive separable measure space $(T, \mathscr{T}, \mu)$ of agents, where T represents the set of agents, the σ-algebra $\mathscr{T}$ represents the collections of allowable coalitions whose economic weights on the market are given by μ. The commodity space is the ℓ-dimensional Euclidean space $\mathbb{R}^\ell$.

We define an **economy** $\mathscr{E}$ such that $\mathscr{E} = \{(X(t), P(t,\cdot,\cdot), e(t)) : t \in T\}$ as follows:

(i) $X(t) \subseteq \mathbb{R}^\ell_+$ denotes the **consumption set** of agent $t \in T$;

[11] Recall that $\Phi(t,\widetilde{x^*}) \subseteq \widetilde{\psi}(t,\widetilde{x^*})$ for all $t \in U_\Phi^{\widetilde{x}^*} = U_\psi^{\widetilde{x}^*}$.

[12] Recall that $U_\psi^{\widetilde{x}^*} = U_\Phi^{\widetilde{x}^*}$.

(ii) $P(t,\cdot,\cdot): L_X \times \Delta \rightrightarrows X(t)$ is the **prefernce correspondence** of agent t, where $\Delta = \left\{p \in \mathbb{R}^\ell_+ : \sum_{k=1}^\ell p^k = 1\right\}$ is the set of all possible prices; and

(iii) $e(t) \in X(t)$ is the **initial endowment** of agent t.

The **budget correspondece** $\mathbb{B} : T \times \Delta \rightrightarrows \mathbb{R}^\ell_+$ is defined by

$$\mathbb{B}(t,p) = \{y \in X(t) : p \cdot y \leq p \cdot e(t)\}.$$

Definition 6.1. A **Walrasian equilibrium** for the exchange economy $\mathscr{E}$ is a pair $(p^*, x^*) \in \Delta \times L_X$ such that

(i) for almost all $t \in T$, we have $x^*(t) \in \mathbb{B}(t,p^*)$, $\mathbb{B}(t,p^*) \cap P(t,x^*,p^*) = \emptyset$; and

(ii) $\int_T x^* d\mu \leq \int_T e\, d\mu$.

The following result is an extension of Theorem 2 in He and Yannelis (2016) to a measure space of agents.

Theorem 6.2. *Suppose that the economy $\mathscr{E}$ satisfies the following:*

(i) *The consumption correspondence $X : T \rightrightarrows \mathbb{R}^\ell$ is integrably bounded and for all $t \in T$, $X(t)$ is a non-empty, convex, closed subset of $\mathbb{R}^\ell$;*

(ii) *$P : T \times L_X \times \Delta \rightrightarrows \mathbb{R}^\ell$ has the property that $x(t) \notin \operatorname{con}P(t,x,p)$ for all $(x,p) \in L_X \times \Delta$ and almost all $t \in T$; and*

(iii) *$\psi : T \times L_X \times \Delta \rightrightarrows \mathbb{R}^\ell$, defined by $\psi(t,x,p) = \mathbb{B}(t,p) \cap \operatorname{con}P(t,x,p)$ for all $(t,x,p) \in T \times L_X \times \Delta$, has the continuous inclusion property if the economy is atomic and the strong continuous inclusion property, otherwise.*

Then $\mathscr{E}$ has a Walrasian equilibrium.

Proof. The idea of the proof is to transform the exchange economy $\mathscr{E}$ to an abstract economy Γ by adding a 'fictitious agent'. For each agent $t \in T$ of the economy $\mathscr{E}$, $p \in \Delta$ and $x \in L_X$, let $A(t,x,p) = \mathbb{B}(t,p)$. In our context, we have to ensure that the added agent is an atom 'τ'. Let $X(\tau) = A(\tau,x,p) = \Delta$ and

$$P(\tau,x,p) = \left\{q \in \Delta : q \cdot \int_T (x(t) - e(t))d\mu > p \cdot \int_T (x(t) - e(t))d\mu\right\}.$$

Define $\widetilde{T} = T \cup \tau$. Note that an abstract economy Γ with $(\widetilde{T}, \widetilde{\mathscr{T}}, \widetilde{\mu})$ as the measure space of agents can be derived from our construction, where $\widetilde{\mathscr{T}} := \mathscr{T} \otimes \{\tau\}$ is the product σ-algebra and the measure $\widetilde{\mu}$ is an extension of μ to $\widetilde{\mathscr{T}}$. Then for any $(t,x,p) \in \widetilde{T} \times L_X \times \Delta$,

$A(t,x,p)$ is a nonempty, convex, and closed set. Furthermore, $A(t,\cdot,\cdot) : L_X \times \Delta \rightrightarrows \mathbb{R}^\ell$ is upper-semicontinuous for all $t \in \widetilde{T}$ and $A(\cdot,x,p) : \widetilde{T} \rightrightarrows \mathbb{R}^\ell_+$ is lower measurable for all $(x,p) \in L_X \times \Delta$.[13] By assumption, $\psi : T \times L_X \times \Delta \rightrightarrows \mathbb{R}^\ell$ has the continuous inclusion property if the economy $\mathscr{E}$ is atomic and the strong continuous inclusion property, otherwise. Hence, for each $(y,q) \in L_X \times \Delta$, there exists a correspondence $F_{(y,q)} : T \times L_X \times \Delta \rightrightarrows \mathbb{R}^\ell$ such that

(i) $F_{(y,q)}(t,x,p) \neq \emptyset$ and $F_{(y,q)}(t,x,p) \subseteq \psi(t,x,p)$ for all $(x,p) \in O^t_{(y,q)}$ and all $t \in U^{(y,q)}_\psi$ for a collection $\{O^t_{(y,q)} : t \in U^{(y,q)}_\psi\}$ of weakly open neighbourhoods of (y,q) in $L_X \times \Delta$ if $U^{(y,q)}_\psi \neq \emptyset$;

(ii) $F_{(y,q)}(t,\cdot) : O^t_{(y,q)} \rightrightarrows \mathbb{R}^\ell$ is lower-semicontinuous for all $t \in U^{(y,q)}_\psi$ and $F_{(y,q)}(t,\cdot) : L_X \times \Delta \rightrightarrows \mathbb{R}^\ell$ is lower-semicontinuous for all $t \notin U^{(y,q)}_\psi$; and

(iii) $F_{(y,q)} : T \times L_X \times \Delta \rightrightarrows \mathbb{R}^\ell$ is jointly lower measurable.

Moreover, if the economy is non-atomic then the following must hold: if $F_{(y,q)} \neq F_{(z,r)}$ for some (y,q), $(z,r) \in L_X \times \Delta$, then any one of the following is true:

(a) The collection $\{F_{(y,q)} : (y,q) \in L_X \times \Delta\}$ and $\{O^t_{(y,q)} : (t,y,q) \in U_\psi\}$ are countable, and the set $\{(t,x,p) : (x,p) \in O^t_{(y,q)}\}$ is $\mathscr{T} \otimes \mathscr{B}_w(L_X \times \Delta)$-measurable.

(b) $U_\psi \in \mathscr{T} \otimes \mathscr{B}_w(L_X \times \Delta)$. The correpondence $\mathbb{I} : T \times L_X \times \Delta \rightrightarrows L_X \times \Delta$, defined by $\mathbb{I}(t,x,p) = \{(y,q) \in U^t_\psi : (x,p) \in O^t_{(y,q)}\}$, is jointly lower measurable and is finite valued. Furthermore, for each fixed $(t,x,p) \in T \times L_X \times \Delta$, the correpondence $H : L_X \times \Delta \rightrightarrows \mathbb{R}^\ell$, defined by $H(y,q) = F_{(y,q)}(t,x,p)$, is continuous, and is contained in $X(t)$ for all $(y,q) \in L_X \times \Delta$.

Furthermore, let $\psi(\tau,x,p) = A(\tau,x,p) \cap P(\tau,x,p)$.[14] It can be easily checked that $\psi(\tau,x,p) = P(\tau,x,p)$ and thus $\psi(\tau,\cdot,\cdot)$ has open graph. Define $O^\tau_{(y,q)} = U^\tau_\psi$ for all $(y,q) \in L_X \times \Delta$ and define $\widetilde{F}_{(y,q)} : \widetilde{T} \times L_X \times \Delta \rightrightarrows \mathbb{R}^\ell$ such that

$$\widetilde{F}_y(t,x,p) = \begin{cases} F_y(t,x,p), & \text{if } (t,x,p) \in T \times L_X \times \Delta; \\ \psi(t,x,p), & \text{otherwise.} \end{cases}$$

It can be readily verified that ψ satisfies (A.4). Further, it can be easily observed that $p \notin P(\tau,x,p)$ for any $(x,p) \in L_X \times \Delta$. Thus, the original economy $\mathscr{E}$ has

[13] The upper hemi-continuity of $\mathbb{B}(t,\cdot)$ follows from the fact the graph of $\mathbb{B}(t,\cdot)$ is closed, and $\mathbb{B}(t,\cdot)$ is compact-valued.

[14] Note that $P(\tau,x,p)$ is a convex set.

been converted to an abstract economy $\Gamma = \langle (\widetilde{T}, \widetilde{\mathscr{T}}, \widetilde{\mu}), X, P, A \rangle$ which satisfies all the conditions of Theorem 4.1. Therefore, by virtue of Theorem 4.1, there exists a point $(p^*, x^*) \in \Delta \times L_X$ such that

(i) $x^*(t) \in A(t, x^*, p^*) = \mathbb{B}(t, p^*)$, $\psi(t, x^*, p^*) = \emptyset$ μ-a.e.; and

(ii) $P(\tau, x^*, p^*) = \psi(\tau, x^*, p^*) = \emptyset$.

Let

$$z = \int_T (x^*(t) - e(t)) d\mu.$$

Then it follows from (i) that $p^* \cdot z \leq 0$, and one can further observe from (ii) that $q \cdot z \leq p^* \cdot z$ for any $q \in \Delta$. Combining these inequalities, we conclude that $q \cdot z \leq 0$ for all $q \in \Delta$. Putting $q = e^k$ the k^{th}-unit vector, we conclude that $z^k \leq 0$. Consequently, $z \in \mathbb{R}^\ell_-$, which implies that

$$\int_T x^* d\mu \leq \int_T e \, d\mu.$$

Therefore, (p^*, x^*) is a Walrasian equilibrium. □

6.2 The existence of Walrasian equilibria without free disposal

In this subsection, we consider the economy $\mathscr{E}$ as it was defined in 6.1.

Definition 6.3. A **non-free disposal Walrasian equilibrium** for the exchange economy $\mathscr{E}$ is a pair $(p^*, x^*) \in (\mathbb{R}^\ell \setminus \{0\}) \times L_X$ such that

(i) for almost all $t \in T$, we have $x^*(t) \in \mathbb{B}(t, p^*)$, $\mathbb{B}(t, p^*) \cap P(t, x^*, p^*) = \emptyset$; and

(ii) $\int_T x^* d\mu = \int_T e \, d\mu$.

Define the set of feasible allocations and price space as follows

$$\mathcal{A} = \left\{ f \in L_X : \int_T f \, d\mu = \int_T e \, d\mu \right\}$$

and

$$\widetilde{\Delta} = \left\{ p \in \mathbb{R}^\ell : \|p\|_1 \leq 1 \right\}.$$[15]

[15] $\|p\|_1 = \sum_{i=1}^{\ell} p_i$ for $p = (p_1, \cdots, p_\ell) \in \mathbb{R}^\ell$.

The following result is an extension of Theorem 3 in He and Yannelis (2016) to a measure space of agents.

Theorem 6.4. *Suppose that the economy $\mathscr{E}$ satisfies following assumptions:*

(i) *The consumption correspondence $X : T \rightrightarrows \mathbb{R}^\ell$ is integrably bounded and for all $t \in T$, $X(t)$ is a non-empty, convex, closed subset of $\mathbb{R}^\ell$;*

(ii) *$P : T \times L_X \times \widetilde{\Delta} \rightrightarrows \mathbb{R}^\ell$ has the property that $x(t) \notin \mathrm{con}P(t, x, p)$ for all $(x, p) \in L_X \times (\mathbb{R}^\ell \setminus \{0\})$ and almost all $t \in T$;*

(iii) *$\psi : T \times L_X \times \widetilde{\Delta} \rightrightarrows \mathbb{R}^\ell$, defined by $\psi(t, x, p) = \mathbb{B}(t, p) \cap \mathrm{con}P(t, x, p)$ for all $(t, x, p) \in T \times L_X \times \widetilde{\Delta}$, has the continuous inclusion property if the economy is atomic and the strong continuous inclusion property, otherwise.*

(iv) *If $x \in \mathcal{A}$ and $p \in \widetilde{\Delta}$ then $x(t) \in \mathrm{bd}P(t, x^*, p^*)$ μ-a.e.*

Then $\mathscr{E}$ has a non-free disposal Walrasian equilibrium.

Proof. Define $\widetilde{\mathbb{B}} : T \times \widetilde{\Delta} \rightrightarrows \mathbb{R}^\ell_+$ by letting

$$\widetilde{\mathbb{B}}(t, p) = \{y \in X(t) : p \cdot y \le p \cdot e(t) + 1 - \|p\|_1\}.$$

As in the proof of Theorem 6.2, we transform the exchange economy $\mathscr{E}$ to an abstract economy Γ by adding a 'fictitious agent'. For each agent $t \in T$ of the economy $\mathscr{E}$, $p \in \widetilde{\Delta}$ and $x \in L_X$, let $A(t, x, p) = \widetilde{\mathbb{B}}(t, p)$. In our context, we have to ensure that the added agent is an atom 'τ'. Let $X(\tau) = A(\tau, x, p) = \widetilde{\Delta}$ and

$$P(\tau, x, p) = \left\{ q \in \widetilde{\Delta} : q \cdot \int_T (x(t) - e(t)) d\mu > p \cdot \int_T (x(t) - e(t)) d\mu \right\}.$$

Define $\widetilde{T} = T \cup \tau$. Repeating the arguments of the proof of Theorem 6.2, one could show that there exists a point $(p^*, x^*) \in \widetilde{\Delta} \times L_X$ such that

(A) $x^*(t) \in A(t, x^*, p^*) = \widetilde{\mathbb{B}}(t, p^*)$, $\psi(t, x^*, p^*) = \emptyset$ μ-a.e.; and

(B) $P(\tau, x^*, p^*) = \psi(\tau, x^*, p^*) = \emptyset$.

From (A), it follows that $p^* \cdot x^*(t) \le p^* \cdot e(t) + 1 - \|p^*\|_1$ μ-a.e. Define

$$z = \int_T (x^*(t) - e(t)) d\mu.$$

We must show that $z = 0$. Suppose that $z \neq 0$. From (B), it follows that $q \cdot z \leq p^* \cdot z$ for any $q \in \widetilde{\Delta}$. Letting $q = \frac{z}{\|z\|_1}$, we note that $q \in \widetilde{\Delta}$ and $p^* \cdot z \geq q \cdot z > 0$. Define $q^* = \frac{p^*}{\|p^*\|_1}$. Since $p^* \cdot z \geq q^* \cdot z$, it follows that $\|p^*\|_1 = 1$. As a result, we have $p^* \cdot x(t) \leq p^* \cdot e(t)$ μ-a.e., which implies that

$$p^* \cdot z = p^* \cdot \int_T (x - e) d\mu \leq 0.$$

This is a contradiction. Consequently, we have $z = 0$, which means

$$\int_T x^* d\mu = \int_T e\, d\mu.$$

Hence, $x^* \in \mathcal{A}$. By (iv), it follows that $x^*(t) \in \mathrm{bd}P(t, x^*, p^*)$ μ-a.e. Since $x^*(t) \notin \mathrm{con}P(t, x^*, p^*)$ μ-a.e., we conclude that $x^*(t) \notin P(t, x^*, p^*)$ μ-a.e. We claim that

$$p^* \cdot x^*(t) = p^* \cdot e(t) + 1 - \|p^*\|_1$$

μ-a.e. Suppose that the claim is false, which means that

$$p^* \cdot x^*(t) < p^* \cdot e(t) + 1 - \|p^*\|_1$$

for all $t \in S$ for some coalition S. Without loss of generality, we assume that $x^*(t) \in \mathrm{bd}P(t, x^*, p^*)$ for all $t \in S$. Therefore, for each $t \in S$, there exists some $y(t) \in P(t, x^*, p^*)$ such that

$$p^* \cdot y(t) < p^* \cdot e(t) + 1 - \|p^*\|_1.$$

Consequently, $y(t) \in \psi(t, x^*, p^*)$, which contradicts (A). Therefore,

$$p^* \cdot x^*(t) = p^* \cdot e(t) + 1 - \|p^*\|_1$$

μ-a.e. on T. Integrating over T yields $\|p^*\|_1 = 1$. Therefore, (p^*, x^*) is a non-free disposal Walrasian equilibrium. □

Remark 6.5. Our existence theorems (Theorem 6.2 and Theorem 6.4) of a competitive equilibrium for an economy with a measure space of agents exhibit preferences that may be interdependent, non ordered, price dependent, discontinuous, and are derived from Theorem 4.1. Thus, our existence theorems can be viewed as extensions of Aumann (1966), Schmeidler (1969) and Shitovitz (1973) that allow for non-ordered, interdependent, price dependent and discontinuous preferences. However, we don't generalize the above theorems as we assume that the consumption sets are bounded. It is an open question if the bound can be relaxed in this very general set up. Independently from our work and using the excess demand approach Otsuka (2024) has proved the existence of a free disposal equilibrium with a continuum of agents with price dependent preferences. He also has a bound on the consumption sets.

Remark 6.6. In view of Theorem 5.4, the existence of a competitive equilibrium similar to Theorem 6.2 and Theorem 6.4 can also be guaranteed for an economy with convexifying effect.

Remark 6.7. It would be of interest to see if the extension of the continuous continuous inclusion property introduced on this paper can be used to obtain generalizations of the existence results in Hou (2008), He (2022), Pastine-Pastine (2023), Ke-Xu (2023), Urbinati (2023), and Podczeck-Yannelis (2024).

7 Appendix A

In this section, we outline the proof of a Carathéodory-type selection theorem for discontinuous correspondences, recently established in Bhowmik and Yannelis (2024).

Theorem 7.1. *Assume that the abstract economy* Γ *satisfies the assumptions* (A.1)-(A.4)*. Then there exists a correspondence* $\Phi : T \times L_X \rightrightarrows \mathbb{R}^\ell$ *satisfying the following:*

(A) $\Phi(t,x) \subseteq \psi(t,x)$ *for all* $(t,x) \in U_\psi$*;*

(B) $U_\psi = U_\Phi$*;*

(C) $\Phi(t,\cdot) : L_X \rightrightarrows \mathbb{R}^\ell$ *is lower-semicontinuous for all* $t \in T$*;*

(D) Φ *is jointly lower measurable; and*

(E) *There exists a Carathéodory-type selection* $f : U_\Phi \to \mathbb{R}^\ell$ *of* $\Phi|_{U_\Phi}$*.*

Proof. For pedagogical purposes, we outline the proof of the purely atomic economy only. The proof regarding the atomless or mixed economy is significantly more complicated and the details can be found in Bhowmik-Yannelis (2024).

Since the correspondence ψ satisfies the continuous inclusion property, for each $y \in L_X$, there exists a correspondence $F_y : T \times L_X \rightrightarrows \mathbb{R}^\ell$ satisfying:

(i) If $U_\psi^y \neq \emptyset$ then there exists a collection $\{O_y^t : t \in U_\psi^y\}$ of weakly open neighbourhoods of y in L_X such that $F_y(t,x) \neq \emptyset$ and $F_y(t,x) \subseteq \psi(t,x)$ for all $x \in O_y^t$ and all $t \in U_\psi^y$;

(ii) $F_y(t,\cdot) : O_y^t \rightrightarrows \mathbb{R}^\ell$ is lower-semicontinuous for all $t \in U_\psi^y$ and $F_y(t,\cdot) : L_X \rightrightarrows \mathbb{R}^\ell$ is lower-semicontinuous for all $t \notin U_\psi^y$; and

(iii) $F_y : T \times L_X \rightrightarrows \mathbb{R}^\ell$ is jointly lower measurable.

The collection of open neigbourhoods defined above is therefore

$$\mathcal{O} = \left\{O_y^t : t \in U_\psi^y \text{ and } y \in L_X\right\}.$$

Then $\mathcal{O}$ can be rewritten as $\mathcal{O} = \{O_y^t : (t, y) \in U_\psi\}$. Since $O_y^t \subseteq U_\psi^t$ for all $y \in U_\psi^t$, we have that U_ψ^t is a weakly open subset of L_X, for all $t \in T$. By Lemma 8.5, we have that L_X is non-empty, convex, weakly compact, and metrizable. Consequently, U_ψ^t is paracompact for all $t \in T$. Since $\{O_y^t : y \in U_\psi^t\}$ is an open cover of U_ψ^t, it has a closed locally finite refinement, say $\mathscr{F}_t = \{V_k^t : k \in K_t\}$, where K_t is an index set and V_k^t is a closed set in L_X for each $k \in K_t$. For each $(t, x) \in T \times L_X$, we define

$$\mathbb{J}(t, x) = \left\{k \in K_t : x \in V_k^t\right\}.$$

Note that $\mathbb{J}(t, x)$ is (possibly empty) finite for all $(t, x) \in T \times L_X$. Moreover, $\mathbb{J}(t, x) \neq \emptyset$ if and only if $x \in U_\psi^t$. We choose an element $y_k^t \in L_X$ such that $V_k^t \subseteq O_{y_k^t}^t$ for all $k \in K_t$ and all $t \in T$. Let $\Phi : T \times L_X \rightrightarrows \mathbb{R}^\ell$ be a correspondence such that

$$\Phi(t, x) = \begin{cases} \Theta(t, x), & \text{if } \mathbb{J}(t, x) \neq \emptyset; \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $\Theta : T \times L_X \rightrightarrows \mathbb{R}^\ell$ is defined by

$$\Theta(t, x) = \text{con}\left(\bigcup\left\{\text{con} F_{y_k^t}(t, x) : k \in \mathbb{J}(t, x)\right\}\right).$$

To verify Condition (A), take an element $(t, x) \in U_\psi$. Since $x \in U_\psi^t$, we have $\mathbb{J}(t, x) \neq \emptyset$. Since $F_{y_k^t}(t, x) \subseteq \psi(t, x)$ for all $k \in \mathbb{J}(t, x)$ and $\psi(t, x)$ is convex, we must have $\Phi(t, x) \subseteq \psi(t, x)$. On the other hand, for each $(t, x) \in U_\psi$, as $\mathbb{J}(t, x) \neq \emptyset$, we have $x \in O_{y_k^t}^t$ for $k \in \mathbb{J}(t, x)$. Hence, $F_{y_k^t}(t, x) \neq \emptyset$ for all $k \in \mathbb{J}(t, x)$. This implies that $\Phi(t, x) \neq \emptyset$, which yields that $U_\psi \subseteq U_\Phi$. Let $(t, x) \in U_\Phi$. Then $\mathbb{J}(t, x) \neq \emptyset$, which implies that $(t, x) \in U_\psi$. Therefore, $U_\Phi \subseteq U_\psi$. Thus, $U_\psi = U_\Phi$, which verifies Condition (B). Lastly, in order to investigate Condition (C) and Condition (D), we take an arbitrary open set W in $\mathbb{R}^\ell$. For each $t \in T$, we define

$$W_t = \{x \in L_X : \Phi(t, x) \cap W \neq \emptyset\}.$$

From the definition of Φ, it follows that $W_t = U_\psi^t \cap \{x \in L_X : \Theta(t, x) \cap W \neq \emptyset\}$. In view of Lemma 8.4 and the fact that the convex hull of a lower-semicontinuous function is lower semicontonous, we conclude that $\Theta(t, \cdot)$ is lower-semicontinuous. Therefore, $\{x \in L_X : \Theta(t, x) \cap W \neq \emptyset\}$ is (weakly) open in L_X. Since U_ψ^t is (weakly) open in L_X, we have that W_t is (weakly) open in L_X. To verify the joint lower measurability of Φ, let $\mathbb{B} = \{t \in T : W_t \neq \emptyset\}$. Since $\mu(T) < \infty$, we have that T is countable and thus, $\mathbb{B}$ is countable. Consequently,

$$\{(t, x) \in T \times L_X : \Phi(t, x) \cap W \neq \emptyset\} = \bigcup\{(\{t\} \times W_t) : t \in \mathbb{B}\}.$$

Since each atom $\{t\}$ belongs to $\mathscr{T}$ and each W_t is (weakly) open, we conclude that $\{(t, x) \in T \times L_X : \Phi(t, x) \cap W \neq \emptyset\}$ is $\mathscr{T} \otimes \mathscr{B}_w(L_X)$-measurable. Therefore, Φ is jointly lower measurable. Finally, by Theorem 8.1, we can guarantee the existence of a Carathéodory-type selection $f : U_\Phi \to \mathbb{R}^\ell$ of $\Phi|_{U_\Phi}$, which verifies Condition (E). □

8 Appendix B

In what follows, we summarize all the results needed for our proofs, the first one of these is a Carathéodory-type selection theorem, due to Kim-Prikry-Yannelis (1987).

Theorem 8.1. *Let $(T, \mathscr{T}, \mu)$ be a complete finite measure space, and $\mathbb{Z}$ be a complete, separable metric space. Suppose that $F : T \times \mathbb{Z} \rightrightarrows \mathbb{R}^\ell$ is a convex (possibly empty) valued correspondence such that:*

(i) *$F(\cdot, \cdot)$ is lower measurable; and*

(ii) *for each $t \in T$, $F(t, \cdot)$ is lower-semicontinuous.*

Let $U_F := \{(t, x) \in T \times \mathbb{Z} : F(t, x) \neq \emptyset\}$, $U_F^t = \{x \in \mathbb{Z} : (t, x) \in U_F\}$ for each $t \in T$, $U_F^x := \{t \in T : (t, x) \in U_F\}$ for each $x \in \mathbb{Z}$. Then there exists a Carathéodory-type selection from $F_{|U}$, i.e. there exists a function $f : U_F \to \mathbb{R}^\ell$ such that $f(t, x) \in F(t, x)$ for all $(t, x) \in U_F$, $f(\cdot, x)$ is measurable on U_F^x for each $x \in \mathbb{Z}$ and $f(t, \cdot)$ is continuous on U_F^t for each $t \in T$. Furthermore, $f(\cdot, \cdot)$ is jointly measurable.

For proofs of the following two lemmata, we refer to Aumann (1965).

Lemma 8.2. *If $(T, \mathscr{T}, \mu)$ be a complete finite measure space and $\Psi : T \rightrightarrows \mathbb{R}^\ell$ is a closed-valued and integrably bounded correspondence then $\int_T \Psi \, d\mu$ is compact.*

Lemma 8.3. *If $(T, \mathscr{T}, \mu)$ be a complete finite atomless measure space and $\Psi : T \rightrightarrows \mathbb{R}^\ell$ is a correspondence then $\int_T \Psi \, d\mu$ is a convex set.*

The next four lemmata can be found in in Aliprantis-Border (2006).

Lemma 8.4. *Let $(T, \mathscr{T}, \mu)$ be a complete finite measure space and X be a topological space. Then the following hold:*

(i) *If* $\Phi_\alpha : X \rightrightarrows \mathbb{R}^\ell$ *is lower-semicontinuous for all* $\alpha \in \mathbb{I}$ *(where* $\mathbb{I}$ *is an index set) then* $\Psi : X \rightrightarrows \mathbb{R}^\ell$*, defined by* $\Psi(t) := \bigcup\{\Phi_\alpha(t) : \alpha \in \mathbb{I}\}$ *for all* $t \in T$*, is lower-semicontinuous.*

(ii) *If* $\Phi_n : T \rightrightarrows \mathbb{R}^\ell$ *is lower measurable for all* $n \in \mathbb{N}$ *then* $\Psi : X \rightrightarrows \mathbb{R}^\ell$*, defined by* $\Psi(t) := \bigcup\{\Phi_n(t) : n \in \mathbb{N}\}$ *for all* $t \in T$*, is lower measurable.*

The proofs of the following four lemmata can be found in Yannelis (1987).

Lemma 8.5. *Let* $(T, \mathscr{T}, \mu)$ *be a complete finite positive separable measure space, and* $X : T \rightrightarrows \mathbb{R}^\ell$ *be an integrably bounded correspondence with measurable graph, such that for all* $t \in T$*,* $X(t)$ *is a nonempty, convex, closed subset of* $\mathbb{R}^\ell$*. Then* L_X *is nonempty, convex, weakly compact, and metrizable.*

Lemma 8.6. *Let* $(T, \mathscr{T}, \mu)$ *be a complete separable measure space, and* $\mathbb{Y}$ *be a separable Banach space. Let* $X : T \rightrightarrows \mathbb{Y}$ *be an integrably bounded, non-empty, convex, weakly compact valued correspondence with a measurable graph. Let* $F : T \times L_X \rightrightarrows \mathbb{Y}$ *be a convex, closed, non-empty valued correspondence such that* $F(t,x) \subseteq X(t)$ *for all* $(t,x) \in T \times L_X$*,* $F(\cdot, x)$ *has a measurable graph for each* $x \in L_X$*,* $F(t,\cdot)$ *is upper-semicontinuous in the sense that the set* $\{x \in L_X : \phi(t,x) \subset V\}$ *is weakly open in* L_X *for every norm open subset* V *of* $\mathbb{Y}$ *for each* $t \in T$*. Then the correspondence* $\Phi : L_X \rightrightarrows L_X$ *defined by*

$$\Phi(x) = \{y \in L_X : y(t) \in F(t,x)\, \mu\text{-a.e.}\}$$

is nonempty valued and weakly upper-semicontinuous.

Lemma 8.7. *Let* X *and* Y *be two linear topological spaces, and* $\Phi : X \rightrightarrows Y$ *be a correspondence such that* G_Φ *is open in* $X \times Y$*. Then the correspondence* $H : X \rightrightarrows Y$*, defined by* $H(x) = \mathrm{con}\Phi(x)$*, has open graph.*

Lemma 8.8. *Let* X *and* Y *be any topological spaces, and* $\Phi : X \rightrightarrows Y$*,* $\Psi : X \rightrightarrows Y$ *be correspondences such that*

(i) G_Φ *is open in* $X \times Y$*.*

(ii) Ψ *is lower-semicontinuous.*

Then the correspondence $F : X \rightrightarrows Y$*, defined by* $F(x) = \Phi(x) \cap \Psi(x)$*, is lower-semicontinuous.*

For proofs of the following two lemmata, we refer to Yannelis (1991a).

Lemma 8.9. *Let* $(T, \mathscr{T}, \mu)$ *be a complete finite measure space and* P *be a metric space. Let* $\Psi : T \times P \rightrightarrows \mathbb{R}^\ell$ *be a non-empty, compact valued correspondence such that for each fixed* $t \in T$*,* $\Psi(t, \cdot)$ *is upper-semicontinuous and that for each* $p \in P$*,* $\Psi(\cdot, p)$ *has a measurable graph. Then* $\int_T \Psi(t, \cdot)\, d\mu$ *is upper-semicontinuous.*

Lemma 8.10. *Let* $(T, \mathscr{T}, \mu)$ *be a complete finite measure space and* P *be a metric space. Let* $\Psi : T \times P \rightrightarrows \mathbb{R}^\ell$ *be an integrably bounded correspondence such that for each fixed* $t \in T$*,* $\Psi(t, \cdot)$ *is lower-semicontinuous and that for each* $p \in P$*,* $\Psi(\cdot, p)$ *has a measurable graph. Then* $\int_T \Psi(t, \cdot)\, d\mu$ *is lower-semicontinuous.*

For proof of the following lemma, we refer to Yannelis (1991b).

Lemma 8.11. *Let* $(T, \mathscr{T}, \mu)$ *be a complete finite measure space and* $\Psi : T \rightrightarrows \mathbb{R}^\ell$ *be a non-empty, convex, compact valued and integrably bounded correpondence. Then* S^1_Ψ *is weakly compact set in* $L_1(\mu, \mathbb{R}^\ell)$*.*

For a proof of the following lemma, we refer to Yannelis-Prabhakar (1983).

Lemma 8.12. *Let* X, Y *be topological spaces and* E *be an open set in* X*. Suppose that* $\Phi : X \rightrightarrows Y$ *is an upper semi-continuous correspondence and* $f : E \to Y$ *be a continuous selection from* $\Phi|_E$*. Let* $\Psi : X \rightrightarrows Y$ *be a correspondence such that*

$$\Psi(x) := \begin{cases} \{f(x)\}, & \textit{if } x \in E; \\ \Phi(x), & \textit{otherwise.} \end{cases}$$

Then Λ *is upper-semicontinuous.*

References

[1] Aliprantis, Charalambos D., and Kim C. Border. "Infinite dimensional analysis: A hitchhiker's guide." Springer, Berlin, 2006.

[2] Anderson, Robert M., Haosui Duanmu, M. Ali Khan, and Metin Uyanik. "Walrasian equilibrium theory with and without free-disposal: theorems and counterexamples in an infinite-agent context." Economic Theory **73** (2022), 1–26.

[3] Anderson, Robert M., Haosui Duanmu, M. Ali Khan, and Metin Uyanik. "On abstract economies with an arbitrary set of players and action sets in locally-convex topological vector spaces." Journal of Mathematical Economics **98** (2022), 102581.

[4] Arrow, Kenneth J., and Gerard Debreu. "Existence of an equilibrium for a competitive economy." Econometrica **22** (1954), 265–290.

[5] Aumann, Robert J. "Existence of competitive equilibria in markets with a continuum of traders." Econometrica **34** (1966), 1-17.

[6] Bagh, Adib, and Alejandro Jofre. "Reciprocal upper semicontinuity and better reply secure games: a comment." Econometrica **74** (2006), 1715–1721.

[7] Barelli, Paulo, and Idione Meneghel. "A note on the equilibrium existence problem in discontinuous games." Econometrica **81** (2013), 813–824.

[8] Bewley, Truman. "An integration of equilibrium theory and turnpike theory." Journal of Mathematical Economics **10** (1982), 233–267.

[9] Bhowmik, Anuj, and Nicholas C. Yannelis. Carathéodory-type selection and random fixed point theorems for discontinuous correpondences, working paper, 2024

[10] Borglin, Anders, and Hans Keiding. "Existence of equilibrium actions and of equilibrium: A note on the 'new'existence theorems." Journal of Mathematical Economics **3** (1976), 313–316.

[11] Carmona, Guilherme. "Symposium on: existence of Nash equilibria in discontinuous games." Economic Theory **48** (2011), 1–4.

[12] Carmona, Guilherme, and Konrad Podczeck. "Existence of Nash equilibrium in ordinal games with discontinuous preferences." Economic Theory **61** (2016), 457–478.

[13] Dasgupta, Partha, and Eric Maskin. "The existence of equilibrium in discontinuous economic games, I: Theory." The Review of Economic Studies **53** (1986), 1–26.

[14] Debreu, Gerard. " social equilibrium existence theorem." Proceedings of the National Academy of Sciences of the USA **38** (1952), 886–893.

[15] Fabian, Marián, Petr Habala, Petr Hájek, Vicente Montesinos Santalucía, Jan Pelant, and Václav Zizler. "Functional analysis and infinite-dimensional geometry". Vol. 8. New York: Springer, 2001.

[16] Gale, David, and Andreu Mas-Colell. "An equilibrium existence theorem for a general model without ordered preferences." Journal of Mathematical Economics **2** (1975), 9–15.

[17] Gale, David, and Andreu Mas-Colell. “Corrections to an equilibrium existence theorem for a general model without ordered preferences.” Journal of Mathematical Economics **6** (1979), 297–298.

[18] He, Wei. “Discontinuous stochastic games.” Econ Theory **73** (2022), 827-–858.

[19] He, Wei, and Nicholas C. Yannelis. “Existence of Walrasian equilibria with discontinuous, non-ordered, interdependent and price-dependent preferences.” Economic Theory **61** (2016), 497–513.

[20] He, Wei, and Nicholas C. Yannelis. “Equilibria with discontinuous preferences: New fixed point theorems.” Journal of Mathematical Analysis and Applications **450** (2017), 1421–1433.

[21] Hou, Ji-Cheng, “A new generalization of the Yannelis–Prabhakar equilibrium existence theorem for abstract economies.” Nonlinear Analysis **68** (2008), 3159–3165.

[22] Ke, Rongzhu, and Xinyi Xu. “The existence of an optimal deterministic contract in moral hazard problems.” Econ Theory **76** (2023), 375—416.

[23] Khan, M. Ali. “Equilibrium points of nonatomic games over a Banach space.” Transactions of the American Mathematical Society **293** (1986), 737–749.

[24] Khan, M. Ali, and Nikolaos S. Papageorgiou. “On Cournot-Nash equilibria in generalized qualitative games with an atomless measure space of agents.” Proceedings of the American Mathematical Society **100** (1987), 505–510.

[25] Khan, M. Ali, and Metin Uyanik. “The Yannelis–Prabhakar theorem on upper semi-continuous selections in paracompact spaces: extensions and applications.” Economic Theory **71** (2021), 799–840.

[26] Khan, M. Ali, and Rajiv Vohra. “Equilibrium in abstract economies without ordered preferences and with a measure space of agents.” Journal of Mathematical Economics **13** (1984), 133–142.

[27] Kim, Taesung, Karel Prikry, and Nicholas C. Yannelis. “Carathéodory-type selections and random fixed point theorems.” Journal of Mathematical Analysis and Applications **122** (1987), 393–407.

[28] Lebrun, Bernard. “Existence of an equilibrium in first price auctions.” Economic Theory **7** (1996), 421–443.

[29] Mas-Colell, Andreu. "An equilibrium existence theorem without complete or transitive preferences." Journal of Mathematical Economics **1** (1974), 237–246.

[30] Mas-Colell, Andreu. "A model of equilibrium with differentiated commodities." Journal of Mathematical Economics **2** (1975), 263–295.

[31] Mas-Colell, Andreu. "On a theorem of Schmeidler." Journal of Mathematical Economics **13** (1984), 201–206.

[32] McKenzie, Lionel W. "On equilibrium in Graham's Model of world trade and other competitive systems." Econometrica **22** (1954), 147–161.

[33] McKenzie, Lionel W. "On the existence of general equilibrium for a competitive economy." Econometrica **27** (1959), 54–71.

[34] McLennan, Andrew, Paulo K. Monteiro, and Rabee Tourky. "Games with discontinuous payoffs: a strengthening of Reny's existence theorem." Econometrica **79** (2011), 1643–1664.

[35] Nash, John F. "Non-cooperative games." Annals of Mathematics **54** (1951), 286–295.

[36] Otsuka, Motoki . "The existence of Walrasian equilibrium: infinitely many commodities, measure space of agents, and discontinuous preferences." Economic Theory Bulletin, forthcoming 2024

[37] Pastine, Ivan, and Tuvana Pastine. "Equilibrium existence and expected payoffs in all-pay auctions with constraints." Economic Theory **75** (2023), 983–1007.

[38] Podczeck, Konrad, and Nicholas C. Yannelis. "Existence of Walrasian equilibria with discontinuous, non-ordered, interdependent and price-dependent preferences, without free disposal, and without compact consumption sets." Economic Theory **73** (2022), 1–8.

[39] Podczeck, Konrad, and Nicholas C. Yannelis. "Existence of Walrasian equilibria with discontinuous, non-ordered, interdependent preferences, without free disposal, and with an infinite-dimensional commodity space." Economic Theory (2024), https://doi.org/10.1007/s00199-024-01553-0.

[40] Prokopovych, Pavlo. "On equilibrium existence in payoff secure games." Economic Theory **48** (2011), 5–16.

[41] Prokopovych, Pavlo. “Majorized correspondences and equilibrium existence in discontinuous games.” Economic Theory **61** (2016), 541–552.

[42] Prokopovych, Pavlo, and Nicholas C. Yannelis. “On strategic complementarities in discontinuous games with totally ordered strategies.” Journal of Mathematical Economics **70** (2017), 147–153.

[43] Reny, Philip J. “On the existence of pure and mixed strategy Nash equilibria in discontinuous games.” Econometrica **67** (1999), 1029–1056.

[44] Reny, Philip J. “Nash equilibrium in discontinuous games.” Economic Theory **61** (2016), 553–569.

[45] Reny, Philip J. “Equilibrium in discontinuous games without complete or transitive preferences.” Econ Theory Bull **4** (2016), 1–4.

[46] Reny, Philip J. “Nash equilibrium in discontinuous games.” Annual Review of Economics **12** (2020), 439–470.

[47] Scalzo, Vincenzo. “On the existence of maximal elements, fixed points and equilibria of generalized games in a fuzzy environment.” Fuzzy Sets and Systems **272** (2015), 126–133.

[48] Schmeidler, David. “Competitive equilibria in markets with a continuum of traders and incomplete preferences.” Econometrica **37** (1969), 578–585.

[49] Schmeidler, David. “Equilibrium points of nonatomic games.” Journal of Statistical Physics **7** (1973), 295–300.

[50] Shafer, Wayne. “ The nontransitive consumer.” Econometrica **42** (1974), 913—919.

[51] Shafer, Wayne. “Equilibrium in economies without ordered preferences or free disposal.” Journal of Mathematical Economics **3** (1976), 135–137.

[52] Shafer, Wayne, and Hugo Sonnenschein. “Equilibrium in abstract economies without ordered preferences.” Journal of Mathematical Economics **2** (1975), 345–348.

[53] Shitovitz, Benyamin. “Oligopoly in markets with a continuum of traders.” Econometrica **41** (1973), 467–501.

[54] Urbinati, Niccolò. "The Walrasian objection mechanism and Mas-Colell's bargaining set in economies with many commodities." Economic Theory **76** (2023), 45–68.

[55] Wu, Xian, and Shikai Shen. "A further generalization of Yannelis–Prabhakar's continuous selection theorem and its applications." Journal of Mathematical Analysis and Applications **197** (1996), 61–74.

[56] Yannelis, Nicholas C. "Equilibria in noncooperative models of competition." Journal of Economic Theory **41** (1987), 96–111.

[57] Yannelis, Nicholas C. "Set-valued functions of two variables in economic theory." In equilibrium theory in infinite dimensional spaces, pp. 36-72. Berlin, Heidelberg: Springer Berlin Heidelberg, 1991a.

[58] N. C. Yannelis, Integration of Banach-Valued Correspondences, in Equilibrium Theory in Infinite Dimensional Space, M. Ali Khan and Nicholas C. Yannelis (eds.), Springer-Verlag, 1991b

[59] Yannelis, Nicholas C., and N. D. Prabhakar. "Existence of maximal elements and equilibria in linear topological spaces." Journal of Mathematical Economics **12** (1983), 233–245.

[60] Yannelis, Nicholas C., and William R. Zame. "Equilibria in Banach lattices without ordered preferences." Journal of Mathematical Economics **15** (1986), 85–110.