Economic Theory **18**, 293–319 (2001)

Economic Theory

An extensive form interpretation of the private core*

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Received: July 21, 2000; revised version: January 16, 2001

Summary. The private core of an economy with differential information, (Yannelis (1991)), is the set of all state-wise feasible and private information measurable allocations which cannot be dominated, in terms of ex ante expected utility functions, by state-wise feasible and private information measurable net trades of any coalition. It is coalitionally Bayesian incentive compatible and also takes into account the information superiority of an individual. We provide a noncooperative extensive form interpretation of the private core for three person games. We construct game trees which indicate the sequence of decisions and the information sets, and explain the rules for calculating ex ante expected payoffs. In the spirit of the Nash programme, the private core is thus shown to be supported by the perfect Bayesian equilibrium of a noncooperative game. The discussion contributes not only to the development of ideas but also to the understanding of the dynamics of how coalitionally incentive compatible contracts can be realized.

Keywords and Phrases: Differential information economy, Private core, Weak fine core, Coalitional Bayesian incentive compatibility, Game trees, Perfect Bayesian equilibrium, Contracts, Nash Programme.

JEL Classification Numbers: 020, 226.

* We would like to thank the referees of this journal for their invaluable comments. In particular one of the referees made extensive suggestions which improved the final version of the paper.

1 Introduction

An economy with differential information consists of a finite set of agents each of which is characterized by a random utility function, a random consumption set, random initial endowments, a private information set and a prior probability distribution. The private core of a differential information economy (see Yannelis (1991)) is the set of all state-wise feasible and private information measurable allocations which cannot be dominated, in terms of expected utility, by any coalition's state-wise feasible and private information measurable net trades.

The private core is not susceptible to the criticism of the traditional rational expectations equilibrium (REE). In particular, the REE does not provide an explanation as to how prices reflect the information asymmetries in the economy. On the contrary the private core not only takes into account the information asymmetries but also rewards agents with "superior" information as shown in Example 3.1 in Section 3. Furthermore it is coalitionally Bayesian incentive compatible (see Koutsougeras and Yannelis, 1993). Hence the private core can be used to explain how incentive compatible contracts are written.

The main purpose of this paper is to provide a noncooperative, extensive form interpretation of the private core. Generally speaking we investigate whether or not cooperative core concepts, i.e. the private core and the weak fine core, defined below, can be supported as a perfect Bayesian equilibrium.

This investigation falls in the area of the Nash programme, which is a research agenda originated by Nash (1953) and emphasized by Binmore (1980a,b). The idea is to provide support and justification of cooperative solutions to economic problems through noncooperative formulations. More generally the issue is the relation between dynamic and static considerations. Our approach provides a dynamic interpretation of the static private core notion. Consequently it helps to explain the dynamics of how incentive compatible contracts are realized.

In our analysis, in order to provide support for the private core, we introduce game trees. They show the prior probability with which nature chooses and make explicit the sequential moves, i.e., which player makes announcements or moves first. They also take into account the private information sets of each player as well as the measurability of decisions.

Given the above structure of the game tree, we specify rules, i.e., the terms of a contract, which imply specific redistributions of the random initial endowments in different events. The rules are a statement as to the consequences of actions by the players under all possible states of nature. Having specified the rules, we obtain the payoffs in terms of quantities and then we are looking for an appropriate refinement of Nash equilibrium for games with imperfect or differential information.

We require an equilibrium concept which adopts a probabilistic approach with respect to the nodes of an information set and reduces to subgame perfect equilibrium in case the information sets are singletons. Such a concept is the Kreps and Wilson (1982) sequential equilibrium and its variants which are either weaker versions or refinements. We adopt here the perfect Bayesian equilibrium, described in Tirole (1988), where also a comparison is made with other, similar type ideas.

A *perfect Bayesian equilibrium* consists of a set of players' optimal behavioral strategies and, consistent with these, a set of beliefs which attach a probability distribution to the nodes of each information set. Consistency requires that the decision from an information set is optimal given the particular player's beliefs about the nodes of this set and the strategies from all other sets, and that beliefs are formed from updating using the available information. If the optimal play of the game enters an information set then updating of beliefs must be Bayesian. Otherwise appropriate beliefs are assigned arbitrarily to the nodes of the set.

The term "implementation" is used below in the sense of realization of an allocation and not in the sense of implementation theory or mechanism design which requires the introduction of a planner. Recent work in this area is by Trockel (2000) which contributes to the Nash programme and casts the implementation discussion in its context.

The main results in this paper are the following. Despite the fact that "pooled" information core allocations, (i.e., the weak fine core), exist under mild assumptions, we construct a game tree, with reasonable rules for calculating payoffs, which shows that a redistribution of this nature cannot be supported as a perfect Bayesian equilibrium. Indeed, such contracts (allocations) need not be Bayesian incentive compatible which suggests a difficulty in implementing them.

On the other hand, we construct a three player example which indicates that the private core, which is Bayesian incentive compatible, can be supported as a perfect Bayesian equilibrium. The above results not only provide a first step into the noncooperative extensive form interpretation of the core of economies with differential information, but also enable us to understand how coalitionally Bayesian incentive compatible contracts are realized.

Finally we provide a generalization of the private core existence result of Yannelis (1991) by relaxing the continuity assumption of the random utility functions. This enables us to include private information sets which not only can be measurable partitions of the exogenously given probability measure space, but can also be sub-σ-algebras.

To the best of our knowledge the present paper is the first attempt to provide a noncooperative foundation for core concepts in economies with differential information.

We note that the complete information works of Lagunoff (1994), Perry and Reny (1994), Serrano (1995) and Serrano and Vohra (1997), which discuss a non-cooperative approach to the core, do not apply to the differentiable economy framework that we are considering here.

The paper is organized as follows. Section 2 contains the definition of the differential information economy. Section 3 contains the core concepts employed in this paper as well as a new core existence result. Section 4 discusses ideas of incentive compatibility on the basis of which core allocations can be classified. Section 5 discusses the non-implementation of the weak fine core and Section 6 the implementation of the private core in extensive form games. Section 7 offers brief concluding remarks. Appendix I proves, under certain conditions, the existence of a private core allocation and Appendix II derives the private core allocations in an explicit example which is used in the text.

2 Differential information economy

Although we shall be concerned with a special model we repeat briefly, for completeness, the notation used and the definition of the private core in a general case. We define below the notion of a finite-agent economy with differential information. Let $(\Omega, \mathscr{F}, \mu)$ be a complete probability measure space and Y be a separable *Banach lattice*[1] with an order continuous norm denoting the commodity space. The positive cone of Y is denoted by Y_+.

A differential information economy $\mathscr{E}$ is a set $\{((\Omega, \mathscr{F}, \mu), X_i, \mathscr{F}_i, u_i, e_i) : i = 1, \ldots, n\}$ where

1. $X_i : \Omega \to 2^{Y_+}$ is the set-valued function giving the *random consumption set* of Agent (Player) i, who is denoted also by Pi,
2. $\mathscr{F}_i$ is a partition (or sub-σ-algebra) of $\mathscr{F}$, denoting the *private information*[2] of Pi,
3. $u_i : \Omega \times Y_+ \to \mathbb{R}$ is the *random utility* function of Pi,
4. $e_i : \Omega \to Y_+$ is the *random initial endowment* of Pi, where $e_i(\cdot)$ is $\mathscr{F}_i$-measurable and *Bochner integrable*[3], and $e_i(\omega) \in X_i(\omega)$ μ-a.e., and
5. μ denotes the common *prior* of all agents.

The *ex ante expected utility* of Pi is given by

$$v_i(x_i) = \int_{\Omega} u_i(\omega, x_i(\omega)) d\mu(\omega). \tag{1}$$

Denote by $E_i(\omega)$ the event in the partition $\mathscr{F}_i$ of Agent i which contains the realized state of nature, $\omega \in \Omega$. The *interim expected utility* function of Agent i is given by

$$v_i(\omega, x_i) = \frac{1}{\mu(E_i(\omega))} \int_{\omega' \in E_i(\omega)} u_i(\omega', x_i(\omega')) d\mu(\omega'), \tag{2}$$

where $\mu(E_i(\omega))$ is assumed to be positive.

Despite the fact that the differential information economy is static, we can provide a two-period interpretation as follows. In the first period agents make contracts in the ex ante stage. In the interim stage, i.e., after they have received a signal[4] as to what is the event containing the realized state of nature, one considers the incentive compatibility of the contract.

[1] See Appendix I.

[2] Following Aumann (1987) we assume that the players' information partitions are common knowledge.

[3] See Appendix I.

[4] A signal to a player is a function from states of nature to the possible observations specific to the player, which induces on Ω a sub-σ-algebra of $\mathscr{F}$.

3 The private core and the weak fine core

First we define the notion of the private core (Yannelis (1991)). We begin with some notation. Denote by $L_1(\mu, Y)$ the space of all equivalence classes of Bochner integrable functions.

L_{X_i} is the set of all Bochner integrable and $\mathscr{F}_i$-measurable selections from the random consumption set of Agent i, i.e.,

$$L_{X_i} = \{x_i \in L_1(\mu, Y) : \ x_i : \Omega \to Y \text{ is } \mathscr{F}_i\text{-measurable and } x_i(\omega) \in X_i(\omega) \ \mu\text{-}a.e.\}$$

and let $L_X = \prod_{i=1}^{n} L_{X_i}$.

Also let

$$\bar{L}_{X_i} = \{x_i \in L_1(\mu, Y) : \ x_i(\omega) \in X_i(\omega) \ \mu\text{-}a.e.\}$$

and let $\bar{L}_X = \prod_{i=1}^{n} \bar{L}_{X_i}$.

An element $x = (x_1, \ldots, x_n) \in \bar{L}_X$ will be called an *allocation*. For any subset of players S, an element $(y_i)_{i \in S} \in \prod_{i \in S} \bar{L}_{X_i}$ will also be called an allocation, although strictly speaking it is an allocation to S.

Definition 3.1. *An allocation $x \in L_X$ is said to be a* **private core allocation** *if*

(i) $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} e_i$ *and*
(ii) *there do not exist coalition S and allocation $(y_i)_{i \in S} \in \prod_{i \in S} L_{X_i}$ such that $\sum_{i \in S} y_i = \sum_{i \in S} e_i$ and $v_i(y_i) > v_i(x_i)$ for all $i \in S$.*

Hence, a private core allocation is feasible, reflects the private information of each agent, i.e., each $x_i(\cdot)$ is $\mathscr{F}_i$-measurable, and has the property that no coalition of agents can redistribute their initial endowments, based on their own private information, and make each of its members better off. It is important to notice that since initial endowments are private information measurable, net trades $x_i(\cdot) - e_i(\cdot)$ are also $\mathscr{F}_i$-measurable.

Observe that despite the fact that a coalition of agents get together they do not necessarily share their own information. On the contrary, the redistributions of the initial endowments are based only on their own private information. This is quite important because the resulting private core allocation has desirable properties, i.e., it is coalitionally incentive compatible, as we shall see below, and takes into account the information superiority of an individual.[5]

[5] See Koutsougeras and Yannelis (1993) and Example 3.1 below. Notice that in Definition 3.1 the ex ante expected utility function is used. The (interim) private core is also defined similarly by replacing (ii) in Definition 3.1 by
(ii) there do not exist coalition S and allocation $(y_i)_{i \in S} \in \prod_{i \in S} L_{X_i}$ such that $\sum_{i \in S} y_i = \sum_{i \in S} e_i$ and $v_i(\omega, y_i) > v_i(\omega, x_i)$ for all $i \in S$ and μ-a.e.
Both private cores (ex ante and interim) exist and also have similar qualitative properties (see Hahn and Yannelis, 2000).

Although several private core existence results can be found in the literature, as for example in Yannelis (1991), Allen (1991), Koutsougeras and Yannelis (1993), Balder and Yannelis (1994), Page (1997) and Lefebvre (2001), among others, the proof of the theorem below appears to be the shortest, simplest and quite general. It improves on the original one of Yannelis (1991).

Theorem 3.1: *Let* $\mathscr{E} = \{((\Omega, \mathscr{F}, \mu), X_i, \mathscr{F}_i, u_i, e_i) : i = 1, \ldots, n\}$ *be a differential information economy satisfying for each* i *the following assumption:*
u_i *is concave, upper semicontinuous (u.s.c.) and integrably bounded.*
Then a private core allocation exists in $\mathscr{E}$.

Proof. See Appendix I.

The theorem in Yannelis (1991) is generalized in the following way. The utility functions need not be weakly continuous, but only u.s.c. in the norm topology. However in the presence of concavity they become weakly u.s.c. (Balder - Yannelis (1993)). The latter enables us to generalize the private information sets $\mathscr{F}_i$ of each agent from partitions to a sub-σ-algebra. Furthermore, we do not need to assume that the dual of Y has the Radon - Nikodym property. In the examples below the σ-algebras will be generated from partitions.

The example below illustrates the private core.

Example 3.1 Consider the following three agents economy, $I = \{1, 2, 3\}$ with one commodity, i.e. $X_i = \mathbb{R}_+$ for each i, and three states of nature $\Omega = \{a, b, c\}$.

The agents are characterized by their initial endowments, their private information and their utility functions. We assume that the structure is

$$\begin{aligned} e_1 &= (5, 5, 0), & \mathscr{F}_1 &= \{\{a, b\}, \{c\}\}; \\ e_2 &= (5, 0, 5), & \mathscr{F}_2 &= \{\{a, c\}, \{b\}\}; \\ e_3 &= (0, 0, 0), & \mathscr{F}_3 &= \{\{a\}, \{b\}, \{c\}\}. \end{aligned}$$

Notice that the initial endowment of each agent is $\mathscr{F}_i$-measurable. It is also assumed that $u_i(\omega, x_i(\omega)) = x_i^{\frac{1}{2}}$, which is a typical strictly concave and monotone function in x_i, and that each state of nature occurs with the same probability, i.e. $\mu(\{\omega\}) = \frac{1}{3}$, for $\omega \in \Omega$. For convenience, in the discussion below expected utilities are multiplied by 3.

It can be shown[6] that a private core allocation of this economy is $x_1 = (4, 4, 1)$, $x_2 = (4, 1, 4)$ and $x_3 = (2, 0, 0)$. Clearly this allocation is feasible and $\mathscr{F}_i$-measurable. It is important to observe that in spite of the fact that Agent 3 has zero initial endowments, her superior information allows him to make a Pareto improvement for the economy as a whole and clearly he was rewarded for doing so. In other words, Agent 3 traded her superior information for actual consumption in state a. In return Agent 3 provided insurance to Agent 1 in state c and to Agent 2 in state b. Notice that if the private information set of Agent 3 is the trivial partition, i.e., $\mathscr{F}'_3 = \{a, b, c\}$, then no trade takes place and clearly in this case she gets zero utility. Thus the private core is sensitive to information asymmetries.

[6] See Appendix II.

Contrary to the private core any rational expectation Walrasian equilibium notion will always give zero to Agent 3 since her budget set is zero in each state. This is so irrespective of whether her private information is the full information partition $\mathscr{F}_3 = \{\{a\}, \{b\}, \{c\}\}$ or the trivial partition $\mathscr{F}_3' = \{a, b, c\}$. Hence the rational expectations equilibrium does not take into account the informational superiority of an agent.

Next we define another core concept, the weak fine core (see Yannelis, 1991, p. 188; Koutsougeras and Yannelis, 1993). This concept is a refinement of the fine core of Wilson (1978). Recall that the fine core notion of Wilson as well as the fine core in Yannelis, and Koutsougeras and Yannelis may be empty in well behaved economies. It is exactly for this reason that we are working with a different concept.

Definition 3.2. *An allocation $x = (x_1, \ldots, x_n) \in \bar{L}_X$ is said to be a* **weak fine core** *allocation if*

(i) *each $x_i(\cdot)$ is $\bigvee_{i=1}^{n} \mathscr{F}_i$-measurable* [7]

(ii) *$\sum_{i=1}^{n} x_i(\omega) = \sum_{i=1}^{n} e_i(\omega)$ μ-a.e. and*

(iii) *there do not exist coalition S and allocation $(y_i)_{i\in S} \in \prod_{i\in S} \bar{L}_{X_i}$ such that $y_i(\cdot) - e_i(\cdot)$ is $\bigvee_{i\in S} \mathscr{F}_i$-measurable for all $i \in S$, $\sum_{i\in S} y_i(\omega) = \sum_{i\in S} e_i(\omega)$ μ-a.e., and $v_i(y_i) > v_i(x_i)$ for all $i \in S$.*

Notice that now in the weak fine core, coalitions of agents are allowed to pool their own information. Identical assumptions with those in Theorem 3.1 and a similar argument shows that a weak fine core allocation exists in $\mathscr{E}$. The example below illustrates this concept.

Example 3.2 Consider Example 3.1 without Agent 3. Then if Agents 1 and 2 pool their own information a possible allocation is $x_1 = x_2 = (5, 2.5, 2.5)$. Notice that this allocation is $\bigvee_{i=1}^{2} \mathscr{F}_i$-measurable and cannot be dominated by any coalition of agents using their pooled information. Hence it is a weak fine core allocation.

4 Incentive compatibility

A careful examination of Example 3.1 indicates that the private core allocation is incentive compatible in the sense that no coalition of agents has an incentive to misreport the realized state of nature and become better off. The argument which supports this conclusion is as follows. Agent 3 can presumably lie to Agents 1 and 2 if the realized state of nature is a since Agent 1 cannot distinguish state a from state b and Agent 2 state a from state c. However, Agent 3 has no incentive

[7] $\bigvee_{i=1}^{n} \mathscr{F}_i$ denotes the smallest σ-algebra containing each $\mathscr{F}_i$.

to do so since only in state a does she get positive consumption. Hence, Agent 3 who would potentially cheat in state a has no incentive to do so.

We could consider the example in more detail. We ask the question whether the coalition $S = \{1, 3\}$ can cheat P2. This is not possible because P3 would become worse off. For suppose that the state of nature is a but S reports c. Then $u_1(e_1(a) + x_1(c) - e_1(c)) = u_1(5 + 1 - 0) > u_1(x_1(b)) = u_1(x_1(a)) = u_1(4)$ but $u_3(e_3(a) + x_3(c) - e_3(c)) = u_3(0) < u_3(x_3(a)) = u_3(2)$. Similarly the coalition $S = \{2, 3\}$ cannot form, and the coalitions $S = \{1, 2\}$, $S = \{1\}$ and $S = \{2\}$ cannot misreport to P3.

Generalizing we have a coalition S and the complementary set which we denote by $I \setminus S$. The members of S will be denoted by i and the members of $I \setminus S$ by j. Suppose that the realized state of nature is ω^*. A member $i \in S$ sees $E_i(\omega^*)$. Obviously not all $E_i(\omega^*)$ need be the same since different i's have different information sets. However they all know from their information that the actual state of nature could be ω^*.

Consider now a state of nature ω' with the following property. For all $j \in I \setminus S$ we have $\omega' \in E_j(\omega^*)$ and for at least one $i \in S$ we have $\omega' \notin E_i(\omega^*)$ (otherwise ω' would be indistinguishable from ω^* for all players so in effect could be considered as the same element of Ω). Now the coalition S decides that each member i will announce that she has seen her own set $E_i(\omega')$ which, of course, definitely contains a lie. On the other hand we have that $\omega' \in \bigcap_{j \notin S} E_j(\omega^*)$ (we also denote $j \in I \setminus S$ by $j \notin S$).

Now the idea is that if all members of $I \setminus S$ believe the statements of the members of S then each $i \in S$ expects to gain. For **coalitional Bayesian incentive compatibility** (CBIC) of an allocation we require that this is not possible.

A formal definition of the notion of CBIC[8] is:

Definition 4.1. *An allocation $x = (x_1, \ldots, x_n) \in \bar{L}_X$ with $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} e_i$ is said to be CBIC if it is not true that there exist coalition S and states ω^*, ω', with ω^* different than ω', and $\omega' \in \bigcap_{j \notin S} E_j(\omega^*)$ such that*

$$\frac{1}{\mu(Z_i(\omega^*))} \int_{\omega \in Z_i(\omega^*)} u_i(\omega, e_i(\omega) + x_i(\omega') - e_i(\omega')) d\mu(\omega)$$
$$> \frac{1}{\mu(Z_i(\omega^*))} \int_{\omega \in Z_i(\omega^*)} u_i(\omega, x_i(\omega)) d\mu(\omega) \qquad (3)$$

for all $i \in S$, where $Z_i(\omega^*) = E_i(\omega^*) \cap (\bigcap_{j \notin S} E_j(\omega^*))$ and $\mu(Z_i(\omega^*))$ is assumed to be positive.

The integrals above can be evaluated since, due to the common knowledge assumption of Section 2, each player knows all the information sets of the other players and therefore can calculate the relevant intersection $Z_i(\omega^*)$.

This definition implies that no coalition of agents has an incentive to misreport the realized state of nature to the complementary set, despite the fact that the latter

[8] See also Krasa and Yannelis (1994), Hahn and Yannelis (2001) for other CBIC concepts.

cannot distinguish the actual state from the misreported one. They do not expect that, by misreporting, each member of the coalition could become better off. If, for example, the realized state of nature is ω^* and for all $j \notin S$, $\omega' \in E_j(\omega^*)$, while for at least $i \in S$ it is true that $\omega^{'} \notin E_i(\omega^*)$, it must be the case that the agents in S have no incentive to report state $\omega^{'}$. I.e., they do not expect that it is possible to become better off if they are believed, by adding to their initial endowment the net trade in state $\omega^{'}$. If $S = \{i\}$ the above definition reduces to *individual Bayesian incentive compatibility* (IBIC).

It has been shown in Koutsougeras - Yannelis (1993) that if the utility functions are monotone and continuous then private core allocations are *always* CBIC. On the other hand the weak fine core allocations are not always CBIC, as the above Example 3.2 with proposed redistribution $x_1 = x_2 = (5, 2.5, 2.5)$ shows.

Indeed, if Agent 1 observes $\{a, b\}$, she has an incentive to report c, as Agent 2 cannot distinguish between a and c. Agent 1 stands to gain if she is believed, which is a possibility as a might be the true state and Agent 2 believes the statement that it is c. In this case Agent 1 keeps the 5 units of the initial endowments in state a, and also gets an additional 2.5 units from Agent 2. In terms of the Definition 4.1, the fact that $u_1(e_1(a) + x_1(c) - e_1(c)) = u_1(5 + 2.5 - 0) > u_1(5) = u_1(x_1(a))$ implies that the proposed allocation is not CBIC. Similarly Agent 2 has an incentive to report b when he observes $\{a, c\}$.

Now in employing game trees in the analysis, as it is done below, we will adopt the definition of IBIC. The equilibrium concept employed will be that of perfect Bayesian equilibrium the application of which is explained below.

A core allocation will be IBIC if there is a profile of optimal behavioral strategies and equilibrium paths along which no player misreports the state of nature he has observed. This allows for the possibility, as we shall see later, that such strategies could imply that players have an incentive to lie from information sets which are not visited by an optimal play. The definition of a play of the game is a directed path from the initial to a terminal node.

The issue is whether core allocations can be obtained as perfect Bayesian equilibria. That is whether the cooperative core solutions can also be supported through an appropriate noncooperative solution concept. The analysis in Sections 5 and 6 below shows that the private core which is CBIC can be supported by a perfect Bayesian equilibrium while for the weak fine core, which may not be CBIC, we find that a reasonable extensive form game does not support it.

5 Non-implementation of the weak fine core in an extensive game

In this section we investigate, by considering sequential decisions, whether in Example 3.2, a particular contract between P1 and P2, with a distribution which is Pareto superior to the initial allocation, will be signed or not.

In particular we consider the weak fine core allocation (5, 2.5, 2.5) in Example 3.2. As we saw in the previous section this is not CBIC which suggests a difficulty in implementing it by means of a contract. We construct a game tree and employ reasonable rules for describing the outcomes of combinations of states of nature and actions of the players. We find that although the Pareto superior allocation (5, 2.5, 2.5) is possible, the optimal strategies of the players imply no trade because of lack of IBIC. Hence there is no advantage in signing such a contract.

One of the issues that has been considered is whether, in order to implement the allocation $(5, 2.5, 2.5)$, the information of P1 and P2 can be pooled into $\mathscr{F}_1 \bigvee \mathscr{F}_2 = \{\{a\}, \{b\}, \{c\}\}$ through the two agents informing each other. The proposed allocation (5, 2.5, 2.5) is measurable with respect to $\mathscr{F}_1 \bigvee \mathscr{F}_2$ and it is a Pareto improvement over the initial endowments.

When the agents form their coalition, they do so in order to sign a contract. The contract depends on their realization that together they could know the state of nature. If each player announces truthfully what he sees, the state of nature would then be common knowledge. Having written the contract, another issue then arises. That is whether the players have an incentive to lie about what they have seen in the interim state. It is this second stage that the game tree is analysing. The game is played before the state is revealed and as the extensive form indicates, in the interim stage each player has an incentive to lie. Therefore the pooling of information does not take place because of lack of incentive compatibility.

We discuss the possible realization of the allocation $(5, 2.5, 2.5)$ through the analysis of a specific sequence of decisions and information sets shown in the game tree in Figure 1. The players are given choices to tell the truth or to lie, i.e., we model the idea that agents truly inform each other about what states of nature they observe, or deliberately aim to mislead their opponent. The issue is what type of behaviour is optimal and therefore whether a proposed contract will be signed or not.

Figures 1 and 2 show that the allocation $(5, 2.5, 2.5)$ will be rejected by the players. It is not IBIC and the proposed contract will not be signed. Notice that vectors at the terminal nodes of a game tree will refer to payoffs of the players, in terms of allocations. The first element will be the payoff to P1, etc.

The explanation of Figure 1 is as follows. Nature chooses states a, b or c with equal probabilities. This choice is flashed on a screen which both players can see. P1 cannot distinguish between a and b, and P2 between a and c . This accounts for the information sets I_1, I_2 and I_2' with more than one node. A player to which such an information set belongs cannot distinguish between these nodes and therefore his decisions are common to all of them. A behavioral strategy of a player is an assignment of a probability distribution per information set that belongs to him over the choices available from that set. This is irrespective of whether a particular play of the game will imply that all these choices will have an effect on the payoffs. Indistinguishable nodes imply the $\mathscr{F}_i$-measurability of decisions.

P1 moves first and has two choices. That is he can either play $A_1 = \{a, b\}$ or $c_1 = \{c\}$, i.e., he can say “I have seen $\{a, b\}$ being unable to distinguish between the two”, or “I have seen c”. Obviously one of these declarations will be true and the other a lie. Following a choice by P1 then P2 is to respond saying that the signal he has seen on the screen is $A_2 = \{a, c\}$ or that it is $b_2 = \{b\}$. One of these statements is of course a lie.

Strictly speaking the notation for choices should vary with the information set but, for simplicity, we do not modify it, as there is no danger of confusion here. Finally notice that the structure of the game tree is such that when P2 is to act he knows exactly what P1 has chosen before him. This is an assumption about the relation between decisions. In general, in forming game trees the sequence of events and the information of the agents must be specified explicitly.

Next, given the sequence of decisions of the players, shown on the tree, we specify the rules for calculating the payoffs, i.e. we specify the terms of the contract. This is a statement of what to do under all possible states of nature and declarations by the players.

The *rules* are:

(i) If the declarations by the two players are incompatible, that is (c_1, b_2) then at least one of the players is lying and, moreover, the opponent of a lying player detects that lie. This is the case when state c occurs and agent 1 reports state c and agent 2 state b. In state a both agents can lie and the lie cannot be detected by either agent (however, the agents are in the events $\{a, b\}$ and $\{a, c\}$, respectively and they get five units of the initial endowments). Therefore, whenever the declarations are incompatible, no trade takes place and the players retain their initial endowments.

(ii) If the declarations are (A_1, A_2) then even if one of the players is lying, this cannot be detected by his opponent who believes that state a has occured and both players have received endowment 5. Hence no trade takes place.

(iii) If the declarations are (A_1, b_2) then a lie can be beneficial and undetected, and P1 is trapped and must hand over half of his endowment to P2. Obviously if his endowment is zero then he has nothing to give.

(iv) If the declarations are (c_1, A_2) then again a lie can be beneficial and undetected. P2 is now trapped and must hand over half of his endowment to P1. Obviously if his endowment is zero then he has nothing to give.

The calculations of payoffs do not require the revelation of the actual state of nature. Optimal decisions from an information set will be denoted by a heavy line. If either decision is optimal then both will be shown with a heavy line. We could assume that a player does not lie if he cannot get a higher payoff by doing so.

Assuming that each player chooses optimally from the information sets which belong to him, the game in Figure 1 folds back to the one in Figure 2. This is achieved by considering the optimal decisions of P2 and applying backward induction. Inspection of Figure 1 reveals that from the information set I_2 he can play b_2 with probability 1. (A heavy line A_2 indicates that this choice also would not affect the analysis). This accounts for the payoff $(2.5, 7.5)$ and the first payoff $(0, 5)$ from left to right in Figure 2. Similarly we undo all other information sets of P2 and we arrive at Figure 2. Inspection of this figure reveals also the optimal strategies of P1.

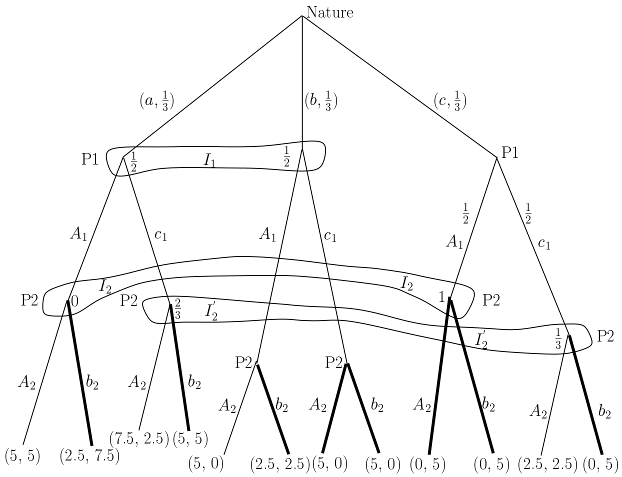

Figure 1

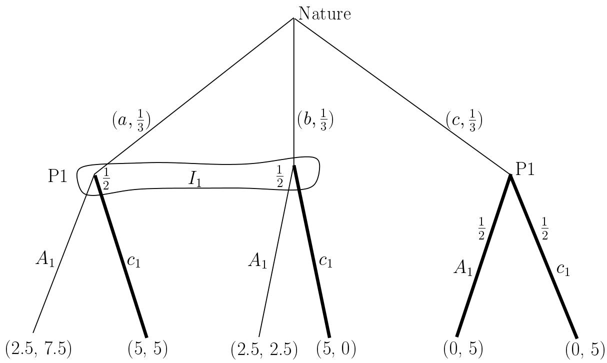

Figure 2

Summarizing, the optimal behavioral strategy for P1 is to play c_1 from I_1, i.e to lie, and from the singleton to play any mixture of options, and we have chosen $(A_1, \frac{1}{2}; c_1, \frac{1}{2})$. This is the meaning of $\frac{1}{2}$ on the branches from the singleton. Optimal behavioral strategy of P2 is to play b_2 with probability 1 from both I_2 and I_2', i.e. to lie, and from the singletons he can either tell the truth or lie, or spin a wheel to decide what to do.

Finally we point out that in Figures 1 and 2 the fractions next to the nodes in the information sets correspond to beliefs of the agents obtained, wherever possible, through Bayesian updating. I.e., they are consistent with the choice of a state by nature and the optimal strategies of the players. Hence strategies and beliefs satisfy the conditions of a perfect Bayesian equilibrium. This is a concept employed in analyzing games with information sets with more than one node. As explained above, it requires that given the beliefs, the strategies are optimal,

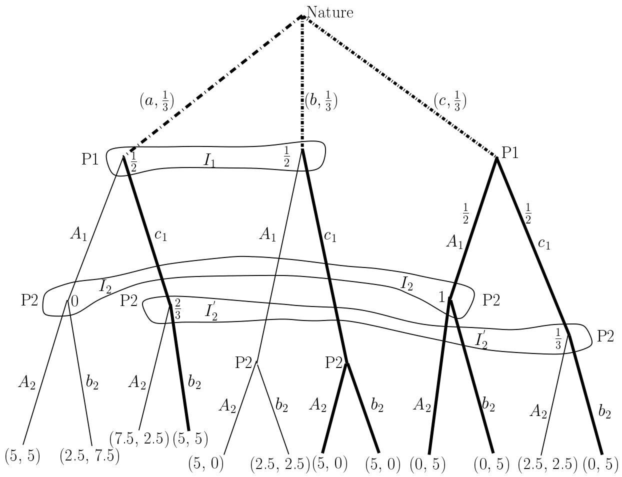

Figure 3

and given the strategies, the beliefs are, wherever possible, obtainable through Bayesian updating.

These probabilities are calculated as follows. We give labels to the nodes of the information sets: From left to right, in I_1, we denote them by j_1 and j_2, in I_2 by n_1 and n_2 and in $I_2^{'}$ by n_3 and n_4. The probabilities attached to the nodes in I_1 follow from the fact that the probability with which nature chooses state a is the same as the one with which it chooses state b. Given the choices by nature, the strategies of the players described above and using the Bayesian formula for updating beliefs we also calculate the conditional probabilities

$$\begin{aligned} Pr(n_1/A_1) &= \frac{Pr(A_1/n_1) \times Pr(n_1)}{Pr(A_1/n_1) \times Pr(n_1) + Pr(A_1/n_2) \times Pr(n_2)} \\ &= \frac{1 \times 0}{1 \times 0 + 1 \times \frac{1}{3} \times \frac{1}{2}} = 0 \end{aligned} \tag{4}$$

and

$$\begin{aligned} Pr(n_3/c_1) &= \frac{Pr(c_1/n_3) \times Pr(n_3)}{Pr(c_1/n_3) \times Pr(n_3) + Pr(c_1/n_4) \times Pr(n_4)} \\ &= \frac{1 \times \frac{1}{3}}{1 \times \frac{1}{3} + 1 \times \frac{1}{2} \times \frac{1}{3}} = \frac{2}{3} \,. \end{aligned} \tag{5}$$

Obviously from the above we obtain $Pr(n_2/A_1) = 1$ and $Pr(n_4/c_1) = \frac{1}{3}$.

Therefore the perfect Bayesian equilibrium obtained above confirms the initial endowments and the decisions to lie imply that the contract $(5, 2.5, 2.5)$ cannot be realized and the players will not sign.

In Figure 3 we indicate, through heavy lines, plays of the game which are the outcome of the choices by nature and the optimal behavioral strategies by the players. The interrupted heavy lines at the beginning of the tree signify that nature does not take an optimal decision, as it has no payoff function, but simply chooses among three alternatives, with equal probabilities. From each such choice the play of the game continues through the optimal decisions by the agents to a specific terminal node. The directed path (a, c_1, b_2) with payoffs (5, 5) occurs with probability $\frac{1}{3}$. The paths (b, c_1, A_2) and (b, c_1, b_2) lead to payoffs (5, 0) and occur with probability $\frac{1}{3}(1-q)$ and $\frac{1}{3}q$, respectively. The values $(1-q)$ and q denote the probabilities with which P2 decides to choose between A_2 and b_2 from the singleton node at the end of (b, c_1). Of course no matter what q is selected this does not affect the payoffs. The paths (c, A_1, b_2) (c, c_1, b_2) lead to payoffs (0, 5) and occur, each, with probability $\frac{1}{3} \times \frac{1}{2}$, as, by assumption, from the singleton node at the end of (c), P1 chooses between A_1 and c_1 with probability $\frac{1}{2}$. This of course is not significant because any other probabilities attached to A_1 and c_1 would not affect the payoffs.

Summarizing, we note that the implied equilibrium paths are as follows. If nature chooses a or b, P1 responds by playing c_1, i.e. he lies. Then P2 lies from $I_2^{'}$ and from the singleton node at the end of (b, c_1) he can tell the truth or lie. The players end up with their initial endowments. If nature chooses c, P1 can tell the truth, or even lie, but P2 will play b_2, i.e. he will lie. Again the players end up with their initial endowments. It follows that for all choices by nature, at least one of the players tells a lie on the optimal play. The players by lying avoid the possibility of having to make a payment to their opponent.

We have constructed an extensive form game and employed reasonable rules for calculating payoffs and shown that the proposed allocation (5, 2.5, 2.5) will not be realized. The same conclusion would have been reached if P2 were assumed to move first.

6 The implementation of private core allocations

Next we investigate the role of P3 in the implementation, or realization, of private core allocations in Example 3.1 of Section 3. We have seen that such core allocations are CBIC, which is a desirable property of the cooperative solution. We shall now show how they can be supported as perfect Bayesian equilibrium of a noncooperative game. This falls into the agenda of the Nash programme.

We use as an example the private core allocation

$$\begin{pmatrix} 4 & 4 & 1 \\ 4 & 1 & 4 \\ 2 & 0 & 0 \end{pmatrix}.$$

The ith line refers to Player i and the columns from left to right to states a, b and c.

When P3 enters the scene he is characterized by $e_3 = (0, 0, 0)$ with $\mathscr{F}_3 = \{\{a\}, \{b\}, \{c\}\}$. P3 announces his observation and this implies that, if he is believed, P1 and P2 will now be able to figure out all states of nature. We shall show how the payoffs of the matrix above will be realized from the optimal decisions of the players in a sequential game.

P1 and P2 see on a screen the announced state but P1 cannot distinguish between states a and b and P2 between a and c. P3 sees the correct state and moves first. However he can either announce exactly what he saw or he can lie. Obviously he can lie in two ways. Following the announcement of P3 it is the turn of P1 to act. When he comes to decide he has his information from the screen and also he knows the strategy that P3 played. Then it is the turn of P2 to act. When he comes to decide he has his information from the screen and he also knows what P3 and P1 played before him. Both P1 and P2, when it is their turn to act, can either tell the truth about what they saw on the screen or they can lie.

We must distinguish between the announcements of the players designed to maximize their expected returns, and the true state of nature. The former, with the players' temptations to lie, cannot be used to determine the true state which is needed for the purpose of making payoffs, which include any imposition of penalties for lying. P3 has a special status but he should also take into account that in the end the lie will be detected and this can affect his payoff. The terms of the contract, which we propose to examine below, take this into account.

The *rules* of calculating payoffs, i.e. the terms of the contract, are as follows: If P3 tells the truth we implement the redistribution in the matrix above which is proposed for this particular choice of nature.
If P3 lies then we look into the strategies of P1 and P2 and decide as follows:

(i) If the declaration of P1 and P2 are incompatible we go to the initial endowments and each player keeps his.
(ii) If the declarations are compatible we expect the players to honour their commitments for the state in the overlap, using the endowments of the true state, provided these are positive. If a player's endowment is zero then no transfer from that agent takes place as he has nothing to give.

We are looking for a perfect Bayesian equilibrium, i.e. a set of optimal behavioral strategies consistent with a set of beliefs. The beliefs are indicated by the probabilities attached to the nodes of the information sets in Figure 4 with arbitrary r, s, q, p and t between 0 and 1. Given these beliefs optimal decisions of P2 are indicated with heavy lines and the tree in Figure 4 folds up to the one in Figure 5. In this, optimal decisions of P1 are indicated with heavy lines. Figure 5 then folds up into Figure 6 which shows with heavy lines optimal decisions of P3.

In summary, an optimal behavioral strategy for P3 is to tell the truth, i.e. to play, with probability 1, a from a, b from b and c from c. An optimal behavioral strategy for P1 is to play A_1 from both I_1^1 and I_1^2, i.e. to tell the truth, and to play c_1 from I_1^3, i.e. to lie. From the singletons he plays c_1, i.e. he tells the truth. Finally optimal behavioral strategy for P2 is to play b_2 from the singletons, i.e. to tell the truth, to play A_2 from I_2^1 and I_2^6, i.e. to tell the truth, and to play b_2 from

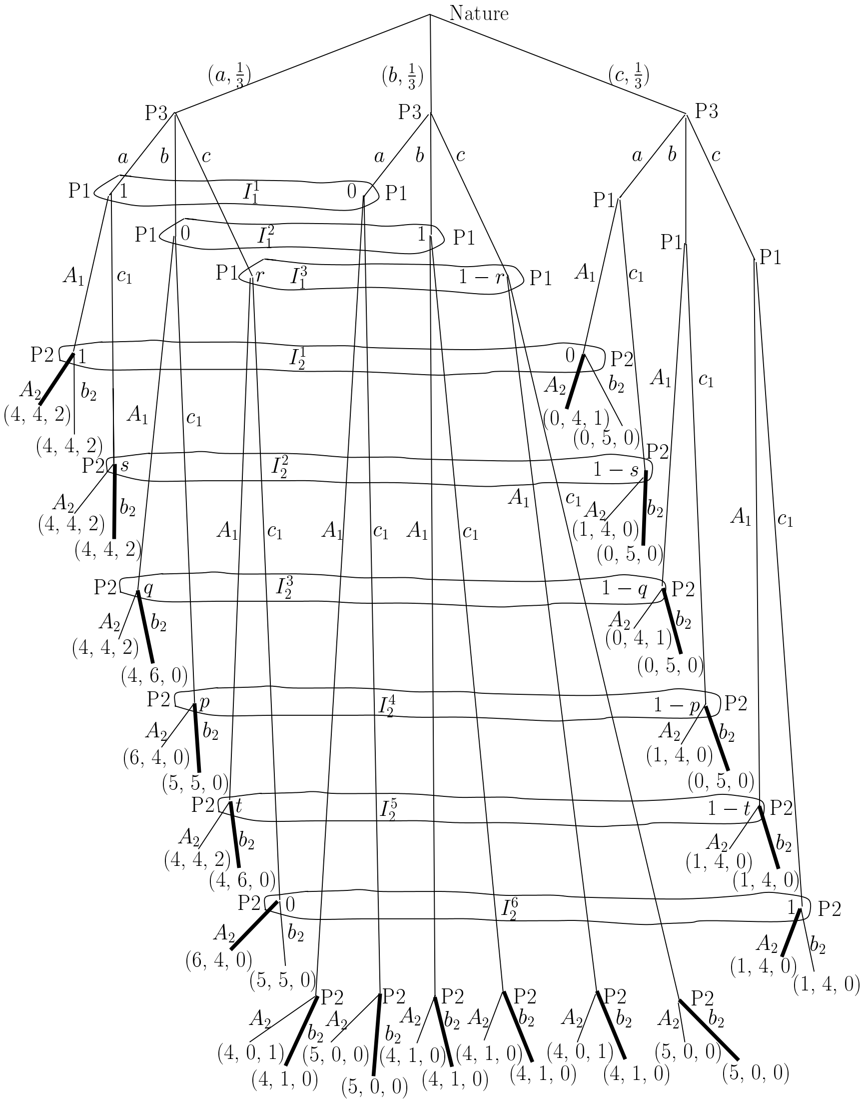

Figure 4

I_2^2, I_2^3, I_2^4 and I_2^5, i.e. to lie. Each player is rational and reaches the conclusion that P3 has no incentive to lie, before any revelation of the actual state of nature.

It is possible to check that the beliefs indicated next to the nodes are consistent with these strategies. Hence optimal behavioral strategies and beliefs form a perfect Bayesian equilibrium. We note that the implied equilibrium paths are as follows. If nature chooses a, P3 follows with a, P1 responds with A_1 and P2 declares A_2, and the payoffs are (4, 4, 2). If nature chooses b, P3 follows with b, P1 responds with A_1 and P2 declares b_2, and the payoffs are now (4, 1, 0).

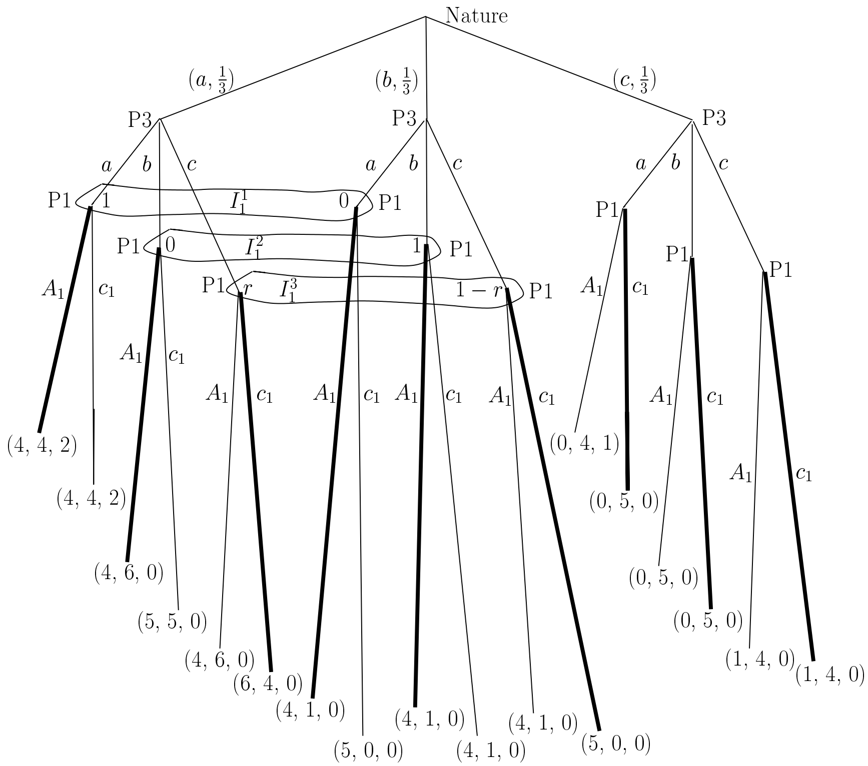

Figure 5

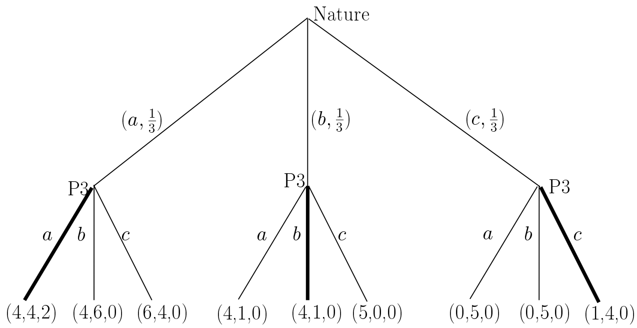

Figure 6. This figure sums up the implications of the optimal strategies used by the players. The payoffs at the end of the heavy lines correspond to these strategies and they are realizable by the equilibrium paths along which no player has an incentive to lie. The private core allocation is incentive compatible

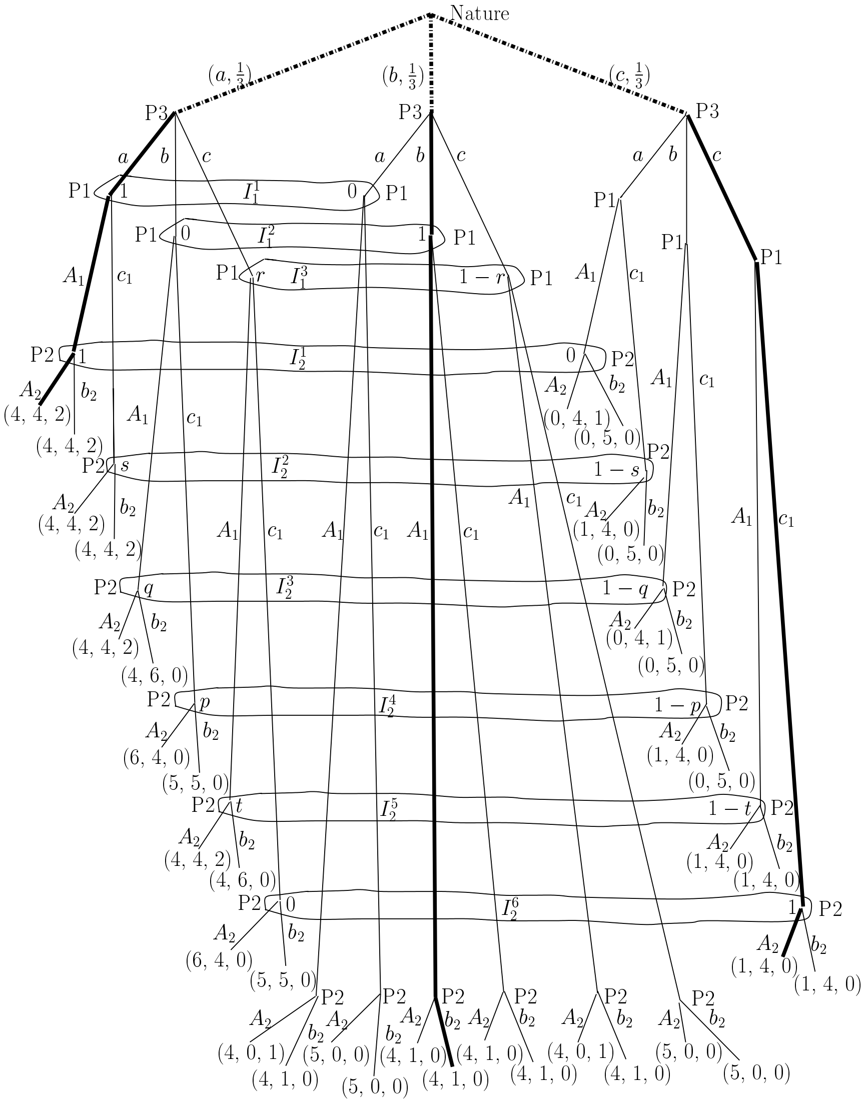

Figure 7

Finally if nature chooses c, P3 plays c, P1 follows with c_1 and P2 responds with A_2, and the payoffs are (1, 4, 0).

Along the optimal paths nobody has an incentive to misrepresent the realized state of nature and hence the private core allocation is incentive compatible. On the other hand the explicit considerations through a game tree show clearly that even optimal behavioral strategies, which of course are fully rational, can imply that players might have an incentive to lie from certain information sets, which though are not visited by the optimal play of the game. For example, P1, although he knows that nature has chosen a or b, has an incentive to declare c_1 from I_1^3, trying to take advantage of a possible lie by P3. Similarly P2, although he knows that nature has chosen a or c, has an incentive to declare b_2 from I_2^2, I_2^3, I_2^4 and I_2^5, trying to take advantage of possible lies by the other players. For example, the right hand side node of I_2^3 is reached by both P3 and P1 lying. Incentive compatibility has now been defined to allow that the optimal behavioral strategies can contain lies, while there must be an optimal play which does not.

In Figure 7 we indicate through heavy lines the equilibrium paths obtained above. Again, the interrupted heavy lines at the beginning of the tree signify that nature does not take an optimal decision, as it has no payoff function, but simply chooses among three alternatives, with equal probabilities. The directed paths (a, a, A_1, A_2) with payoffs (4, 4, 2), (b, b, A_1, b_2) with payoffs (4, 1, 0) and (c, c, c_1, A_2) with payoffs (1, 4, 0) occur, each, with probability $\frac{1}{3}$. It is clear that nobody lies on the optimal paths and that the proposed reallocation is incentive compatible and hence it will be realized.

Off the equilibrium strategies even P3 has considered the possibility of lying. For example when nature chooses b he would consider playing a, hoping that P1 will respond with A_1 and P2 with A_2. However such a move is dismissed because he knows that the other players are rational.

Analogous conclusions as above would have been reached if, following the announcement of P3, it was assumed that P2 moves first.

7 Concluding remarks

We consider the area of incomplete and differential information and how it is modeled important for the development of economic theory. Efforts are being made in breaking new ground using formulations which are promising but rather difficult. It is hoped that the use of game trees in the analysis helps in the development of ideas in that it makes them more discussable.

Our discussion in Section 5 suggests that core notions which may not be CBIC, i.e., the weak fine core, cannot easily be supported as a perfect Bayesian equilibrium. On the other hand, as we saw in Section 6, the private core which is CBIC can be supported as a perfect Bayesian equilibrium. The discussion above provides a noncooperative interpretation or foundation of the private core while making, through the game tree, the individual decisions transparent. In this way a better and possibly deeper understanding of how CBIC contracts are formed is obtained.

The positive result for the private core is not a general theorem but rather a 3-agent differential information economy example. However we believe that graph theory techniques may be adopted to construct a general result. We have not attempted this since it would complicate the technical analysis while it is not certain that it would advance our economic insights or knowledge very much. At the moment we leave this as an open question.

Appendix I: Proof of Theorem 3.1

Before we engage in the proof of Theorem 3.1, we will need some definitions. Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space, and X be a Banach space. Following Diestel and Uhl (1977), the function $f : \Omega \to X$ is called *simple* if there exist $x_1, x_2, \ldots, x_n$ in X and $A_1, A_2, \ldots, A_n$ in $\mathscr{F}$ such that $f = \sum_{i=1}^{n} x_i \mathscr{X}_{A_i}$ where $\mathscr{X}_{A_i}$ denotes the indicator function. A function $f : \Omega \to X$ is said to be μ-*measurable* if there exists a sequence of simple functions $f_n : \Omega \to X$ such that $\lim\limits_{n\to\infty} \| f_n(\omega) - f(\omega) \| = 0$ for almost all $\omega \in \Omega$. A μ-measurable function $f : \Omega \to X$ is *Bochner integrable* if there exists a sequence of simple functions $\{f_n : n = 1, 2, ...\}$ such that

$$\lim_{n\to\infty} \int_{\Omega} \| f_n(\omega) - f(\omega) \| \, d\mu(\omega) = 0. \tag{I.1}$$

In this case, for each $A \in \mathscr{F}$, we define the integral to be

$$\int_A f(\omega) d\mu(\omega) = \lim_{n\to\infty} \int_A f_n(\omega) d\mu(\omega). \tag{I.2}$$

The integral is of course independent of the approximating sequence of simple functions.[9]

[9] Let $\Phi = \{f_n : n = 1, \ldots, n\}$ be a sequence of simple functions from Ω to X for which $\lim \int f_n(\omega) d\mu(\omega)$ exists with respect to the norm topology and take this limit to define a quantity $I(\Phi)$. We can certainly use linearity, particularly in the form

$$I(\Phi - \mathscr{G}) = I(\Phi) - I(\mathscr{G}), \tag{I.3}$$

for *any* two such sequences.

We have defined that a μ-measurable function is Bochner integrable if there exists a sequence Φ for which

$$\lim_{n\to\infty} \int_{\Omega} \|f_n(\omega) - f(\omega)\| d\mu(\omega) = 0. \tag{I.4}$$

Now we argue that if two sequences Φ, $\mathscr{G}$ both satisfy this, for some given f, then $I(\Phi) = I(\mathscr{G})$. This will establish that the value obtained only depends upon f and so can be used to define its integral. We proceed as follows.

$$\begin{aligned}
\|I(\Phi) - I(\mathscr{G})\| &= \|I(\Phi - \mathscr{G})\| \quad \text{from (I.3)} \\
&= \| \lim_{n\to\infty} \int_{\Omega} \left[f_n(\omega) - g_n(\omega)\right] d\mu(\omega)\| \quad \text{from (I.4)} \\
&= \lim_{n\to\infty} \| \int_{\Omega} \left[f_n(\omega) - g_n(\omega)\right] d\mu(\omega)\| \\
&\leq \lim_{n\to\infty} \int_{\Omega} \|f_n(\omega) - g_n(\omega)\| d\mu(\omega) \quad \text{see Note * below} \qquad \text{(I.5)} \\
&= \lim_{n\to\infty} \int_{\Omega} \| \left[f_n(\omega) - f(\omega)\right] - \left[g_n(\omega) - f(\omega)\right] \| d\mu(\omega) \\
&\leq \lim_{n\to\infty} \int_{\Omega} \|f_n(\omega) - f(\omega)\| d\mu(\omega) + \int_{\Omega} \|g_n(\omega) - f(\omega)\| d\mu(\omega) \\
&= 0.
\end{aligned}$$

Then $\|I(\Phi) - I(\mathscr{G})\| = 0$ implies $I(\Phi) = I(\mathscr{G})$.

It can be shown (see Diestel and Uhl, 1977), Theorem 2, pp. 45) that if $f : \Omega \to X$ is a μ-measurable function, then f is Bochner integrable if and only if $\int_\Omega \| f(\omega) \| \, d\mu < \infty$. It is important to note that the *Dominated Convergence Theorem* holds for Bochner integrable functions. In particular, if $\{f_n : \Omega \to X : n = 1, 2, ...\}$ is a sequence of Bochner integrable functions such that $\lim_{n\to\infty} f_n(\omega) = f(\omega)$ μ-a.e., and $\| f_n(\omega) \| \leq g(\omega)$ μ-a.e., where $g : \Omega \to R$ is an integrable function, then f is Bochner integrable and $\lim \int_\Omega \| f_n(\omega) - f(\omega) \| \, d\mu(\omega) = 0$, (see Diestel and Uhl, 1977), Theorem 3, pp. 45).

Denote by $L_p(\mu, X)$ with $1 \leq p < \infty$ the space of equivalence classes of X-valued Bochner integrable functions $x : \Omega \to X$ normed by

$$\| x \|_p = \left(\int_\Omega \| x(\omega) \|^p \, d\mu(\omega)\right)^{\frac{1}{p}} < \infty. \tag{I.6}$$

It is a standard result that normed by the functional $\| \, . \, \|_p$ above, $L_p(\mu, X)$ becomes a Banach space (see Diestel and Uhl, 1977, p. 50). It is also well-known that $L_q(\mu, X^*)$ is the dual of $L_p(\mu, X)$, where $1 \leq p < \infty$ and $1/p + 1/q = 1$, and the value $w \cdot x$ of $x \in L_p(\mu, X)$ at $w \in L_q(\mu, X^*)$ is defined by

$$w \cdot x = \int_\Omega [w(\omega) \cdot x(\omega)] d\mu(\omega). \tag{I.7}$$

Recall that $\sigma(L_p(\mu, X), L_q(\mu, X^*))$ is defined as the weakest topology on $L_p(\mu, X)$ for which a net $\{x^\lambda : \lambda \in \Lambda\}$ converges to x if and only if $\{w \cdot x^\lambda\} \to w \cdot x$ for all $w \in L_q(\mu, X^*)$. We call this topology as *weak topology* and the convergence as *weak convergence*. A function $f : X \to R$ is *weakly upper semicontinuous* if $\limsup f(x^\lambda) \leq f(x)$, *weakly lower semicontinuous* if $\liminf f(x^\lambda) \geq f(x)$, and weakly continuous if it is both weakly upper semicontinuous and weakly lower semicontinuous, whenever $\{x^\lambda\} \to x$ weakly.

We now define a Banach lattice (see Aliprantis and Burkinshaw, 1985). A Banach space X is a Banach lattice if there is an ordering $\geq$ on X with the following properties:

(i) $x \geq y$ implies $x + z \geq y + z$ for every $z \in X$,
(ii) $x \geq y$ implies $\lambda x \geq \lambda y$ for $\lambda \in \mathbb{R}_+$,
(iii) for all $x, y \in X$, there exist a supremum $x \vee y$ and an infimum $x \wedge y$,
(iv) $| x | \geq | y |$ implies $\| x \| \geq \| y \|$ for every $x, y \in X$.

If X is a Banach lattice[10] then for any $x, y \in X$, define the *order interval* $[x, y] = \{z \in X : x \leq z \leq y\}$. Note that $[x, y]$ is convex and norm closed, hence weakly closed (Mazur's Theorem). Cartwright (1974) has shown that if X is a Banach lattice with an *order continuous norm*[11] (or equivalently has weakly

Note * : This inequality can be used since it only involves the finite sum employed in the definition of the integral of a simple function.

[10] An example of a Banach lattice is $\mathbb{R}^n$ with the usual vector partial ordering, the sum of moduli as norm, and absolute value of an element, the vector of absolute values of its coordinates.

[11] $\{x^\lambda\} \downarrow 0$ means that $\{x^\lambda : \lambda \in \Lambda\}$ is a decreasing net with $\inf x^\lambda = 0$. A Banach lattice X is said to have an *order continuous norm* if $\{x^\lambda\} \downarrow 0$ in X implies $\| x^\lambda \| \downarrow 0$. If X is a Banach lattice, X has an order continuous norm if and only if any order interval is weakly compact.

compact order intervals), then $L_p(\mu, X)$ with $1 \leq p < \infty$ has weakly compact order intervals as well. With the above preliminaries out of the way we can proceed with the proof.

Proof of Theorem 3.1. For each $i = 1, 2, \ldots, n$ let L_{X_i} be the set of all Bochner integrable and $\mathscr{F}_i$-measurable selections from the consumption set correspondence $X_i : \Omega \to 2^{Y_+}$ of Player i, i.e.

$$L_{X_i} = \{x_i \in L_1(\mu, Y) : \ x_i(\cdot) \text{ is } \mathscr{F}_i\text{-measurable and } x_i(\omega) \in X_i(\omega)\ \mu\text{-}a.e.\}. \tag{I.8}$$

This means that for each agent we select from her consumption correspondence an element per ω and form a function. We require this function to be in $L_1(\mu, Y)$, and measurable with respect to the agent's information partition.

Since by assumption each $e_i : \Omega \to Y$ is $\mathscr{F}_i$-measurable and Bochner integrable, it follows that $e_i \in L_{X_i}$ for all i. Therefore each L_{X_i} is non-empty and so is $L_X = \prod_{i=1}^{n} L_{X_i}$.

For each i, define the correspondence $P_i : L_{X_i} \to 2^{L_{X_i}}$ by

$$P_i(x_i) = \{y_i \in L_{X_i} : \ v_i(y_i) > v_i(x_i)\}. \tag{I.9}$$

Since for each i, and each fixed $\omega \in \Omega$, $u_i(\omega, \cdot)$ is concave, upper semicontinuous (u.s.c.) and integrably bounded, by Theorem 2.8 in Balder and Yannelis, $v_i(\cdot)$ is weakly-u.s.c. Hence, the set

$$P_i^{-1}(y_i) = \{x_i \in L_{X_i} : \ y_i \in P_i(x)\} = \{x_i \in L_{X_i} : v_i(y_i) > v_i(x_i)\} \tag{I.10}$$

is weakly open in L_{X_i}. Notice that since for any fixed $\omega \in \Omega$, $u_i(\omega, \cdot)$ is concave the set $P_i(x_i)$ for all $x_i \in L_{X_i}$ is convex and also $x_i \notin P_i(x_i)$ for all $x_i \in L_{X_i}$. Hence the correspondence $P_i : L_{X_i} \to 2^{L_{X_i}}$ is convex valued and irreflexive.

We now have an infinite dimensional commodity space economy

$$\bar{\mathscr{E}} = \{(L_{X_i}, P_i, e_i) : \ i = 1, 2, \ldots, n\} \tag{I.11}$$

where

(a) L_{X_i} denotes the consumption set of Pi,
(b) $P_i : L_{X_i} \to 2^{L_{X_i}}$ is the preference correspondence of Pi, and
(c) $e_i \in L_{X_i}$, is the initial endowments of Pi.

In the new economy that has been constructed, a good is also characterized by the state of nature, and $v_i(x_i)$, on which the preference correspondence is based, can be thought of as a utility, rather than an expected utility, function. It is as if uncertainty and information partitions have vanished from the scene. However they are present since L_{X_i}, the consumption set of Agent i, takes into account the information partition $\mathscr{F}_i$.

We show that a core allocation exists in $\bar{\mathscr{E}}$, i.e., there exists $x^* \in L_X$ satisfying the following two conditions:

(1) $\sum_{i=1}^{n} x_i^* = \sum_{i=1}^{n} e_i$, and

(2) there do not exist coalition S and allocation $(y_i)_{i\in S} \in \prod_{i\in S} L_{X_i}$ such that $\sum_{i\in S} y_i = \sum_{i\in S} e_i$, $y_i \in P_i(x_i^*)$ for all $i \in S$.

It can easily be checked that the existence of a core allocation in $\overline{\mathscr{E}}$ implies the existence of a private core allocation in the original economy $\mathscr{E} = \{((\Omega, \mathscr{F}, \mu), X_i, \mathscr{F}_i, u_i, e_i) : i = 1, \ldots, n\}$.

Let $\mathscr{A}$ be the set of all finite dimensional subspaces of $L_1(\mu, Y)$ containing the initial endowments. For each $\alpha \in \mathscr{A}$ define $L_{X_i}^{\alpha} = L_{X_i} \cap \alpha$ and $P_i^{\alpha} : L_{X_i}^{\alpha} \to 2^{L_{X_i}^{\alpha}}$ by $P_i^{\alpha}(x_i) = P_i(x_i) \cap \alpha$. We have constructed an economy $\overline{\mathscr{E}}^{\alpha} = \{(L_{X_i}^{\alpha}, P_i^{\alpha}, e_i) : i = 1, 2, \ldots, n\}$ in a finite dimensional commodity space where

(1′) $L_{X_i}^{\alpha}$ is the consumption set of Pi,

(2′) $P_i^{\alpha} : L_{X_i}^{\alpha} \to 2^{L_{X_i}^{\alpha}}$ is the preference correspondence of Pi,

(3′) $e_i \in L_{X_i}^{\alpha}$ is the initial endowment of Pi.

The economy constructed is finite dimensional in that each consumption set can be spanned by a finite number of vectors. For every such, finite dimensional, α-economy one can prove the existence of a core allocation. This implies in the limit, as the number of dimensions tends to infinity, the existence of a core allocation for $\overline{\mathscr{E}}$, which has been approximated through the net of economies.

It can easily be checked that for each $\alpha \in \mathscr{A}$, $\overline{\mathscr{E}}^{\alpha}$ satisfies all the assumptions of Florenzano's (1989) core existence theorem and therefore there exists $x^{\alpha} \in \prod_{i=1}^{n} L_{X_i}^{\alpha} = L_X^{\alpha}$ such that

(4′) $\sum_{i=1}^{n} x_i^{\alpha} = \sum_{i=1}^{n} e_i$

(5′) and it is not true that there exist $S \subset \{1, 2, \ldots, n\}$ and $(y_i)_{i\in S} \in \prod_{i\in S} L_{X_i}^{\alpha}$ such that $\sum_{i\in S} y_i = \sum_{i\in S} e_i$ and $y_i \in P_i^{\alpha}(x_i^{\alpha})$ for all $i \in S$.

From (4′) it follows that for each $\alpha \in \mathscr{A}$ we have that every $x_i^{\alpha} \in [0, \sum_{i=1}^{n} e_i]$. Since by assumption Y is a Banach lattice with an order continuous norm by the Cartwright theorem so is $L_1(\mu, Y)$ and therefore we can conclude that the order interval $[0, \sum_{i=1}^{n} e_i]$ in $\sum_{i=1}^{n} L_{X_i}$ is weakly compact.

Direct the set $\mathscr{A}$ by inclusion so that $\{(x_1^{\alpha}, x_2^{\alpha}, \ldots, x_n^{\alpha}) : \alpha \in \mathscr{A}\}$ forms a net in $\prod_{i=1}^{n} L_{X_i}$. Since each x_i^{α} lies in $[0, \sum_{i=1}^{n} e_i]$ which is weakly compact we can extract a subnet

$$\{(x_1^{\alpha(m)}, x_2^{\alpha(m)}, \ldots, x_n^{\alpha(m)}) : m \in M\},$$

(where M is directed by "$\geq$"), from the net $\{(x_1^{\alpha} x_2^{\alpha}, \ldots, x_n^{\alpha}) : \alpha \in \mathscr{A}\}$ which converges weakly to some vector $(x_1, x_2, \ldots, x_n)$ in $[0, \sum_{i=1}^{n} e_i]$.

We will show that $(x_1, x_2, \ldots, x_n)$ is a core allocation for the economy $\bar{\mathscr{E}}$. Notice that since for each $m \in M$, $\sum_{i=1}^{n} x_i^{\alpha(m)} = \sum_{i=1}^{n} e_i$ and $x_i^{\alpha(m)}$ converges weakly to $x_i \in L_X$ we have that $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} e_i$, i.e., $(x_1, x_2, \ldots, x_n)$ is a feasible allocation.

In order to complete the proof we must show that:

($\star$) It is not true that there exists $S \subset \{1, 2, \ldots, n\}$ and $(y_i)_{i \in S} \in \prod_{i \in S} L_{X_i}$ such that $\sum_{i \in S} y_i = \sum_{i \in S} e_i$ and $y_i \in P_i(x_i)$ for all $i \in S$.

Suppose that ($\star$) is not true, then there exist coalition S and $(y_i)_{i \in S} \in \prod_{i \in S} L_{X_i}$ such that $\sum_{i \in S} y_i = \sum_{i \in S} e_i$ and $y_i \in P_i(x_i)$ for all $i \in S$. Since $x_i^{\alpha(m)}$ converges weakly to x_i and P_i has weakly open lower sections, there exists $m_0 \in M$ such that $y_i \in P_i(x_i^{\alpha(m)})$ for all $m \geq m_0$, and for all $i \in S$. Choose $m_1 \geq m_0$ so that if $m \geq m_1$, $y_i \in L_{X_i}^{\alpha(m)}$ for all $i \in S$. Then $y_i \in P_i^{\alpha(m)}(x_i^{\alpha(m)})$, for all $m \geq m_1$, and for all $i \in S$, a contradiction to $(5')$, which means that ($\star$) holds.

Finally, the fact that $\bar{\mathscr{E}}$ has been derived from the original economy $\mathscr{E}$ by integrating over the states of nature implies that a core allocation in the former is also a private core[12] allocation in the latter, and this completes the proof of Theorem 3.1.

Appendix II: The private core allocations of Example 3.1

In this section we show that the redistribution

$$\begin{pmatrix} 4 & 4 & 1 \\ 4 & 1 & 4 \\ 2 & 0 & 0 \end{pmatrix}.$$

where again the ith line refers to Player i and the columns from left to right to states a, b and c, is a private core allocation.

An $\mathscr{F}_i$-measurable redistribution of the endowments of the three agents above is given by

$$\begin{pmatrix} 5-\varepsilon & 5-\varepsilon & \delta_1 \\ 5-\delta & \varepsilon_1 & 5-\delta \\ \varepsilon+\delta & \varepsilon_2 & \delta_2 \end{pmatrix}$$

with $\varepsilon = \varepsilon_1 + \varepsilon_2$ and $\delta = \delta_1 + \delta_2$.

We show below that the private core allocations are obtained from

Problem

Maximize $\mathscr{U}_3 = (\varepsilon + \delta)^{\frac{1}{2}} + \varepsilon_2^{\frac{1}{2}} + \delta_2^{\frac{1}{2}}$

[12] This can easily be shown by contradiction. I.e. one picks a core allocation x in the economy $\bar{\mathscr{E}}$ and supposes that it is not a core allocation in $\mathscr{E}$ and reaches a contradiction.

Subject to

$$
\begin{aligned}
\mathscr{U}_1 &= 2(5-\varepsilon)^{\frac{1}{2}} + \delta_1^{\frac{1}{2}} \geq \alpha_1^{\frac{1}{2}} \\
\mathscr{U}_2 &= 2(5-\delta)^{\frac{1}{2}} + \varepsilon_1^{\frac{1}{2}} \geq \alpha_2^{\frac{1}{2}}, \\
&\varepsilon_1 + \varepsilon_2 = \varepsilon \leq 5 \text{ and } \delta_1 + \delta_2 = \delta \leq 5, \\
&\varepsilon_i, \delta_i \geq 0
\end{aligned}
$$

for Pareto optimality, and $\alpha_1^{\frac{1}{2}}, \alpha_2^{\frac{1}{2}} \geq 2(5^{\frac{1}{2}}) = 20^{\frac{1}{2}}$ for individual rationality. We shall not give characterizations of optimality through Lagrange or Kuhn-Tucker conditions because the utility functions, although continuous on their domains of definition, are not differentiable at the origin.

The solution to the problem exists because of the compactness of the feasible set, which follows from the fact that the values of all variables are bounded between 0 and 5 and the set defined by the utility constraints is closed, and the maximum is unique due to the concavity of the functions. Pareto optimality of the solution follows from the fact that there is no possible improvement to the values of all three utility functions, because if there were then we could increase the value of $\mathscr{U}_3$ without violating the constraints. Individual rationality follows from the fact that the initial endowments of the players imply utility $2(5^{\frac{1}{2}}) = 20^{\frac{1}{2}}$. Finally it is not possible for any pair of traders to redistribute their initial endowments and become better off, while retaining measurability. Hence the solution to the Problem is in the core.

Next we note that the solution to the problem above always satisfies the utility constraints of P1 and P2 with equality. For suppose, say, the first constraint was satisfied with an inequality. Then it would be possible to increase ε and ε_2, without disturbing measurability, and thus increase $\mathscr{U}_3$.

The question arises whether there exist core allocations which cannot be captured as solutions to a problem of the above type. Consider any allocation in the core and formulate the above problem with $\mathscr{U}_1$, $\mathscr{U}_2$ taking the corresponding values. From the fact that it is maximized, we should get for $\mathscr{U}_3$ at least the value of the proposed allocation, and if we actually obtain a higher one then it must be for a different allocation for at least one of the utilities, say $\mathscr{U}_1$. Now through concavity we can improve the proposed values of $\mathscr{U}_3$ and $\mathscr{U}_1$ and then through a redistribution from $\mathscr{U}_1$ to $\mathscr{U}_2$ and $\mathscr{U}_3$, by a small increase in ε and ε_1, we can improve all utilities in relation to the proposed allocation, which therefore was not Pareto optimal.

We shall now discuss properties of the private core allocations. We shall call symmetric allocations those with $\varepsilon = \delta$ and $\varepsilon_i = \delta_i$. First we consider the case when $\alpha_1 = \alpha_2 = \alpha$. This condition implies that the solution is symmetric. For otherwise it would not be unique. A further restriction on the symmetric solutions is when $\varepsilon_1 = \varepsilon$. In order to investigate this we look at the function $y = 2(5-\varepsilon)^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}}$. It is routine to show that it is a strictly concave function attaining its maximum value 5 at $\varepsilon = 1$. We are interested in the values of ε for which $y \geq 20^{\frac{1}{2}}$.

Suppose now that the common value of α is equal to 25 which is the maximum possible such value, since $\alpha^{\frac{1}{2}} = 2(5-\varepsilon)^{\frac{1}{2}} + \varepsilon_1^{\frac{1}{2}} \leq y = 2(5-\varepsilon)^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} \leq (25)^{\frac{1}{2}}$. Then we must have $\varepsilon_1 = \varepsilon = 1$, for otherwise the constraint will not be satisfied. The implied value for $\mathscr{U}_3$ is $2^{\frac{1}{2}}$ and this confirms that the redistribution at the beginning of this appendix is a private core allocation.

Next let the common admissible value of α be less than 25. We investigate whether it is now possible that the solution implies $\varepsilon_1 = \varepsilon$. In such a case the structure of the function y above would mean that there are two such values of ε, one smaller and one greater than 1. But then by strict concavity of the functions we could obtain feasible ε which would satisfy the constraint and increase the value of the objective function. It follows that although the solution is symmetric we do not have $\varepsilon_1 = \varepsilon$ which would have implied the corner solution $\varepsilon_2 = 0$.

Finally we look at the case where $\alpha_1 \neq \alpha_2$. Obviously the solution cannot be symmetric. The question arises whether we should have $\varepsilon_1 = \varepsilon$ and $\delta_1 = \delta$. On a (δ, ε) plane we consider the iso-level curves $2(5-\varepsilon)^{\frac{1}{2}} + \delta^{\frac{1}{2}} = 5$ and $2(5-\delta)^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} = 5$. In this plane the first one is a concave and the second a convex function. Their unique common point is (1, 1). Now consider a slightly lower in value iso-level curve of the second type while the one of the first type stays the same. The two curves cross at a point with $\varepsilon, \delta > 1$ and $\varepsilon < \delta$. However there is no obvious reason why in the solution of the maximization problem we should have both $\varepsilon_1 = \varepsilon$ and $\delta_1 = \delta$.

References

Aliprantis, C. D., Burkinshaw, O.: Positive operators. New York London: Academic Press 1985

Allen., B.: Market games with asymmetric information and non-transferable utility: representation results and the core. Mimeo (1991)

Aumann. R. J.: Correlated equilibria as an expression of Bayesian rationality. Econometrica **55**, 1–18 (1987)

Balder, E. J., Yannelis, N. C. : On the continuity of expected utility. Economic Theory **3**, 625–643 (1993)

Binmore, K.: Nash bargaining theory I, ICERD Discussion Paper No 9, London School of Economics, pp. 1–40 (1980a). Published in: Binmore, K., Dasgupta, P. (eds.) The economics of bargaining, pp. 27–46. Oxford: Blackwell 1987

Binmore, K.: Nash bargaining theory II, ICERD Discussion Paper No 14, London School of Economics, pp. 1–28 (1980b). Published in: Binmore, K., Dasgupta, P. (eds.) The economics of Bargaining, pp. 61–76. Oxford: Blackwell 1987

Cartwright, D.: The order completeness of some spaces of vector valued functions. Bull. Aust. Math. Soc. **11**, 57–61 (1974)

Diestel, J., Uhl, J. J.: Vector measures. In: Mathematical surveys, vol. 15. Providence, RI: American Mathematical Society 1997

Florenzano, M.: On the nonemptiness of the core of a coalitional production economy without ordered preferences. Journal of Mathematical Analysis and Applications **141**(2), 207–219 (1989)

Hahn, G., Yannelis, N.C.: Coalitional Bayesian Nash implementation in differential information economies. Economic Theory (this issue) (2001)

Koutsougeras, L., Yannelis, N.C.: Incentive compatibility and information superiority of the core of an economy with differential information. Economic Theory **3**, 195–216 (1993)

Krasa, S., Yannelis, N. C.: The value allocation of an economy with differential information. Econometrica **62**, 881–900 (1994)

Kreps, M.D., Wilson, R.: Sequential equilibrium. Econometrica **50**, 889–904 (1982)

Lagunoff, R.: A simple noncooperative core story. Games and Economic Behaviour **7**, 54–61 (1994)
Lefebvre, I.: An alternative proof of the non-emptiness of the private core. Economic Theory (this issue) (2001)
Nash, J.F.: Two-person cooperative games. Econometrica **21**, 128–40 (1953)
Page, F.: Market games with differential information and infinite dimensional commodity spaces: the core. Economic Theory **9**, 151–159 (1997)
Perry, M., Reny, P. J.: A non-cooperative view of coalition formation and the core. Econometrica **62**, 795–817 (1994)
Serrano, R.: A market to implement the core. Journal of Economic Theory **67**, 285–294 (1995)
Serrano, R., Vohra, R.: Non-cooperative implementation of the core. Social Choice and Welfare **14**, 513–525 (1997)
Tirole, J.: The theory of industrial organization. Cambridge, MA London: MIT Press 1988
Trockel, W.: Implementations of the Nash solution based on its Walrasian characterization (revised version of former IMW working Paper No. 306). Economic Theory **16**, 277–294 (2000)
Wilson, R.: Information, efficiency, and the core of an economy. Econometrica **46**, 807–816 (1978)
Yannelis, N. C.: The core of an economy with differential information. Economic Theory **1**, 183–198 (1991)