A Bayesian equilibrium existence theorem

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Abstract. We generalize the social equilibrium existence theorem of Debr(
(1952) to a Bayesian setting. Our new Bayesian result includes as a speci
case not only the existence results of Debreu (1952) and Nash (1951) but al
the recent Bayesian equilibrium existence result of Kim-Yannelis (1997).

Key words: Bayesian abstruct economy, Bayesian equilibrium, Bochner
tegration, Fatou's lemma in infinite dimensions, weak compactness

1. Introduction

The equilibrium existence result of Nash (1951) for a normal form ga
was generalized by Debreu (1952) to a social system or abstract e
omy. The abstract economy was the main tool to prove the existen
a Walrasian equilibrium in concrete economy as Arrow-Debreu (
showed. There have been several generalization of the Nash equilil
existence theorems to a Bayesian framework (see Kim-Yannelis (1
for an extensive list of references).

The propose of this paper is to present a new generalization
Debreu (1952) existence theorem for a Bayesian abstract econo
social system). This theorem generalizes the results of Debreu (
Nash (1951), Kim-Yannelis (1997) and Yannelis [1998], also, ext
recent related result of Aliprantis-Tourky-Yannelis (2001).

We would like to comment on the difficulties that one faces in (
generalize the theorems of Debreu [1952] and Nash [1951] to a B

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abstract economy. There are two main technical obstacles that one must overcome. First, the strategy set is a random set-valued correspondence with weakly compact, convex values, and we are working with the set of all private information measurable and Bochner integrable selections from that set. The latter set must be shown that it is weakly compact. This is done by using an extension of Diestel's theorem on weak compactness (see Theorem 3.1 in Yannelis [1991b]). Secondly, one must show that the set of all Bochner integrable selections from an upper semicontinuous, (u.s.c.) correspondence, is also u.s.c. This result is similar to the one that integration preserves u.s.c. and one way to prove this theorem is by means of the Fatou Lemma in infinite dimensions (e.g., Yannelis [1991b, Section 5]). In addition to the two above main difficulties there are also other secondary technical details entailing to the measurability properties of certain set-valued functions. Those details are not routine and the Aumann measurable selection theorem, and the projection theorem will turn out to be of extreme importance in carrying out the arguments.

The paper proceeds as follows: The next Section contains definitions and some mathematical preliminaries. The main Bayesian equilibrium existence theorem and its proof are contained in Section 3. Finally, a comparison with the related literature and some remarks are given in Section 4.

2. Mathematical preliminaries

2.1 Notation

2^A denotes the set of all nonempty subsets of the set A.

$\setminus$ denotes the set theoretic subtraction.

$\Re^\ell$ denotes the ℓ-fold Cartesian product of the set of real numbers $\Re$.

$\emptyset$ denotes the empty set.

A^c denotes the complement of the set A.

clA denotes the closure of the set A.

2.2 Definitions

Let X and Y be sets. The graph G_ϕ of a correspondence $\phi : X \to 2^Y$ is the set $G_\phi = \{(x, y) \in X \times Y : y \in \phi(x)\}$. If X and Y are topological spaces, $\phi : X \to 2^Y$ is said to be *lower-semicontinuous* (l.s.c) if the set $\{x \in X : \phi(x) \cap V \neq \emptyset\}$ is open in X for every open subset V of Y; $\phi : X \to 2^Y$ is said to be *upper-semicontinuous* (u.s.c) if the set $\{x \in X : \phi(x) \subset V\}$ is open in X for every open subset V of Y.

If (X, α) and (Y, β) are measurable spaces and $\phi : X \to 2^Y$ is a correspondence, ϕ is said to have a *measurable graph* if G_ϕ belongs to the product σ-algebra $a \otimes \beta$. We are often interested in the situation where (X, α) is a measurable space, Y is a topological space and $\beta = \beta(Y)$ is the Borel σ-algebra of Y. For a correspondence ϕ from a measurable space into a topological space, if we say that ϕ has a measurable graph, it is understood that the topological space is endowed with its Borel σ-algebra (unless specified otherwise). In the same setting as above, i.e., (X, α), a measurable space and Y a topological space, ϕ is said to be *lower measurable* if $\{x \in X : \phi(x) \cap V \neq \emptyset\} \in \alpha$ for every V open in Y.

We now define the notion of a Bochner integrable function. We will follow closely Diestel-Uhl (1977). Let (T, τ, μ) be a finite measure space and X be a Banach space. A function $f : T \to X$ is called *simple* if there exist $x_1, x_2, \ldots, x_n$ in X and $\alpha_1, \alpha_2, \ldots, \alpha_n$ in τ such that $f = \sum_{i=1}^n x_i \chi_{\alpha_i}$, where $\chi_{\alpha_i}(t) = 1$ if $t \in \alpha_i$ and $\chi_{\alpha_i}(t) = 0$ if $t \notin \alpha_i$. A function $f : T \to X$ is said to be *μ-measurable* if there exists a sequence of simple functions $f_n : T \to X$ such that $\lim_{n\to\infty} \| f_n(t) - f(t) \| = 0$ for almost all $t \in T$. A μ-measurable function $f : T \to X$ is said to be *Bochner integrable* if there exists a sequence of simple functions $\{f_n : n = 1, 2, \ldots, \}$ such that

$$\lim_{n\to\infty} \int_T \| f_n(t) - f(t) \| \, d\mu(t) = 0.$$

In this case we define for each $E \in \tau$ the *integral* to be $\int_E f(t)\, d\mu(t) = \lim_{n\to\infty} \int_E f_n(t)\, d\mu(t)$. It can be shown [see Diestel-Uhl (1977), Theorem 2, p.45] that, if $\phi : T \to X$ is a μ-measurable function then f is *Bochner integrable* if and only if $\int_T \| f(t) \| \, d\mu(t) < \infty$. It is important to note that the *Dominated Convergence Theorem* holds for Bochner integrable functions, in particular, if $f_n : T \to X$ $(n = 1, 2, \ldots)$ is a sequence of Bochner integrable functions such that $\lim_{n\to\infty} f_n(t) = f(t)$ μ-a.e., and $\| f_n(t) \| \leq g(t)$ μ-a.e., where $g \in L_1(\mu, \Re)$, then f is Bochner integrable and $\lim_{n\to\infty} \int_T \| f_n(t) - f(t) \| \, d\mu(t) = 0$.

For $1 \leq p < \infty$, we denote by $L_p(\mu, X)$ the space of equivalence classes of X-valued Bochner integrable functions $x : T \to X$ normed by

$$\| x \|_p = \left(\int_T \| x(t) \|_p \, d\mu(t) \right)^{1/p}.$$

It is a standard result that normed by the functional $\| \cdot \|_p$ above, $L_p(\mu, X)$ becomes a Banach space [see Diestel-Uhl (1977), p.50]. We denote by S_ϕ^p the *set of all selections from* $\phi : T \to 2^X$ that belong to the space $L_p(\mu, X)$, i.e.,

$$S_\phi^p = \{x \in L_p(\mu, X) : x(t) \in \phi(t) \ \ \mu - a.e.\}.$$

We will also consider the set $S^1_\phi = \{x \in L_1(\mu, X) : x(t) \in \phi(t) \ \mu - a.e.\}$, i.e., S^1_ϕ is the *set of all Bochner integrable selections from* $\phi(\cdot)$. Recall that the correspondence $\phi : T \to 2^X$ is said to be *integrably bounded* if there exists a map $h \in L_1(\mu, \Re)$ such that $\sup\{\| x \| : x \in \phi(t)\} \leq h(t) \ \mu - a.e.$ Moreover, note that if T is a complete measure space, X is a separable Banach space and $\phi : T \to 2^X$ is an integrably bounded, nonempty valued correspondence having a measurable graph, then by virtue of the Aumann measurable selection theorem (see the next Section) we can conclude that S^1_ϕ is nonempty.

2.3 Preliminary theorems

The results below will be useful for the proof of our main theorem. We refer the reader to Yannelis [1991a, 1991b] for more details and further references..

Aumann measurable selection theorem. *Let (T, τ, μ) be a complete finite measure space, Y be a complete, separable metric space and $\phi : T \to 2^Y$ be a nonempty valued correspondence with a measurable graph, i.e., $G_\phi \in \tau \otimes \beta(Y)$. Then there is a measurable function $f : T \to Y$ such that $f(t) \in \phi(t) \ \mu - a.e.$*

Diestel's theorem. *Let (T, τ, μ) be a complete finite measure space, X be a separable Banach space and $\phi : T \to 2^X$ be an integrally bounded, convex, weakly compact and nonempty valued correspondence. Then S^1_ϕ is weakly compact in $L_1(\mu, X)$.*

Projection theorem. *Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite measure space and Y be a complete, separable metric space. If H belongs to $\mathcal{F} \otimes \beta(Y)$, its projection $Proj_\Omega(H)$ belongs to $\mathcal{F}$.*

It follows from the projection theorem that if a correspondence has a measurable graph then it is also lower measurable. The reverse is also true if the correspondence is closed valued (see Castaing-Valadier (1977, p. 80, Theorem III.30]).

The following Lemma (see also Yannelis [1991b, Theorem 5.4] for a similar argument) will be of fundamental importance for the proof of the main theorem in the next Section.

Lemma 2.1. *Let Y be a separable space, $(\Omega, \mathcal{F}, \mu)$ be a complete finite measure space and $X : \Omega \to 2^Y$ be an integrably bounded, nonempty, convex valued correspondence such that for all $\omega \in \Omega, X(\omega)$ is a weakly compact, convex subset of Y. Denote by L_X the set $\{x \in L_1(\mu, Y) : x(\omega) \in X(\omega) \ \mu - a.e.\}$. Let $\phi : \Omega \times L_X \to 2^Y$ be a nonempty, closed, convex valued correspondence such that $\phi(\omega, x) \subset X(\omega)$ for all $(\omega, x) \in \Omega \times L_X$. Assume that for each fixed $x \in L_X, \phi(\cdot, x)$ has a measurable graph and that for each fixed $\omega \in \Omega$, $\phi(\omega, \cdot) : L_X \to 2^Y$ is u.s.c. in the sense that the set $\{x \in L_X : \phi(\omega, x) \subset V\}$ is weakly open in L_X for every norm open subset V of Y. Define the correspondence $\Phi : L_X \to 2^{L_X}$ by*

$$\Phi(x) = \{y \in L_X : y(\omega) \in \phi(\omega, x) \ \mu - a.e.\}.$$

Then Φ is weakly u.s.c., i.e., the set $\{x \in L_X : \Phi(x) \subset V\}$ is weakly open in L_X for every weakly open subset V of L_X.

Proof. By Diestel's Theorem L_X as well as $\Phi(x)$ for each $x \in L_X$, endowed with the weak topology is compact. Since the weak topology of a weakly compact subset of a separable Banach space is metrizable (Dunford and Schwartz [1958, p.434]), L_X is a compact, metrizable space. Thus, it suffices to show that if x and $\{x_n : n = 1, 2, \dots\}$ belong to $L_X, \{x_n : n = 1, 2, \dots\}$ converges weakly to x, and V is a relatively weakly open subset of L_X containing $\Phi(x)$, then $\Phi(x_n) \subset V$ for all sufficiently large n. For if $U = \{z \in L_X : \Phi(z) \subset V\}$ is not relatively weakly open in L_X, we can pick some $x \in U$ such that every neighborhood of x in the (relative) weak topology of L_X contains an $\tilde{x} \notin U$. Hence, we can construct a sequence $\{x_n\}$ converging weakly to x such that $x_n \in L_X$ and $\Phi(x_n) \not\subset V$.

Let $\mathcal{B}$ and $\tilde{\mathcal{B}}$ denote the closed unit balls in Y and L_X, respectively and let $\varepsilon > 0$. It will suffice to show that for a suitable $n_0, \Phi(x_n) \subset \Phi(x) + \varepsilon\tilde{\mathcal{B}}$ for $n \geq n_0$. This is so, because every weakly open neighborhood of the weakly compact set $\phi(x)$ contains the norm neighborhood $\Phi(x) + \varepsilon\tilde{\mathcal{B}}$ for a suitable $\varepsilon > 0$.

We begin in finding the suitable n_0. Since for each $\omega \in \Omega, \phi(\omega, \cdot)$ is u.s.c, we can find a minimal N_ω such that

$$\phi(\omega, x_n) \subset \Phi(\omega, x) + \frac{\varepsilon}{3\mu(\Omega)\mathcal{B}} \qquad \text{for all } n \geq N_\omega. \tag{3.1}$$

Let $\varepsilon/3\mu(\Omega) = \delta_1$. By assumption, for fixed x and n, the correspondences $\phi(\cdot, x) : \Omega \to 2^Y$ and $\phi(\cdot, x_n) : \Omega \to 2^Y$ have measurable graphs. Thus by the projection theorem the set $\mathcal{Q}$ defined below belongs to $\mathcal{F}$:

$$\mathcal{Q} = proj_\Omega\{(\omega, y) \in \Omega \times Y : (\omega, y) \in G_{\phi(\cdot, x_n)} \cap (G_{\phi(\cdot, x) + \delta_1 \mathcal{B}})^c\} \in \mathcal{F}.$$

Note that

$$\begin{aligned} \mathcal{Q} &= \{\omega \in \Omega : \phi(\omega, x_n) \not\subset \phi(\omega, x) + \delta_1\mathcal{B}\} \\ &= \{\omega \in \Omega : \phi(\omega, x_n) \setminus \phi(\omega, x) + \delta_1\mathcal{B}) \neq \emptyset\}. \end{aligned}$$

This will enable us to conclude that N_ω is a measurable function of ω. This is clearly so, since,

$$\{\omega \in \Omega : N_\omega = m\} = \bigcap_{n \geq m} \{\omega \in \Omega : \phi(\omega, x_n) \subset \phi(\omega, x) + \delta_1 \mathcal{B}\} \cap \{\omega \in \Omega : \phi(\omega, x_{m-1}) \not\subset \phi(\omega, x) + \delta_1 \mathcal{B}\}.$$

We are now ready to choose the suitable n_0. Since $X(\cdot)$ is integrably bounded, there exists $g \in L_1(\mu, \Re)$ such that $\sup\{\| x \| : x \in X(\omega)\} \leq g(\omega)$. Pick δ_2 such that if $\mu(A) < \delta_2$, $(A \subset \Omega)$ then $\int_A g(\omega) d\mu(\omega) < \varepsilon/3$. Since N_ω is a measurable function of ω, we can choose n_0 such that $\mu(\{\omega \in \Omega : N_\omega \geq n_0\}) < \delta_2$. This is the desired n_0.

Let $n \geq n_0$ and $y \in \Phi(x_n)$. We must show that $y \in \Phi(x) + \varepsilon\tilde{\mathcal{B}}$.

Since $\phi(\cdot, x)$ has a measurable graph and it is nonempty valued by the Aumann measurable selection theorem, there is a measurable selection $z_1 : \Omega \to Y, z_1(\omega) \in \phi(\omega, x)$ $\mu - a.e.$ in Ω.

The correspondence

$$\psi(\omega) = (\{y(\omega)\} + \delta_1 \mathcal{B}) \cap \phi(\omega, x)$$

has a measurable graph and by (3.1) is nonempty valued for $\omega \in \Omega_0 = \{\omega : N_\omega \leq n_0\}$.

Thus by the Aumann measurable selection theorem, there is a measurable function $z_2 : \Omega \to Y$ such that $z_2(\omega) \in \psi(\omega)$ $\mu - a.e.$ in Ω_0. Let,

$$z(\omega) = z_1(\omega) \quad \text{for } \omega \notin \Omega_0 \qquad \text{and} \qquad z(\omega) = z_2(\omega) \quad \text{for } \omega \in \Omega_0.$$

Then $z(\omega) \in \phi(\omega, x)$ $\mu - a.e.$ and thus $z \in \Phi(x)$. If we show that $\| z - y \| < \varepsilon$, the proof of the Lemma would be complete.

However, this is easy to check. Indeed,

$$\begin{aligned} \| z - y \| &= \int_{\Omega \setminus \Omega_0} \| z_1(\omega) - y(\omega) \| d\mu(\omega) + \int_{\Omega_0} \| z_2(\omega) - y(\omega) \| d\mu(\omega) \\ &< 2 \int_{\Omega \setminus \Omega_0} g(\omega) d\mu(\omega) + \int_{\Omega_0} \delta_1 d\mu(\omega) \\ &< \frac{2}{3}\varepsilon + \delta_1 \mu(\Omega) = \frac{2}{3}\varepsilon + \frac{\varepsilon}{3\mu(\Omega)} \cdot \mu(\Omega) = \varepsilon. \end{aligned}$$

The proof of the lemma is now complete.

3. A Bayesian social equilibrium existence theorem

Let $(\Omega, \mathcal{F}, \mu)$ be a complete, finite, separable measure space, where Ω denotes the set of *states* of nature of the world and the σ-algebra $\mathcal{F}$, denotes the set of *events*. Let Y, denote the *strategy* or *commodity* space, where Y is a separable Banach space.

A *Bayesian abstract economy* (or *social system*) *with differential information* is a set $G = \{(X_t, u_t, \mathcal{F}_t, A_t, q_t) : t \in T\}$, where

1. $X_t : \Omega \to 2^Y$ is the *action* (*strategy*) set-valued function of agent t,
2. for each $\omega \in \Omega, u_t(\omega, \cdot) : \prod_{s \in T} X_s(\omega) \to \Re$ is the *random utility function* of agent t,
3. $\mathcal{F}_t$ is a sub σ-algebra of $\mathcal{F}$ which denotes the *private information* of agent t,
4. for each $\omega \in \Omega, A_t(\omega, \cdot) : \prod_{s \in T} X_s(\omega) \to 2^Y$ is the *random constraint set-valued function* of agent t, where for all $(\omega, x) \in \Omega \times \prod_{t \in T} X_t(\omega)$, $A_t(\omega, x) \subset X_t(\omega)$,
5. $q_t : \Omega \to \Re_{++}$ is the *prior* of agent t, which is a Radon-Nikodym derivative such that $\int_\Omega q_t(\omega) d\mu(\omega) = 1$.

Let $L_{X_t} = \{x \in L_1(\mu, Y) : x_t : \Omega \to Y$ is $\mathcal{F}_t$-measurable and $x_t(\omega) \in X_t(\omega)$ $\mu - a.e.\}$. Denote by L_X the set $\prod_{t \in T} L_{X_t}$ and by $L_{X_{-t}}$ the set $\prod_{s \neq t} L_{X_s}$. An element x_t of L_{X_t} is called a *strategy* for player (agent) t. The typical element of L_{X_t} is denoted by $\tilde{x}_t$ and that of $X_t(\omega)$ by $x_t(\omega)$ (or x_t). We assume that for each $t \in T$, there exists a countable partition $\prod_t$ of Ω. Moreover, the σ-algebra $\mathcal{F}_t$ is generated by $\prod_t$. For each $\omega \in \Omega$, let $E_t(\omega) \in \prod_t$ denote the smallest set in $\mathcal{F}_t$ containing ω and assume that for each t,

$$\int_{\omega' \in E_t(\omega)} q_t(\omega') d\mu(\omega') > 0.$$

The *interim expected utility* of agent t, $V_t(\omega, \cdot, \cdot) : L_{X_{-t}} \times X_t(\omega) \to \Re$ is defined as

$$V_t(\omega, \tilde{x}_{-t}, x_t) = \int_{\omega' \in E_t(\omega)} u_t(\omega', \tilde{x}_{-t}(\omega'), x_t) q_t(\omega' | E_t(\omega)) d\mu(\omega')$$

where

$$q_t(\omega' | E_t(\omega)) = \begin{cases} 0 & \text{if } \omega' \notin E_t(\omega) \\ \frac{q_t(\omega')}{\int_{\bar{\omega} \in E_t(\omega)} q_t(\bar{\omega}) d\mu(\bar{\omega})} & \text{if } \omega' \in E_t(\omega). \end{cases}$$

The function $V_t(\omega, \tilde{x}_{-t}, x_t)$ is interpreted as the (interim) *expected utility* of agent t, using his own action (*strategy*) x_t, where the state of nature is ω and the other agents employ the strategy profile $\tilde{x}_{-t}$.

A *Bayesian equilibrium* for G is a strategy profile $\tilde{x}^* \in L_X$ such that for all $t \in T$,

(i) $x_t^*(\omega) \in A_t(\omega, \tilde{x}^*)$ $\mu - a.e.$

(ii) $V_t(\omega, \tilde{x}^*_{-t}, \tilde{x}_t^*(\omega)) = \max_{y_t \in A_t(\omega, \tilde{x}^*)} V_t(\omega, \tilde{x}^*_{-t}, y_t)$ $\mu - a.e.$

Assumptions

(A.1)

(a) $X_t : \Omega \to 2^Y$ is a nonempty, convex, weakly compact-valued and integrably bounded correspondence.

(b) $X_t : \Omega \to 2^Y$ is $\mathcal{F}_t$-lower measurable, i.e., for every open subset V of Y, the set, $\{\omega \in \Omega : X_t(\omega) \cap V \neq \phi\}$ belongs to $\mathcal{F}_t$.[1]

(A.2)

(a) For each fixed x, $u_t(\cdot, x) : \Omega \to \Re$ is $\mathcal{F}$-measurable.

(b) For each $\omega \in \Omega$ and $x_{-t} \in \prod_{s \neq t} X_s(\omega)$, $u_t(\omega, x_{-t}, \cdot) : X_t(\omega) \to \Re$ is concave.

(c) u_t is integrably bounded.

(d) For each $\omega \in \Omega$, $u_t(\omega, \cdot, \cdot) : \prod_{s \neq t} X_s(\omega) \times X_t(\omega) \to \Re$ is continuous where $X_s(\omega)$, $s \neq t$, is endowed with the weak topology and $X_t(\omega)$ with the norm topology.

(A.3)

(a) For each $(\omega, x) \in \Omega \times \prod_{t \in T} X_t(\omega)$, $A_t(\omega, x)$ is lower measurable, i.e., for every open subset V of Y, the set $\{(\omega, x) \in \Omega \times \prod_{t \in T} X_t(\omega) : A_t(\omega, x) \cap V \neq \phi\}$ belongs to $\mathcal{F} \otimes \beta(\prod_{t \in T} X_t(\omega))$.

(b) For each $\omega \in \Omega$, $A_t(\omega, \cdot) : \prod_{t \in T} X_t(\omega) \to 2^Y$ is a weakly continuous correspondence with closed, convex, nonempty values.

Theorem 3.1. *Let T be a countable set. Let $G = \{(X_t, u_t, A_t, \mathcal{F}_t, q_t) : t \in T\}$ be a Bayesian abstract econom satisfying (A.1) - (A.3). Then there exists a Bayesian equilibrium for G.*

Proof. It follows from the Lebesgue dominated convergence theorem that $V_t(\omega, \cdot, \cdot) : L_{X_{-t}} \times X_t(\omega) \to \Re$ is continuous where $L_{X_s}, (s \neq t)$ is endowed with the weak topology and $X_t(\omega)$ with the norm topology, (see for example Lemma A.1, p.345 in Kim-Yannelis [1997]). Moreover, for each $(\tilde{x}_{-t}, x_t) \in L_{X_{-t}} \times X_t(\omega)$, $V_t(\cdot, \tilde{x}_{-t}, x_t) : \Omega \to \Re$ is $\mathcal{F}_t$-measurable and from the concavity of u_t in x_t it follows that V_t is also concave in x_t.

For each $t \in T$, define $\bar{A}_t : \Omega \times L_X \to 2^Y$ by $\bar{A}_t(\omega, \tilde{x}) = A_t(\omega, x)$. For each $t \in T$, define the correspondence

[1] Notice that in the presence of assumption A.1(a) the $\mathcal{F}_t$-lower measurability of $X_t(\cdot)$ is equivalent to the $\mathcal{F}_t$-measurable graph of $X_t(\cdot)$. This is also the case for the lower measurability assumption of the random constraint correspondence (assumption, A.3(a), below).

$\phi_t : \Omega \times L_{X_{-t}} \to 2^Y$ by

$$\phi_t(\omega, \tilde{x}_{-t}) = \{x_t \in \bar{A}_t(\omega, \tilde{x}) : V_t(\omega, \tilde{x}_{-t}, x_t) = \max_{y_t \in \bar{A}_t(\omega, \tilde{x})} V_t(\omega, \tilde{x}_{-t}, y_t)\}$$

It is easy to see that ϕ_t is nonempty valued. Indeed, since all the values of A_t are contained in the weakly compact valued correspondence X_t and A_t is (norm) closed and convex, so weakly closed, we can conclude that $\bar{A}_t$ is weakly compact valued. Since V_t is norm u.s.c. in $X_t(\omega)$ and also concave, it is weakly u.s.c. (Balder-Yannelis [1993, p. 629]) and therefore we can conclude that ϕ_t is nonempty valued. Moreover, it follows from Berge's maximum theorem that for each fixed $\omega \in \Omega$, $\phi_t(\omega, \cdot)$ is u.s.c. in the sense that the set $\{x_{-t} \in L_{X_{-t}} : \phi_t(\omega, x_{-t}) \subset V\}$ is weakly open in $L_{X_{-t}}$ for every norm open subset V of Y.

For each $t \in T$, define $\Phi_t : L_{X_{-t}} \to 2^{L_{X_t}}$ by

$$\Phi_t(\tilde{x}_{-t}) = \{\tilde{x}_t \in L_{X_t} : \tilde{x}_t(\omega) \in \phi_t(\omega, \tilde{x}_{-t}) \ \mu - a.e.\}$$

We must first show that Φ_t is nonempty valued. It follows from Lemma III.39 in Castaing-Valadier [1977] that for each fixed $\tilde{x}_{-t}$, $\phi_t(\cdot, \tilde{x}_{-t})$ has a measurable graph. Moreover, ϕ_t is nonempty valued. Hence, by the Aumann measurable selection theorem for each fixed $\tilde{x}_{-t} \in L_{X_{-t}}$ there exists an $\mathcal{F}_t$-measurable function $f_t : \Omega \to Y$ such that $f_t(\omega) \in \phi_t(\omega, \tilde{x}_{-t})$ $\mu - a.e.$. Since for each $(\omega, \tilde{x}_{-t}) \in \Omega \times L_{X_{-t}}$, $\phi_t(\omega, \tilde{x}_{-t})$ is contained in the integrably bounded correspondence $X_t(\cdot)$, then $f_t \in L_{X_t}$ and we conclude that $f_t \in \Phi_t(\tilde{x}_{-t})$ for every $\tilde{x}_{-t} \in L_{X_{-t}}$. Thus, Φ_t is nonempty valued and it is clearly convex valued since ϕ_t is so. A similar argument which the above can be followed to show that each L_{X_t} is nonempty and therefore so is L_X. By Diestel's Theorem L_{X_t} is a weakly compact subset of $L_1(\mu, Y)$. By virtue of Theorem 3 in Dunford-Schwartz [1958, p. 434], we conclude that L_{X_t} is metrizable. Since T is a countable set, the set $L_{X_{-t}}$ is also metrizable. Also, $L_{X_{-t}}$ is weakly compact. By Lemma 2.1, Φ_t is weakly u.s.c. Define $\Phi : L_X \to 2^{L_X}$ by

$$\Phi(\tilde{x}) = \prod_{t \in T} \Phi_t(\tilde{x}_{-t}).$$

One can easily check that Φ is also a weakly u.s.c. correspondence with convex, closed, nonempty valued. Since the set L_X is weakly compact, convex and nonempty, by the Fan-Glicksberg fixed point theorem there exists an $\tilde{x}^* \in L_X$ such that $\tilde{x}^* \in \Phi(\tilde{x}^*)$. One can easily verify that $\tilde{x}^*$ is a Bayesian equilibrium for G.

4. Related literature and concluding remarks

4.1 Related literature

First it should be noted that if for each $\omega \in \Omega, E_i(\omega) = \{\omega\}$ then $V_i(\omega, x) = u_i(\omega, x)$ and our theorem reduces to a random equilibrium. Moverover, if for each $\omega \in \Omega$, we set $X_t(\omega) = X_t$, where X_t is a compact, convex, nonempty subset of Y and also set for each $(\omega, x) \in \Omega \times \prod_{t \in T} X_t, A_t(\omega, x) = A_t(x)$ and $u_t(\omega, x) = u_t(x)$ we can deduce from our main result, a generalization of the deterministic existence result of Debreu (1952) for an abstract economy with an infinite dimensional strategy space and with a countable number of players.

It should be also noted that by setting for all $(\omega, x) \in \Omega \times \prod_{t \in T} X_t$, $A_t(\omega, x) = X_t(\omega)$ we obtain as a corollary, Theorem 4.1 of Kim-Yannelis (1997), as well as Theorem 4.1 in Yannelis (1998). Finally, our theorem generalizes Theorem 4.1 in Aliprantis-Tourky-Yannelis (2001) from a finite dimensional strategy space to an infinite dimensional one. Moreover, the infinite dimensional equilibrium result in Aliprantis-Tourky-Yannelis (2001) has a slightly stronger assumption on the random constraint correspondences which imply ours. In particular, in Aliprantis-Tourky-Yannelis it is assumed that:

1. $A_t : \Omega \times \prod_{s \in T} X_s \to 2^Y$ is u.s.c.
2. There exists a correspondence $\tilde{A}_t : \Omega \times \prod_{s \in T} X_s \to 2^Y$ such that:
   (a) $cl\tilde{A}_t(\omega, x) = A_t(\omega, x)$ for all $(\omega, x) \in \Omega \times \prod_{t \in T} X_t$
   (b) for each $\omega \in \Omega$, $\tilde{A}_t(\omega, \cdot)$ has open lower sections and it is convex, nonempty valued.

In view of Proposition 3.1 in Yannelis (1987) we can easily see that assumptions (1) and (2) above imply A.3(b), but the reverse is not true.

Finally, it is important to point out that the method of proof in Aliprantis-Tourky-Yannelis (ATY) is different that the one in this paper. In particular, ATY make use of Caratheodory-type selections. To be more precise, the present proof follows the footsteps of the one of Nash (1950)[2] and Debreu (1952) which relies on the fixed point of the best reply correspondence. However, the ATY follows the Yannelis-Prabhakar (1983) approach which relys on a continuous selection theorem. Of course, the mathematical argument needed to provide Bayesian generalizations following either approach are far from trivial.

[2] Nash (1950) attributes the method of proof (fixed point of the best reply mapping) to David Gale. The best reply mapping proof simplifies the original one of Nash (1951) which makes use of an ingenious mapping on which the Brouwer fixed point theorem is applied.

1. Notice that in the ...
   Bayesian abstract economy G, the interim ...
   One may want to consider the *ex ante utility function* of agent ..., de-fined as,

$$V_t(\tilde{x}_{-t}, x_t) = \int_\Omega u_t(\omega, \tilde{x}_{-t}(\omega), x_t(\omega)) d\mu(\omega).$$

   A Bayesian equilibrium can be defined in terms of the ex ante expected utility and the proof of our theorem covers this case as well.
2. The social equilibrium existence theorem of Debreu, was the main mathematical tool to prove the existence of a Walrasian equilibrium as shown by Arrow-Debreu (1954). In a similar fashion, one can show that the abstract Bayesian equilibrium theorems can be used to prove the existence of an expectations (Radner) equilibrium for an economy with differential information (see for example Radner (1968) or Einy-Moreno-Shitovitz (2001), among others). We hope to take there details up in a subsequent paper.
3. By imposing a measurability assumption for a mapping from the set of agent to the strategy set (e.g., Kim-Yannelis [1997, B.4, p. 342]) we can employ a similar argument with the one in this paper to extend our existence result to a measure space of agents (see also Balder (2001) for such results).
4. In Section 3 the random constraint set valued function was defined as a mapping from $\Omega \times \prod_{t \in T} X_t(\omega)$ to 2^Y. The Theorem 3.1 and its proof remain unchanged if the random constraint correspondence i... defined as, $A_t : \Omega \times L_X \to 2^Y$.

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