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A Theory of Value with Non-linear Prices[1]

Equilibrium Analysis beyond Vector Lattices

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This paper presents a new theory of value with a personalized pricing system that naturally induces a family of non-linear prices. This affords a coordinate free theory of value in which the analysis is without any lattice theoretic considerations. When commodity bundles are perfectly decomposable the generalized prices become linear and the analysis specializes to the Walrasian model. This happens, for instance, whenever the commodity space is a vector lattice and consumption sets coincide with the positive cone. Our approach affords theorems on the existence of equilibrium and provides a value-based characterization of Pareto optimality and

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Edgeworth equilibrium where the Walrasian linear price-based characterization fails. The analysis has applications in the finite as well as the infinite dimensional setting. *Journal of Economic Literature* Classification Numbers: C62, C71, D46, D51, D61.

Key Words: personalized prices; arbitrage-free equilibrium; rational equilibrium; welfare theorems; Edgeworth equilibrium; existence of equilibrium; economies with differential information; discriminatory pricing.

1. INTRODUCTION

General equilibrium theory describes the equilibrium or disequilibrium arising from the interaction of all economic agents in markets. The main abstractions in the Arrow–Debreu–McKenzie model of general equilibrium are the notions of commodities and prices—the classical references include the works of Arrow and Hahn [11], Arrow and Debreu [13], Debreu [20], and McKenzie [35]. Commodities define the universe of discourse within which the constraints, motivations, and choices of agents are defined. A linear price system summarizes the information concerning relative scarcities and at equilibrium approximates the possibly non-linear primitive data of the economy.

Of the important theoretical questions that arise in connection with the Walrasian model of general equilibrium two are most fundamental. The first concerns the existence of at least one competitive equilibrium under reasonable economic assumptions. The second deals with the relationship between optimality and price equilibrium, i.e., the decentralization (and characterization) of optimal allocations as interpreted in the fundamental theorems of welfare economics. These two questions have been to a large extent settled in the case of economies with finitely many commodities—the existence of equilibrium requires an application of Brouwer's fixed point theorem and the validity of the welfare theorems can be established by separating hyperplane arguments.

Unfortunately, the techniques and analysis developed for economies with finitely many commodities are not readily applicable to models with infinitely many commodities—for instance infinite horizon models, economies with uncertainty, and models with commodity differentiation. The problems associated with this deficiency have been the subject of extensive research during the second half of the twentieth century. Early results in this direction include the works of Bewley [16], Debreu [19], Hurwicz [32], Kurz and Majumdar [33], McFadden [37], Majumdar [38], Malinvaud [39], Mas-Colell [40], Peleg [50], Peleg and Yaari [51, 52], and Radner [54].

It is clear that one of the major differences between economic models with finite and infinite dimensional commodity spaces is the absence of interior points in the positive cone of infinite dimensional commodity spaces. In fact, if one assumes that initial endowments are interior points in their respective consumption sets, then many of the classical finite dimensional results extend verbatim to infinite dimensional settings. However, unless the commodity space is a majorizing subspace of some $C(\Omega)$-space, the positive cone of an infinite dimensional commodity space has an empty interior.[2] Moreover, in spaces where the positive cone lacks interior points every lower bounded consumption set lacks interior points as well.[3] Therefore, the transition from finite dimensional models to infinite dimensional settings required new mathematical concepts and techniques.

The 1980s saw the emergence of a new approach to the study of infinite dimensional equilibrium theory. In their paper [3], Aliprantis and Brown proposed that a Riesz commodity-price duality is the appropriate setting for infinite dimensional analysis. This departure from the ubiquitous space (and model) specific analysis in the literature emphasized the rich lattice theoretic structure that is shared by the prevalent models in economics. The most important application of this approach was presented in the work of Mas-Colell [41] who proved the existence of equilibrium in economies with a vector lattice commodity space. As highlighted by Mas-Colell [41, p. 1040], a "major surprise" of this new analysis was the relevance of the lattice structure to the existence of equilibrium and the validity of the welfare theorems. This is in sharp contrast to the finite dimensional theory, where Debreu [21] shows that it is possible to answer the main optimality questions using appropriate cones in a coordinate free manner. Of course, Debreu's remarks [21, p. 259] concerning the "coordinate free theory" can be understood as a *vector lattice free analysis*.[4]

Mas-Colell's work was quickly extended in various important directions —see for instance Aliprantis *et al.* [4], Mas-Colell and Richard [43], and Yannelis and Zame [62]—and has spawned a large literature (for references see [5]). There are, however, several shortcomings that have been prevalent in this literature. First, the existence of equilibrium and the welfare theorems are obtained by imposing very strong assumptions on the consumption sets. Indeed, the benefit of the lattice theoretic windfall has come at the cost of strong assumptions on consumption sets. All these results require that consumption sets span the commodity space, and thus preclude models where location matters and economies with differential information—where consumption sets can be very small. In fact, most of the important results require that consumption sets coincide with the positive cone of the

[2] See [2, Sect. 8.5].

[3] For further discussion see [8].

[4] Once a basis has been selected, the convex cone generated by the basis induces a vector lattice structure on the commodity space and the basis becomes a Yudin basis for the commodity space. For details see [6].

commodity space. The second shortcoming of the lattice theoretic framework is associated with the interpretation of the order structure of the commodity space. As is apparent from the pioneering works of Debreu [20], McKenzie [36], and Nikaidô [49], the positive cone of the commodity space is the free disposal technology. Therefore, by assuming that the commodity space is a vector lattice one precludes a rich class of free disposal technologies (for more details see Monteiro and Tourky [48]).

This paper addresses the limitations of the vector lattice approach and presents a new coordinate free theory of value. In our coordinate free analysis we assume that the commodity space is simply an ordered vector space which *need not* be a vector lattice. In this very general context, we introduce an alternate theory of value that arises from a personalized pricing system which induces a non-linear value function. Our analysis specializes to the standard Walrasian model whenever the commodity space is a vector lattice and consumption sets coincide with the positive cone.

Our personalized prices are introduced by means of a discriminating Walrasian auctioneer. The auctioneer assigns to each (price taking) consumer i a personal *linear* price p_i. Now the list of linear personal prices $p=(p_1, p_2, ..., p_m)$ induces a natural value function ψ_p on the commodity space by calculating for each commodity bundle x the maximum revenue that can be achieved by decomposing the bundle x into consumable allocations, where each consumer pays the price assigned to her. That is, if $\mathscr{A}_x$ denotes the set of all x-feasible allocations,[5] then the value of ψ_p at the bundle x is given by

$$\psi_p \cdot x = \sup_{y \in \mathscr{A}_x} [p_1 \cdot y_1 + p_2 \cdot y_2 + \cdots + p_m \cdot y_m].$$

The function ψ_p as defined above (which we shall refer to as a *generalized price*) is always concave—super-additive and positively homogeneous—but may fail to be linear. This often happens even with finitely many commodities; see Example 9.6. We shall see that as commodity bundles "become more and more" decomposable the generalized prices "become more and more" linear. In fact, when we have perfect divisibility (which is expressed by a consumption decomposition property), the personalized prices become linear—and we are at the classical Arrow–Debreu–McKenzie world. In particular, if the commodity space is a vector lattice and the consumption sets coincide with the positive cone, then the generalized prices are linear.

Given the generalized prices, the main concern of this work is focused on answering the following fundamental question:

[5] An m-tuple $y=(y_1, y_2, ..., y_m)$ is an *x-feasible allocation* if y_i belongs the consumption set of each consumer i and $\sum_{i=1}^m y_i \leqslant x$.

- *Can we characterize optimality notions* (*Pareto optimality, Walrasian equilibrium, Edgeworth equilibrium*) *in terms of generalized prices*?

Non-linear prices allow for arbitrage opportunities. Consequently, in the presence of non-linearity the standard supporting properties do not characterize allocative efficiency. Indeed, there are suboptimal allocations that can be supported by super-additive prices. Therefore, we introduce two new notions of arbitrage-free equilibria—the *personalized valuation equilibrium* and the *personalized equilibrium*—in terms of generalized prices. We prove their existence and show that personalized valuation equilibria characterize Pareto optimality and that personalized equilibria characterize Edgeworth equilibria, i.e., allocations that belong to the core of every replica economy. In particular, we show that an allocation is Pareto optimal if and only if it is supported by a generalized price and the following arbitrage-free condition holds:

- *Combining the grand coalition's allocation presents no opportunity for arbitrage.*

Moreover, we establish that an allocation is an Edgeworth equilibrium if and only if it is supported by some generalized price such that the following coalition-based arbitrage-free condition holds:

- *Coalitions—in the limit of all replication of the economy—have no "nominal" incentive to sell their total endowments.*

We also show that an allocation is a Walrasian equilibrium if and only if it is an personalized equilibrium for which the generalized price is linear (and continuous). These results generalize the existence of Walrasian equilibrium results in the literature. In particular, we generalize and provide new proofs of existence of equilibrium results of Aliprantis *et al.* [4], Mas-Colell [41], Mas-Colell and Richard [43], Podczeck [53], Yannelis and Zame [62], and the Edgeworth equivalence results of Aliprantis *et al.* [4], and Tourky [56].

Our approach provides a value-based characterization of Pareto optimal and Edgeworth allocations where the (Walrasian) uniform price-based characterization fails. The analysis has new applications in both the finite and the infinite dimensional settings. In particular, Monteiro and Tourky [48] recently showed that the lattice theoretic properties in Mas-Colell's work [41] are relevant to the existence of equilibrium problem even when the commodity space is finite dimensional. They provided a surprising example of an economy with three commodities. In that example, all of Mas-Colell's assumptions hold except that the ordering of the commodity space is not a lattice. However, in that economy there is no Walrasian equilibrium and the second theorem of welfare economics fails for linear prices. On the other hand, our personalized equilibrium exists in their example and our generalized welfare theorems hold, i.e., our generalized prices decentralize the optimal allocations that cannot be supported by a uniform linear price system.

The results in the present paper are obtained for uniformly proper economies where consumption sets may lack interior points and the commodity space need not be a vector lattice. The results allow for consumption sets that are much smaller than the positive orthant of the commodity space. Furthermore, since our generalized prices are functions of schedules of individual prices, the analysis in this paper avoids a serious paradox in the theory of Walrasian equilibrium with differential information. That is: *How can the existence of private information be reconciled with the presence of a Walrasian price system that conveys full information*? In fact, we establish that our analysis applies even when each consumer's personal price contains no more information than the his private information.

The paper is organized as follows. The standard model of general equilibrium is outlined in Section 2. Section 3 introduces and studies the properties of the generalized prices. The definitions of personalized equilibria and their economic interpretation are included in Section 4. Our main results are contained in subsequent sections.

The central ingredient of our analysis is the new notion of *rational equilibrium*, which is a "convexification" of the notions of Pareto optimality and individual rationality. This class of allocations is larger than the class of Edgeworth equilibria and is studied in Section 6. Section 9 contains three applications of our main result. First, we establish that the Walrasian model of general equilibrium is a special case of our model. Second, we apply our results to differential information economies, and the paper concludes by offering another interpretation of our personalized pricing system and relating our model to a discriminatory price auction which mimics the U.S. Treasury Bill Auction.

Non-linear prices arise naturally in the presence of price discrimination, progressive income tax tariffs, and land markets, and they have been the subject of much literature (see for instance Arrow and Hurwicz [12], Berliant and Dunz [15], Guesnerie and Seade [31], Spence [55], and Villamil [58].) We note, therefore, that there is a broad range of other applications of the analysis presented in this paper.

2. THE MODEL

We shall only deal with pure exchange economies. The commodity space is a Hausdorff locally convex ordered topological vector space L equipped with a linear topology τ such that:

(a) *The positive cone* L_+ *of* L, *is proper* (*i.e.*, $L_+ \cap (-L_+) = \{0\}$), *generating* (*i.e.*, $L_+ - L_+ = L$), *convex, and* τ*-closed; and*

(b) *The order intervals of* L *are* τ*-bounded.*

The *topological dual* of (L, τ) (i.e., the vector space of all τ-continuous linear functionals on L) will be denoted L'. As usual, the *algebraic dual* of L (i.e., the vector space of all linear functionals on L) is denoted L^*. The *order dual* of L (i.e., the vector space of all order bounded linear functionals[6] on L) is denoted $L^\sim$. Since every order interval of L is τ-bounded, it follows that $L' \subseteq L^\sim \subseteq L^*$. As usual, if $\psi: A \to \mathbb{R}$ is a real-valued function defined on a subset of L, then we shall denote the value $\psi(x)$ by $\psi \cdot x$, i.e., we write $\psi \cdot x = \psi(x)$.

There are m consumers; we let $I = \{1, ..., m\}$ for the set of consumers and designate the arbitrary consumer by the index i. The bundle $\omega_i \in L$ is the ith consumer's *initial endowment.* As usual, $\omega = \sum_{i=1}^m \omega_i$ is the *total endowment.*

The consumption set of consumer i is X_i. Throughout this paper, for each consumer i, we assume that:

- *the consumption set* X_i *is a convex* τ*-closed subcone of* L_+, *and*
- $0 < \omega_i \in X_i$.

It should be clear that $\prod_{i=1}^m X_i \subseteq L_+^m$ is a closed convex cone of $(L, \tau)^m$. Also, since each consumption set is a τ-closed subcone of L_+ and $\omega_i \in X_i$, it follows that each X_i is non-empty and the entire half-ray $\{\alpha\omega_i : \alpha \geqslant 0\}$ lies in X_i.

The correspondence $P_i: X_i \twoheadrightarrow X_i$ denotes the ith consumer's preference. The following conditions on preferences will play a critical role in our discussion.

(A1) For a preference correspondence $P_i: X_i \twoheadrightarrow X_i$, we distinguish the properties.

(1) P_i is *irreflexive*, i.e., $x \notin P_i(x)$ for each $x \in X_i$.

(2) P_i is *convex-valued*, i.e., $P_i(x)$ is a convex set for each $x \in X_i$.

(3) P_i is *strictly monotone*, i.e., $x \in X_i$ implies $x + y \in P_i(x)$ for each $y \in X_i \backslash \{0\}$.

(4) P_i has *open values* in X_i relative to a linear topology on L.

(5) P_i has *weakly open lower sections*, i.e., for each $y \in X_i$ the lower section $P_i^{-1}(y) = \{x \in X_i : y \in P_i(x)\}$ is weakly open in X_i.

[6] A linear functional on L is said to be *order bounded* if it carries order intervals of L to bounded subsets of $\mathbb{R}$.

Notice that if some P_i is strictly monotone, then from $x+\alpha\omega_i \in P_i(x)$ for each $\alpha > 0$, it follows that $x \in \overline{P_i(x)}$ for each $x \in X_i$. In particular, if P_i is strictly monotone, then P_i is a non-empty-valued correspondence. We let

$$\mathscr{A}_\omega = \left\{ (y_1, y_2, ..., y_m) \in \prod_{i=1}^{m} X_i : \sum_{i=1}^{m} y_i \leqslant \omega \right\}.$$

The members of $\mathscr{A}_\omega$ will be referred to as *feasible allocations* or simply as *allocations*. Clearly, $\mathscr{A}_\omega$ is a non-empty, convex and weakly closed subset of L^m. Moreover, in view of $\mathscr{A}_\omega \subseteq [0, \omega]^m$, it should be noticed that if the order interval $[0, \omega]$ is a weakly compact subset of L, then $\mathscr{A}_\omega$ is also a weakly compact subset of L^m.

The following compactness property of the set of all allocations will be used for establishing existence of equilibria.

(A2) (Compactness). The non-empty convex set $\mathscr{A}_\omega$ is weakly compact.

The notion of properness that will be employed in this work is the one introduced by Tourky [56]. It is weaker than Mas-Colell's uniform properness condition [41].[7]

(A3) (Properness). The economy is said to be v-*proper*, where the vector $v = (v_1, ..., v_m)$ is an allocation satisfying $v_i > 0$ for each i, if for each i there exists another correspondence $\hat{P}_i \colon X \twoheadrightarrow L$ (which is convex-valued if P_i is also convex-valued) such that for each $x \in X_i$:

(i) the vector $x + v_i$ is a τ-interior point of $\hat{P}_i(x)$; and

(ii) $\hat{P}_i(x) \cap X = P_i(x)$.

We conclude by defining the standard notions of optimality and equilibrium.

DEFINITION 2.1. An allocation $(x_1, x_2, ..., x_m)$ is said to be:

(1) *individually rational*, if $\omega_i \notin P_i(x_i)$ for each $i \in I$,

(2) *weakly Pareto optimal*, if there is no allocation $(y_1, y_2, ..., y_m)$ satisfying $y_i \in P(x_i)$ for each $i \in I$,

[7] We show in [8] that when preferences are complete preorderings, Mas-Colell's uniform properness condition [41] is strictly stronger than our properness condition. We also show that when preferences have open lower sections and are irreflexive, then the Yannelis–Zame extreme desirability condition [62] is strictly stronger than Mas-Colell's uniform properness. In this case the preference maps are (weakly) majorized by convex valued mappings that have open graphs (with weakly open lower-sections) and satisfy our properness condition. Therefore, the existence of equilibrium result in this paper implies results for economies that satisfy the properness conditions of Mas-Colell and of Yannelis and Zame; see also Subsection 9.1. For related conditions on preferences see Araujo and Monteiro [10] and Podczeck [53].

(3) a *core allocation*, if it cannot be blocked by any allocation in the sense that there is no allocation $(y_1, y_2, ..., y_m)$ and a coalition $S \subseteq I$ such that

(a) $\sum_{i \in S} y_i \leqslant \sum_{i \in S} \omega_i$, and
(b) $y_i \in P_i(x_i)$ for all $i \in S$, and

(4) an *Edgeworth equilibrium* if it belongs to the core every r-fold replica economy.[8]

Observe that an allocation $(x_1, x_2, ..., x_m)$ is weakly Pareto optimal if and only if $\mathscr{A}_\omega \cap \prod_{i=1}^m P_i(x_i) = \varnothing$. Also, the following simple property should be obvious: *Every core allocation (in particular every Edgeworth equilibrium) is individually rational and weakly Pareto optimal.*

DEFINITION 2.2. An allocation $(x_1, x_2, ..., x_m)$ is said to be:

(1) a *Walrasian valuation equilibrium*, if there exists some continuous price $q \in L'$ satisfying $q \cdot \omega \neq 0$ and

$$x \in P_i(x_i) \Rightarrow q \cdot x \geqslant q \cdot x_i,$$

(2) a *Walrasian quasi-equilibrium*, if there exists some continuous price $q \in L'$ satisfying $q \cdot \omega \neq 0$, $q \cdot x_i = q \cdot \omega_i$ for each i, and

$$x \in P_i(x_i) \Rightarrow q \cdot x \geqslant q \cdot \omega_i,$$

(3) a *Walrasian equilibrium*, if there exists some price $q \in L'$ satisfying $q \cdot \omega \neq 0$, $q \cdot x_i = q \cdot \omega_i$ for each i, and

$$x \in P_i(x_i) \Rightarrow q \cdot x > q \cdot \omega_i.$$

3. GENERALIZED PRICES

We shall call an arbitrary linear functional $p = (p_1, p_2, ..., p_m)$ on L^m a *list of personalized prices* (or simply a *list of prices*). In this section, we define a value function that is naturally induced by such a list of personalized prices. The domain, C, of this value function will be the convex cone generated in L by $\bigcup_{i=1}^m X_i$, i.e.,

$$C = X_1 + X_2 + \cdots + X_m.$$

[8] An Edgeworth equilibrium [4] represents an allocation in the limit of all replications of the economy that satisfies the equal treatment property and which is in the core of that economy; see Debreu and Scarf [22].

Clearly, C is a convex subcone of L_+ and $\omega \in C$. The vector space generated by C is denoted by M, i.e., $M = C - C$.

For each commodity bundle $x \in L_+$, we let $\mathscr{A}_x$ denote the set of all allocations when the total endowment is x, i.e.,

$$\mathscr{A}_x = \left\{ y = (y_1, y_2, ..., y_m) \in \prod_{i=1}^{m} X_i : \sum_{i=1}^{m} y_i \leqslant x \right\}.$$

Clearly, each $\mathscr{A}_x$ is a nonempty, convex and closed subset of $(L, \tau)^m$; and hence also a weakly closed subset of L^m. Notice also that for each $x, y \in L_+$ and all $\alpha \geqslant 0$, we have

$$\mathscr{A}_x + \mathscr{A}_y \subseteq \mathscr{A}_{x+y} \qquad \text{and} \qquad \mathscr{A}_{\alpha x} = \alpha \mathscr{A}_x.$$

DEFINITION 3.1. The *generalized price* of an arbitrary list of personalized prices $p = (p_1, p_2, ..., p_m)$ is the function $\psi_p \colon C \to [0, \infty]$ defined by

$$\psi_p \cdot x = \sup_{y \in \mathscr{A}_x} [p_1 \cdot y_1 + p_2 \cdot y_2 + \cdots + p_m \cdot y_m].$$

Clearly, if $p \in (L^\sim)^m$ (in particular, if $p \in (L')^m$), then ψ_p is a real-valued function. The value $\psi_p \cdot x$ is the maximum value that one can obtain by decomposing the bundle x into consumable allocations, where each consumer i pays the price p_i assigned to her. For obvious reasons, we shall call any allocation $(y_1, y_2, ..., y_m) \in \mathscr{A}_\omega$ that maximizes value, i.e., $\psi_p \cdot \omega = \sum_{i=1}^{m} p_i \cdot y_i$, a *maximizing allocation* for the list of personalized prices $p = (p_1, p_2, ..., p_m)$.

LEMMA 3.2. *If the economy satisfies the compactness property* (A2), *then maximizing allocations exist for any list of continuous personalized prices.*

Proof. Let $p = (p_1, p_2, ..., p_m) \in (L')^m$ be a list of continuous personalized prices. Now notice that the function $R \colon L^m \to \mathbb{R}$, defined by

$$R(x_1, x_2, ..., x_m) = \sum_{i=1}^{m} p_i \cdot x_i,$$

is weakly (i.e., $\sigma(L^m, (L')^m$-) continuous. If the non-empty convex set $\mathscr{A}_\omega$ is weakly compact (which is precisely what the compactness property (A2) asserts), then R attains its maximum on $\mathscr{A}_\omega$. Clearly, any maximizer of R on $\mathscr{A}_\omega$ is a maximizing allocation for the list of personalized prices p. ∎

The basic properties of the generalized prices are included in the next result.

LEMMA 3.3. *If $p=(p_1, p_2, ..., p_m) \in (L^{\sim})^m$ is a list of order bounded personalized prices, then its generalized price $\psi_p: C \to [0, \infty)$ is a non-negative real-valued function such that*:

(1) ψ_p *is* monotone, *i.e., $x, y \in C$ with $x \leqslant y$ implies $\psi_p \cdot x \leqslant \psi_p \cdot y$,*

(2) ψ_p *is* super-additive, *i.e., $\psi_p \cdot x + \psi_p \cdot y \leqslant \psi_p \cdot (x+y)$ for all $x, y \in C$,*

(3) ψ_p *is* positively homogeneous, *i.e., $\psi_p \cdot (\alpha x) = \alpha(\psi_p \cdot x)$ for all $\alpha \geqslant 0$ and $x \in C$,*

(4) *if $p_1 = p_2 = \cdots = p_m = q \geqslant 0$, then $\psi_p \cdot x = q \cdot x$ for all $x \in C$, i.e., $\psi_p = q$, and*

(5) *if $x \in X_i$, then $p_i \cdot x \leqslant \psi_p \cdot x$.*

Proof. Item (3) is straightforward.

(1) Assume $x, y \in C$ satisfy $x \leqslant y$. If $z = (z_1, z_2, ..., z_m) \in \mathscr{A}_x$, then the inequalities $\sum_{i=1}^m z_i \leqslant x \leqslant y$ imply $z \in \mathscr{A}_y$, i.e., $\mathscr{A}_x \subseteq \mathscr{A}_y$. This easily yields $\psi_p \cdot x \leqslant \psi_p \cdot y$.

(2) Assume $u = (u_1, u_2, ..., u_m) \in \mathscr{A}_x$ and $v = (v_1, v_2, ..., v_m) \in \mathscr{A}_y$. Then, from the inclusion $\mathscr{A}_x + \mathscr{A}_y \subseteq \mathscr{A}_{x+y}$, it follows that $u + v \in \mathscr{A}_{x+y}$. So,

$$\sum_{i=1}^{m} p_i \cdot x_i + \sum_{i=1}^{m} p_i \cdot y_i = \sum_{i=1}^{m} p_i \cdot (x_i + y_i) \leqslant \psi_p \cdot (x+y),$$

from which the desired inequality follows.

(4) Assume $p_1 = p_2 = \cdots = p_m = q \geqslant 0$ and let $x \in C$. If a decomposition $z = (z_1, z_2, ..., z_m) \in \mathscr{A}_x$, then $\sum_{i=1}^m p_i \cdot z_i = q \cdot (\sum_{i=1}^m z_i) \leqslant q \cdot x$, and so $\psi_p \cdot x \leqslant q \cdot x$. On the other hand, if we write $x = \sum_{i=1}^m x_i$ with $x_i \in X_i$ for each i, then we have $q \cdot x = q \cdot (\sum_{i=1}^m x_i) = \sum_{i=1}^m q \cdot x_i \leqslant \psi_p \cdot x$. This implies $\psi_p \cdot x = q \cdot x$ for each $x \in C$.

(5) Fix $x \in X_i$. Since $0 \in X_j$ for each j implies $(0, ..., 0, x, 0, ..., 0)$ is in $\mathscr{A}_x$, where $x \in X_i$ occupies the ith position, we see that $p_i \cdot x \leqslant \psi_p \cdot x$. This completes the proof. ∎

We now introduce the decomposability property of the consumption sets. This property provides a "measure" of the linearity of the generalized prices. This property is not used in our major results.

(A3) (Consumption Decomposability). The economy has the *Consumption Decomposability Property* if for each $x, y \in C$ we have $\mathscr{A}_x + \mathscr{A}_y = \mathscr{A}_{x+y}$.

When the consumption sets coincide with the positive cone L_+, the Consumption Decomposability Property is equivalent to the Riesz Decomposition Property. (Recall that the Riesz Decomposition Property asserts that for all $x, y \in L_+$ the following equality holds $[0, x] + [0, y] = [0, x+y]$.)

LEMMA 3.4. *If $X_i = L_+$ for each consumer i, then the economy has the Consumption Decomposability Property if and only if L has the Riesz Decomposition Property.*

Proof. Notice that $C = L_+$. Assume first that the economy has the Consumption Decomposability Property and let three vectors $x, y, z \in L_+$ satisfy $0 \leqslant z \leqslant x + y$. Then $u = (z, 0, 0, ..., 0) \in \mathscr{A}_{x+y} = \mathscr{A}_x + \mathscr{A}_y$. So, there exist $v = (v_1, v_2, ..., v_m) \in \mathscr{A}_x$ and $w = (w_1, w_2, ..., w_m) \in \mathscr{A}_y$ satisfying $u = v + w$. The latter implies $v_i = w_i = 0$ for $i = 2, 3, ..., m$, and so the vectors $v_1, w_1 \in L_+$ satisfy $0 \leqslant v_1 \leqslant x$, $0 \leqslant w_1 \leqslant y$ and $v_1 + w_1 = z$. This shows that L has the Riesz Decomposition Property.

For the converse, suppose that L satisfies the Riesz Decomposition Property and let $x, y \in L_+$. We must show $\mathscr{A}_{x+y} \subseteq \mathscr{A}_x + \mathscr{A}_y$. So, choose a decomposition $(u_1, u_2, ..., u_m) \in \mathscr{A}_{x+y}$, i.e., $\sum_{i=1}^m u_i \leqslant x + y$. Since L has the Riesz Decomposition Property, there exist $x_1, y_1 \geqslant 0$ satisfying $0 \leqslant x_1 \leqslant x$, $0 \leqslant y_1 \leqslant y$ and $\sum_{i=1}^m u_i = x_1 + y_1$. To complete the proof, now invoke [7, Theorem 1.15, p. 14]. ∎

THEOREM 3.5. *For a vector subspace $\mathscr{P}$ of $L^\sim$ the following statements are equivalent.*

(1) *For each non-zero list of prices $p = (p_1, p_2, ..., p_m) \in \mathscr{P}^m$ the generalized price $\psi_p\colon C \to [0, \infty)$ is additive—and hence it has a unique linear extension to $M = C - C$.*

(2) *For each $x, y \in C$ we have $\mathscr{A}_{x+y} \subseteq \overline{\mathscr{A}_x + \mathscr{A}_y}$, where the bar denotes $\sigma(L^m, \mathscr{P}^m)$-closure.*

Proof. Let $\mathscr{P}$ be a vector subspace of $L^\sim$.

(1) $\Rightarrow$ (2). Suppose by way of contradiction that there exist $x, y \in C$ and some $z = (z_1, z_2, ..., z_m) \in \mathscr{A}_{x+y}$ such that $z \notin \overline{\mathscr{A}_x + \mathscr{A}_y}$. Since $\sigma(L^m, \mathscr{P}^m)$ is a locally convex topology on L^m, there exists some non-zero list of prices $p = (p_1, p_2, ..., p_m) \in \mathscr{P}^m$ which strongly separates z and $\overline{\mathscr{A}_x + \mathscr{A}_y}$; see [2, Corollary 5.59, p. 194]. That is, there exists some $\varepsilon > 0$ satisfying

$$\sum_{i=1}^m p_i \cdot z_i \geqslant \varepsilon + \sum_{i=1}^m p_i \cdot u_i$$

for all $(u_1, u_2, ..., u_m) \in \overline{\mathscr{A}_x + \mathscr{A}_y}$. This easily implies

$$\psi_p \cdot (x+y) \geqslant \sum_{i=1}^{m} p_i \cdot z_i \geqslant \varepsilon + \psi_p \cdot x + \psi_p \cdot y > \psi_p \cdot x + \psi_p \cdot y,$$

which contradicts the additivity of ψ_p.

(2) $\Rightarrow$ (1). Let $p = (p_1, p_2, ..., p_m) \in \mathscr{P}^m$. The function $\psi_p: C \to [0, \infty]$, defined by

$$\psi_p \cdot x = \sup_{z \in \mathscr{A}_x} [p_1 \cdot z_1 + p_2 \cdot z_2 + \cdots + p_m \cdot z_m],$$

is real-valued, positively homogeneous and super-additive. To see that ψ_p is additive, let $x, y \in C$, and fix $z = (z_1, z_2, ..., z_m) \in \mathscr{A}_{x+y} \subseteq \overline{\mathscr{A}_x + \mathscr{A}_y}$. Then there exist two nets $\{(u_1^\alpha, ..., u_m^\alpha)\} \subseteq \mathscr{A}_x$ and $\{(v_1^\alpha, ..., v_m^\alpha)\} \subseteq \mathscr{A}_y$ such that

$$(u_1^\alpha + v_1^\alpha, ..., u_m^\alpha + v_m^\alpha) \xrightarrow{\sigma(L^m, \mathscr{P}^m)} z.$$

In particular, we have $\lim_\alpha \sum_{i=1}^m p_i \cdot (u_i^\alpha + v_i^\alpha) = \sum_{i=1}^m p_i \cdot z_i$. Now taking into account that

$$\sum_{i=1}^{m} p_i \cdot (u_i^\alpha + v_i^\alpha) = \sum_{i=1}^{m} p_i \cdot u_i^\alpha + \sum_{i=1}^{m} p_i \cdot v_i^\alpha \leqslant \psi_p \cdot x + \psi_p \cdot y,$$

we see that $\sum_{i=1}^m p_i \cdot z_i \leqslant \psi_p \cdot x + \psi_p \cdot y$. Since $z = (z_1, z_2, ..., z_m) \in \mathscr{A}_{x+y}$ is arbitrary, we conclude that $\psi_p \cdot (x+y) \leqslant \psi_p \cdot x + \psi_p \cdot y$. Consequently, $\psi_p \cdot (x+y) = \psi_p \cdot x + \psi_p \cdot y$, so that ψ_p is additive on C.

Now we leave it to the reader to verify that if for each $x \in M$ we write $x = a - b$ with $a, b \in C$, then the formula

$$\psi_p \cdot x = \psi_p \cdot a - \psi_p \cdot b$$

is independent of the representation of x as a difference of two vectors of C and defines a (unique) linear extension of ψ_p to all of M. (Notice also that ψ_p is a positive linear functional on M with respect to the generating cone C.) ∎

Consumption decomposability guarantees linearity of the generalized prices.

Corollary 3.6. *If the Consumption Decomposability Property* (A4) *is satisfied, then every generalized price is additive—and hence it has a unique linear extension to* $M = C - C$.

Regarding continuity and extendability of the generalized prices, we have the following result—which also shows that our analysis reduces to a uniform price setting whenever the commodity space has the Riesz Decomposition Property. For more results in this direction see [9].

THEOREM 3.7. *Assume that consumption sets coincide with the positive cone (i.e., $X_i = L_+$ for each i) and let $p = (p_1, p_2, ..., p_m) \in (L^\sim)^m$ be a list of personalized prices.*

(1) *If the generalized price ψ_p has a τ-continuous linear extension to all of L, then the linear functional $q = (\bigvee_{i=1}^m p_i)^+$ exists in $L^\sim$, is τ-continuous, and $\psi_p = q$.*

(2) *If the space L has the Riesz Decomposition Property and the linear functional $q = (\bigvee_{i=1}^m p_i)^+$ (which exists in $L^\sim$) is τ-continuous, then we have $\psi_p = q$ (and so ψ_p has a τ-continuous linear extension to all of L).*

Proof. For every i, $X_i = L_+$ and $p \in (L^\sim)^m$.

(1) Let $q \in L'$ be a τ-continuous linear extension of ψ_p to all of L. Since $C = L_+$, it should be clear that $q \geqslant 0$. Note that if $x \in L_+ = X_i$, then $p_i \cdot x \leqslant \psi_p \cdot x = q \cdot x$, and so $q \geqslant p_i$ for each i. To see that q is the least upper bound of the set $\{p_1, ..., p_m\}$ in $L^\sim$, assume that some $0 \leqslant \pi \in L^\sim$ satisfies $\pi \geqslant p_i$ for each i. Then for each $x \in L_+$ we have

$$\pi \cdot x = \psi_{(\pi, \pi, ..., \pi)} \cdot x \geqslant \psi_p \cdot x = q \cdot x.$$

Thus, $\pi \geqslant q$, and so $q = (\bigvee_{i=1}^m p_i)^+$ holds in $L^\sim$.

(2) Assume that L has the Riesz Decomposition Property. In this case $L^\sim$ is a Riesz space and $q = (\bigvee_{i=1}^m p_i)^+$ is given by the Riesz–Kantorovich formula,

$$q \cdot x = \sup \left\{ \sum_{i=1}^m p_i \cdot x_i : x_i \in L_+ \text{ for each } i \text{ and } \sum_{i=1}^m x_i \leqslant x \right\},$$

for all $x \in L_+$; see [7, Theorem 1.13, p. 12]. Clearly, $q \cdot x = \psi_p \cdot x$ for every point $x \in L_+$ and if $q \in L'$, then q is a τ-continuous linear extension of ψ_p to all of L. This completes the proof. ∎

4. PERSONALIZED EQUILIBRIA

We are now ready to introduce our basic concepts of equilibria.

DEFINITION 4.1. An allocation $(x_1, x_2, ..., x_m)$ is said to be a:

(1) *personalized valuation equilibrium*, whenever there exists some list of personalized prices $p = (p_1, p_2, ..., p_m) \in (L^\sim)^m$ such that

(i) $\psi_p \cdot \omega > 0$,

(ii) $y \in P_i(x_i) \Rightarrow \psi_p \cdot y \geqslant \psi_p \cdot x_i$, and

(iii) the following arbitrage-free condition holds

$$\psi_p \cdot \omega = \sum_{i=1}^{m} \psi_p \cdot x_i;$$

(2) *personalized quasi-equilibrium*, whenever there exists some list of personalized prices $p = (p_1, p_2, ..., p_m) \in (L^\sim)^m$ such that

(a) $\psi_p \cdot \omega > 0$,

(b) $y \in P_i(x_i) \Rightarrow \psi_p \cdot y \geqslant \psi_p \cdot x_i$, and

(c) for each $(\alpha_1, \alpha_2, ..., \alpha_m) \in \mathbb{R}^m_+$ we have

$$\psi_p \cdot \left(\sum_{i=1}^{m} \alpha_i \omega_i \right) \leqslant \sum_{i=1}^{m} \alpha_i \psi_p \cdot x_i,$$

(3) *personalized equilibrium*, whenever there exists a list of personalized prices $p = (p_1, p_2, ..., p_m) \in (L^\sim)^m$ such that

(a) $\psi_p \cdot \omega > 0$,

(b) $y \in P_i(x_i) \Rightarrow \psi_p \cdot y > \psi_p \cdot x_i$, and

(c) for each $(\alpha_1, \alpha_2, ..., \alpha_m) \in \mathbb{R}^m_+$ we have

$$\psi_p \cdot \left(\sum_{i=1}^{m} \alpha_i \omega_i \right) \leqslant \sum_{i=1}^{m} \alpha_i \psi_p \cdot x_i.$$

It should be clear that

Personalized Equilibrium $\Rightarrow$ *Personalized Quasi-equilibrium*

$\Rightarrow$ *Personalized Valuation Equilibrium.*

Conditions (ii) and (b) in the definitions of a personalized valuation equilibrium and a personalized equilibrium are analogous to the standard supporting properties in the Arrow–Debreu–McKenzie model of general equilibrium.

Condition (iii) in the definition of a personalized valuation equilibrium is a non-arbitrage condition which states the following:

- *The grand coalition cannot combine the commodity bundles and obtain a bundle with a greater valued than the sum of the original values.*

To give an interpretation to condition (c) in the definition of a personalized equilibrium consider an economy with a continuum of agents indexed by the unit interval $[0, 1]$ equipped with the Lebesgue measure λ. We define the m order intervals

$$J_1 = \left[0, \frac{1}{m}\right], J_2 = \left(\frac{1}{m}, \frac{2}{m}\right], ..., J_m = \left(\frac{m-1}{m}, 1\right].$$

We let the total endowment ω: $[0, 1] \to L$ be the measurable simple function defined by $\omega(t) = \omega_i$ for $t \in J_i$. Also, we can identify any vector $(x_1, x_2, ..., x_m)$ with the simple measurable function x: $[0, 1] \to L$ defined by $x(t) = x_i$ for $t \in J_i$. In the next lemma, the integral of simple functions is defined in the obvious way.

LEMMA 4.2. *For a list of order bounded prices $p = (p_1, p_2, ..., p_m)$ and an allocation $x = (x_1, x_2, ..., x_m)$ the following statements are equivalent*:

(1) *For each $(\alpha_1, \alpha_2, ..., \alpha_m) \in \mathbb{R}^m_+$ we have*

$$\psi_p \cdot \left(\sum_{i=1}^m \alpha_i \omega_i\right) \leqslant \sum_{i=1}^m \alpha_i \psi_p \cdot x_i.$$

(2) *For each measurable coalition S of the continuum economy we have*

$$\psi_p \cdot \left[\int_S \omega(t)\, d\lambda(t)\right] \leqslant \int_S [\psi_p \cdot x(t)]\, d\lambda(t).$$

Proof. Assume that (1) holds and let $S \subseteq [0, 1]$ be any measurable coalition. Then

$$\begin{aligned}\psi_p \cdot \left[\int_S \omega(t)\, d\lambda(t)\right] &= \psi_p \cdot \left[\sum_{i=1}^m \lambda(S \cap J_i)\, \omega_i\right] \\ &\leqslant \sum_{i=1}^m \lambda(S \cap J_i)\, \psi_p \cdot x_i \\ &= \int_S [\psi_p \cdot x(t)]\, d\lambda(t).\end{aligned}$$

Therefore $(1) \Rightarrow (2)$.

Next, assume that (2) holds and let $(\alpha_1, \alpha_2, ..., \alpha_m)$ be any vector in $\mathbb{R}^m_+$. Choose a positive integer n such that for each i we have $\alpha_i/n \leqslant \lambda(J_i) = 1/m$.

Also for each i pick a measurable set $S_i \subseteq J_i$ with $\lambda(S_i) = \alpha_i/n$ and consider the coalition $S = \bigcup_{i=1}^m S_i$. We have

$$\psi_p \cdot \left(\sum_{i=1}^m \frac{\alpha_i}{n} \omega_i \right) = \psi_p \cdot \left[\int_S \omega(t) \, d\lambda(t) \right] \leqslant \int_S \left[\psi_p \cdot x(t) \right] d\lambda(t) = \sum_{i=1}^m \frac{\alpha_i}{n} \psi_p \cdot x_i.$$

From the homogeneity of ψ_p we get $\psi_p \cdot (\sum_{i=1}^m \alpha_i \omega_i) \leqslant \sum_{i=1}^m \alpha_i \psi_p \cdot x_i$. So, $(2) \Rightarrow (1)$. ▮

Using Lemma 4.2 we can now give the following coalition based, arbitrage free, interpretation of condition (c) in the definition of a personalized equilibrium:

- *Coalitions—in the limit of all replications of the economy—cannot combine their commodity bundles and obtain a bundle with a greater valued than the coalitions' endowment.*

We conclude this section by presenting some relationships between Walrasian equilibria and personalized equilibria. In view of Theorem 3.7 the next result also states that we are in the Arrow–Debreu–McKenzie world whenever our generalized price is additive.

THEOREM 4.3. *The following statements are true*:

(1) *Every Walrasian equilibrium with positive equilibrium price is an personalized equilibrium.*

(2) *Every personalized equilibrium with an additive τ-continuous generalized price is a Walrasian equilibrium.*

Proof. The validity of (1) is an easy consequence of Lemma 3.3(4). To establish (2), let $x = (x_1, x_2, ..., x_m)$ be a personalized equilibrium with respect to some τ-continuous additive generalized price ψ_p. Notice that $\psi_p \cdot \omega_i \leqslant \psi_p \cdot x_i$ for each i. A quick glance at the inequality

$$\sum_{i=1}^m \psi_p \cdot x_i = \psi_p \cdot \left(\sum_{i=1}^m x_i \right) \leqslant \psi_p \cdot \omega = \psi_p \cdot \left(\sum_{i=1}^m \omega_i \right) = \sum_{i=1}^m \psi_p \cdot \omega_i,$$

shows that $\psi_p \cdot \omega_i = \psi_p \cdot x_i$ for each i. Now any τ-continuous linear extension of ψ_p to all of L guarantees that x is a Walrasian equilibrium for this price system. ▮

5. WEAK PARETO OPTIMALITY

In this section we shall establish that under certain general conditions the notions of a personalized valuation equilibrium and that of a weakly Pareto optimal allocation coincide.

THEOREM 5.1. *We have the following relationships between weakly Pareto optimal allocations and personalized valuation equilibria*:

(1) *If properties* (A1.2), (A1.3), *and* (A3) *are satisfied, then every weakly Pareto optimal allocation* $x \in \mathscr{A}_\omega$ *is a personalized valuation equilibrium with respect to some list of continuous prices* $p \in (L')^m$. *Moreover, the allocation* x *is a maximizing allocation for* p.

(2) *If* (A1.4) *is satisfied, then every personalized valuation equilibrium is weakly Pareto optimal.*

Proof. We begin with the analogue of the second welfare theorem.

(1) Let $x = (x_1, x_2, ..., x_m)$ be a weakly Pareto optimal allocation. This means that $\mathscr{A}_\omega \cap \prod_{i=1}^m P_i(x_i) = \varnothing$. Now from (A1.2) and the properness assumption (A3), there exist convex-valued correspondences $\hat{P}_i \colon \hat{X}_i \twoheadrightarrow L$ (for $i \in I$) whose values have non-empty interior satisfying

$$\hat{P}_i(x_i) \cap X_i = P_i(x_i).$$

for each i. Since $\mathscr{A}_\omega \subseteq \prod_{i=1}^m X_i$, it follows that $\mathscr{A}_\omega \cap \prod_{i=1}^m \hat{P}_i(x_i) = \varnothing$.

Since both sets $\mathscr{A}_\omega$ and $\prod_{i=1}^m \hat{P}_i(x_i)$ are non-empty and convex, and each $\hat{P}_i(x_i)$ has a non-empty interior, there exists a non-zero list of continuous personalized prices $p = (p_1, p_2, ..., p_m) \in (L')^m \subseteq (L^\sim)^m$ such that

$$\sum_{i=1}^m p_i \cdot y_i \leqslant \sum_{i=1}^m p_i \cdot z_i \tag{$\star$}$$

for all $(y_1, y_2, ..., y_m) \in \mathscr{A}_\omega$ and all $(z_1, z_2, ..., z_m) \in \prod_{i=1}^m \hat{P}_i(x_i)$. Fix a $v \in \mathscr{A}_\omega$ for which the economy is v-proper and note that $x + v$ is an interior point of $\prod_{i=1}^m \hat{P}_i(x_i)$. This, combined with ($\star$) and the facts that $(x_1, x_2, ..., x_m) \in \mathscr{A}_\omega$ and p is non-zero, yields $\sum_{i=1}^m p_i \cdot (v_i + x_i) > \sum_{i=1}^m p_i \cdot x_i$, which in view of $v \in \mathscr{A}_\omega$ implies $\psi_p \cdot \omega \geqslant \underline{\sum_{i=1}^m p_i \cdot v_i > 0}$.

Next, notice that since (A1.3) implies $x \in \prod_{i=1}^m P_i(x_i)$, it follows from ($\star$) that

$$\sum_{i=1}^m p_i \cdot y_i \leqslant \sum_{i=1}^m p_i \cdot x_i$$

for all $y = (y_1, y_2, ..., y_m) \in \mathscr{A}_\omega$. Taking into account that $x \in \mathscr{A}_\omega$, the latter inequality implies

$$\psi_p \cdot \omega = \sum_{i=1}^m p_i \cdot x_i. \tag{$\star\star$}$$

Thus, x is a maximizing allocation for the list of personalized prices p. Now the monotonicity and super-additivity of ψ_p imply the validity of the inequality $\sum_{i=1}^m \psi_p \cdot x_i \leqslant \psi_p \cdot (\sum_{i=1}^m x_i) \leqslant \psi_p \cdot \omega$. So, from (★★), we get

$$\sum_{i=1}^{m} \psi_p \cdot x_i \leqslant \sum_{i=1}^{m} p_i \cdot x_i. \tag{★★★}$$

Since 0 belongs to each consumption set, it is easy to see that $p_i \cdot z \leqslant \psi_p \cdot z$ for all $z \in X_i$. In particular, $p_i \cdot x_i \leqslant \psi_p \cdot x_i$, which in view of (★★★) yields $\psi_p \cdot x_i = p_i \cdot x_i$ for each i, and so $\sum_{i=1}^m \psi_p \cdot x_i = \psi_p \cdot \omega$.

Next, fix $z \in P_i(x_i)$, and let $y_j = x_j + v_j \in \hat{P}(x_j)$ for each $j \in I$. If for each $0 < \alpha < 1$, we define $(u_1, u_2, ..., u_m) \in \prod_{i=1}^m \hat{P}_i(x_i)$ by letting $u_i = z$ and $u_j = \alpha x_j + (1-\alpha)\, y_j = x_j + (1-\alpha)\, v_j \in \hat{P}_j(x_j)$ for $j \neq i$, then from (★) it follows that

$$\sum_{j=1}^{m} p_j \cdot u_j = p_i \cdot z + \alpha \sum_{j \neq i} p_j \cdot x_j + (1-\alpha) \sum_{j \neq i} p_j \cdot y_j \geqslant \sum_{i=1}^{m} p_i \cdot x_i.$$

Letting $\alpha \uparrow 1$, we easily get $p_i \cdot z \geqslant p_i \cdot x_i$ for each $i \in I$. From this, we infer that

$$\psi_p \cdot z \geqslant p_i \cdot z \geqslant p_i \cdot x_i = \psi_p \cdot x_i$$

for each i. In other words, we have shown that x is a personalized valuation equilibrium.

(2) Suppose that $x = (x_1, x_2, ..., x_m)$ is a personalized valuation equilibrium. This means that there exists some $p = (p_1, p_2, ..., p_m) \in (L^{\sim})^m$ such that $\sum_{i=1}^m \psi_p \cdot x_i = \psi_p \cdot \omega > 0$ and

$$y \in P_i(x_i) \Rightarrow \psi_p \cdot y \geqslant \psi_p \cdot x_i.$$

Next, assume by way of contradiction that there exists some allocation $z = (z_1, z_2, ..., z_m) \in \mathscr{A}_\omega$ such that $z_i \in P_i(x_i)$ for all $i \in I$. Since ψ_p is supporting, we have $\psi_p \cdot z_i \geqslant \psi_p \cdot x_i$ for each i, and so

$$\sum_{i=1}^{m} \psi_p \cdot z_i \geqslant \sum_{i=1}^{m} \psi_p \cdot x_i = \psi_p \cdot \omega \geqslant \sum_{i=1}^{m} \psi_p \cdot z_i.$$

This implies $\sum_{i=1}^m \psi_p \cdot z_i = \sum_{i=1}^m \psi_p \cdot x_i = \psi_p \cdot \omega > 0$ and $\psi_p \cdot z_i = \psi_p \cdot x_i$ for each i. In particular, for some $j \in I$ we have $\psi_p \cdot x_j > 0$. Since by A1.4 the set $P_j(x_j)$ is open for some linear topology on L, $\alpha z_j \in X_j$ for each $0 < \alpha < 1$,

$z_j \in P_j(x_j)$, and $\lim_{\alpha \uparrow 1} \alpha z_j = z_j$ for any linear topology on L, we can find some $0 < \alpha < 1$ such that $\alpha z_j \in P_j(x_j)$. Therefore, the homogeneity of ψ_p yields

$$\psi_p \cdot (\alpha z_j) = \alpha \psi_p \cdot z_j < \psi_p \cdot z_j = \psi_p \cdot x_j,$$

contrary to the supporting property of ψ_p. The proof of the theorem is now complete. ∎

6. RATIONAL ALLOCATIONS

Here we shall isolate a class of allocations—whose members will be referred to as rational allocations—that lies between the classes of weakly Pareto optimal allocations and Edgeworth equilibria. The notion of a rational allocation is in essence a "combination" of the concepts of weak Pareto optimality and individual rationality. In order to introduce the notion of a rational allocation we need some notation. Throughout the rest of this paper the letter $\mathscr{L}$ will denote the following non-empty, closed, and convex subset of L^m:

$$\mathscr{L} = \left\{ y \in L^m : \sum_{i=1}^{m} y_i \leqslant \omega \right\}.$$

Clearly, $\mathscr{A}_\omega \subseteq \mathscr{L}$. Now let $x = (x_1, x_2, ..., x_m)$ be an allocation. With this allocation we associate the m vectors $\theta_x^1, \theta_x^2, ..., \theta_x^m$ of L^m defined by

$$\begin{aligned} \theta_x^1 &= (\omega_1, x_2, x_3, ..., x_{m-1}, x_m), \\ \theta_x^2 &= (x_1, \omega_2, x_3, ..., x_{m-1}, x_m), \\ &\vdots \\ \theta_x^m &= (x_1, x_2, x_3, ..., x_{m-1}, \omega_m). \end{aligned}$$

Let $\Theta_x = \{\theta_x^1, \theta_x^2, ..., \theta_x^m\}$, and let

$$\mathscr{Z}_x = [\operatorname{co}(\Theta_x \cup \mathscr{L})] \cap \prod_{i=1}^{m} X_i.$$

Clearly, $\mathscr{Z}_x$ is non-empty and convex, and $\mathscr{A}_\omega \subseteq \mathscr{Z}_x \subseteq \prod_{i=1}^m X_i$. Now we can introduce the notion of a rational allocation.

Definition 6.1. An allocation $x = (x_1, x_2, ..., x_m)$ is said to be *rational* if

$$\mathscr{Z}_x \cap \prod_{i=1}^{m} P_i(x_i) = \operatorname{co}(\Theta_x \cup \mathscr{L}) \cap \prod_{i=1}^{m} P_i(x_i) = \varnothing.$$

Since $\mathcal{A}_\omega \subseteq \mathcal{Z}_x$, it should be clear that every rational allocation is weakly Pareto optimal. We shall see next that when the initial endowments can be decomposed into non-zero consumable bundles, rational allocations are individually rational. But first, let us introduce one more condition.

(A5) For each i there exists some $(z_1, z_2, ..., z_m) \in \mathcal{A}_{\omega_i}$ satisfying $z_j > 0$ for every $j \in I$.

Notice that (A5) is satisfied when one of the following conditions holds:

- $X_i = X_j$ for every $i, j \in I$. (Note that $(\frac{1}{m}\omega_i, \frac{1}{m}\omega_i, ..., \frac{1}{m}\omega_i) \in \mathcal{A}_{\omega_i}$.)
- There exists $\varepsilon > 0$ such that $\varepsilon\omega_i \geqslant \omega$ for every $i \in I$. (In this case, observe that $(\frac{1}{\varepsilon}\omega_1, \frac{1}{\varepsilon}\omega_2, ..., \frac{1}{\varepsilon}\omega_m) \in \mathcal{A}_{\omega_i}$.)

Lemma 6.2. *If conditions* (A1.3), (A1.4), *and* (A5) *are satisfied, then every rational allocation is individually rational.*

Proof. Let $x = (x_1, x_2, ..., x_m)$ be a rational allocation, in other words $\mathcal{Z}_x \cap \prod_{i=1}^m P_i(x_i) = \varnothing$. Suppose by way of contradiction that x is not individually rational. This means that $\omega_i \in P_i(x_i)$ for some $i \in I$; without loss of generality we can assume that $\omega_1 \in P_1(x_1)$.

By (A5) there exists some $(z_1, z_2, ..., z_m) \in \mathcal{A}_{\omega_1}$ satisfying $z_i > 0$ for each i. Since $P_1(x_1)$ is (by (A1.4)) an open subset of X_1 for some linear topology on L, $\alpha(x_1 + z_1) + (1 - 2\alpha)\,\omega_1 \in X_1$ for all $0 < \alpha < \frac{1}{2}$, and

$$\lim_{\alpha \downarrow 0}\,[\alpha(x_1 + z_1) + (1 - 2\alpha)\,\omega_1] = \omega_1 \in P_1(x_1),$$

for any linear topology on L, there exists some $0 < \alpha < \frac{1}{2}$ such that

$$\alpha(x_1 + z_1) + (1 - 2\alpha)\,\omega_1 \in P_1(x_1). \tag{$\dagger$}$$

Notice that the vector

$$u = (x_1 - \omega_1 + z_1, x_2 + z_2, x_3 + z_3, ..., x_m + z_m) \in L^m$$

belongs to $\mathcal{L}$. This implies that the vector

$$\begin{aligned} v &= \alpha u + (1 - \alpha)\,\theta_x^1 \\ &= (\alpha(x_1 + z_1) + (1 - 2\alpha)\,\omega_1, x_2 + \alpha z_2, x_3 + \alpha z_3, ..., x_m + \alpha z_m), \end{aligned}$$

belongs to $\mathcal{Z}_x$. From the strict monotonicity of preferences (A1.3), it follows that $x_i + \alpha z_i \in P_i(x_i)$ for all $i \geqslant 2$. Now a glance at ($\dagger$) guarantees that $v \in \prod_{i=1}^m P_i(x_i)$, and so $v \in \mathcal{Z}_x \cap \prod_{i=1}^m P_i(x_i) = \varnothing$, a contradiction. Therefore, x is individually rational. ∎

Rational allocations are personalized quasi-equilibria and when the initial endowments are decomposable in the sense of (A5) they are also personalized equilibria. The details are included in the next result.

LEMMA 6.3. *If conditions* (A1.2), (A1.3), (A1.4), *and* (A3) *are satisfied, then a rational allocation* $x = (x_1, x_2, ..., x_m)$

(1) *is always a personalized quasi-equilibrium with respect to a list of continuous personalized prices* $(p_1, p_2, ..., p_m) \in (L')^m$, *and*

(2) *if, in addition* (A5) *holds, then it is also a personalized equilibrium with respect to a list of continuous personalized prices* $(p_1, p_2, ..., p_m)$ *in* $(L')^m$.

Moreover, in both cases x *is a maximizing allocation for the list of personalized prices* p.

Proof. We begin with the first statement.

(1) Let $x = (x_1, x_2, ..., x_m)$ be a rational allocation, in other words $\mathscr{Z}_x \cap \prod_{i=1}^m P_i(x_i) = \emptyset$. From (A1.2) and the properness Condition (A3), for each i there is a convex-valued correspondence $\hat{P}_i: \hat{X}_i \twoheadrightarrow L$ whose values have non-empty interior such that $\mathscr{Z}_x \cap \prod_{i=1}^m \hat{P}_i(x_i) = \emptyset$.

Since the sets $\mathscr{Z}_c$ and $\prod_{i=1}^m \hat{P}_i(x_i)$ are both non-empty and convex, and each $\hat{P}_i(x_i)$ has a non-empty interior, there exists (by the separation theorem) a non-zero list of personalized prices $p = (p_1, p_2, ..., p_m) \in (L')^m$ such that

$$\sum_{i=1}^m p_i \cdot h_i \leqslant \sum_{i=1}^m p_i \cdot z_i \tag{$\star$}$$

for all $(h_1, h_2, ..., h_m) \in \mathscr{Z}_x$ and all $(z_1, z_2, ..., z_m) \in \prod_{i=1}^m \hat{P}_i(x_i)$. Taking into account that $\mathscr{A}_\omega \subseteq \mathscr{Z}_x$, we see (as in the proof of part (1) of Theorem 5.1) that x is a personalized valuation equilibrium and a maximizing allocation with respect to p.

We now show that if $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m) \in \mathbb{R}^m_+$, then

$$\sum_{i=1}^m \alpha_i \psi_p \cdot x_i \geqslant \psi_p \cdot \left(\sum_{i=1}^m \alpha_i \omega_i \right). \tag{$\star\star$}$$

To this end, let $\omega_\alpha = \sum_{i=1}^m \alpha_i \omega_i$ and let $z = (z_1, z_2, ..., z_m) \in \mathscr{A}_{\omega_\alpha}$. Then, it is easy to see that

$$u = (x_1 - \alpha_1 \omega_1 + z_1, z_2 - \alpha_2 \omega_2 + z_2, ..., x_m - \alpha_m \omega_m + z_m) \in \mathscr{L}.$$

For simplicity, we let $\beta = \sum_{i=1}^m \alpha_i$ and $\beta_j = \sum_{i \neq j} \alpha_i$ for each j. Now consider the vector $h = (h_1, h_2, ..., h_m)$ defined by

$$h = \frac{1}{1+\beta}\, u + \frac{\alpha_1}{1+\beta}\, \theta_x^1 + \frac{\alpha_2}{1+\beta}\, \theta_x^2 + \cdots + \frac{\alpha_m}{1+\beta}\, \theta_x^m \in L^m,$$

and note that $h \in \mathrm{co}(\Theta_x \cup \mathscr{L})$. An easy computation shows that

$$h_j = \frac{1+\beta_j}{1+\beta}\, x_j + \frac{1}{1+\beta}\, z_j \in X_j,$$

and so $h \in \prod_{i=1}^m X_i$. Therefore, $h \in \mathscr{Z}_x = \overline{[\mathrm{co}(\Theta_x \cup \mathscr{L})]} \cap \prod_{i=1}^m X_i$. Next, notice that (A1.3), implies $x \in \overline{\prod_{i=1}^m P_i(X_i)}$, and so from (★) it follows that

$$\begin{aligned}
\sum_{i=1}^m p_i \cdot x_i &\geqslant \sum_{i=1}^m p_i \cdot h_i \\
&= \frac{1+\beta_1}{1+\beta}\, p_1 \cdot x_1 + \frac{1+\beta_2}{1+\beta}\, p_2 \cdot x_2 + \cdots + \frac{1+\beta_m}{1+\beta}\, p_m \cdot x_m \\
&\quad + \frac{1}{1+\beta} \sum_{i=1}^m p_i \cdot z_i.
\end{aligned}$$

Now observing that $\beta - \beta_j = \alpha_j$, we get $\frac{1}{1+\beta} \sum_{i=1}^m \alpha_i p_i \cdot x_i \geqslant \frac{1}{1+\beta} \sum_{i=1}^m p_i \cdot z_i$, or

$$\sum_{i=1}^m \alpha_i p_i \cdot x_i \geqslant \sum_{i=1}^m p_i \cdot z_i.$$

Since $z \in \mathscr{A}_{\omega_\alpha}$, where $\omega_\alpha = \sum_{i=1}^m \alpha_i \omega_i$, was arbitrarily chosen, we see that

$$\sum_{i=1}^m \alpha_i p_i \cdot x_i \geqslant \psi_p \cdot \left(\sum_{i=1}^m \alpha_i \omega_i \right).$$

Finally, taking into account that $\psi_p \cdot x_i \geqslant p_i \cdot x_i$ for each i, the validity of (★★) follows.

(2) Once again, let $x = (x_1, x_2, ..., x_m)$ be a rational allocation. From (1) above there is a list of continuous prices $p = (p_1, p_2, ..., p_m) \in (L')^m$ such that x is a personalized quasi-equilibrium. We want to show that p can be chosen so that $\psi_p \cdot x_i > 0$ for each $i \in I$. Notice first that (as in the proof of Theorem 5.1) we can choose the list of continuous personalized prices p such that, on one hand, x is a maximizing allocation for p and, on the other hand, for each i we have

$$p_i \cdot x_i = \psi_p \cdot x_i \qquad \text{and} \qquad y \in P_i(x_i) \Rightarrow p_i \cdot y \geqslant p_i \cdot x_i.$$

Since $\sum_{i=1}^{m} \psi_p \cdot x_i \geqslant \psi_p \cdot \omega > 0$, there exists some $j \in I$ such that $p_j \cdot x_j = \psi_p \cdot x_j > 0$. Now using (A1.4), for each $y \in P_j(x_j)$ we can find $0 < \lambda < 1$ such that $\lambda y \in P_j(x_j)$. Thus,

$$p_j \cdot y \geqslant \lambda p_j \cdot y \geqslant p_j \cdot x_j > 0.$$

So, $p_j \cdot y > 0$, from which it follows that $p_j \cdot y > \lambda p_j \cdot y$ and $p_j \cdot y > p_j \cdot x_j$. Now from the strict monotonicity of preferences (A1.3), if $0 < z \in X_j$, then $x_j + z \in P_j(x_j)$. Therefore, $p_j \cdot (x_j + z) > p_j \cdot x_j > 0$ so that $p_j \cdot z > 0$.

From (A5), for each i there is some $z \in X_j$ such that $0 < z \leqslant \omega_i$. We conclude that

$$\psi_p \cdot x_i \geqslant \psi_p \cdot \omega_i \geqslant \psi_p \cdot z \geqslant p_j \cdot z > 0.$$

Now let $y \in P_i(x_i)$. There exists some $0 < \lambda < 1$ such that $\lambda y \in P_i(x_i)$. Thus, $\psi_p \cdot y \geqslant \lambda \psi_p \cdot y \geqslant \psi_p \cdot x_i > 0$. This implies $\psi_p \cdot y > \psi_p \cdot x_i$, and so x is a personalized equilibrium. ∎

The next result is a converse of Lemma 6.3.

Lemma 6.4. *If* (A1.4) *is satisfied, then a personalized quasi-equilibrium is a rational allocation.*

Proof. Assume that $x = (x_1, x_2, ..., x_m)$ is a personalized quasi-equilibrium. This guarantees the existence of some $p = (p_1, p_2, ..., p_m) \in (L^\sim)^m$ such that

(i) $\psi_p \cdot \omega > 0$,

(ii) $y \in P_i(x_i) \Rightarrow \psi_p \cdot y \geqslant \psi_p \cdot x_i$, and

(iii) for each $(\alpha_1, \alpha_2, ..., \alpha_m) \in \mathbb{R}_+^m$ we have

$$\sum_{i=1}^{m} \alpha_i \psi_p \cdot x_i \geqslant \psi_p \cdot \left(\sum_{i=1}^{m} \alpha_i \omega_i \right).$$

Now assume by way of contradiction that x is not a rational allocation. That is, that there exists some $z = (z_1, z_2, ..., z_m) \in \mathscr{Z}_x \cap \prod_{i=1}^{m} P_i(x_i)$. Notice first that since $\sum_{i=1}^{m} \psi_p \cdot x_i \geqslant \psi_p \cdot \omega > 0$, there exists some i for which $\psi_p \cdot x_i > 0$. Since $\psi_p \cdot z_i \geqslant \psi_p \cdot x_i > 0$, it follows that $z_i > 0$ for this particular i. Now taking into account $z_i \in P_i(x_i)$, it follows from (A1.4) that there exists some $0 < \lambda < 1$ satisfying

$$\lambda z_i \in P_i(x) \qquad \text{and} \qquad \psi_p \cdot z_i > \lambda \psi_p \cdot z_i = \psi_p \cdot (\lambda z_i) \geqslant \psi_p \cdot x_i > 0. \qquad (\dagger)$$

Next, observe that z is the convex combination of a point $u \in \mathscr{L}$ and the vectors $\theta_x^1, \theta_x^2, ..., \theta_x^m$ defined before Definition 6.1. That is, we can write

$z=\alpha u+\sum_{j=1}^{m}\beta_j\theta_x^j$, where $\alpha+\sum_{j=1}^{m}\beta_j=1$ and $\alpha, \beta_1, \beta_2, ..., \beta_m \geqslant 0$. From this and the definition of the vectors θ_x^j, it follows that

$$\begin{aligned}\sum_{i=1}^{m}(z_i-\omega_i)&=\alpha\sum_{i=1}^{m}(u_i-\omega_i)+\sum_{j=1}^{m}\sum_{i=1}^{m}\beta_j[(\theta_x^j)_i-\omega_i]\\ &\leqslant\sum_{j=1}^{m}\sum_{i=1}^{m}\beta_j[(\theta_x^j)_i-\omega_i]\\ &\leqslant-\sum_{j=1}^{m}\beta_j(x_j-\omega_j).\end{aligned}$$

This inequality can be rewritten as

$$\sum_{i=1}^{m}z_i+\sum_{i=1}^{m}\beta_i x_i\leqslant\sum_{i=1}^{m}\beta_i\omega_i+\sum_{i=1}^{m}\omega_i.$$

By the monotonicity of ψ_p, we get

$$\psi_p\cdot\left(\sum_{i=1}^{m}z_i+\sum_{i=1}^{m}\beta_i x_i\right)\leqslant\psi_p\cdot\left(\sum_{i=1}^{m}\beta_i\omega_i+\sum\nolimits_{i=1}^{m}\omega_i\right). \tag{$\star$}$$

However, from the super additivity of ψ_p, the definition of a personalized quasi-equilibrium, and (†), we have

$$\begin{aligned}\psi_p\cdot\left(\sum_{i=1}^{m}z_i+\sum_{i=1}^{m}\beta_i x_i\right)&\geqslant\sum_{i=1}^{m}\psi_p\cdot z_i+\sum_{i=1}^{m}\beta_i\psi_p\cdot x_i\\ &>\sum_{i=1}^{m}\psi_p\cdot x_i+\sum_{i=1}^{m}\beta_i\psi_p\cdot x_i\\ &\geqslant\psi_p\cdot\left(\sum_{i=1}^{m}\beta_i\omega_i+\sum\nolimits_{i=1}^{m}\omega_i\right),\end{aligned}$$

which contradicts ($\star$). Therefore, x is a rational allocation. ∎

The following theorem summarizes the results in this section.

THEOREM 6.5. *We have the following interrelationships between rational allocations and personalized quasi-equilibria*:

(1) *If conditions* (A1.2), (A1.3), (A1.4), *and* (A3) *are satisfied, then a rational allocation is a personalized quasi-equilibrium—which is a personalized equilibrium if* (A5) *is also valid.*

(2) *If* (A1.4) *is satisfied, then a personalized quasi-equilibrium is a rational allocation.*

7. EDGEWORTH EQUILIBRIA

In this section we shall show that the personalized equilibria under certain conditions coincide with the Edgeworth equilibria. We will also show that if the Decomposition Property (A5) does not hold, then rational allocations and personalized quasi-equilibria need not be Edgeworth equilibria (or even individually rational).

Let us first isolate another useful class of allocations.

DEFINITION 7.1. An allocation $(x_1, x_2, ..., x_m)$ is a *strong-Edgeworth equilibrium* if

$$\left[\operatorname{co} \bigcup_{i=1}^{m} [P_i(x_i) - \omega_i]\right] \cap (-L_+) = \varnothing.$$

Now using the Debreu–Scarf proof [22], we can prove that Edgeworth equilibria are strong-Edgeworth equilibria.

LEMMA 7.2. *Strong-Edgeworth equilibria are Edgeworth equilibria. Conversely, if* (A1.2) *and* (A1.4) *are satisfied, then an Edgeworth equilibrium is a strong-Edgeworth equilibrium.*

Proof. Let $x = (x_1, x_2, ..., x_m)$ be a strong-Edgeworth equilibrium. Assume by way of contradiction that x is not an Edgeworth equilibrium. This means that there exists an r-fold replica of the original economy, an allocation

$$(y_{11}, y_{12}, ..., y_{1r}, y_{21}, y_{22}, ..., y_{2r}, ..., y_{r1}, y_{r2}, ..., y_{rr}),$$

and a coalition S in this r-fold replica economy such that:

(1) $\sum_{(i,j) \in S} y_{ij} \leqslant \sum_{(i,j) \in S} \omega_{ij}$, and

(2) $y_{ij} \in P_{ij}(x_{ij})$ for all $(i, j) \in S$,

where $\omega_{ij} = \omega_i$, $x_{ij} = x_i$, and $P_{ij}(x_{ij}) = P_i(x_i)$ for all (i, j). If n is the number of consumers in S, then

$$\sum_{(i,j) \in S} \frac{1}{n} (y_{ij} - \omega_{ij}) \in \left[\operatorname{co} \bigcup_{i=1}^{m} [P_i(x_i) - \omega_i]\right] \cap (-L_+) = \varnothing,$$

a contradiction. We conclude that x is an Edgeworth equilibrium.

Now assume that (A1.4) is satisfied. Let $x=(x_1, x_2, ..., x_m)$ be an Edgeworth equilibrium and assume by way of contradiction that it fails to be a strong-Edgeworth equilibrium. This means that

$$\left[\operatorname{co} \bigcup_{i=1}^{m} [P_i(x_i)-\omega_i]\right] \cap (-L_+) \neq \emptyset.$$

Therefore, there exist $y_i \in P_i(x_i)$ and $\lambda_i \geqslant 0$ $(i=1, ..., m)$ such that

$$\sum_{i=1}^{m} \lambda_i = 1 \qquad \text{and} \qquad \sum_{i=1}^{m} \lambda_i(y_i-\omega_i) \leqslant 0. \tag{$\star$}$$

Next, consider the set $S=\{i\colon \lambda_i>0\}$ and note that $S\neq\emptyset$. From ($\star$), it follows that $\sum_{i=1}^m \lambda_i y_i \leqslant \sum_{i=1}^m \lambda_i \omega_i$ or

$$\sum_{i\in S} \lambda_i y_i \leqslant \sum_{i\in S} \lambda_i \omega_i. \tag{$\star\star$}$$

Now for each positive integer n, let n_i be the smallest integer greater than or equal to $n\lambda_i$, that is, $0 \leqslant n_i - n\lambda_i < 1$; clearly, $(n\lambda_i/n_i)\uparrow 1$ as $n\to\infty$ for each $i\in S$. Since for each $i\in S$ we have $\lim_{n\to\infty}(n\lambda_i/n_i)\, y_i = y_i \in P_i(x_i)$, each $P_i(x_i)$ is (according to Condition (A1.4)) open in X_i for some linear topology on L, and $(n\lambda_i/n_i)\, y_i \in X_i$ for each n, there exists some n large enough satisfying

$$z_i = \frac{n\lambda_i}{n_i}\, y_i \in P_i(x_i) \qquad \text{for each} \quad i\in S. \tag{$\star\star\star$}$$

Taking into account ($\star\star$), we see that

$$\sum_{i\in S} n_i z_i = \sum_{i\in S} n\lambda_i y_i \leqslant \sum_{i\in S} n\lambda_i \omega_i \leqslant \sum_{i\in S} n_i \omega_i.$$

The preceding inequality and ($\star\star\star$) show that the allocation $(x_1, ..., x_m)$ can be blocked by a coalition in the $(\sum_{i\in S} n_i)$-fold replica of the economy, which is a contradiction. This completes the proof of the theorem. ∎

Strong-Edgeworth equilibria are rational allocations.

LEMMA 7.3. *If conditions* (A1.3) *and* (A1.4) *are satisfied, then every strong-Edgeworth equilibrium is a rational equilibrium.*

Proof. Let $x=(x_1, x_2, ..., x_m)$ be a strong-Edgeworth equilibrium. Now let us suppose by way of contradiction that there exists some vector

$$z=(z_1, z_2, ..., z_m)\in \mathscr{Z}_x \cap \prod_{i=1}^{m} P_i(x_i).$$

Notice first that $z_i>0$ for each $i\in I$. Otherwise, $z_i=0$ for some i easily implies that $z_i-\omega_i=-\omega_i$ belongs to $[\text{co}\bigcup_{i=1}^{m}[P_i(x_i)-\omega_i]]\cap(-L_+)=\varnothing$, which is impossible.

Next, observe that z is the convex combination of a point $u\in\mathscr{L}$ and the vectors $\theta_x^1, \theta_x^2, ..., \theta_x^m$ defined before Definition 6.1. That is, we can write $z=\alpha u+\sum_{j=1}^{m}\beta_j\theta_x^j$, where $\alpha+\sum_{j=1}^{m}\beta_j=1$ and $\alpha, \beta_1, \beta_2, ..., \beta_m\geqslant 0$. From this and the definition of the vectors θ_x^j, it follows that

$$\begin{aligned}\sum_{i=1}^{m}(z_i-\omega_i)&=\alpha\sum_{i=1}^{m}(u_i-\omega_i)+\sum_{j=1}^{m}\sum_{i=1}^{m}\beta_j[(\theta_x^j)_i-\omega_i]\\&\leqslant\sum_{j=1}^{m}\sum_{i=1}^{m}\beta_j[(\theta_x^j)_i-\omega_i]\\&\leqslant-\sum_{j=1}^{m}\beta_j(x_j-\omega_j).\end{aligned}$$

If $J=\{j\in I:\beta_j>0\}$, then the last inequality can be re-written as

$$\sum_{i=1}^{m}(z_i-\omega_i)+\sum_{j\in J}\beta_j(x_j-\omega_j)\leqslant 0. \tag{1}$$

Since $z_i\in P_i(x_i)$ and each $P_i(x_i)$ is (in view of (A1.4)) open in X_i for some linear topology on L, it follows from $\lim_{\lambda\uparrow 1}\lambda z_i=z_i$ that there exists some $0<\lambda<1$ such that $\lambda z_i\in P_i(x_i)$ for all $i\in I$. Next, rewrite (1) as

$$\sum_{i=1}^{m}(\lambda z_i-\omega_i)+\sum_{j\in J}\beta_j(x_j-\omega_j)+\sum_{i=1}^{m}(1-\lambda)\,z_i\leqslant 0,$$

and conclude that

$$\begin{aligned}&\sum_{i=1}^{m}(\lambda z_i-\omega_i)+\sum_{j\in J}\beta_j\left(x_j+\frac{1-\lambda}{\beta_j}z_j-\omega_j\right)\\&\quad=\sum_{i=1}^{m}(\lambda z_i-\omega_i)+\sum_{j\in J}\beta_j(x_j-\omega_j)+\sum_{j\in J}(1-\lambda)\,z_i\\&\quad\leqslant\sum_{i=1}^{m}(\lambda z_i-\omega_i)+\sum_{j\in J}\beta_j(x_j-\omega_j)+\sum_{i=1}^{m}(1-\lambda)\,z_i\\&\quad\leqslant 0.\end{aligned}$$

Now let $\gamma = \sum_{j \in J} \beta_j \geqslant 0$ (with $\gamma = 0$ if $J = \varnothing$), and note that the preceding inequality yields

$$\sum_{i=1}^{m} \frac{1}{\gamma + m} (\lambda z_i - \omega_i) + \sum_{j \in J} \frac{\beta_j}{\gamma + m} \left(x_j + \frac{1-\lambda}{\beta_j} z_j - \omega_j \right) \leqslant 0. \tag{2}$$

Since $0 < ((1-\lambda)/\beta_j)\, z_j \in X_j$ for each $j \in J$, it follows from the monotonicity condition (A1.3) that $x_j + ((1-\lambda)/\beta_j)\, z_j \in P_j(x_j)$ for each $j \in J$. From the equality $\sum_{i=1}^{m}(1/\gamma + m) + \sum_{j \in J}(\beta_j/\gamma + m) = 1$ and (2), we obtain

$$\sum_{i=1}^{m} \frac{1}{\gamma + m} (\lambda z_i - \omega_i) + \sum_{j \in J} \frac{\beta_j}{\gamma + m} \left(x_j + \frac{1-\lambda}{\beta_j} z_j - \omega_j \right)$$
$$\in \left[\operatorname{co} \bigcup_{i=1}^{m} [P_i(x_i) - \omega_i] \right] \cap (-L_+) = \varnothing,$$

which is impossible. This contradiction completes the proof. ∎

Personalized equilibria are strong-Edgeworth equilibria—and thus under minor assumptions they are also Edgeworth equilibria.

LEMMA 7.4. *Every personalized equilibrium is a strong-Edgeworth equilibrium.*

Proof. Let $x = (x_1, x_2, ..., x_m)$ be a personalized equilibrium. This guarantees the existence of some list of prices $p = (p_1, p_2, ..., p_m) \in (L^{\sim})^m$ such that

(i) $y \in P_i(x_i) \Rightarrow \psi_p \cdot y > \psi_p \cdot x_i$, and

(ii) for every $(\alpha_1, \alpha_2, ..., \alpha_m) \in \mathbb{R}^m_+$ we have

$$\sum_{i=1}^{m} \alpha_i \psi_p \cdot x_i \geqslant \psi_p \cdot \left(\sum_{i=1}^{m} \alpha_i \omega_i \right).$$

Now assume by way of contradiction that x is not a strong-Edgeworth equilibrium. That is, there exist non-negative scalars $\alpha_1, \alpha_2, ..., \alpha_m$ with $\sum_{i=1}^{m} \alpha_i = 1$ and some $(z_1, z_2, ..., z_m) \in \prod_{i=1}^{m} P_i(x_i)$ satisfying

$$\sum_{i=1}^{m} \alpha_i z_i \leqslant \sum_{i=1}^{m} \alpha_i \omega_i.$$

The super-additivity and monotonicity of ψ_p imply

$$\sum_{i=1}^{m} \alpha_i \psi_p \cdot z_i = \sum_{i=1}^{m} \psi_p \cdot (\alpha_i z_i) \leqslant \psi_p \cdot \left(\sum_{i=1}^{m} \alpha_i z_i \right) \leqslant \psi_p \cdot \left(\sum_{i=1}^{m} \alpha_i \omega_i \right). \tag{$\bigstar$}$$

Now a glance at (i) and (ii) yields

$$\sum_{i=1}^{m} \alpha_i \psi_p \cdot z_i > \sum_{i=1}^{m} \alpha_i \psi_p \cdot x_i \geqslant \psi_p \cdot \left(\sum_{i=1}^{m} \alpha_i \omega_i \right),$$

contrary to (★). Therefore, x is a strong-Edgeworth equilibrium. ▮

Combining Lemmas 6.3, 7.2, 7.3, and 7.4 we obtain the following result regarding the relationships among various notions of equilibria that we have introduced.

THEOREM 7.5. *We have the following interrelationships between personalized equilibria, Edgeworth equilibria, and rational allocations*:

(1) *If conditions* (A1.2), (A1.3), (A1.4), (A3), *and* (A5) *are satisfied, then every Edgeworth equilibrium is a personalized equilibrium.*

(2) *If conditions* (A1.2), (A1.3), (A1.4), (A3), *and* (A5) *are satisfied, then a rational allocation is an Edgeworth equilibrium.*

(3) *A personalized equilibrium is an Edgeworth equilibrium.*

(4) *If conditions* (A1.2), (A1.3), *and* (A1.4) *are satisfied, then an Edgeworth equilibrium is a rational allocation.*

We also have the following equivalence result regarding the relationships among the various notions of equilibria that we have introduced before.

COROLLARY 7.6. *Assume* (A1.2), (A1.3) (A1.4), (A3), *and* (A5) *hold. Then for an allocation x the following statements are equivalent.*

(1) *x is an Edgeworth equilibrium.*

(2) *x is a personalized equilibrium.*

(3) *x is a rational equilibrium.*

(4) *x is a strong-Edgeworth equilibrium.*

Before moving on to the next section, we give an example which brings out the role of the Decomposition Assumption (A5). The example presents a rational allocation that is a personalized quasi-equilibrium but fails to be a personalized equilibrium and individually rational—in particular it is neither a strong-Edgeworth equilibrium nor an Edgeworth equilibrium.

EXAMPLE 7.7. The commodity space is $\mathbb{R}^2$. There are two consumers with characteristics:

Consumer 1. $X_1 = \{(x, 0) : x \geqslant 0\}$, $\omega_1 = (1, 0)$, and $P_1 \colon X_1 \twoheadrightarrow X_1$ defined by

$$P_1(x, 0) = \{(s, 0) \in X_1 : s > x\}.$$

Consumer 2. $X_2 = \{(0, y) : y \geqslant 0\}$, $\omega_2 = (0, 1)$, and $P_2 \colon X_2 \twoheadrightarrow X_2$ defined by

$$P_2(0, y) = \{(0, t) \in X_2 :\ t > y\}.$$

Clearly, $\omega = \omega_1 + \omega_2 = (1, 1)$ and $C = X_1 + X_2 = \mathbb{R}^2_+$. A straightforward verification shows that (A1), (A2), (A3), and (A4) are true. Also, an easy argument shows that (A5) is not satisfied.

Now consider the allocation $x = (x_1, x_2) = ((1, 0), (0, 0))$. We claim that x is a personalized quasi-equilibrium with respect to the list of personalized prices $p = ((1, 0), (0, 0))$. To see this, we need to verify that $\psi_p \cdot (x, y) = x$ holds for each $(x, y) \in \mathbb{R}^2_+$; and so the generalized price ψ_p is a linear price here. Now the properties

(1) $\psi_p \cdot \omega = 1 > 0$,

(2) $(s, 0) \in P_1((1, 0)) \Rightarrow \psi_p \cdot (s, 0) = s > 1 = \psi_p \cdot (1, 0)$,

(3) $(0, t) \in P_2((0, 0)) \Rightarrow \psi_p \cdot (0, t) = 0 = \psi_p \cdot (0, 0)$, and

(4) $\psi_p \cdot (\alpha_1(1, 0) + \alpha_2(0, 0)) = \alpha_1 = \alpha_1 \psi_p \cdot (1, 0) + \alpha_2 \psi_p \cdot (0, 0)$,

guarantee that x is indeed a personalized quasi-equilibrium.

From $\omega_2 \in P_2((0, 0))$, we see that x is not individually rational. Hence, it is neither a core allocation nor an Edgeworth equilibrium. By Lemma 7.4, x cannot be a personalized equilibrium.

8. EXISTENCE OF EQUILIBRIA

In this section we shall establish the existence of strong-Edgeworth equilibria. Our proof will be based upon a limiting argument that uses the following fixed point theorem. This theorem can be found in the work of Toussaint [57] and Yannelis and Prabhakar [59]. The finite dimensional version of this theorem can be found in the proof of Gale and Mas-Colell [28].

THEOREM 8.1. *Assume that X_i $(i = 0, 1, ..., m)$ is a non-empty compact convex subset of a Hausdorff topological vector space L_i and let $X = \prod_{i=0}^m X_i$. For each i let $P_i \colon X \twoheadrightarrow X_i$ be a convex-valued correspondence which is:*

(1) *irreflexive, i.e., for each* $x=(x_0, x_1, ..., x_m) \in X$ *we have* $x_i \notin P_i(x)$, *and*

(2) *has open lower sections, i.e., the set* $P_i^{-1}(y)=\{x \in X: \ y \in P_i(x)\}$ *is open in* X *for each* $y \in X_i$.

Then there exists some $x \in X$ *with* $P_i(x)=\emptyset$ *for all* i.

For our proof of the existence of strong-Edgeworth equilibrium we shall employ the following simple result from general topology.

LEMMA 8.2. *Let* S *be a subset of a topological space and let some point* $x \in S$. *Then* x *is an interior point of* S *if and only if every net of* X *that converges to* x *lies eventually in* S.

And now we are ready to prove the existence of strong-Edgeworth equilibria; for related results see [4, 23, 27, 60].

THEOREM 8.3. *If* (A1.1), (A1.2), (A1.4), (A1.5), *and* (A2) *hold, then the economy has a strong-Edgeworth equilibrium.*

Proof. Since the compact convex set $\mathrm{co}\{\omega_1, \omega_2, ..., \omega_m\}$ is disjoint from the τ-closed convex cone $-L_+$, there exists (by the strict separation theorem) some positive $q \in L'$ satisfying $q \cdot \omega_i > 0$ for each i. This implies that there is a weak*-dense subset T of L'_+ such that: if $p \in T$, then $p \cdot \omega_i > 0$ for each $i \in I$. (To see this, notice that if $q \in L'_+$, then $[\alpha q + (1-\alpha)\, p] \cdot \omega_i > 0$ holds for each i and all $0 < \alpha < 1$.)

Let $\mathscr{D}$ be the family of all convex hulls of the finite subsets of T. Also, for each i, let $\mathscr{B}_i$ be the family of all convex hulls of the finite subsets of X_i that contain 0 and ω_i. Put $\mathscr{B} = \prod_{i=1}^m \mathscr{B}_i$ and direct both $\mathscr{D}$ and $\mathscr{B}$ by "superset" inclusion. For each $(\alpha, \beta) \in \mathscr{D} \times \mathscr{B}$, where $\beta = \prod_{i=1}^m \beta_i$, define the correspondences $\eta_i^{\alpha\beta}, \gamma_i^{\alpha\beta} \colon \alpha \twoheadrightarrow \beta_i$ $(i=1, 2, ..., m)$ by

$$\eta_i^{\alpha\beta}(p) = \{x \in \beta_i : p \cdot x < p \cdot \omega_i\} \qquad \text{and} \qquad \gamma_i^{\alpha\beta}(p) = \{x \in \beta_i : p \cdot x \leqslant p \cdot \omega_i\}.$$

Clearly, both correspondences are convex-valued and, in view of

$$0 \in \eta_i^{\alpha\beta}(p) \subseteq \gamma_i^{\alpha\beta}(p), \tag{a}$$

they are also both non-empty-valued.

Following the proof in Gale–Mas-Colell [28], for each (α, β) in $\mathscr{D} \times \mathscr{B}$, where $\beta = \prod_{i=1}^m \beta_i$, let us define the convex-valued correspondences $\Phi_i^{\alpha\beta} \colon \alpha \times \beta \twoheadrightarrow \beta_i$ $(i=1, 2, ..., m)$ by

$$\Phi_i^{\alpha\beta}(p, x_1, ..., x_m) = \begin{cases} \eta_i^{\alpha\beta}(p) & \text{if} \quad x_i \notin \gamma_i^{\alpha\beta}(p), \\ \eta_i^{\alpha\beta}(p) \cap P_i(x_i) & \text{if} \quad x_i \in \gamma_i^{\alpha\beta}(p), \end{cases}$$

and the convex-valued correspondence $\Phi_0^{\alpha\beta}\colon \alpha\times\beta \twoheadrightarrow \alpha$ by

$$\Phi_0^{\alpha\beta}(p, x_1, ..., x_m) = \left\{q\in\alpha : q\cdot\left(\sum_{i=1}^{m} x_i - \omega\right) > p\cdot\left(\sum_{i=1}^{m} x_i - \omega\right)\right\}.$$

(Notice that we use here that each P_i is convex-valued; condition (A1.2).)

Now fix some $(\alpha, \beta)\in\mathscr{D}\times\mathscr{B}$, where $\beta=\prod_{i=1}^{m}\beta_i$. Let $X_0=\alpha$ and $X_i=\beta_i$ for each $i=1, ..., m$. We claim that the convex-valued correspondences $\Phi_i^{\alpha\beta}\colon X=\prod_{i=0}^{m} X_i \twoheadrightarrow X_i$ $(i=0, 1, ..., m)$ satisfy the assumptions of Theorem 8.1. To see this, note first that they are irreflexive (for this we must use Condition (A1.1)) and that $\Phi_0^{\alpha\beta}$ has τ-open lower sections.

What needs verification is the τ-openness of the lower sections of the correspondences $\Phi_i^{\alpha\beta}$ for $i=1, 2, ..., m$. To establish this, fix i and let $(p, x_1, x_2, ..., x_m)\in(\Phi_i^{\alpha\beta})^{-1}(y)$, i.e., $y\in\Phi_i^{\alpha\beta}(p, x_1, x_2, ..., x_m)$. Also, assume that a net $\{(p^\mu, x_1^\mu, ..., x_m^\mu)\}$ of $\alpha\times\beta$ satisfies

$$(p^\mu, x_1^\mu, ..., x_m^\mu) \xrightarrow[\mu]{} (p, x_1, x_2, ..., x_m).$$

Each β_i lies in a finite dimensional subspace of L, so we have $x_i^\mu \xrightarrow[\mu]{\sigma(L, L')} x_i$ for each $i=1, ..., m$. Similarly, $p^\mu \xrightarrow[\mu]{\sigma(L', L)} p$.

By Lemma 8.2 it suffices to show that $(p^\mu, x_1^\mu, ..., x_m^\mu)\in(\Phi_i^{\alpha\beta})^{-1}(y)$ holds true for all μ eventually large. We distinguish two cases.

Case I. $x_i\in\gamma_i^{\alpha\beta}(p)$. This implies $y\in\eta_i^{\alpha\beta}(p)\cap P_i(x_i)$. In particular, we have $p\cdot y<p\cdot\omega_i$. Since (according to Condition (A1.5)) $P_i(x_i)$ has weakly open lower sections in X_i and $x_i\in P_i^{-1}(y)$, there exists some μ_0 such that $p^\mu\cdot y<p^\mu\cdot\omega_i$ and $x_i^\mu\in P_i^{-1}(y)$ for all $\mu\geqslant\mu_0$. That is, $y\in\eta_i^{\alpha\beta}(p^\mu)\cap P_i(x_i^\mu)$ for all $\mu\geqslant\mu_0$. Now assume $\mu\geqslant\mu_0$. If $x_i^\mu\in\gamma_i^{\alpha\beta}(p^\mu)$, then

$$y\in\eta_i^{\alpha\beta}(p^\mu)\cap P_i(x_i^\mu)=\Phi_i^{\alpha\beta}(p^\mu, x_1^\mu, ..., x_m^\mu).$$

If $x_i^\mu\notin\gamma_i^{\alpha\beta}(p^\mu)$, then

$$y\in\eta_i^{\alpha\beta}(p^\mu)=\Phi_i^{\alpha\beta}(p^\mu, x_1^\mu, ..., x_m^\mu).$$

In other words, $(p^\mu, x_1^\mu, ..., x_m^\mu)\in(\Phi_i^{\alpha\beta})^{-1}(y)$ for all $\mu\geqslant\mu_0$.

Case II. $x_i\notin\gamma_i^{\alpha\beta}(p)$. Notice that $x_i\notin\gamma_i^{\alpha\beta}(p)$ is equivalent to $p\cdot x_i>p\cdot\omega_i$. In this case, we also have $y\in\eta_i^{\alpha\beta}(p)$ or $p\cdot y<p\cdot\omega_i$, and $y\in P_i(x_i)$ or $x_i\in P_i^{-1}(y)$. Since the α and the β both lie in finite dimensional spaces and the valuation is jointly continuous on finite dimensional spaces and $P_i^{-1}(y)$

is a weakly open subset of X_i, it follows that there exists some index μ_1 such that

$$p^\mu \cdot x_i^\mu > p^\mu \cdot \omega_i, \qquad p^\mu \cdot y < p^\mu \cdot \omega_i, \qquad \text{and} \qquad x_i^\mu \in P_i^{-1}(y)$$

for all $\mu \geqslant \mu_1$. Therefore, $x_i^\mu \notin \gamma_i(p^\mu)$ and

$$y \in \eta_i^{\alpha\beta}(p^\mu) \cap P_i(x_i^\mu) = \Phi_i^{\alpha\beta}(p^\mu, x_1^\mu, ..., x_m^\mu),$$

for all $\mu \geqslant \mu_1$.

By Theorem 8.1 for each $(\alpha, \beta) \in \mathscr{D} \times \mathscr{B}$ there is some $(p^{\alpha\beta}, x_1^{\alpha\beta}, ..., x_m^{\alpha\beta})$ in $\alpha \times \beta$ such that

$$\Phi_i^{\alpha\beta}(p^{\alpha\beta}, x_1^{\alpha\beta}, ..., x_m^{\alpha\beta}) = \varnothing, \qquad \text{for each} \quad i = 0, 1, ..., m.$$

Since each $\eta_i^{\alpha\beta}(p^{\alpha\beta})$ is, in view of (a), never empty, the definition of $\Phi_i^{\alpha\beta}$ implies that $x_i^{\alpha\beta} \in \gamma_i^{\alpha\beta}(p^{\alpha\beta})$ and

$$\Phi_i^{\alpha\beta}(p^{\alpha\beta}, x_1^{\alpha\beta}, ..., x_m^{\alpha\beta}) = \eta_i^{\alpha\beta}(p^{\alpha\beta}) \cap P_i(x_i^{\alpha\beta}) = \varnothing, \tag{b}$$

for each $i = 1, ..., m$. It follows that

$$p^{\alpha\beta} \cdot x_i^{\alpha\beta} - p^{\alpha\beta} \cdot \omega_i \leqslant 0. \tag{c}$$

Next, we claim that $z \in P_i(x_i^{\alpha\beta}) \cap \beta_i$ implies $p^{\alpha\beta} \cdot z > p^{\alpha\beta} \cdot \omega_i$. Indeed, if $z \in P_i(x_i^{\alpha\beta}) \cap \beta_i$, then $p^{\alpha\beta} \cdot z < p^{\alpha\beta} \cdot \omega_i$ cannot be true. Otherwise, we must have $z \in \eta_i^{\alpha\beta}(p^{\alpha\beta})$ which violates (b). Moreover, $p^{\alpha\beta} \cdot z = p^{\alpha\beta} \cdot \omega_i$ cannot be true either. Otherwise, if equality holds, then (since, according to Condition (A1.4), $P_i(x_i^{\alpha\beta})$ is an open subset of X_i for some linear topology) for some $0 < \lambda < 1$ we must have $\lambda z \in P_i(x_i^{\alpha\beta})$, which in view of $p^{\alpha\beta} \cdot \omega_i > 0$ (recall here that $p^{\alpha\beta} \in T$) implies $p^{\alpha\beta} \cdot (\lambda z) < p^{\alpha\beta} \cdot z = p^{\alpha\beta} \cdot \omega_i$, contrary to (b) again. Since $p^{\alpha\beta} \cdot (-x) \leqslant 0$ for each $x \in L_+$, the preceding conclusion shows that

$$\text{co}\left[\bigcup_{i=1}^{m} ([P_i(x_i^{\alpha\beta}) \cap \beta_i] - \omega_i)\right] \cap (-L_+) = \varnothing. \tag{d}$$

Since $\Phi_0^{\alpha\beta}(p^{\alpha\beta}, x_1^{\alpha\beta}, ..., x_m^{\alpha\beta}) = \varnothing$, it follows from the definition of $\Phi_0^{\alpha\beta}$ and (c) that for each $q \in \alpha$ we have

$$q \cdot \left[\sum_{i=1}^{m} (x_i^{\alpha\beta} - \omega_i)\right] \leqslant p^{\alpha\beta} \cdot \left[\sum_{i=1}^{m} (x_i^{\alpha\beta} - \omega_i)\right] = \sum_{i=1}^{m} [p^{\alpha\beta} \cdot x_i^{\alpha\beta} - p^{\alpha\beta} \cdot \omega_i] \leqslant 0. \tag{e}$$

Now fix $\beta \in \mathscr{B}$ and consider the net $\{(x_1^{\alpha\beta}, x_2^{\alpha\beta}, ..., x_m^{\alpha\beta})\}_{\alpha \in \mathscr{D}}$ of the τ-compact subset β. If $(x_1^\beta, ..., x_m^\beta) \in \beta$ is a τ-accumulation point of this net, then it follows from (e) that $q \cdot [\sum_{i=1}^m (x_i^\beta - \omega_i)] \leqslant 0$ for each $q \in T$, and consequently for all $q \in L'_+$. This implies $\sum_{i=1}^m (x_i^\beta - \omega_i) \in -L_+$ or that $(x_1^\beta, ..., x_m^\beta)$ is an allocation.[9] Now we claim that

$$\operatorname{co}\left[\bigcup_{i=1}^m ([P_i(x_i^\beta) \cap \beta_i] - \omega_i)\right] \cap (-L_+) = \varnothing. \tag{f}$$

To see this, assume by way of contradiction that

$$\operatorname{co}\left[\bigcup_{i=1}^m ([P_i(x_i^\beta) \cap \beta_i] - \omega_i)\right] \cap (-L_+) \neq \varnothing.$$

So, there exist $u_i \in P_i(x_i^\beta) \cap \beta_i$ $(i = 1, 2, ..., m)$ and a convex combination u of the $u_i - \omega_i$ satisfying $u = \sum_{i=1}^m \lambda_i(u_i - \omega_i) \leqslant 0$. Since P_i has weakly open lower sections, we see that the u_i belong to $P_i(x_i^{\alpha\beta}) \cap \beta_i$ for all eventually large α, which contradicts (d).

Finally, since (in view of Condition A2) $\mathscr{A}_\omega$ is a weakly compact set and the net $\{(x_1^\beta, x_2^\beta, ..., x_m^\beta)\}_{\beta \in \mathscr{B}}$ lies in $\mathscr{A}_\omega$, there exists an accumulation point $(x_1, ..., x_m)$ of this net in $\mathscr{A}_\omega$. In particular, we have $\sum_{i=1}^m (x_i - \omega_i) \leqslant 0$. Also since $P_i^{-1}(y)$ is weakly open in X_i for each $y \in X_i$, we infer (as before) from (f) that

$$\operatorname{co}\left[\bigcup_{i=1}^m [P_i(x_i) - \omega_i]\right] \cap (-L_+) = \varnothing.$$

This shows that the allocation $(x_1, x_2, ..., x_m)$ is a strong-Edgeworth equilibrium. ∎

9. APPLICATIONS

In this section we provide some applications of our main results. Subsection 9.1 demonstrates that the most important results on the existence of equilibrium in the literature are easy consequences of our analysis. Subsection 9.2 indicates how one can apply our results to differential information economies. Subsection 9.3 offers an alternative interpretation to our model in terms of a discriminatory price auction for the total endowment of resources.

[9] Here we use the following fact: *a vector $x \in L$ satisfies $x \in -L_+$ if and only if $p \cdot x \leqslant 0$ for each $p \in L'_+$*. To see this, assume $p \cdot x \leqslant 0$ for all $p \in L'_+$. If $x \notin -L_+$, then (by the separation theorem) there exists some non-zero $q \in L'_+$ such that $q \cdot x > 0$, which is a contradiction.

9.1. *The Walrasian Model*

Our objective here is to demonstrate that the most important results on the existence of equilibrium in the literature are easy consequences of our analysis. Throughout the discussion in this section, we assume that:

(1) L is a vector lattice,

(2) $X_i = L_+$ for each i, and

(3) L' is a vector sublattice of the order dual $L^\sim$.

In this case, the generalized price ψ_p of a list of order bounded personalized prices $p = (p_1, p_2, ..., p_m)$ coincides with the linear functional

$$\left(\bigvee_{i=1}^{m} p_i\right)^+ = p_1 \vee p_2 \vee \cdots \vee p_m \vee 0 \in L^\sim,$$

i.e., $\psi_p \cdot x = (\bigvee_{i=1}^m p_i)^+ (x)$ for each $x \in L_+$. Notice that ψ_p need not be τ-continuous. However, if $p_i \in L'$ for each i, then (since L' is a vector sublattice of $L^\sim$) we automatically have $\psi_p = (\bigvee_{i=1}^m p_i)^+ \in L'$. Moreover, it should be noticed that in this case:

(α) (A4) (see Lemma 3.4) and (A5) are automatically true, and

(β) (A2) is true if and only if the order interval $[0, \omega]$ is weakly compact.

Let us recall the standard properties of a preference relation (i.e., a complete and transitive relation) $\succcurlyeq$ on L_+.

(a) a bundle $v > 0$ is *desirable* for $\succcurlyeq$ if $x + \alpha v \succ x$ for each $x \in L_+$ and all $\alpha > 0$,

(b) $\succcurlyeq$ is *convex*, if the set $\{y \in L_+ : y \succcurlyeq x\}$ is convex for each $x \in L_+$,

(c) $\succcurlyeq$ is *monotone* (resp. *strictly monotone*), if $x > y \geqslant 0$ implies $x \succcurlyeq y$ (resp. $x \succ y$),

(d) $\succcurlyeq$ is *locally non-satiated*, if for each $x \in L_+$ and each τ-neighborhood V of x there exists some $y \in V \cap L_+$ such that $y \succ x$,

(e) $\succcurlyeq$ is *upper-semicontinuous* (resp. *lower-semicontinuous*) if for each $x \in L_+$ the set $\{y \in L_+ : y \succcurlyeq x\}$) (resp. the set $\{y \in L_+ : x \succcurlyeq y\}$) is closed, and

(f) $\succcurlyeq$ is *continuous*, if it is both upper and lower semicontinuous.

Now let us make the connection between the above standard assumptions and our properties. If $\succcurlyeq$ is a preference on L_+, then we define its *strict preference correspondence* $P\colon L_+ \twoheadrightarrow L_+$ via the formula

$$P(x) = \{ y \in L_+ : y \succ x \}.$$

We have the following relationships between the properties of $\succcurlyeq$ and the properties of its strict preference correspondence P.

Non-emptiness of the values. If $\succcurlyeq$ is locally non-satiated (in particular if $\succcurlyeq$ has a desirable bundle), then $P(x) \neq \varnothing$ for each $x \in L_+$.

Transitivity. P is a transitive correspondence.

(A1.1) *Irreflexivity.* P is irreflexive.

(A1.2) *Convexity.* If $\succcurlyeq$ is convex, upper-semicontinuous and locally non-satiated, then $P(x)$ is a non-empty convex set for each $x \in L_+$.

If $y_1 \succ x$, $y_2 \succ x$ and $0 \leqslant \alpha \leqslant 1$, then the set

$$\{ z \in L_+ : y_1 \succ z \text{ and } y_2 \succ z \}$$

is a neighborhood of x in L_+, and so there exists some $z_1 \in L_+$ satisfying $y_1 \succ z_1 \succ x$ and $y_1 \succ z_1 \succ x$. Now notice that $\alpha y_1 + (1-\alpha)\, y_2 \succcurlyeq z_1 \succ x$.

(A1.3) *Strict monotonicity.* If $\succcurlyeq$ is strictly monotone, then P is strictly monotone.

(A1.4) *Open values.* If $\succcurlyeq$ is lower-semicontinuous, then P has τ-open values in L_+.

This follows immediately from the identity

$$P(x) = \{ y \in L_+ : y \succ x \} = L_+ \backslash \{ z \in L_+ : x \succcurlyeq z \}.$$

(A1.5) *Weakly open lower sections.* If $\succcurlyeq$ is convex and upper-semicontinuous, then P has weakly open lower sections.

To see this, let $y \in L_+$. Then we have

$$\begin{aligned} P^{-1}(y) &= \{ x \in L_+ : y \in P(x) \} \\ &= \{ x \in L_+ : y \succ x \} = L_+ \backslash \{ z \in L_+ : z \succcurlyeq y \}. \end{aligned}$$

Since the set $\{ z \in L_+ : z \succcurlyeq y \}$ is a τ-closed convex subset, it follows that it is also weakly closed—recall here that the τ-closed convex subsets of L coincide with the weakly closed convex subsets of L. This implies that $P^{-1}(y)$ is a weakly open subset of L_+.

(A3) *Properness.* If $\succcurlyeq$ is ω-uniformly τ-proper in the sense of Mas-Colell [41, 42], then P is $(\frac{1}{m}\omega, \frac{1}{m}\omega, ..., \frac{1}{m}\omega)$-proper as in (A3); for a proof see [8].

In the first application of our results, we shall obtain as corollaries the second welfare theorems of Mas-Colell [41] and Mas-Colell and Richard [43] (see also [1]). ▮

Corollary 9.1. *If* (A1.2) (A1.3), *and* (A3) *are valid, then every weakly Pareto optimal allocation is a Walrasian valuation equilibrium.*

Proof. Let $x=(x_1, x_2, ..., x_m)$ be a weakly Pareto optimal allocation. According to Theorem 5.1(1), there exists some $(p_1, p_2, ..., p_m) \in (L')^m$ such that x is a personalized valuation equilibrium with respect to the generalized price $0 < \psi_p = (\bigvee_{i=1}^m p_i)^+ \in L'$. This implies immediately that x is a Walrasian valuation equilibrium with respect to the non-zero τ-continuous linear price ψ_p. ▮

The well-known existence of equilibrium theorems of Bewley [16], Mas-Colell [41] and Mas-Colell and Richard [43] (which include as corollaries the classical Arrow–Debreu–McKenzie theorem) also follow from our results—with both order and unordered preferences.

Corollary 9.2. *If* (A1), (A2), *and* (A3) *hold, then the economy has a Walrasian equilibrium.*

Proof. By Theorem 8.3 there exists a strong-Edgeworth equilibrium $x=(x_1, x_2, ..., x_m)$. This strong-Edgeworth equilibrium x is, by Theorem 7.6, also a personalized equilibrium. In turn, by Theorem 6.3(2), x is a personalized equilibrium with respect to a non-zero list of continuous personalized prices $p=(p_1, p_2, ..., p_m) \in (L')^m$. Clearly $\psi_p = (\bigvee_{i=1}^m p_i)^+$ is a non-zero τ-continuous positive linear functional on L such that: $y \in P_i(x_i)$ implies $\psi_p \cdot y > \psi_p \cdot x_i$.

Now notice that from $\psi_p \cdot \omega_i \leqslant \psi_p \cdot x_i$, $\sum_{i=1}^m x_i \leqslant \omega = \sum_{i=1}^m \omega_i$, and the linearity of ψ_p, it follows that $\psi_p \cdot x_i = \psi_p \cdot \omega_i$. Since $\psi_p \cdot \omega > 0$, the above show that $(x_1, x_2, ..., x_m)$ is a Walrasian equilibrium with respect to the (linear) price ψ_p. ▮

The infinite dimensional analogue of the Debreu–Scarf core equivalence theorem [22]—as formulated in [4]—is also true with or without local solidness and with or without ordered preferences (see Peleg and Yaari [52] for the commodity space of real sequences, Aliprantis *et al.* [4] for topological vector lattices, and Tourky [56] for vector lattices with vector lattice price spaces).

COROLLARY 9.3. *If* (A1) *and* (A3) *hold, then an allocation is an Edgeworth equilibrium if and only if it is a Walrasian equilibrium.*

Proof. Let x be an allocation. If x is a Walrasian equilibrium, then it should be obvious that it is also an Edgeworth equilibrium. For the converse, assume that x is an Edgeworth equilibrium. Then, by Theorem 7.6, x is a personalized equilibrium. Now a glance at the proof of Corollary 9.2 guarantees that x is a Walrasian equilibrium. ▮

9.2. *Economies with Differential Information*

In this section, we shall indicate how one can apply our results to differential information economies; see [61]. To do this, we need to describe a differential information economy in the framework of our analysis. First, let us look at its commodity space. The commodity space depends on an underlying Banach space Y of physical commodities. We shall assume that Y is an ordered Banach space with topological dual Y'. We shall also assume that Y' possesses the Radon–Nikodým property. The nature of the (exogenous) uncertainty is given by a probability space $(\Omega, \mathscr{F}, \mu)$. For instance, if there are n physical commodities, then $Y = \mathbb{R}^n$ and is ordered by its canonical ordering. Now the commodity space is the space $L_1(\mu, Y)$, the Banach space of all Bochner integrable functions (equivalence classes) from Ω into Y. The space $L_1(\mu, Y)$ is an ordered Banach space with positive cone

$$L_1^+(\mu, Y) = \{x \in L_1(\mu, Y) : x(\omega) \geqslant 0 \text{ for } \mu\text{-a.e. } \omega \in \Omega\}.$$

As usual, the norm dual of $L_1(\mu, Y)$ is identified with the space $L_\infty(\mu, Y')$; see [18, Theorem 1, p. 98].

The m consumers have asymmetric information. The ith individual's *private information* is given by a sub σ-algebra $\mathscr{F}_i$ of $\mathscr{F}$. For simplicity we assume that $\bigvee_{i=1}^m \mathscr{F}_i = \mathscr{F}$. The ith consumer's consumption is limited by his incomplete information. That is, the ith consumer can only choose $\mathscr{F}_i$-measurable consumption plans and his consumption set is

$$X_i = \{x \in L_1^+(\mu, Y) : x \text{ is } \mathscr{F}_i\text{-measurable}\}.$$

Clearly, each X_i is a closed convex subcone of $L_1^+(\mu, Y)$. Alternatively, one can think of X_i as being the positive cone of the space $L_1(\mu \mid \mathscr{F}_i, Y)$.

The *initial endowment* ω_i of each consumer i is a vector in X_i. That is, ω_i is also individually measurable. Each consumer i is also given a *state* *dependent utility function* $u_i\colon \Omega \times Y_+ \to \mathbb{R}$. We assume the standard conditions which guarantee that the *ex ante expected utility* function $U_i\colon X_i \to \mathbb{R}$ of consumer i is given by

$$U_i(x) = \int_\Omega u_i(\omega, x(\omega))\, d\mu(\omega).$$

The expected utility function U_i for consumer i induces in the standard way a preference correspondence $P_i\colon X_i \twoheadrightarrow X_i$. That is,

$$P_i(x) = \{y \in X_i : U_i(y) > U_i(x)\}.$$

Now the *differential information economy* $\mathscr{E}$ is the m-tuple

$$\mathscr{E} = \{(X_i, \omega_i, P_i, \mathscr{F}_i, (\Omega, \mathscr{F}, \mu)) : i = 1, 2, ..., m\}.$$

Before stating our result, let us define the notion of an individually measurable list of personalized prices. We shall say that a list of personalized prices $(p_1, p_2, ..., p_m) \in L_\infty(\mu, Y')^m$ is *individually measurable* whenever p_i is $\mathscr{F}_i$-measurable for each i. This means that p_i contains no more information than the ith consumer's private information.

DEFINITION 9.4. Let $\mathscr{E}$ be a differential information economy. An allocation $x \in \prod_{i=1}^m X_i$ is said to be:

(1) *An individually measurable personalized valuation equilibrium*, if there exists a list of individually measurable personalized prices $p \in L_\infty^+(\mu, Y')^m$ for which x is a personalized valuation equilibrium.

(2) *An individually measurable personalized quasi-equilibrium*, if there exists a list of individually measurable personalized prices p in $L_\infty^+(\mu, Y')^m$ for which x is a personalized quasi-equilibrium.

(3) *An individually measurable personalized equilibrium*, if there exists a list of individually measurable personalized prices $p \in L_\infty^+(\mu, Y')^m$ for which x is a personalized equilibrium.

Notice that the individually measurable personalized equilibrium is in the spirit of Lucas [34]. In particular, in this differential information economy each agent has a personalized price that reflects her private information. Despite the fact that each agent has a different price valuation, the generalized price (equilibrium value in the economy) is the same for every consumer and can be seen as arising from the behavior of a fictitious auctioneer who maximizes revenue. Such an interpretation of our notion of personalized equilibrium will be the subject of our next application.

Furthermore, since agents maximize ex ante utility functions, consumers do not update their personalized prices using the generalized price. To include updating or signaling one needs to use interim utility functions.

We are now ready to interpret our results in the differential information setting.

THEOREM 9.5. *If $\mathscr{E}$ is a differential information economy satisfying* (A1.2), (A1.3), *and* (A3), *then the following hold true*:

(1) *Every Pareto optimal allocation is an individually measurable personalized valuation equilibrium.*

(2) *An Edgeworth equilibrium is an individually measurable personalized quasi-equilibrium.*

(3) *A Walrasian equilibrium with respect to a price $p \in L_\infty^+(\mu, Y')$ is an individually measurable personalized equilibrium for the list of personalized prices*

$$(E(p \mid \mathscr{F}_1), E(p \mid \mathscr{F}_2), ..., E(p \mid \mathscr{F}_m)).$$

Proof. Let $p = (p_1, p_2, ..., p_m) \in L_\infty(\mu, Y')^m$ be a list of personalized prices. Denote by q_i the conditional expectation $E(p_i \mid \mathscr{F}_i)$ for $i = 1, 2, ..., m$; and let $q = (q_1, q_2, ..., q_m)$. Now notice that $q_i \cdot x = p_i \cdot x$ for every $x \in X_i = L_1^+(\mu \mid \mathscr{F}_i, Y)$. It is therefore easy to see that $\psi_p = \psi_q$. Thus, an allocation that is a personalized (valuation) equilibrium for ψ_p must also be a personalized (valuation) equilibrium for ψ_q. Statements (1) and (2) now follow easily from Theorems 5.1 and 7.5.

For statement (3) notice that for each $x \in X_i$ and $j \neq i$ we have

$$E(p \mid \mathscr{F}_i) \cdot x = p \cdot x \geqslant E(p \mid \mathscr{F}_j) \cdot x.$$

Furthermore, letting $q = (E(p \mid \mathscr{F}_1), E(p \mid \mathscr{F}_2), ..., E(p \mid \mathscr{F}_m))$ it is easy to see that for each i and each $x \in X_i$ we have $\psi_q \cdot x = E(p \mid \mathscr{F}_i) \cdot x = p \cdot x$. ∎

We close this section with a simple illustration of a differential information economy with non-linear generalized prices.

EXAMPLE 9.6. In this model there is one physical commodity and there are two states of the world, i.e., $\Omega = \{a, b\}$. Therefore, the commodity space is $\mathbb{R}^2$.

There are two consumers. The first consumer can distinguish between realizations of states a and b and the second consumer cannot distinguish

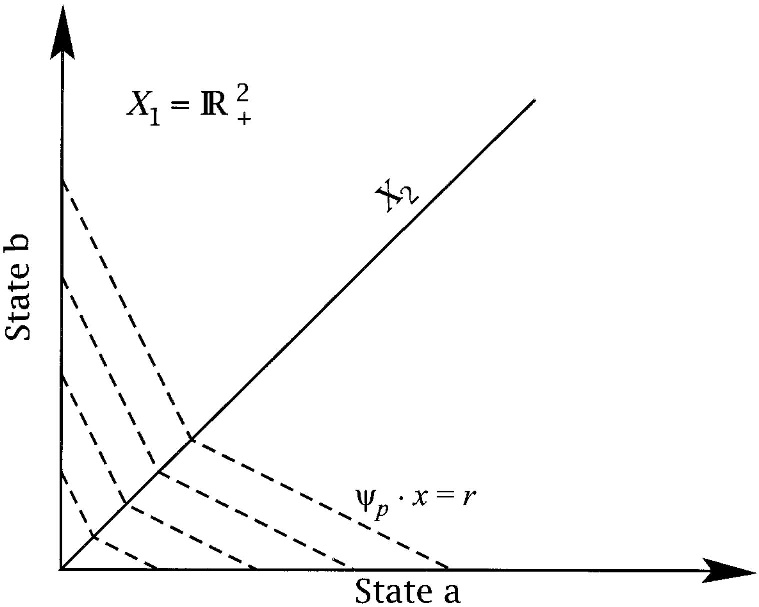

FIG. 1. There are two states of the world and one physical commodity. The first consumer can distinguish between the two states his consumption set is $\mathbb{R}^2_+$. The second cannot distinguish between the two states and her consumption set is the 45° line. In this case ψ_p need not be linear.

between realizations of states a and b. That is $\mathscr{F}_1 = \{\emptyset, \{a, b\}, \{a\}, \{b\}\}$ and $\mathscr{F}_2 = \{\emptyset, \{a, b\}\}$. This clearly implies that $X_1 = \mathbb{R}^2_+$ and

$$X_2 = \{(x, x) \in \mathbb{R}^2_+ : x \geqslant 0\}.$$

The consumption cones X_1 and X_2 can be seen in Fig. 1.

Now if the list of personalized prices is $p = ((1, 1), (2, 1))$, then a simple exercise shows that ψ_p is given by

$$\psi_p \cdot (x, y) = x + y + \min\{x, y\}.$$

This functional is non-linear and its level curves are shown in Fig. 1. If, on the other hand, the list of personalized prices is $p = ((1, 3), (2, 2))$, then ψ_p is now a linear functional given by the vector (1,3), i.e.,

$$\psi_p \cdot (x, y) = x + 3y.$$

In this case ψ_p is not the pointwise supremum of the list of personalized prices.

9.3. *Discriminatory Price Auctions*

There is another interesting interpretation of the model studied in this paper. In this interpretation, the principal real-world analogue of our model is the Treasury Bill Auction. In such an auction, the auctioneer announces an available quantity of bills and each player bids a pair comprising a price and an asking quantity. The auctioneer then orders the bids according to their prices, and a quantity of bills no more than the asking quantity is given to each player starting with the player who bids the highest price. Most importantly, each player pays the price she bids.[10] The Treasury Bill Auction has been the subject of theoretical, empirical, and experimental investigation and debate; see for instance Back and Zender [14], Bolten [17], Friedman [24–26], Goswami *et al.* [30], Goldstein [29], Menezes [44], and Menezes and Monteiro [45]. For surveys that compare the various types of auctions see Milgrom [46, 47].

Our model can be viewed as an abstraction of the Treasury Bill Auction to the case of a general equilibrium model with more than one commodity. Consumers, who do not know the total endowment of resources, bid a pair comprising a consumption set and a linear price system. The auctioneer divides the total endowment into a consumable allocation and each consumer pays the price she bids; see Fig. 2 where very general consumption sets are shown. Taking a list of price-bids as given, we calculate a value for each commodity bundle. This value is the maximum revenue that the auctioneer can obtain by dividing the commodity bundle into consumable allocations. This value function is precisely our generalized prices, which need not be linear. We call an allocation that maximizes revenue for the auctioneer an *auctioneer's allocation.*

Now we can reinterpret our results. As we have already shown, an allocation is weakly Pareto optimal if and only if there exists a list of price-bids for which the allocation is an auctioneer's allocation and the following supporting property holds:

($\bigstar$) *Consumers have no incentive to re-auction their assigned bundle. That is, given the list of price-bids, the revenue from re-auctioning an assigned bundle is less than the value of any preferred bundle.*

[10] This is a simplified description of the Treasury's "multi-price competitive" bid procedure. In fact, the Treasury accepts two types of bids for the bill and bond auctions and auctions of 3-year and 10-year notes. First, the "competitive" bid, where each bid comprises a quantity yield pair and successful bidders purchase securities at the price equivalent to their yield bid. Second, the "non-competitive" bid, where the investor agrees to accept a price equivalent to the weighted average yield of accepted competitive bids and in return is guaranteed a security —most small investors are in this category.

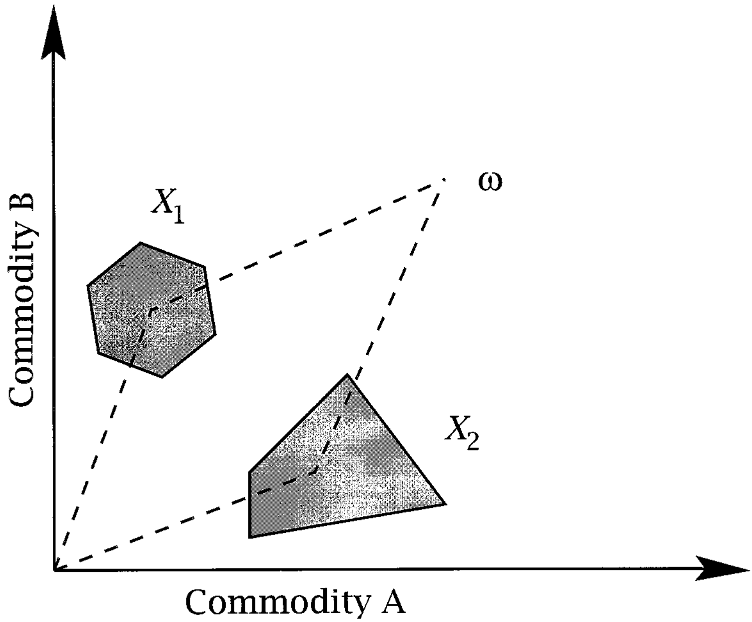

FIG. 2. Agents bid a pair comprising a consumption set and a linear price system. The auctioneer divides the total endowment into a consumable allocations and maximizes his revenue.

Furthermore, an allocation is an Edgeworth equilibrium if and only if there exists some list of price-bids for which the allocation is an auctioneer's allocation, the supporting condition (★) is satisfied, and the following condition holds:

(★★) *Coalitions—in the limit of all replications of the economy—have no "nominal" incentive to auction their total endowments. That is, given the list of price-bids, the revenue from auctioning the coalition's endowments is not greater than the total revenue from re-auctioning each member's assigned bundle.*

Interestingly, condition (★★) suggests that if all commodities are auctioned in a discriminatory price auction, perfect competition prevails (there are many consumers), and collusion is pervasive (in the above sense), then allocative outcomes will be in the core.[11]

[11] Following the 1991 Salomon Brothers scandal the Treasury began experimenting with uniform price auctions. The rationale, due to Milton Friedman, behind a uniform price auction is that it will encourage more (less specialized) bidders to participate. This would increase the demand and reduce the incentive to collude.

This auction theoretic interpretation leaves open the question of whether our personalized equilibria are Nash implementable. That is, can the auctioneer—who may prefer Pareto optimal or Walrasian allocations but does not know individual preferences—design an auction whose choice space contains lists of price-bids and whose Nash equilibria are auction equilibria? Given the scope of the results in the present paper, such an investigation is likely to lead to a particularly palatable underpinning to Walrasian equilibrium.

Finally, let us draw a brief connection between our non linear prices and the revenue function of the Treasury Bill auctioneer. In a discriminatory price Treasury Bill Auction with m players, the auctioneer announces a minimum price $p_0 > 0$ and an available quantity of bills $Q > 0$. Subsequently, each player i bids a pair (p_i, Q_i), where $p_i \geqslant p_0$ is her price-bid and $0 < Q_i \leqslant Q$ is her asking quantity. After receiving the bids, the auctioneer orders them according to their prices—without loss of generality we can assume that $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_m \geqslant p_0$. Afterwards, a quantity x_i of bills (less than or equal to the asking quantity of bills Q_i) is given to each player i which is determined by the auctioneer following certain rules that will be described below.

We shall say that an amount A of treasury bills is *distributed proportionally* (or that it is *proportionally rationed*) among a group of bidders S if each bidder $i \in S$ receives the amount of bills $(Q_i / \sum_{j \in S} Q_j) A$.

The total quantity of treasury bills handed out to the players by the auctioneer is

$$\sum_{i=1}^{m} x_i = \min \left\{ Q, \sum_{i=1}^{m} Q_i \right\}.$$

We distinguish three cases.

Case I. $\sum_{i=1}^{m} Q_i \leqslant Q$. In this case each bidder i gets the quantity of treasury bills $x_i = Q_i$ and the total revenue of the auctioneer is $\sum_{i=1}^{m} p_i x_i = \sum_{i=1}^{m} p_i Q_i$.

Case II. $Q_1 > Q$. In this case the auctioneer distributes proportionally the amount Q among the group of bidders with the highest bidding price p_1. The remaining bidders get zero. The revenue of the auctioneer in this case is $\sum_{i=1}^{m} p_i x_i = p_1 Q$.

Case III. $Q_1 \leqslant Q$ and $\sum_{i=1}^{m} Q_i > Q$. Let ℓ be the unique integer $1 \leqslant \ell < m$ such that $\sum_{i=1}^{\ell} Q_i \leqslant Q$ and $\sum_{i=1}^{\ell+1} Q_i > Q$. We distinguish two subcases.

(1) $p_\ell > p_{\ell+1}$. In this case, the auctioneer hands out the bills

$$x_1 = Q_1, x_2 = Q_2, ..., x_\ell = Q_\ell,$$

gives the quantity of bills $Q-\sum_{i=1}^{\ell} Q_i$ proportionally to the group of players with the price $p_{\ell+1}$, and gives nothing to the remaining bidders.

(2) $p_\ell = p_{\ell+1}$. In this case, the list of prices looks like

$$p_1 \geqslant p_2 \geqslant \cdots \geqslant p_{k-1} > p_k = p_{k+1} = \cdots = p_\ell$$
$$= p_{\ell+1} = \cdots = p_{\ell+r} > p_{\ell+r+1} \geqslant \cdots \geqslant p_m.$$

The auctioneer now hands out the following bills to the players,

$$x_1 = Q_1,\, x_2 = Q_2,\, ...,\, x_{k-1} = Q_{k-1},$$

distributes the quantity of bills $Q-\sum_{i=1}^{k-1} Q_i$ proportionally to the group of players with price p_ℓ, and gives zero to the remaining bidders.

Notice that in both subcases (1) and (2) above the auctioneer gets the revenue

$$\sum_{i=1}^{m} p_i x_i = \sum_{i=1}^{\ell} p_i Q_i + p_{\ell+1} \left(Q - \sum_{i=1}^{\ell} Q_i \right).$$

It turns out that in all cases the revenue obtained by the auctioneer is the maximum revenue that the auctioneer can achieve.

Lemma 9.7. *The vector of bills $(x_1, x_2, ..., x_m)$ maximizes the revenue of the auctioneer; i.e., if another vector of treasury bills $(y_1, y_2, ..., y_m)$ satisfies $0 \leqslant y_i \leqslant Q_i$ for each i and $\sum_{i=1}^{m} y_i \leqslant Q$, then*

$$\sum_{i=1}^{m} p_i y_i \leqslant \sum_{i=1}^{m} p_i x_i.$$

Moreover, if $p_1 > p_2 > \cdots > p_m$ holds, then $(x_1, x_2, ..., x_m)$ is the only maximizer of the revenue.

Proof. Assume that a vector of bills $(y_1, y_2, ..., y_m)$ satisfies $0 \leqslant y_i \leqslant Q_i$ and $\sum_{i=1}^{m} y_i \leqslant Q$. If $Q_1 > Q$, then

$$\sum_{i=1}^{m} p_i y_i \leqslant p_1 \left(\sum_{i=1}^{m} y_i \right) \leqslant p_1 Q = \sum_{i=1}^{m} p_i x_i.$$

Also, if $\sum_{i=1}^{m} Q_i \leqslant Q$, then $x_i = Q_i$ for each i and $\sum_{i=1}^{m} p_i y_i \leqslant \sum_{i=1}^{m} p_i x_i$ is trivially true.

So, we need to consider the case $Q_1 \leqslant Q$ and $\sum_{i=1}^{m} Q_i > Q$. Let ℓ be the unique integer $1 \leqslant \ell < m$ satisfying $\sum_{i=1}^{\ell} Q_i \leqslant Q$ and $\sum_{i=1}^{\ell+1} Q_i > Q$. Then $\sum_{i=\ell+1}^{m} y_i \leqslant Q - \sum_{i=1}^{\ell} y_i$, and so

$$\begin{aligned}
\sum_{i=1}^{m} p_i y_i &= \sum_{i=1}^{\ell} p_i y_i + \sum_{i=\ell+1}^{m} p_i y_i \\
&\leqslant \sum_{i=1}^{\ell} p_i y_i + p_{\ell+1}\left(\sum_{i=\ell+1}^{m} y_i\right) \leqslant \sum_{i=1}^{\ell} p_i y_i + p_{\ell+1}\left(Q - \sum_{i=1}^{\ell} y_i\right) \\
&= \sum_{i=1}^{\ell} p_i y_i + p_{\ell+1}\left(Q - \sum_{i=1}^{\ell} Q_i\right) + p_{\ell+1}\left[\sum_{i=1}^{\ell} (Q_i - y_i)\right] \\
&\leqslant \sum_{i=1}^{\ell} p_i y_i + p_{\ell+1}\left(Q - \sum_{i=1}^{\ell} Q_i\right) + \sum_{i=1}^{\ell} p_i (Q_i - y_i) \\
&= \sum_{i=1}^{\ell} p_i Q_i + p_{\ell+1}\left(Q - \sum_{i=1}^{\ell} Q_i\right) = \sum_{i=1}^{m} p_i x_i.
\end{aligned}$$

For the last part, assume $p_1 > p_2 > \cdots > p_m > 0$. In this case, observe that the auctioneer uses the following recursive process to determine the quantities of treasury bills that must give to the players:

(1) Player 1 is given a quantity of $x_1 = \min\{Q_1, Q\}$ bills and pays $p_1 x_1$.

(2) If players 1, 2, ..., $i-1$ are given the quantity of bills $x_1, x_2, ..., x_{i-1}$, then player $i > 1$ is given the quantity of bills

$$x_i = \begin{cases} Q_i & \text{if } \; Q_i \leqslant Q - \sum_{j<i} x_j, \\ Q - \sum_{j<i} x_j & \text{if } \; 0 < Q - \sum_{j<i} x_j < Q_i, \\ 0 & \text{if } \; Q - \sum_{j<i} x_j \leqslant 0. \end{cases}$$

Now let $z = (z_1, z_2, ..., z_n)$ satisfy $0 \leqslant z_i \leqslant Q_i$ for each i, $\sum_{i=1}^{n} z_i \leqslant Q$, and

$$\sum_{i=1}^{m} p_i z_i = \sum_{i=1}^{m} p_i x_i. \tag{$\star$}$$

We shall show that $z_i \leqslant x_i$ holds for each i, which, in view of our assumption about the prices and ($\star$), will guarantee that $z_i = x_i$ for each i.

To establish this, assume by way of contradiction that $z_i > x_i$ holds for some i, and let $k = \min\{i : z_i > x_i\}$. From $z_1 \leqslant \{Q_1, Q\} = x_1$, it follows that

$1 < k \leqslant m$. We claim that for some $1 \leqslant j < k$ we must have $z_j < x_j$. If this claim can be established, then we get a contradiction as follows. Pick some $\varepsilon > 0$ satisfying $0 < z_j + \varepsilon < x_j$ and $z_k - \varepsilon > x_k \geqslant 0$, and for each i define

$$y_i = \begin{cases} z_i + \varepsilon & \text{if} \quad i = j, \\ z_k - \varepsilon & \text{if} \quad i = k, \\ z_i & \text{otherwise.} \end{cases}$$

Clearly, $0 \leqslant y_i \leqslant Q_i$ for each i and $\sum_{i=1}^m y_i = \sum_{i=1}^m z_i \leqslant Q$. Now note that $p_j > p_k$ implies

$$\sum_{i=1}^m p_i y_i = \sum_{i=1}^m p_i z_i + (p_j - p_k)\,\varepsilon > \sum_{i=1}^m p_i z_i = \sum_{i=1}^m p_i x_i,$$

which is a contradiction.

To finish the proof, we shall prove that $z_j < x_j$ must be true for some $1 \leqslant j < k$. If this is not the case, then $z_i = x_i$ holds for all $1 \leqslant i < k$. Now let us consider the quantity of bills $Q - \sum_{i<k} x_i \geqslant 0$. If $Q - \sum_{i<k} x_i \geqslant Q_k$, then $z_k > x_k = Q_k$, which is impossible. So, $0 \leqslant Q - \sum_{i<k} x_i < Q$. If $Q - \sum_{i<k} x_i = 0$, then $x_k = 0$, and so

$$\sum_{i=1}^m z_i \geqslant \sum_{i<k} z_i + z_k = \sum_{i<k} x_i + z_k = Q + z_k > Q,$$

which is also impossible. Finally, if $0 < Q - \sum_{i<k} x_i < Q$, then $x_k = Q - \sum_{i<k} x_i$, and

$$\sum_{i=1}^m z_i \geqslant \sum_{i<k} x_i + z_k > \sum_{i<k} x_i + x_k = Q,$$

which is a contradiction too. Hence, $z_j < x_j$ must be true for some $1 \leqslant j < k$, and the proof of the theorem is complete. ∎

The Treasury Bill Auction Lemma 9.7 can now be restated using the preceding terminology as follows. (The quantities Q_i, x_i, and p_i are as above. L is our commodity space and X_i denotes a consumption set.)

THEOREM 9.8. *If $L = \mathbb{R}$ and $X_i = [0, Q_i]$ for each i, then the total revenue of the Treasury Bill Auction is equal to the value of Q under ψ_p. That is,*

$$\psi_p \cdot Q = \sum_{i=1}^m p_i \cdot x_i.$$

In particular, $x=(x_1, x_2, ..., x_m)$ *is an auctioneer's allocation for the list of price-bids* $p=(p_1, p_2, ..., p_m)$. *Moreover, if* $p_1>p_2> \cdots >p_m$ *holds, then* x *is the only auctioneer's allocation for* p.

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