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# On monotone pure-strategy Bayesian-Nash equilibria of a generalized contest ☆

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ABSTRACT

We introduce a new approach to studying the existence of a monotone pure-strategy Bayesian-Nash equilibrium in an $n$-player single-prize contest model that covers both perfectly and imperfectly discriminating contests. The contestants have continua of possible types and bids, atomless type distributions, and their valuations and costs might depend not only on their own bids and types but also on other bidders' bids and types. Many, quite different contests are covered by our generalized contest model and equilibrium existence in monotone pure strategies in them follows from this paper's results.


## 1. Introduction

This paper investigates the existence of a monotone pure-strategy Bayesian-Nash equilibrium in an $n$-player single-prize contest model in which the contestants have continua of possible types and bids, atomless type distributions, and their valuations and costs might depend not only on their own bids and types but also on other bidders' bids and types. The model covers both perfectly and imperfectly discriminating contests. The two classes of contests are quite different from the point of view of equilibrium existence. In perfectly discriminating contests, winners' bid ties tend to be discontinuity points of their ex-post payoff functions, whereas in imperfectly discriminating contests, the contestants' ex-post payoff functions have only one point of discontinuity, the zero vector of bids. However, since the only difference between the two types of contests is in the probabilities of success, their many essential features are similar from the point of view of economic theory, which has found reflection in this paper's encompassing results.

In contests with incomplete information, affiliation of types is not much of help in establishing the single-crossing property of each contestant's interim payoff function in her own bid and type. Instead, Krishna and Morgan's (1997) monotonicity condition – Condition M in Siegel's (2014) terminology – is often employed. According to Condition M (Condition WM), each contestant's fictitious valuation (i.e., the product of her valuation and the conditional density of the other contestants' types

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given her type) is strictly increasing (resp., weakly increasing) in her own type.[1] If a contestant's valuation does not depend on bids and her cost depends only on her own bid, then Condition WM implies that the contestant's interim payoff function has increasing differences in her own bid and type.[2]

In order to study monotone pure-strategy equilibria of contests in which the contestants' valuations depend not only on types but also on bids, we additionally assume increasing differences of each contestant's fictitious valuation in her own bid and type. If the contestants' costs depend on types and/or other players' bids, then additional conditions regarding the cost functions are needed as well. We provide three possible options to choose from for the conditions, with one of them requiring the log-supermodularity of the density function of the contestants' types.

This paper's approach to establishing equilibrium existence in contests extends that employed by Prokopovych and Yannelis (2019) for studying equilibrium existence in first-price auctions with incomplete information. First, we prove the interim payoff security of the contest and the continuity of each contestant's interim payoff function in the other contestants' strategies and her own type. Then, we provide conditions under which each contestant's interim payoff function exhibits increasing differences in her bid and type.[3] The most important of the conditions is Condition WM. If, for example, an all-pay auction does not satisfy the condition, the single-crossing property tends to fail for its bidders' interim payoff functions when their types are highly correlated.

We consider the truncated multi-valued selections of the ex-ante approximate best-reply correspondences that are defined on the Cartesian product of the other contestants' sets of nondecreasing strategies and consist of interim nondecreasing approximate best replies. Since the multi-valued selections possess the local intersection property and their values are nonempty $H$-convex sets in a compact $H$-space of uniformly bounded, nondecreasing strategies, the existence of a monotone pure-strategy approximate interim Bayesian-Nash equilibrium follows from a generalization of Browder's (1968) fixed-point theorem for correspondences with open-lower sections (Theorem 1).[4] $H$-convexity, not contractibility itself, is used in this paper since the contestants' ex-post payoff functions are not continuous in bids.[5] Then, in Theorem 2, we show that every converging sequence of monotone pure-strategy interim $\frac{1}{k}$-equilibria converges to a pure-strategy Bayesian-Nash equilibrium, as $k$ tends to infinity, without making any additional assumptions on the strict monotonicity of the contestants' payoff functions in own type. Finally, we establish the better-reply security of the ex-ante generalized contest game in nondecreasing strategies. However, in general, Reny's (1999) equilibrium existence theorem can not be applied to the generalized contest since each contestant's ex-ante payoff function is not necessarily quasiconcave in her own strategy.

The structure of the paper is as follows. Section 2 provides an overview of existing literature on equilibrium existence in contests with incomplete information. Section 3 describes the generalized contest model studied in this paper. The three examples of contests in Section 4 illustrate the basic premises of this paper's model. Section 5 investigates continuity-related properties of the contestants' interim payoff and value functions. A number of properties of the interim and ex-ante approximate best-reply correspondences are studied in Section 6. Section 7 presents two equilibrium existence results establishing that the generalized contest has a monotone pure-strategy Bayesian-Nash equilibrium. In Section 8, we show that, though the generalized contest is better-reply secure, a direct application of Reny's (1999) theorem might be hampered by the absence of the quasiconcavity of each contestant's ex-ante payoff function in her own strategy. The Appendix contains a number of proofs.

## 2. Related literature on equilibrium existence in Bayesian games

In this section, we describe several strands of literature on equilibrium existence in Bayesian games most closely related to this paper's results.

First-order optimality conditions are often employed to characterize bidding behavior and market outcomes in Bayesian auction games with continuous type and bid spaces. Amann and Leininger (1996) study the existence and uniqueness of a monotone differentiable Bayesian-Nash equilibrium in a class of asymmetric two-player all-pay auctions with independent private values. Krishna and Morgan (1997) extend Milgrom and Weber's (1982) seminal analysis of symmetric auctions with affiliated values to all-pay auctions with several bidders. For a large class of two-player bidding games with a reserve price and affiliated types, Lizzeri and Persico (2000) characterize equilibrium inverse bidding functions as the solution to a system of differential equations and investigate the existence and uniqueness of a monotone differentiable equilibrium. Lu and Parreiras (2017) propose conditions weaker than Conditions WM and M for the existence of a monotone pure-strategy equilibrium in a two-player asymmetric all-pay auction and extend Amann and Leininger's (1996) results to the case of interdependent valuations and correlated types.

[1] Fictitious valuations can be interpreted as contestants' valuations in a related fictitious contest with independent types (see Lu and Parreiras, 2017, p. 80).

[2] See, e.g., Vives (1990), Vives (1999), and Van Zandt and Vives (2007) for applications of the property of increasing differences in Bayesian games.

[3] There is no need in employing tieless increasing differences in our model, in contrast to tieless single-crossing needed for first-price auctions (see Reny and Zamir, 2004).

[4] See, e.g., Yannelis and Prabhakar (1983), Yannelis (1991), McLean (2021), and Khan and Uyanik (2021) for some applications and extensions of Browder's (1968) fixed-point theorem in economics.

[5] See Reny (2011) for general equilibrium existence results employing contractibility of best-reply sets in Bayesian games.

Since many classical results in auction theory were first established using derivatives, the literature on auctions with continuous bid and type spaces is considerably more voluminous than that for discrete auctions, even though real-world auctions are discrete. Games with discrete bid and/or type spaces are often considered and used as approximations to continuous games, even though these two classes of games may have drastically different outcomes. For example, in the case of complete information – when type spaces are singletons – all-pay auctions typically have no pure-strategy Nash equilibrium if the bidders have different values. This is also the case for all-pay auctions with discrete type spaces. Consequently, equilibrium existence in them is usually studied in behavioral or distributional strategies. Endogenous tie-breaking rules are also employed to achieve the existence of pure-strategy Bayesian-Nash equilibria in all-pay auctions (see, e.g., Araujo and de Castro, 2009).

Studies of contests with discrete type spaces often benefit from the techniques developed for the analysis of games with complete information.[6] Siegel (2014) and Rentschler and Turocy (2016) rely on indifference conditions to solve for a behavioral equilibrium in asymmetric two-player all-pay auctions with discrete type spaces. Einy et al. (2015) describe a class of better-reply secure, generalized concave Tullock contests with at most countable sets of types to which Reny's (1999) equilibrium existence theorem can be applied. Ewerhart and Quartieri (2020) apply Nikaido and Isoda's (1955) equilibrium existence theorem to the agent normal form of truncated continuous games and then provide sufficient conditions for the existence of a unique pure-strategy Bayesian-Nash equilibrium in imperfectly discriminating contests with finite type spaces and budget caps on the sets of admissible efforts.

Finite-action approximations of Bayesian games with continuous type and action spaces are often used in the studies of equilibrium existence in Bayesian games. If the action spaces are finite, the ex-post payoff functions are continuous in actions, which makes the equilibrium existence problem considerably more tractable. Athey (2001) employs Kakutani's (1941) fixed point theorem to establish the existence of monotone pure-strategy equilibria in finite-action approximations of Bayesian games. McAdams (2003) extends Athey's results to settings with multidimensional type and action spaces. After establishing the contractability of the players' sets of monotone best replies in Bayesian games where the ex-post payoff functions are continuous in actions, Reny (2011) employs Eilenberg-Montgomery's (1946) fixed-point theorem to study the existence of monotone pure-strategy equilibria in such games (see also Meneghel and Tourky, 2020). Another way to handle the possible emptiness of the values of best-reply correspondences in discontinuous games lies in making use of approximate best-reply correspondences. Under certain conditions, they contain a multi-valued selection consisting of interim approximate monotone best replies and possessing the local intersection property (Prokopovych and Yannelis, 2019). Though the selections need not be contractible-valued, their values are $H$-convex; that is, they consist of families of contractible subsets. Then the existence of pure-strategy monotone approximate equilibria follows from Horvath's (1987) extension of Browder's (1968) fixed-point theorem.

If the ex-ante normal-form game of a Bayesian game is aggregate upper semicontinuous and payoff secure in behavioral strategies and the sets of behavioral strategies are endowed with topologies in which they are compact, then Reny's (1999) equilibrium existence theorem implies that the game has a behavioral-strategy equilibrium. Identifying sufficient conditions on the primitives of a Bayesian game for applicability of the theorem might be quite challenging. He and Yannelis (2016) and Carbonell-Nicolau and McLean (2018) extend Reny's (1999) results to behavioral-strategy equilibria of Bayesian games and illustrate their findings on perfectly and imperfectly discriminating contests. He and Yannelis (2015) establish the existence of a pure-strategy Bayesian-Nash equilibrium in games with quasiconcave ex-ante payoff functions and countable sets of types and provide sufficient conditions for the purification of a behavioral-strategy equilibrium of a Bayesian game. Using results by He and Yannelis (2015) and Carbonell-Nicolau and McLean (2018), Brookins and Ryvkin (2016) establish the existence of a pure-strategy Bayesian-Nash equilibrium in all-pay group contests and contests with lottery contest success functions. Haimanko (2021, 2022) studies the existence of behavioral- and pure-strategy Bayesian-Nash equilibria in a large class of imperfectly discriminating single-prize contests with the absolute continuity of information and without it. Olszewski and Siegel (2022a) provide a number of conditions to facilitate the application of Reny's (1999) equilibrium existence theorem to Bayesian games with ties in distributional strategies, including first-price auctions, Hotelling models, and all-pay contests.

Complementarity conditions have found a number of applications in contest theory. Wasser (2013) applies Athey's (2001) equilibrium existence results to show the existence of a monotone pure-strategy Bayesian-Nash equilibrium in imperfectly discriminating contests with a continuous contest success function. Ewerhart (2014) extends Wasser's (2013) results to discontinuous imperfectly discriminating contests, using the fact that they can be approximated by a sequence of truncated continuous contests, each of which has a monotone pure-strategy equilibrium. By invoking the weak single-crossing property and Reny's (2011) results on the existence of a monotone pure-strategy equilibrium in games with continuous payoff functions and multi-dimensional types, Brookins and Ryvkin (2016) establish the existence of a monotone pure-strategy equilibrium in group-level private information contests. Olszewski and Siegel (2016) and Bodoh-Creed and Hickman (2018) approximate equilibria of large all-pay contests with many prizes and players whose ex-post payoffs satisfy the strict single-crossing property. Relaxing the property makes the notion of approximation weaker and the set of approximating mechanisms larger (Olszewski and Siegel, 2022b).

[6] For some additional information about contests with complete information, see Tullock (1980); Hillman and Riley (1989); Baye et al. (1996); Kaplan et al. (2002); and Siegel (2009).

The next section describes the contest model studied in this paper.

## 3. The model

We consider an $n$-player contest $\Gamma = (B_i, T_i, f, u_i)_{i\in I}$, where $I = \{1, \ldots, n\}$, $n \geq 2$, is the set of contestants (bidders); $B_i = [0, \overline{b}]$, $\overline{b} > 0$, is the set of bids, or effort levels, available to contestant $i$; $T_i = [0, 1]$ is contestant $i$'s set of types. The joint probability density function $f$ of the contestants' types is a continuous function from $T$ to $(0, +\infty)$. The bid cap $\overline{b}$ is introduced in this model for technical reasons and is so high that no contestant has any incentive to bid it, irrespective of her type and the other contestants' strategies. Denote $I_{-i} = I\backslash\{i\}$, $B = \prod_{i\in I} B_i$, and $T = \prod_{i\in I} T_i$. Also denote the marginal density function of $t_i$ by $f_i(t_i) = \int_{T_{-i}} f(t_i, t_{-i})dt_{-i}$ and the conditional density function of $t_{-i}$ given $t_i$ by $f_{-i}(t_{-i}|t_i) = \frac{f(t_i,t_{-i})}{f_i(t_i)}$. Further on, the terms 'increasing' and 'nondecreasing' are considered synonymous.

Each contestant $i$'s ex-post payoff function $u_i : B \times T \to \mathbb{R}$ is defined, for every $(b, t) \in B \times T$, by

$$u_i(b; t) = p_i(b)W_i(b; t) - C_i(b; t),$$

where the functions $W_i : B \times T \to [0, +\infty)$, $C_i : B \times T \to \mathbb{R}$ and $p_i : B \to [0, 1]$ have the following properties:
(i) $p_i$ is Borel measurable, increasing in $b_i$, and, additionally, $\sum_{j\in I} p_j(b) = 1$ for every $b \in B$;
(ii) $p_i$ is continuous at every $b \in B$ such that $b_i \neq b_j$ for all $j \in I_{-i}$; if $p_i$ is discontinuous at some $b = (b_i, b_{-i}) \in B$, then it is continuous at any $(b'_i, b_{-i})$ with $b'_i \in (b_i, \overline{b}]$;
(iii) $W_i$ and $C_i$ are continuous on $B \times T$;
(iv) $W_i(b; t) > 0$ for every $b \in B$ and every $t \in T$ with $t_i \in (0, 1]$;
(v) Condition WM: $\widetilde{W}_i(b; t) = W_i(b; t_i, t_{-i}) f_{-i}(t_{-i}|t_i)$ is increasing in $t_i$ for every $(b, t_{-i}) \in B \times T_{-i}$;
(vi) $\widetilde{W}_i$ has increasing differences in $(b_i, t_i)$ for every $(b_{-i}, t_{-i}) \in B_{-i} \times T_{-i}$; that is, for every pair of bids $b_i$ and $b'_i$ in $B_i$ with $b_i < b'_i$ and every $(b_{-i}, t_{-i}) \in B_{-i} \times T_{-i}$, $\Delta\widetilde{W}_i(b'_i, b_i, b_{-i}; t_i, t_{-i}) = \widetilde{W}_i(b'_i, b_{-i}; t_i, t_{-i}) - \widetilde{W}_i(b_i, b_{-i}; t_i, t_{-i})$ is increasing in $t_i$;
(vii) $C_i(0; b_{-i}; t) \leq 0$ for all $(b_{-i}, t) \in B_{-i} \times T$;
(viii) $W_i(\overline{b}, b_{-i}; t) - C_i(\overline{b}, b_{-i}; t) < 0$ for all $(b_{-i}, t) \in B_{-i} \times T$.

Conditions (i) and (ii) hold true in most models of perfectly and imperfectly discriminating contests.[7] The continuity of the $W_i$'s and $C_i$'s on $B \times T$, assumed in (iii), can be replaced, if needed, by less demanding conditions. However, in applications, it is often possible to redefine discontinuous valuation and cost functions in order to meet the continuity conditions, taking advantage of the fact that the contest success function is discontinuous (see Example 2 below). Condition (iv) is satisfied naturally in most contest games and plays an important role for equilibrium existence. Without it, an asymmetric first-price all-pay auction with no behavioral equilibria can be constructed similar to the one described by Lebrun (1996, p. 422) for first-price sealed-bid auctions.

Condition (v) is borrowed from the literature on all-pay auctions (see, e.g., Krishna and Morgan, 1997, Theorem 2; Siegel, 2014, Condition WM). Condition (vi) is added to the model since the contestants' valuations might depend not only on types but also on bids.

Though (viii) is not burdensome, it excludes some related auction games from consideration, such as finite-horizon wars of attrition.

A useful feature of this paper's model is that it contains neither conditions regarding the game's aggregate upper semi-continuity in bids nor conditions regarding the strict monotonicity of each contestant's valuation and/or cost in her own type. Such conditions are often necessary in proofs establishing that the limit of a converging sequence of $\varepsilon$-equilibria of a game is an exact equilibrium of it.

In contests with type-dependent costs, it might be the case that each contestant's cost function does not depend on the other contestants' types and bids (see, e.g., Cohen et al., 2008; Fey, 2008; Ryvkin, 2010; Wasser, 2013; Ewerhart, 2014). If this is the case, our results are valid without assuming the logsupermodularity of the density function $f$ or an analogue of Condition (vi) for the cost functions.

**Assumption 1.** Each $C_i$ does not depend on the other contestants' bids and types and has decreasing differences in $(b_i, t_i)$.

If $C_i$ depends only on $b_i$ and $t_i$ and is twice continuously differentiable, then the fact that $C_i$ has decreasing differences in $(b_i, t_i)$ is equivalent to $\frac{\partial^2 C}{\partial b_i \partial t_i} \leq 0$ (see, e.g., Cohen et al., 2008). It is also worth noting that the cost function $C_i : B_i \times T_i \to \mathbb{R}$ defined by $C_i(b_i, t_i) = t_i b_i$ does not possess this property.

[7] Though the model considered in this paper is quite general, it does not cover contest games in which the contest success function also depends on types (see, e.g. Ewerhart and Quartieri, 2020).

In general, the cost functions might depend on the other contestants' types and/or bids. In this case, in order to show that each contestant's interim payoff function has increasing differences in her own bid and type, we will employ either Assumption 2 or Assumption 3.

Denote by $\widetilde{C}_i$ the contestant $i$'s fictitious cost function defined by $\widetilde{C}_i(b,t) = C_i(b;t) f_{-i}(t_{-i}|t_i)$ for every $(b,t) \in B \times T$.

**Assumption 2.** Each fictitious cost function $\widetilde{C}_i$ has decreasing differences in $(b_i, t_i)$ for every $(b_{-i}, t_{-i}) \in B_{-i} \times T_{-i}$.

Along with conditions concerning decreasing differences of the cost functions, Assumption 3 requires the log-supermodularity of the density function $f$.

**Assumption 3.** Each $C_i$ has decreasing differences in $(b_i, t)$ and decreasing differences in $(b_i, b_{-i})$ and the joint probability density function $f$ of the contestants' types satisfies the following affiliation condition: $f(t \wedge t') f(t \vee t') \geq f(t) f(t')$ for all $t$ and $t'$ in $T$, where $\wedge$ and $\vee$ denote the componentwise minimum and maximum of $t$ and $t'$, respectively.

We now introduce some notation and definitions. Denote by $L_i$ ($S_i$) the set of equivalent classes of Lebesgue measurable functions (resp., nondecreasing functions) from $T_i$ to $B_i$, equipped with the metric $d_1(s_i, s_i') = \int_{T_i} |s_i(t_i) - s_i'(t_i)| \, dt_i$ for all $s_i$, $s_i' \in L_i$ (resp., $s_i$, $s_i' \in S_i$). As is conventional, we treat the elements of the metric spaces as functions, not equivalence classes of functions. Denote $L = \prod_{i \in I} L_i$ and $L_{-i} = \prod_{j \in I_{-i}} L_j$. The Cartesian products $S$ and $S_{-i}$ are defined in a similar manner. The products are equipped with a product metric that induces the product topology on them.

Contestant $i$'s interim payoff and value functions, $V_i : B_i \times L_{-i} \times T_i \to \mathbb{R}$ and $\overline{V}_i : L_{-i} \times T_i \to \mathbb{R}$, and her ex-ante payoff and value functions, $V_i^* : L \to \mathbb{R}$ and $\overline{V}_i^* : L_{-i} \to \mathbb{R}$, are defined as usual:

$$V_i(b_i, s_{-i}; t_i) = \int_{T_{-i}} u_i(b_i, s_{-i}(t_{-i}); t_i, t_{-i}) f_{-i}(t_{-i}|t_i) dt_{-i},$$

$$\overline{V}_i(s_{-i}; t_i) = \sup_{b_i \in B_i} V_i(b_i, s_{-i}; t_i),$$

$$V_i^*(s) = \int_{T_i} V_i(s_i(t_i), s_{-i}; t_i) f_i(t_i) dt_i,$$

$$\overline{V}_i^*(s_{-i}) = \sup_{s_i \in L_i} V_i^*(s_i, s_{-i}).$$

A strategy profile $s \in L$ constitutes an interim $\varepsilon$-equilibrium ($\varepsilon > 0$) of the game $\Gamma$ if, for each $i \in I$ and for almost all $t_i \in T_i$,

$$V_i(s_i(t_i), s_{-i}; t_i) > \overline{V}_i(s_{-i}; t_i) - \varepsilon.$$

A strategy profile $s \in L$ constitutes a Bayesian-Nash equilibrium of the game $\Gamma$ if $V_i^*(s_i, s_{-i}) = \overline{V}_i^*(s_{-i})$ for each $i \in I$.

The existence of a monotone pure-strategy Bayesian-Nash equilibrium in the generalized contest will be shown in two steps. In the first step, we will show the existence of a monotone pure-strategy approximate equilibrium for every $\varepsilon > 0$, and, in the second step, we will prove that every convergent sequence of monotone pure-strategy $\frac{1}{k}$-equilibria ($k = 1, 2, \ldots$) tends to a pure-strategy Bayesian-Nash equilibrium of the game.

## 4. Motivating examples

In this section, three examples of contests illustrate that the numerous conditions made in the above contest model are not overly burdensome. All of the contests described are particular cases of the generalized contest.

**Example 1.** Consider the following endogeneous-prize contest model. Let $T_i = [0,1]$ and $B_i = [0,10]$ for all $i \in I = \{1,2\}$. The bidders' types are independently uniformly distributed on $[0,1]$. The functions $W_i$ and $C_i$ are defined as follows:

$$W_i(b;t) = t_i \sqrt{1 + b_1 + b_2} \text{ and } C_i(b;t) = b_i$$

for every $(b,t) \in B \times T$ and each $i \in \{1,2\}$.

Most contest success functions satisfy Conditions (i) and (ii). For example, the all-pay-auction success function is often used in endogeneous-prize contests. It is defined as follows:

$$\overline{p}_i(b) = \begin{cases} 1 \text{ if } b_i > b_{-i}, \\ \frac{1}{2} \text{ if } b_1 = b_2, \\ 0 \text{ if } b_i < b_{-i} \end{cases}$$

for each $i \in I$ and every $b \in B$ (see, e.g., Kaplan et al., 2002; Cohen et al., 2008). Another contest success function often used in endogeneous-prize contests is the Tullock contest success function (see, e.g., Chung, 1996; Shaffer, 2006). Let $r > 0$.[8] For each $i \in I$ and every $b \in B$, the function is defined as follows:

$$\widetilde{p}^r_i(b) = \begin{cases} \frac{b_i^r}{b_1^r + b_2^r} \text{ if } b_1 + b_2 > 0, \\ \frac{1}{2} \text{ if } b_1 + b_2 = 0. \end{cases}$$

The functions $\overline{p}_i$ and $\widetilde{p}^r_i$ have different sets of discontinuities on $B$. The function $\overline{p}_i$ is discontinuous on the diagonal $\{b \in B : b_i = b_{-i}\}$, whereas $\widetilde{p}^r_i$ has only one point of discontinuity, the zero vector.[9] Consequently, in the latter case, it might be possible to approximate the contest by a sequence of truncated contests with continuous contest success functions (see, e.g., Ewerhart, 2014; Einy et al., 2015; Ewerhart and Quartieri, 2020; Haimanko, 2021, 2022).

The contest satisfies Condition (v) of the model since the bidders' types are independently uniformly distributed on $[0, 1]$ and each $W_i$ is increasing in $t_i$. Condition (vi) is satisfied for each contestant $i$ since for every pair of bids $b_i$ and $b'_i$ in $B_i$ satisfying the inequality $b_i < b'_i$ and every $(b_{-i}, t_i) \in B_{-i} \times T_i$,

$$\Delta\widetilde{W}_i(b'_i, b_i, b_{-i}; t_i) = (\sqrt{1 + b'_i + b_{-i}} - \sqrt{1 + b_i + b_{-i}})t_i$$

is increasing in $t_i$. The cost functions satisfy any of Assumptions 1-3. It is not difficult to verify that the rest of the model's conditions are also satisfied.

**Example 2.** Consider the following innovation contest with spillovers, in which each firm $i$'s expenditure on R&D, $b_i$, benefits the rival (see, e.g., Baye et al., 2005, 2012; Chowdhury and Sheremeta, 2011). Let $I = \{1, 2\}$, $T_1 = T_2 = [0, 1]$, and $B_1 = B_2 = [0, 10]$. The types are independently distributed according to the uniform distribution on $[0, 1]$. For every $(b, t) \in B \times T$, firm $i$'s ex-post payoff is given by

$$u_i(b_i, b_{-i}; t) = \begin{cases} t_i - b_i + \frac{1}{2}b_{-i} \text{ if } b_i > b_{-i}, \\ \frac{1}{2}t_i - b_i + \frac{3}{8}b_{-i} \text{ if } b_i = b_{-i}, \\ -b_i + \frac{1}{4}b_{-i} \text{ if } b_i < b_{-i}. \end{cases}$$

The contest success function of the game is the all-pay-auction success function $\overline{p}_i$. In order to formalize the contest in our model's parlance, put $W_i(b; t) = t_i + \frac{1}{4}b_{-i}$ and $C_i(b; t) = b_i - \frac{1}{4}b_{-i}$ for all $(b, t) \in B \times T$ and each $i \in \{1, 2\}$. In essence, each contestant's cost function $C_i$ represents the costs she incurs when she loses the contest. The cost functions satisfy Assumptions 2 and 3.

Since each $W_i$ is increasing in $t_i$ and does not depend on $b_i$, the contest satisfies Conditions (v) and (vi). The rest of the model's conditions can be easily verified.

**Example 3.** Contests in which bidders are uncertain about the other bidders' costs constitute an important class of contests with incomplete information (see, e.g., Fey, 2008; Ryvkin, 2010; Wasser, 2013; Ewerhart, 2014). Let $I = \{1, \ldots, n\}$, $T_1 = \ldots = T_n = [0, 1]$, and $B_1 = \ldots = B_n = [0, 10]$. The value of the prize to each contestant is normalized to unity; that is, $W_i(b; t) = 1$ for all $(b, t) \in B \times T$ and all $i \in I$. The contestants' types are drawn from independent probability distributions with continuous, strictly positive probability density functions $f_i$ defined on $[0, 1]$. Contestant $i$'s impact function, $h_i$, converts her effort $b_i$ into her effective output. Each function $h_i$ is nonnegative, continuous, strictly increasing, with $h_i(0) = 0$. The logit contest success function, $p^h_i : B \to [0, 1]$, is defined by

$$p^h_i(b) = \begin{cases} \frac{h_i(b_i)}{\Sigma_{j=1}^n h_i(b_j)} \text{ if } \Sigma_{j=1}^n h_i(b_j) > 0, \\ \frac{1}{n}, \text{otherwise}. \end{cases}$$

Each contestant $i$'s cost function $C_i : B_i \times T_i \to \mathbb{R}$ is defined by $C_i(b_i; t_i) = (2 - t_i)b_i$. Since it has decreasing differences in $(b_i, t_i)$, Assumption 1 is satisfied.

[8] In the case $r = 1$, this contest success function is also known as the lottery contest success function (see Tullock, 1980).

[9] See, e.g., Levine and Mattozzi (2021) for a number of general results regarding contest success functions.

Then, each contestant $i$'s ex-post expected payoff, $u_i : B \times T_i \to \mathbb{R}$, is given by

$$u_i(b, t_i) = \frac{h(b_i)}{\Sigma_{j=1}^n h(b_j)} - (2 - t_i)b_i.$$

It is not difficult to verify that this contest satisfies Conditions (i)-(viii).

We now modify this contest in three respects by assuming that: (i) types of the bidders are drawn not independently but from a joint probability distribution with a density function $f : T \to (0, +\infty)$; (ii) the value of the prize to each contestant $i$ is $t_i$; and (iii) each contestant $i$'s cost function, $C_i : B_i \times T \to \mathbb{R}$, depends not only on her type but also on the other contestants' types and is defined by $C_i(b_i; t) = (2 - \frac{\Sigma_{j=1}^n t_j}{n})b_i$. Then, each contestant $i$'s ex-post expected payoff, $u_i : B \times T_i \to \mathbb{R}$, is given by

$$u_i(b, t_i) = \frac{h(b_i)}{\Sigma_{j=1}^n h(b_j)} t_i - (2 - \frac{\Sigma_{j=1}^n t_j}{n})b_i.$$

Condition (v) is satisfied when each fictitious valuation $\widetilde{W}_i(t) = t_i f_{-i}(t_{-i}|t_i)$ is increasing in $t_i$ for each $t_{-i} \in T_{-i}$. If the joint probability density function $f : T \to (0, +\infty)$ is log-supermodular, then Assumption 3 obtains. One can verify that the rest of the conditions of our model are also satisfied.

If $f$ is not log-supermodular, then we need to verify whether each $\widetilde{C}_i(b_i; t) = C_i(b_i; t_i)_i f_{-i}(t_{-i}|t_i)$ has decreasing differences in $(b_i, t_i)$ for every $t_{-i} \in T_{-i}$. For example, let $I = \{1, 2\}$, and let $f(t_1, t_2) = \frac{t_1+t_2+2}{3}$ for all $(t_1, t_2) \in T_1 \times T_2$. Since $\frac{f(t_1', t_2)}{f(t_1, t_2)} = \frac{t_1'+t_2+2}{t_1+t_2+2}$ is strictly decreasing in $t_2$ for every $t_1'$ and $t_1$ in $T_1$ with $t_1' > t_1$, $f$ is not log-supermodular. For each $i \in I$ and every $t \in T$, we have $f_{-i}(t_{-i}|t_i) = \frac{t_i+t_{-i}+2}{t_i+2.5}$. Though the conditional density function $f_{-i}$ is not increasing in $t_i$ for all $t_{-i} \in (0.5, 1]$, each fictitious valuation $\widetilde{W}_i(t) = t_i f_{-i}(t_{-i}|t_i)$ is increasing in $t_i$ for all $t_{-i} \in [0, 1]$; that is, Condition WM is satisfied. To verify whether Assumption 2 is satisfied, fix $i \in I$ and some $b_i'$ and $b_i$ in $B_i$ with $b_i' > b_i$. It is not difficult to see that, for every $t_{-i} \in T_{-i}$,

$$\begin{aligned} &(C_i(b_i'; t_i) - C_i(b_i; t_i)) f_{-i}(t_{-i}|t_i) \\ &= (b_i' - b_i)(2 - \frac{t_i + t_{-i}}{2}) \frac{t_i + t_{-i} + 2}{t_i + 2.5}, \end{aligned}$$

is decreasing in $t_i$. Verifying the rest of the model's conditions is straightforward.

## 5. Interim payoff functions

Payoff security is one of the most important conditions of Reny's (1999) seminal equilibrium existence theorem. In discontinuous normal-form games, payoff functions are rarely lower semicontinuous in the other players' strategies. However, they are often transfer lower semicontinuous in them. Such games are called payoff secure by Reny (1999). Our approach to handling equilibrium existence in the generalized contest also relies on the fact that each contestant's interim payoff function is payoff secure in the other contestants' strategies and her own type.

Let $\mu$ denote the Lebesgue measure. The next property of the interim payoff functions is quite intuitive and usually holds in auction games.

**Lemma 1.** *If, in the contest $\Gamma$, for some $i \in I$, $b_i \in B_i$, and $s_{-i} \in L_{-i}$, $\mu(t_j \in T_j : b_i = s_j(t_j)) = 0$ for all $j \in I_{-i}$, then the interim payoff function $V_i$ is jointly continuous at $(b_i, s_{-i}, t_i)$ for every $t_i \in T_i$.*

The proof of this property for the contest model is similar to the proof of Lemma 1 of Prokopovych and Yannelis (2019) for first-price auctions, since each contestant $i$'s ex-post payoff function $u_i(b; t)$ is continuous at every $(b, t) \in B \times T$ such that $b_i \neq b_j$ for all $j \in I_{-i}$ in both of the classes of games. We will also need the following corollary (see Prokopovych and Yannelis, 2019, Corollary 2).

**Corollary 1.** *If, in the contest $\Gamma$, for some $i \in I$, a strategy profile $s \in L$ satisfies the following properties: (i) $s_i$ is a step function; (ii) $\mu(t_j \in T_j : s_i(t_i) = s_j(t_j)) = 0$ for each $j \in I_{-i}$ and for every $t_i \in T_i$, then, for every $\varepsilon > 0$, there exists an open neighborhood $\mathcal{N}_{L_{-i}}(s_{-i})$ of $s_{-i}$ in $L_{-i}$ such that $V_i(s_i(t_i), s'_{-i}; t_i) > V_i(s_i(t_i), s_{-i}; t_i) - \varepsilon$ for every $s'_{-i} \in \mathcal{N}_{L_{-i}}(s_{-i})$ and every $t_i \in T_i$.*

In order to establish the interim payoff security of the contest $\Gamma$, we need to show that each interim payoff function $V_i$ is transfer lower semicontinuous in $(s_{-i}, t_i)$. By definition, the transfer lower semicontinuity of $V_i$ in $(s_{-i}, t_i)$ means that for every $\varepsilon > 0$ and every $(b_i, s_{-i}, t_i) \in B_i \times L_{-i} \times T_i$, there exist a bid $\widetilde{b}_i \in B_i$ and an open neighborhood $\mathcal{N}_\varepsilon(s_{-i}, t_i)$ of $(s_{-i}, t_i)$ in $L_{-i} \times T_i$ such that $V_i(\widetilde{b}_i, s'_{-i}; t_i') > V_i(b_i, s_{-i}; t_i) - \varepsilon$ for every $(s'_{-i}, t_i') \in \mathcal{N}_\varepsilon(s_{-i}, t_i)$.

**Proposition 1.** *The contest $\Gamma$ is interim payoff secure.*

The proof of Proposition 1 can be found in the Appendix.

The transfer lower semicontinuity of a contestant's interim payoff function in the other contestants' strategies and own type implies the lower semicontinuity of her interim value function since it is the value function of the interim payoff function (see, e.g., Prokopovych, 2011, Lemma 4). In the next proposition, we show that each interim value function $\overline{V}_i$ is also upper semicontinuous on its domain.

**Proposition 2.** *In the contest $\Gamma$, each interim value function $\overline{V}_i : L_{-i} \times T \to \mathbb{R}$ is continuous.*

The proof of Proposition 2 is provided in the Appendix.

In order to be able to confine attention to the compact sets of nondecreasing strategies, we show, in the next proposition, that each contestant's interim payoff function has increasing differences in her own bid and type if, additionally, one of Assumptions 1 or 2 holds.[10]

**Proposition 3.** *Consider, in the contest $\Gamma$, some $i \in I$ and some $s_{-i} \in L_{-i}$. If one of Assumptions 1 or 2 holds, holds, then the interim payoff function $V_i$ has increasing differences in $(b_i, t_i)$.*

**Proof.** Let $b_i$ and $b_i'$ be two bids in $B_i$ such that $b_i < b_i'$, and suppose that Assumption 2 holds. Define $\Delta V_i : T_i \to \mathbb{R}$ by $\Delta V_i(t_i) = V_i(b_i', s_{-i}; t_i) - V_i(b_i, s_{-i}; t_i)$. We need to show that $\Delta V_i$ is an increasing function.

For every $t_i \in T_i$, we have

$$\Delta V_i(t_i) = \int_{T_{-i}} (p(b_i', s_{-i}(t_{-i})) - p(b_i, s_{-i}(t_{-i})))\widetilde{W}_i(b_i', s_{-i}(t_{-i}); t_i, t_{-i})dt_{-i}$$
$$+ \int_{T_{-i}} p(b_i, s_{-i}(t_{-i}))\Delta\widetilde{W}_i(b_i', b_i, s_{-i}(t_{-i}); t_i, t_{-i})dt_{-i} +$$
$$\int_{T_{-i}} (\widetilde{C}_i(b_i, s_{-i}(t_{-i}); t_i, t_{-i}) - \widetilde{C}_i(b_i', s_{-i}(t_{-i}); t_i, t_{-i}))dt_{-i}$$

Since $\widetilde{W}_i$ is increasing in $t_i$ for every $(b, t_{-i}) \in B \times T_{-i}$, $\widetilde{W}_i$ has increasing differences and $\widetilde{C}_i$ has decreasing differences in $(b_i, t_i)$ for every $(b_{-i}, t_{-i}) \in B_{-i} \times T_{-i}$, each of the three terms in the sum above is increasing in $t_i$, which completes the proof of this case.

If, instead of Assumption 2, each $C_i$ satisfies Assumption 1, then the third term of the sum above is increasing in $t_i$ since it is equal to $C_i(b_i; t_i) - C_i(b_i'; t_i)$, which completes the proof. $\square$

Since Assumption 3 also implies that the third term of the sum in the proof of Proposition 3 is increasing in $t_i$ when the other contestants use nondecreasing strategies, we have the following proposition.

**Proposition 4.** *Consider, in the contest $\Gamma$, some $i \in I$ and some $s_{-i} \in S_{-i}$. If Assumption 3 holds, then the interim payoff function $V_i$ has increasing differences in $(b_i, t_i)$.*

**Remark.** The condition that each fictitious valuation $\widetilde{W}_i$ is increasing in $t_i$ (Condition WM) is often employed in the literature on all-pay auctions. Its importance is due to the fact that it implies that each contestant's interim value function has increasing differences in her own bid and type when her valuation does not depend on bids and her cost function depends only on her own bid.

## 6. Approximate best-reply correspondences

Before studying properties of the ex-ante approximate best-reply correspondences, first we need to take a look at the interim approximate best-reply correspondences. For each contestant $i$ and $\varepsilon > 0$, her truncated interim $\varepsilon$-best-reply correspondence, $M_i^\varepsilon : S_{-i} \times T_i \twoheadrightarrow B_i$, is defined as follows:

$$M_i^\varepsilon(s_{-i}; t_i) = \{b_i \in B_i : V_i(b_i, s_{-i}; t_i) > \overline{V}_i(s_{-i}; t_i) - \varepsilon\},$$

[10] In many other games, the easy-to-handle property of increasing differences needs to be relaxed to the single-crossing property (see, e.g., Milgrom and Shannon, 1994; Amir, 1996; Athey, 2001; Reny and Zamir, 2004; and Quah and Strulovici, 2012).

where the word 'truncated' reflects the fact that each $M_i^\varepsilon$ is defined on $S_{-i} \times T_i$, not on $L_{-i} \times T$.

The next proposition shows that the truncated interim approximate best-reply correspondences have single-valued selections possessing a number of favorable properties. In it, the part of the single-crossing property called by Reny (2011) the weak single-crossing property – not the stronger property of increasing differences – is used.[11]

**Proposition 5.** *Consider, in the contest $\Gamma$, some $i \in I$, some $\varepsilon > 0$, and some strategy subprofile $s_{-i} \in S_{-i}$. If one of Assumptions 1, 2, or 3 holds, then the correspondence $M_i^\varepsilon(s_{-i}; \cdot)$ has a single-valued selection $\widetilde{s}_i \in S_i$ possessing the following properties: (i) $\widetilde{s}_i$ is a step function; (ii) $\mu(t_j \in T_j : \widetilde{s}_i(t_i) = s_j(t_j)) = 0$ for each $j \in I_{-i}$ and every $t_i \in T_i$.*

The proof of Proposition 5 is provided in the Appendix.

A number of important properties of the ex-ante value functions follow from Proposition 5.

**Corollary 2.** *If one of Assumptions 1, 2, or 3 holds in the contest $\Gamma$, then*

$$\overline{V}_i^*(s_{-i}) = \int\limits_{T_i} \overline{V}_i(s_{-i}; t_i) f_i(t_i) dt_i = \sup_{s_i \in S_i} V_i^*(s_i, s_{-i})$$

*for every $s_{-i} \in S_{-i}$. Therefore, $\overline{V}_i^*$ is continuous on $S_{-i}$.*

**Proof.** It follows from Proposition 5 that $\overline{V}_i^*(s_{-i}) = \sup_{s_i \in S_i} V_i^*(s_i, s_{-i})$ for every $s_{-i} \in S_{-i}$. Assume, by contraction, that, for some $\varepsilon > 0$ and some $s_{-i} \in S_{-i}$, $\int_{T_i} \overline{V}_i(s_{-i}; t_i) f_i(t_i) dt_i > \overline{V}_i^*(s_{-i}) + \varepsilon$. Then, for any single-valued nondecreasing selection $\widetilde{s}_i$ of $M_i^{\frac{\varepsilon}{2}}(s_{-i}; \cdot)$, we have

$$V_i^*(\widetilde{s}_i, s_{-i}) > \int\limits_{T_i} \overline{V}_i(s_{-i}; t_i) f_i(t_i) dt_i - \frac{\varepsilon}{2} > \overline{V}_i^*(s_{-i}) + \frac{\varepsilon}{2},$$

a contradiction. Then, the continuity of $\overline{V}_i^*$ on $S_{-i}$ follows from the fact that $\overline{V}_i$ is continuous on $S_i \times T_i$ by Proposition 2. □

Proposition 5 also implies that the ex-ante approximate best-reply correspondences are nonempty-valued.

Now we introduce truncated multi-valued selections of the ex-ante approximate best-reply correspondences that possess two desirable properties, namely they have the local intersection property and their values are $H$-convex sets.

For each $i \in I$ and every $\varepsilon > 0$, define the correspondence $\widetilde{M}_i^\varepsilon : S_{-i} \twoheadrightarrow S_i$ as follows:

$$\widetilde{M}_i^\varepsilon(s_{-i}) = \{s_i \in S_i : V_i(s_i(t_i), s_{-i}; t_i) > \overline{V}_i(s_{-i}; t_i) - \varepsilon \text{ for almost all } t_i \in T_i\}.$$

The following proposition states that each $\widetilde{M}_i^\varepsilon$ has the local intersection property; that is, for every $s_{-i} \in S_{-i}$, there exist a strategy $\widetilde{s}_i \in S_i$ and an open neighborhood $\mathcal{N}_{S_{-i}}(s_{-i})$ of $s_{-i}$ in $S_{-i}$ such that $\widetilde{s}_i \in \widetilde{M}_i^\varepsilon(s'_{-i})$ for every $s'_{-i} \in \mathcal{N}_{S_{-i}}(s_{-i})$.

**Proposition 6.** *If one of Assumptions 1, 2, or 3 holds in the contest $\Gamma$, then each correspondence $\widetilde{M}_i^\varepsilon : S_{-i} \twoheadrightarrow S_i$ defined above has the local intersection property.*

The proof of Proposition 6 is similar to the proof of Proposition 7 of Prokopovych and Yannelis (2019). It is provided in the Appendix for the reader's convenience.

Since the values of each $\widetilde{M}_i^\varepsilon$ need not be convex, we will employ a kind of generalized convexity. For this purpose, we interpret the sets of nondecreasing strategies as $H$-spaces (see, e.g., Horvath, 1987; Tarafdar, 1992; and Prokopovych and Yannelis, 2019). By definition, the strategy set $S_i$, $i \in I$, being an $H$-space means that for every finite number of nondecreasing strategies, $A = \{s_{1i}, \ldots, s_{ki}\}$ in $S_i$, there exists a contractible subset of $S_i$, $F_A$, such that $F_A \subset F_{A'}$ whenever $A$ is contained in another finite subset $A'$ of $S_i$. More specifically, in our contest model, the set $F_A$ is the minimal set in $S_i$ containing $A$ and satisfying the following two conditions: If $s_i$ and $s'_i$ are in $F_A$, then the function $s_i \vee s'_i$ defined by $(s_i \vee s'_i)(t_i) = \max\{s_i(t_i), s'_i(t_i)\}$ for all $t_i \in T_i$ is also in $F_A$; and for every $\tau \in (0, 1)$, $s_i \mathcal{X}_{[0,1-\tau]} + (s_i \vee s'_i)\mathcal{X}_{(1-\tau,1]}$ also belongs to $F_A$, where $\mathcal{X}_D$ is the indicator function of the set $D$. A homotopy $h : [0, 1] \times F_A \to F_A$ that continuously shrinks $F_A$ to the strategy $\overline{s}_i \in S_i$ defined by $\overline{s}_i(t_i) = \max\{s_{1i}(t_i), \ldots, s_{ki}(t_i)\}$ for all $t_i \in T_i$ is proposed and studied by Reny (2011):

$$h(\tau, s_i)(t_i) = \begin{cases} s_i(t_i), \text{ if } t_i \le 1 - \tau \text{ and } \tau < 1, \\ \overline{s}_i(t_i), \text{ otherwise} \end{cases}$$

[11] The property employed in the proof is even slightly weaker that the weak single-crossing property. See, e.g., the definition of the quasimonotonicity of a function in Lizzeri and Persico (2000).

for any $\tau \in T_i$ and $s_i \in F_A$. The function $h$ acts continuously from $[0,1] \times F_A$ to $F_A$, $h(0, s_i) = s_i$, and $h(1, s_i) = \overline{s}_i$.[12]

Since every piecewise combination of two interim $\varepsilon$-best replies is also an interim $\varepsilon$-best reply, the values of each $\widetilde{M}_i^{\varepsilon}$ are $H$-convex; that is, for every $s_{-i} \in S_{-i}$ and each finite set $A$ in $\widetilde{M}_i^{\varepsilon}(s_{-i})$, $F_A \subset \widetilde{M}_i^{\varepsilon}(s_{-i})$. Consequently, a Browder-type (1968) fixed-point theorem can be employed to establish the existence of approximate pure-strategy equilibria in the contest.

## 7. Approximate and exact Bayesian-Nash equilibria

This section contains two equilibrium existence results, the first one studying the existence of monotone pure-strategy approximate Bayesian-Nash equilibria and the second one studying the existence of monotone pure-strategy Bayesian-Nash equilibria in the generalized contest. The existence of a monotone approximate pure-strategy Bayesian-Nash equilibrium in the generalized contest follows from the following variant of Horvath's (1987) extension of Browder's (1968) fixed-point theorem for a family of correspondences: If each $S_i$ is a compact $H$-space, and for every $\varepsilon > 0$ and each $i \in I$, $\widetilde{M}_i^{\varepsilon}$ has the local intersection property and its values are $H$-convex and nonempty, then there exists a strategy profile $\overline{s} \in S$ such that $\overline{s}_i \in \widetilde{M}_i^{\varepsilon}(\overline{s}_{-i})$ for each $i \in I$ (see, e.g., Tarafdar, 1992, Theorem 2.3; and Prokopovych and Yannelis, 2019, Corollary 1).

**Theorem 1.** *If one of Assumptions 1, 2, or 3 holds in the contest $\Gamma$, then it has a monotone interim $\varepsilon$-equilibrium in pure strategies for every $\varepsilon > 0$.*

Another important question to be answered is whether a convergent sequence of nondecreasing $\frac{1}{k}$-equilibria of the contest $\Gamma$, as $k$ tends to infinity, tends to a pure-strategy Bayesian equilibrium of the contest. It turns out that this is so without any additional assumptions.

**Theorem 2.** *If one of Assumptions 1, 2, or 3 holds in the contest $\Gamma$, then it has a monotone pure-strategy Bayesian-Nash equilibrium.*

**Proof.** By Theorem 1, for each $k \in \{1, 2, \ldots\}$, there exists a nondecreasing interim $\frac{1}{k}$-equilibrium $s^k = (s_1^k, \ldots, s_n^k)$ of $\Gamma$ such that $V_i(s_i^k(t_i), s_{-i}^k; t_i) > \overline{V}_i(s_{-i}^k; t_i) - \frac{1}{k}$ for almost all $t_i \in T_i$ and each $i \in I$. Assume, without loss of generality, that for each $i \in I$, the sequence $\{s_i^k\}_{k=1}^{\infty}$ converges pointwise on $T_i$ to a nondecreasing strategy $\overline{s}_i$ and $\overline{s}_i(1) < \overline{b}$ for all $i \in I$ (see Condition (viii)). Since, by Proposition 2, $\overline{V}_i$ is continuous on $L_{-i} \times T_i$, we have $\lim_k \overline{V}_i(s_{-i}^k; t_i) = \overline{V}_i(\overline{s}_{-i}; t_i)$ for all $t_i \in T_i$ and each $i \in I$ and, therefore, $\lim_k V_i(s_i^k(t_i), s_{-i}^k; t_i) = \overline{V}_i(\overline{s}_{-i}; t_i)$ for almost all $t_i \in T_i$ and all $i \in I$. Then, by the dominated convergence theorem,

$$\lim_k \int_{T_i} V_i(s_i^k(t_i), s_{-i}^k; t_i) f_i(t_i) dt_i = \int_{T_i} \overline{V}_i(\overline{s}_{-i}; t_i) f_i(t_i) dt_i$$

Since, by Corollary 2, $\overline{V}_i^*(\overline{s}_{-i}) = \int_{T_i} \overline{V}_i(\overline{s}_{-i}; t_i) f_i(t_i) dt_i$, we have

$$\lim_k V_i^*(s_i^k, s_{-i}^k) = \overline{V}_i^*(\overline{s}_{-i}).$$

For each $i \in I$, denote by $\widetilde{V}_i^*(\overline{s})$ the expected payoff contestant $i$ would get at $\overline{s}$ if the contest success function $p_i$ is replaced with its upper-semicontinuous envelope $\widetilde{p}_i$, defined by $\widetilde{p}_i(b) = \limsup_{b' \to b} p_i(b')$ for all $b \in B$. Since

$$\lim_k V_i^*(s_i^k, s_{-i}^k) \leq \int_{T_{-i}} (\limsup{}_k p_i(s^k(t)) W_i(\overline{s}(t); t) - C_i(\overline{s}(t); t)) f(t) dt$$

and

$$\limsup{}_k p(s^k(t)) \leq \limsup_{b \to \overline{s}(t)} p(b) = \widetilde{p}_i(\overline{s}(t)) \text{ for all } t \in T,$$

we have $\lim_k V_i^*(s_i^k, s_{-i}^k) \leq \widetilde{V}_i^*(\overline{s})$.

In order to show that $\widetilde{V}_i^*(\overline{s}) = \lim_k V_i^*(s_i^k, s_{-i}^k)$ for all $i \in I$, fix some $i \in I$ and some $\varepsilon \in (0,1)$. Define $H_i : B \times T \to [0, +\infty)$ by $H_i(b; t) = W_i(b; t) + |C_i(b; t)|$. It follows from the uniform continuity of $H_i$ on $B \times T$ that there exists $\delta \in (0, \overline{b} - \overline{s}_i(1))$ such that $\left| H_i(b_i', b_{-i}; t) - H_i(b; t) \right| < \varepsilon$ for every $(b, t) \in B \times T$ and every $b_i' \in B_i$ such that $\left| b_i' - b_i \right| < 2\delta$. Denote by $s_i^{\delta}$ contestant $i$'s strategy defined by $s_i^{\delta}(t_i) = \overline{s}_i(t_i) + \delta$ for all $t_i \in T_i$. We need to show that $p_i(s_i^{\delta}(t_i), \overline{s}_{-i}(t_{-i})) \geq \widetilde{p}_i(\overline{s}_i(t_i), \overline{s}_{-i}(t_{-i}))$ for every $t = (t_i, t_{-i}) \in T$. If $p_i$ is continuous at $(\overline{s}_i(t_i), \overline{s}_{-i}(t_{-i}))$ for some $t \in T$, then the inequality holds trivially. Consider the case when $\widetilde{p}_i(\overline{s}_i(t_i), \overline{s}_{-i}(t_{-i})) > p_i(\overline{s}_i(t_i), \overline{s}_{-i}(t_{-i}))$ for some $t \in T$. Then, pick any sequence $\{b^m\}_{m=1}^{\infty}$ in $\{b_i' \in$

[12] A new, related type of generalized convexity with a number of applications to auction games with discrete bid sets is proposed by Meneghel and Tourky (2020).

$B_i : \left|b_i' - \overline{s}_i(t_i)\right| < \delta\} \times B_{-i}$ that converges to $(\overline{s}_i(t_i), \overline{s}_{-i}(t_{-i}))$ and satisfies the inequality $\lim_m p_i(b^m) > p_i(\overline{s}_i(t_i), \overline{s}_{-i}(t_{-i}))$. By Condition (ii), $p_i$ is continuous at $(s_i^\delta(t_i), \overline{s}_{-i}(t_{-i}))$ and $\lim_m p_i(b^m) \le \lim_m p_i(s_i^\delta(t_i), b_{-i}^m) = p_i(s_i^\delta(t_i), \overline{s}_{-i}(t_{-i}))$. Therefore, $\widetilde{p}_i(\overline{s}(t)) \le p_i(s_i^\delta(t_i), \overline{s}_{-i}(t_{-i}))$ for every $t \in T$. Since $W_i(b;t) > 0$ for every $(b,t) \in B \times T$ with $t_i \in (0,1]$, we have $\widetilde{V}_i^*(\overline{s}) - \varepsilon < V_i^*(s_i^\delta, \overline{s}_{-i}) \le \overline{V}_i^*(\overline{s}_{-i})$. Since it is true for every $\varepsilon > 0$, we conclude that $\widetilde{V}_i^*(\overline{s}) = \lim_k V_i^*(s_i^k, s_{-i}^k) = \overline{V}_i^*(\overline{s}_{-i})$.

Now we need to show that the sequence $\{s^k\}_{k=1}^\infty$ has a subsequence $\{\widehat{s}^k\}_{k=1}^\infty$ such that $\lim_k p_i(\widehat{s}^k(t)) = \widetilde{p}_i(\overline{s}(t))$ for almost all $t \in T$ and each $i \in I$. With this end in mind, let us show that the sequence $\{p_i(s^k(\cdot))\}$ converges to $\widetilde{p}_i(\overline{s}(\cdot))$ in measure for each $i \in I$. Fix some $i \in I$ and some $\varepsilon > 0$. Denote

$$T(i,\varepsilon,k) = \{t \in T : \left|\widetilde{p}_i(\overline{s}(t)) - p_i(s^k(t))\right| > \varepsilon\}.$$

In order to show that $\lim_k \mu(T(i,\varepsilon,k)) = 0$, assume, by contradiction, that there is a subsequence of $\{s^k\}$, again denoted by $\{s^k\}$, for which $\lim_k \mu(T(i,\varepsilon,k)) = \gamma > 0$. Since $\widetilde{p}_i(b) = \limsup_{b' \to b} p_i(b')$ for all $b \in B$ and each sequence $\{s_j^k\}$, $j \in I$, converges to $\overline{s}_j$ pointwise, we have

$$\lim_k \mu\{t \in T : p_i(s^k(t)) - \widetilde{p}_i(\overline{s}(t)) > \varepsilon\} = 0$$

and, therefore,

$$\lim_k \mu\{t \in T : \widetilde{p}_i(\overline{s}(t)) - p_i(s^k(t)) > \varepsilon\} = \gamma.$$

Since $W_i(b;t) > 0$ for every $(b,t) \in B \times T$ with $t_i \in (0,1]$, $\widetilde{V}_i^*(\overline{s})$ is strictly larger than $V_i^*(s_i^k, s_{-i}^k)$, a contradiction. The convergence of each sequence $\{p_i(s^k(\cdot))\}$ to $\widetilde{p}_i(\overline{s}(\cdot))$ in measure implies that the sequence $\{s^k\}$ has a subsequence $\widehat{s}^k$ such that each sequence $\{p_i(\widehat{s}^k(\cdot))\}$ converges to $\widetilde{p}_i(\overline{s}(\cdot))$ almost everywhere on $T$; that is, $\lim_k p_i(\widehat{s}^k(t)) = \widetilde{p}_i(\overline{s}(t))$ for almost all $t \in T$ and each $i \in I$.

Then, for almost all $t \in T$,

$$\sum_{i \in I} \widetilde{p}_i(\overline{s}(t)) = \sum_{i \in I} \lim_k p_i(\widehat{s}^k(t)) = \lim_k \sum_{i \in I} p_i(\widehat{s}^k(t)) = 1.$$

Since $1 = \sum_{i \in I} p_i(\overline{s}(t)) \le \sum_{i \in I} \widetilde{p}_i(\overline{s}(t)) = 1$ for almost all $t \in T$ and $p_i(\overline{s}(t)) \le \widetilde{p}_i(\overline{s}(t))$ for every $t \in T$ and each $i \in I$, we have $p_i(\overline{s}(t)) = \widetilde{p}_i(\overline{s}(t))$ for almost all $t \in T$ and each $i \in I$. Therefore, $V_i^*(\overline{s}) = \widetilde{V}_i^*(\overline{s}) = \overline{V}_i^*(\overline{s}_{-i})$ for each $i \in I$, which completes the proof. □

## 8. Some remarks on the better-reply security of the ex-ante normal form of the generalized contest

Reny's (1999) equilibrium existence theorem is an important tool for establishing equilibrium existence in normal-form games. It is applicable to the better-reply secure games in which each player's set of strategies is a compact convex subset of a topological vector space and each player's payoff function is quasiconcave in her own strategy. In this section, the better-reply security of the ex-ante normal form of the generalized contest is established. Then, an example of an all-pay auction illustrates that, in general, the contestants' ex-ante payoff functions are neither quasiconcave nor $H$-quasiconcave in own strategies.

We begin with showing that the generalized contest is ex-ante payoff secure in nondecreasing pure strategies, which is closely related to the fact that each $\widetilde{M}_i^\varepsilon$ has the local intersection property.

**Proposition 7.** *If one of Assumptions 1, 2, or 3 holds in the contest $\Gamma$, then it is ex-ante payoff secure in nondecreasing pure strategies.*

**Proof.** We need to prove the normal-form game $(S_i, V_i^*)_{i \in I}$ is payoff secure. Fix some $\varepsilon > 0$, some $i \in I$, and some $s \in S$. To prove the payoff security of the ex-ante game, it suffices to find $\widetilde{s}_i \in S_i$ such that $V_i^*(\widetilde{s}_i, s_{-i}') > V_i^*(s_i, s_{-i}) - \varepsilon$ for all $s_{-i}'$ in some open neighborhood $\mathcal{N}_{S_{-i}}(s_{-i})$ in $S_{-i}$.

By Proposition 5, the correspondence $M_i^{\frac{\varepsilon}{2}}(s_{-i}; \cdot)$ has a single-valued selection $\widetilde{s}_i \in S_i$ possessing the following properties: (i) $\widetilde{s}_i$ is a step function; (ii) $\mu(t_j \in T_j : \widetilde{s}_i(t_i) = s_j(t_j)) = 0$ for each $j \in I_{-i}$ and every $t_i \in T_i$. Since $\widetilde{s}_i(t_i) \in M_i^{\frac{\varepsilon}{2}}(s_{-i}; t_i)$ for every $t_i \in T_i$, we have $V_i^*(\widetilde{s}_i, s_{-i}) > V_i^*(s_i, s_{-i}) - \frac{\varepsilon}{2}$. On the other hand, by Corollary 1, there exists an open neighborhood $\mathcal{N}_{S_{-i}}(s_{-i})$ of $s_{-i}$ in $S_{-i}$ such that $V_i(\widetilde{s}_i(t_i), s_{-i}'; t_i) > V_i(\widetilde{s}_i(t_i), s_{-i}; t_i) - \frac{\varepsilon}{2}$ for all $s_{-i}' \in \mathcal{N}_{S_{-i}}(s_{-i})$ and all $t_i \in T_i$. Then $V_i^*(\widetilde{s}_i, s_{-i}') > V_i^*(\widetilde{s}_i, s_{-i}) - \frac{\varepsilon}{2}$ for all $s_{-i}' \in \mathcal{N}_{S_{-i}}(s_{-i})$ and, therefore, $V_i^*(\widetilde{s}_i, s_{-i}') > V_i^*(s_i, s_{-i}) - \varepsilon$ for all $s_{-i}' \in \mathcal{N}_{S_{-i}}(s_{-i})$. □

The next proposition establishes the better-reply security of the generalized contest in nondecreasing pure strategies.

**Proposition 8.** *If one of Assumptions 1, 2, or 3 holds in the contest $\Gamma$, then it is ex-ante better-reply secure in nondecreasing pure strategies.*

**Proof.** Since, by Proposition 7, the game is ex-ante payoff secure in nondecreasing pure strategies, the contest $\Gamma$ is better-reply secure if the following property holds: For every sequence $\{s^k\}_{k=1}^{\infty}$, $s^k \in S$, converging to some nonequilibrium strategy profile $\bar{s} \in S$ such that $\lim_k V_i^*(s^k)$ exists for each $i \in I$, there exists $j \in I$ for which $\overline{V}_j^*(\bar{s}_{-\hat{j}}) > \lim_k V_j^*(s^k)$.

Assume, by contradiction, that there exists a sequence $\{s^k\}_{k=1}^{\infty}$, $s^k \in S$, converging to some nonequilibrium strategy profile $\bar{s} \in S$ such that $\overline{V}_i^*(\bar{s}_{-j}) \leq \lim_k V_i^*(s^k)$ for each $i \in I$. Without loss of generality, each sequence $\{s_i^k\}_{k=1}^{\infty}$ converges pointwise on $T_i$ to $\bar{s}_i$. Since $\lim_k V_i^*(s^k) \leq \lim_k \overline{V}_i^*(s_{-i}^k)$ and $\overline{V}_i^*$ is continuous on $S_{-i}$ for each $i \in I$ by Corollary 2, we have $\overline{V}_i^*(\bar{s}_{-i}) = \lim_k \overline{V}_i^*(s_{-i}^k) = \lim_k V_i^*(s^k)$ for each $i \in I$. By repeating the argument provided in the proof of Theorem 2, one can establish that $V_i^*(\bar{s}) = \widetilde{V}_i^*(\bar{s}) = \overline{V}_i^*(\bar{s}_{-i})$ for each $i \in I$; that is, $\bar{s}$ is a Bayesian-Nash equilibrium of the contest game, a contradiction. □

According to Reny (1999, Remark 3.1), the better-reply security of the generalized contest implies that the limit strategy profile of any convergent sequence of nondecreasing pure-strategy $\frac{1}{k}$-equilibria of a game, as $k$ tends to $\infty$, is a pure-strategy Bayesian-Nash equilibrium of it. Consequently, the proofs of Theorem 2 and Proposition 8 are closely related to each other.

An important condition for the application of Reny's (1999) equilibrium existence theorem to normal-form games is the quasiconcavity of each player's payoff function in her own strategy. Since quasiconcavity might not be preserved under integration, the condition is rather demanding in the context of Bayesian games. It is satisfied, for example, if each player's ex-post payoff function is concave in her own action, which is not the case in many auction games, since concave functions are continuous in the interior of their domains.[13] However, in Tullock-type contests, ex-post payoff functions are discontinuous only at the zero vector of bids, which makes the quasiconcavity condition usable and Reny's (1999) equilibrium existence theorem applicable to generalized concave contests (see Einy et al., 2015). In the framework of our contest model, the concavity of contestant $i$'s payoff function $u_i$ in $b_i$ follows, for example, if $W_i$ depends only on types, $C_i$ is convex in $b_i$, and $p_i$ is concave in $b_i$.

Usually, in all-pay auctions, ex-post payoff functions are neither concave nor quasiconcave in own actions. So it is reasonable to attempt to use $H$-quasiconcavity instead of quasiconcavity. The contest game $\Gamma$ is $H$-quasiconcave in nondecreasing strategies if for each $i \in I$, every finite set of nondecreasing strategies $A = \{s_i^1, \ldots, s_i^m\}$, every $s_i \in F_A$, and every $s_{-i} \in S_{-i}$, we have

$$V_i^*(s_i, s_{-i}) \geq \min\{V_i^*(s_i^1, s_{-i}), \ldots, V_i^*(s_i^m, s_{-i})\}.$$

The next example of an all-pay auction illustrates that, in general, the generalized contest is neither quasiconcave nor $H$-quasiconcave in nondecreasing pure strategies.

**Example 4.** Consider the following two-bidder all-pay auction. Let $T_i = [0, 1]$ and $B_i = [0, 3]$ for all $i \in I = \{1, 2\}$. The bidders' types are independently uniformly distributed on $[0, 1]$. Bidder $i$'s probability of winning the item is given by the all-pay-auction success function $\overline{p}_i$. The functions $W_i$ and $C_i$ are defined as follows:

$$W_i(b; t) = 2 \text{ and } C_i(b; t) = b_i$$

for every $(b, t) \in B \times T$ and each $i \in \{1, 2\}$. The existence of a monotone pure-strategy Bayesian-Nash equilibrium in this all-pay auction follows from Theorem 2.

In order to verify that the ex-ante normal form of the all-pay auction is not quasiconcave in monotone pure strategies, consider the following strategies:

$$s_1^1(t_1) = \begin{cases} 0 \text{ for all } t_1 \in [0, \frac{1}{3}], \\ \frac{7}{20} \text{ for all } t_1 \in (\frac{1}{3}, \frac{2}{3}], \\ \frac{3}{4} \text{ for all } t_1 \in (\frac{2}{3}, 1]; \end{cases}$$

and

$$s_2^1(t_2) = \begin{cases} \frac{1}{5} \text{ for all } t_2 \in [0, \frac{1}{2}], \\ \frac{4}{5} \text{ for all } t_2 \in (\frac{1}{2}, 1]; \end{cases}$$

$$s_2^2(t_2) = \begin{cases} \frac{2}{5} \text{ for all } t_2 \in [0, \frac{1}{2}], \\ \frac{3}{5} \text{ for all } t_2 \in (\frac{1}{2}, 1]. \end{cases}$$

Contestant 2's strategy $\frac{1}{2}s_1^2 + \frac{1}{2}s_2^2$ is as follows:

[13] See Example 1 of He and Yannelis (2015) explaining the importance of the concavity condition for equilibrium existence results extending Reny's (1999) theorem to Bayesian games.

$$(\frac{1}{2}s_1^2+\frac{1}{2}s_2^2)(t_2)=\begin{cases}\frac{3}{10} \text{ for all } t_2\in[0,\frac{1}{2}],\\ \frac{7}{10} \text{ for all } t_2\in(\frac{1}{2},1].\end{cases}$$

Then $V_2^*(s_2^1,s_1^1)=\frac{1}{2}(\frac{1}{3}2-\frac{1}{5})+\frac{1}{2}(2-\frac{4}{5})=\frac{5}{6}$, $V_2^*(s_2^2,s_1^1)=\frac{1}{2}(\frac{2}{3}2-\frac{2}{5})+\frac{1}{2}(\frac{2}{3}2-\frac{3}{5})=\frac{5}{6}$, and $V_2^*(\frac{1}{2}s_1^2+\frac{1}{2}s_2^2,s_1^1)=\frac{1}{2}(\frac{1}{3}2-\frac{3}{10})+\frac{1}{2}(\frac{2}{3}2-\frac{7}{10})=\frac{1}{2}$. Therefore, the ex-ante normal form $(S_i,V_i^*)_{i\in I}$ of the all-pay auction is not quasiconcave.

The game $(S_i,V_i^*)_{i\in I}$ is also not $H$-quasiconcave. For the sake of simplicity, define $s_1^2\in S_1$ as follows: $s_1^2(t_1)=0$ for all $t_1\in[0,1]$. The join of bidder 2's strategies $s_2^1$ and $s_2^2$ is equal to

$$(s_2^1\vee s_2^2)(t_2)=\begin{cases}\frac{2}{5} \text{ for all } t_2\in[0,\frac{1}{2}],\\ \frac{4}{5} \text{ for all } t_2\in(\frac{1}{2},1].\end{cases}$$

Then bidder 2's corresponding ex-ante payoffs are the following: $V_2^*(s_2^1,s_1^2)=\frac{1}{2}(2-\frac{1}{5})+\frac{1}{2}(2-\frac{4}{5})=\frac{3}{2}$; $V_2^*(s_2^2,s_1^2)=\frac{1}{2}(2-\frac{2}{5})+\frac{1}{2}(2-\frac{3}{5})=\frac{3}{2}$; and $V_2^*(s_2^1\vee s_2^2,s_1^2)=\frac{1}{2}(2-\frac{2}{5})+\frac{1}{2}(2-\frac{4}{5})=\frac{7}{5}$. Since $V_2^*(s_2^1\vee s_2^2,s_1^2)<\min\{V_2^*(s_2^1,s_1^1),V_2^*(s_2^2,s_1^1)\}$, bidder 2's ex-ante payoff function is not $H$-quasiconcave in her own strategy.

Therefore, the quasiconcavity condition is a major obstacle to a direct application of Reny's (1999) equilibrium existence theorem to the generalized contest.

## 9. Conclusions

This paper introduces a generalized model of perfectly and imperfectly discriminating contests that possesses a monotone pure-strategy Bayesian-Nash equilibrium under quite mild conditions. In the generalized contest, the contestants have continua of possible types and bids, atomless type distributions, their valuations and costs might depend not only on her own bid and/or type but also on other contestants' bids and types, and the conditions imposed on the contest success function are applicable to both the all-pay-auction success function and the Tullock-type contest success functions.

The proof proceeds via studying a number of continuity-related properties of the contestants' interim payoff and value functions. A Browder-type fixed-point theorem is employed to establish the existence of a monotone pure-strategy approximate interim Bayesian-Nash equilibrium. Then it is shown that every converging sequence of monotone pure-strategy $\frac{1}{k}$-equilibria tends to a pure-strategy Bayesian-Nash equilibrium as $k$ tends to infinity.

**Declaration of competing interest**

The paper is theoretical, without any use of data. The authors do not have any interests to disclose.

**Data availability**

No data was used for the research described in the article.

## Appendix A

The Appendix contains the proofs of several auxiliary results.

### *A.1. Proof of Proposition 1*

**Proof.** Fix some $\varepsilon>0$, some $i\in I$, and some $(b_i,s_{-i},t_i)\in B_i\times L_{-i}\times T_i$. We need to show that there exist a bid $\widetilde{b}_i\in B_i$ and an open neighborhood $\mathcal{N}_\varepsilon(s_{-i},t_i)$ of $(s_{-i},t_i)$ in $L_{-i}\times T_i$ such that $V_i(\widetilde{b}_i,s'_{-i};t'_i)>V_i(b_i,s_{-i};t_i)-\varepsilon$ for every $(s'_{-i},t'_i)\in\mathcal{N}_\varepsilon(s_{-i},t_i)$.

If $V_i(b_i,s_{-i};t_i)\le 0$, then put $\widetilde{b}_i$ equal to 0 and pick any open neighborhood of $(s_{-i},t_i)$ in $L_{-i}\times T_i$. Then $V_i(0,s'_{-i};t'_i)>V_i(b_i,s_{-i};t_i)-\varepsilon$ for every $(s'_{-i},t'_i)$ in the neighborhood, since $W_i(0,b_{-i};t)\ge 0$ and $C_i(0,b_{-i};t)\le 0$ for every $(b_{-i},t)\in B_{-i}\times T$. Consider the case $V_i(b_i,s_{-i};t_i)>0$. Since $W_i(\overline{b},b_{-i};t)-C_i(\overline{b},b_{-i};t)<0$ for all $(b_{-i},t)\in B_{-i}\times T$, we have $b_i\in[r,\overline{b})$. If $\mu(t_j\in T_j:b_i=s_j(t_j))=0$ for all $j\in I_{-i}$, then put $\widetilde{b}_i=b_i$. By Lemma 1, there exists an open neighborhood $\mathcal{N}_\varepsilon(s_{-i},t_i)$ of $(s_{-i},t_i)$ in $L_{-i}\times T_i$ such that $V_i(\widetilde{b}_i,s'_{-i};t'_i)>V_i(b_i,s_{-i};t_i)-\varepsilon$ for every $(s'_{-i},t'_i)\in\mathcal{N}_\varepsilon(s_{-i},t_i)$. If $\mu(t_j\in T_j:b_i=s_j(t_j))>0$ for some $j\in I_{-i}$, then choose $\widetilde{b}_i\in(b_i,\overline{b})$ satisfying the following conditions: (a) $\mu(t_j\in T_j:\widetilde{b}_i=s_j(t_j))=0$ for all $j\in I_{-i}$; and (b) $V_i(\widetilde{b}_i,s_{-i};t_i)>V_i(b_i,s_{-i};t_i)-\frac{\varepsilon}{2}$, which is possible since $p_i$ is increasing in $b_i$, $W_i$ and $C_i$ are uniformly continuous on $B\times T$, and the values of $W_i$ are nonnegative. Then, by Lemma 1, there exists a neighborhood $\mathcal{N}_\varepsilon(s_{-i},t_i)$ of $(s_{-i},t_i)$ in $L_{-i}\times T_i$ such that $V_i(\widetilde{b}_i,s'_{-i};t'_i)>V_i(\widetilde{b}_i,s_{-i};t_i)-\frac{\varepsilon}{2}$ for every $(s'_{-i},t'_i)\in\mathcal{N}_\varepsilon(s_{-i},t_i)$, and, therefore, $V_i(\widetilde{b}_i,s'_{-i};t'_i)>V_i(b_i,s_{-i};t_i)-\varepsilon$ for every $(s'_{-i},t'_i)\in\mathcal{N}_\varepsilon(s_{-i},t_i)$. $\square$

### A.2. Proof of Proposition 2

**Proof.** Fix some $i \in I$. To show the upper semicontinuity of $\overline{V}_i$ on $L_{-i} \times T_i$, consider some $(s^0_{-i}, t^0_i) \in L_{-i} \times T_i$ and a sequence $\{(s^k_{-i}, t^k_i)\}$, $(s^k_{-i}, t^k_i) \in L_{-i} \times T_i$, $k = 1, 2, \ldots$, converging to it. Without loss of generality, the corresponding sequence $\{\overline{V}_i(s^k_{-i}; t^k_i)\}$ is also convergent. Assume, by contradiction, that $K = \lim_k \overline{V}_i(s^k_{-i}; t^k_i) - \overline{V}_i(s^0_{-i}; t^0_i) > 0$. Pick a sequence $\{b^k_i\}$ in $B_i$ such that $V_i(b^k_i, s^k_{-i}; t^k_i) > \overline{V}_i(s^k_{-i}; t^k_i) - \frac{1}{k}$ for $k = 1, 2, \ldots$. Without loss of generality, the sequence $\{b^k_i\}$ tends to some $b^0_i$. Clearly, $b^0_i < \overline{b}$. Since $\lim_k V_i(b^k_i, s^k_{-i}; t^k_i) > \overline{V}_i(s^0_{-i}; t^0_i) \geq V_i(b^0_i, s^0_{-i}; t^0_i)$, it must be the case that $\mu(t_j \in T_j : b^0_i = s^0_j(t_j)) > 0$ for some $j \in I_{-i}$. Pick $\delta > 0$ satisfying the following: (a) $b^0_i + \delta < \overline{b}$; (b) $\mu(t_j \in T_j : b^0_i + \delta = s^0_j(t_j)) = 0$ for each $j \in I_{-i}$; (c) $W_i(b^0_i + \delta, b_{-i}; t) > W_i(b_i, b_{-i}; t) - \frac{K}{4}$ and $C_i(b^0_i + \delta, b_{-i}; t) > C_i(b_i, b_{-i}; t) - \frac{K}{4}$ for every $(b_i, b_{-i}, t) \in (B_i \cap (b^0_i - \delta, b^0_i + \delta)) \times B_{-i} \times T$. It is possible to satisfy (c) since both $W_i$ and $C_i$ are uniformly continuous on $B \times T$. Assume also, without loss of generality, that $\left|b^0_i - b^k_i\right| < \delta$ for each $k \in \{1, 2, \ldots\}$. Since $W_i(b; t) \geq 0$ for every $(b, t) \in B \times T$ and $p_i$ is increasing in $b_i$, we have $V_i(b^0_i + \delta, s^k_{-i}; t^k_i) > V_i(b^k_i, s^k_{-i}; t^k_i) - \frac{K}{2}$ for each $k \in \{1, 2, \ldots\}$. Finally, by sending $k$ to infinity, we get $V_i(b^0_i + \delta, s^0_{-i}; t^0_i) > \lim_k \overline{V}_i(s^k_{-i}; t^k_i) - K = \overline{V}_i(s^0_{-i}; t^0_i)$, which is impossible. $\square$

### A.3. Proof of Proposition 5

**Proof.** For every $t_i \in T_i$, choose a bid $b_i(t_i) \in B_i$ satisfying the following two conditions: (a) $\mu(t_j \in T_j : b_i(t_i) = s_j(t_j)) = 0$ for each $j \in I_{-i}$; and (b) $V_i(b_i(t_i), s_{-i}; t_i) > \overline{V}_i(s_{-i}; t_i) - \varepsilon$. Since $V_i(b_i(t_i), s_{-i}; \cdot)$ and $\overline{V}_i(s_{-i}; \cdot)$ are continuous, there exists an open ball $\mathcal{B}(t_i; \delta_i(t_i))$ in $T_i$ with center $t_i$ and radius $\delta_i(t_i)$ such that $V_i(b_i(t_i), s_{-i}; t'_i) > \overline{V}_i(s_{-i}; t'_i) - \varepsilon$ for every $t'_i \in \mathcal{B}(t_i; \delta_i(t_i))$. Pick a finite minimal subcover $\{\mathcal{B}(t_{il}; \frac{\delta_i(t_{il})}{2})\}_{l \in \{1, \ldots, m\}}$ of the open cover $\{\mathcal{B}(t_i; \frac{\delta_i(t_i)}{2})\}_{t_i \in [0,1]}$ of $T_i = [0, 1]$. Without loss of generality, $t_{i1} < \ldots < t_{im}$. If $m = 1$, then define $\widetilde{s}_i$ by $\widetilde{s}_i(t_i) = b_i(t_{i1})$ for all $t_i \in T_i$. Consider the case $m > 1$. Denote $T_{i1} = [0, t_{i1} + \frac{\delta_i(t_{i1})}{2})$, $T_{il} = [t_{i(l-1)} + \frac{\delta_i(t_{i(l-1)})}{2}, t_{il} + \frac{\delta_i(t_{il})}{2})$ for $l = 2, \ldots, m-1$, and $T_{im} = [t_{i(m-1)} + \frac{\delta_i(t_{i(m-1)})}{2}, 1]$. Denote by $s^1_i$ the strategy defined as follows: $s^1_i(t_i) = b_i(t_{il})$ for every $t_i \in T_{il}$ and each $l \in \{1, \ldots, m\}$. If the strategy $s^1_i$ is nondecreasing, then it is a single-valued selection from $M^\varepsilon_i(s_{-i}, \cdot)$ satisfying (i) and (ii); that is, $\widetilde{s}_i$ can be defined by $\widetilde{s}_i(t_i) = s^1_i(t_i)$ for all $t_i \in T_i$. If $s^1_i$ is not increasing, then it can be modified as follows.

Let $l_1, \ldots, l_k$ be all the indices in $\{1, \ldots, m\}$, in increasing order, such that for each $j \in \{1, \ldots, k\}$, $b_i(t_{il_j}) = \min\{b_i(t_{i1}), \ldots, b_i(t_{im})\}$; that is, the same minimal bid is initially chosen on the intervals $T_{l_1}, \ldots, T_{l_k}$. If $l_1 = 1$, then define the strategy $s^{21}_i$ by $s^{21}_i(t_i) = s^1_i(t_i)$ for all $t_i \in T_i$. Assume that $l_1 > 1$. If for each $l \in \{1, \ldots, l_1 - 1\}$ and every $t_i \in T_{il}$, $V_i(s^1_i(t_i), s_{-i}; t_i) \leq V_i(b_i(t_{il_1}), s_{-i}; t_i)$, then put $s^{21}_i(t_i) = b_i(t_{il_1})$ for all $t_i \in T_1 \cup \ldots \cup T_{l_1}$ and $s^{21}_i(t_i) = s^1_i(t_i)$ for the other $t_i$'s in $T_i$. If there exist $l \in \{1, \ldots, l_1 - 1\}$ and $t'_i \in T_{il}$ such that $V_i(s^1_i(t'_i), s_{-i}; t'_i) > V_i(b_i(t_{il_1}), s_{-i}; t'_i)$, then, by the property of increasing differences of $V_i$ in $(b_i, t_i)$, we can put $s^{21}_i(t_i) = s^1_i(t'_i)$ for all $t_i \in T_{il_1}$ and $s^{21}_i(t_i) = s^1_i(t_i)$ for the other $t_i$'s in $T_i$; thereby replacing the bid $b_i(t_{il_1})$ on $T_{l_1}$ with a higher bid. Then repeat the procedure for interval $l_2$ and the strategy $s^{21}_i$, with the resulting strategy denoted by $s^{22}_i$. In $k$ steps, an $\varepsilon$-best-reply strategy $s^2_i = s^{2k}_i$ will be constructed in which either the bid $b_i(t_{il_1})$ is not made at all by any of contestant $i$'s types or the following holds true: if $s^2_i(t'_i) = b_i(t_{il_1})$ for some $t'_i \in T_i$, then $s^2_i(t_i) = b_i(t_{il_1})$ for every $t_i \in [0, t'_i]$. If $s^2_i$ is a nondecreasing strategy, then we have constructed a single-valued selection from $M^\varepsilon_i(s_{-i}, \cdot)$ satisfying (i) and (ii). Otherwise, denote $t^1_i = \min\{t_i \in T_i : s^2_i(t_i) \neq b_i(t_{il_1})\}$. We need to repeat this procedure for the smallest bid recommended by $s^2_i$ on the segment $[t^1_i, 1]$, with contestant $i$'s types in $[0, t^1_i)$, when $t^1_i > 0$, bidding $b_i(t_{il_1})$. In a finite number of steps, a nondecreasing $\varepsilon$-best-reply of contestant $i$ to the strategy subprofile $s_{-i}$ satisfying (i) and (ii) will be constructed. $\square$

### A.4. Proof of Proposition 6

**Proof.** Consider some correspondence $\widetilde{M}^\varepsilon_i$. Fix a strategy subprofile $s_{-i} \in S_{-i}$. By Proposition 5, the correspondence $M^{\frac{\varepsilon}{3}}_i(s_{-i}; \cdot)$ has a single-valued selection $\widetilde{s}_i \in S_i$ possessing the following properties: (i) $\widetilde{s}_i$ is a step function; (ii) $\mu(t_j \in T_j : \widetilde{s}_i(t_i) = s^0_j(t_j)) = 0$ for each $j \in I_{-i}$ and every $t_i \in T_i$. Then, by Corollary 1, there exists an open neighborhood $\mathcal{N}^1_{S_{-i}}(s_{-i})$ of $s_{-i}$ in $S_{-i}$ such that $V_i(\widetilde{s}_i(t_i), s'_{-i}; t_i) > V_i(\widetilde{s}_i(t_i), s_{-i}; t_i) - \frac{\varepsilon}{3}$ for all $s'_{-i} \in \mathcal{N}^1_{S_{-i}}(s_{-i})$ and all $t_i \in T_i$. Therefore, $V_i(\widetilde{s}_i(t_i), s'_{-i}; t_i) > \overline{V}_i(s_{-i}; t_i) - \frac{2\varepsilon}{3}$ for all $s'_{-i} \in \mathcal{N}^1_{S_{-i}}(s_{-i})$ and all $t_i \in T_i$.

Since, by Proposition 2, $\overline{V}_i$ is continuous on $S_{-i} \times T_i$ and the set $S_{-i} \times T_i$ is compact in $L_{-i} \times T_i$, $\overline{V}_i$ is uniformly continuous on $S_{-i} \times T_i$. Pick an open neighborhood $\mathcal{N}^2_{S_{-i}}(s_{-i})$ of $s_{-i}$ in $S_{-i}$ such that $\overline{V}_i(s_{-i}; t_i) > \overline{V}_i(s'_{-i}; t_i) - \frac{\varepsilon}{3}$ for all $s'_{-i} \in \mathcal{N}^2_{S_{-i}}(s_{-i})$ of $s_{-i}$ in $S_{-i}$ and all $t_i \in T_i$. Then $\widetilde{s}_i \in \widetilde{M}^\varepsilon_i(s'_{-i})$ for all $s'_{-i} \in \mathcal{N}^1_{S_{-i}}(s_{-i}) \cap \mathcal{N}^2_{S_{-i}}(s_{-i})$. $\square$

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