Hedging the Smirk

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Abstract

This article presents a simple nonparametric method for inferring deltas and gammas from implicit volatility patterns. An illustration indicates that Black-Scholes deltas and gammas are substantially biased in the presence of the sort of smirks and smiles evident in stock index options.
Post-'87 implicit volatility patterns in stock index options indicate substantial deviation from the Black-Scholes assumption of a lognormal distribution. Out-of-the-money put options trade at high implicit volatilities relative to at-the-money options, which are in turn higher than those from in-the-money puts and out-of-the-money calls. Figure 1 shows the typical volatility “smirk” pattern, using 1-month S&P 500 futures options on May 17, 1995. And while the overall level of volatilities has varied substantially since 1987, with shocks such as the mini-crash of October 13, 1989 and the Kuwait crisis having considerable impact, the relatively high pricing of out-of-the-money puts has been a persistent feature of post-crash options prices. This volatility pattern indicates the market perceives negative skewness in stock returns; customers are willing to pay substantially for the downside risk protection offered by out-of-the-money put options.

![Figure 1. The volatility “smirk.” Put options on S&P 500 futures: May 17, 1995.](image-url)
Given that the Black-Scholes assumption of constant implicit volatilities across all strike prices is egregiously violated, what are the appropriate deltas and gammas to use in hedging option positions? A parametric approach would take alternate negatively skewed distributions, such as the Bates (1991) jump-diffusion model with negative-mean jumps or a stochastic volatility model with negative correlations between price and volatility shocks. Such multiparameter models can be fitted to observed option prices, and the deltas, gammas and other derivatives can be computed given the parameter estimates. A difficulty is that inferring implicit parameters can be computationally expensive -- especially for American options with no closed-form solutions.

This article points out that for a broad class of option pricing models, the appropriate deltas and gammas for hedging option positions can be inferred directly from the pattern of implicit volatilities across different strike prices. The key assumption is that the stochastic process of the underlying asset price exhibits constant returns to scale, so that option prices are homogeneous of degree one in the underlying asset price and the strike price. This assumption first appeared in Merton's (1973) derivation of option pricing properties, and is satisfied by most European and American option pricing models. Examples include the Black-Scholes assumption of geometric Brownian motion, Merton's (1976) jump-diffusion process, and most stock and stock index option models with stochastic volatility or stochastic interest rates. The assumption rules out "level illusion:" whether the S&P 500 index is at 500 or 1000 is irrelevant for the distribution of stock market returns.

Conditional upon homogeneity, Euler's theorem indicates that an option's delta can be inferred directly from the option's sensitivity to the strike price:
3

\[ \Delta = \frac{\partial O}{\partial S} = \frac{1}{S} (O - XO_x) , \]  

(1)

where \( O \) is the option price, \( S \) is the underlying asset price or futures price, and \( O_x = \partial O/\partial X \).

Similarly, the option's gamma can be computed as

\[ \Gamma = \frac{\partial^2 O}{\partial S^2} = \left( \frac{X}{S} \right)^2 O_{xx} . \]  

(2)

For the Chicago Mercantile Exchange settlement prices for American options on S&P 500 futures, expressions (1) and (2) can be implemented directly to compute appropriate deltas and gammas. Option settlement prices are determined synchronously with each other and with the futures settlement price, so that \( O_x \) and \( O_{xx} \) can be computed numerically off observed option settlement prices:

\[
O_x \approx \frac{O(F, X + \Delta X) - O(F, X - \Delta X)}{2\Delta X}, \\
O_{xx} \approx \frac{O(F, X + \Delta X) - 2O(F, X) + O(F, X - \Delta X)}{(\Delta X)^2} . \]  

(3)

For example, the deltas and gammas associated with June 1995 put options on May 17, 1995 have the following values:

\[ \Delta = O_x, \Gamma = O_{xx}. \]

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1 Euler's theorem states that a homogeneous function satisfies \( SO_s + XO_x = O \).

Rearranging yields the expression for the delta \( \Delta = O_x = \partial O/\partial S \). Similarly, since \( O_s \) and \( O_x \) are homogeneous of degree zero,

\[
SO_{ss} + XO_{sx} = 0, \quad SO_{xs} + XO_{xx} = 0 .
\]

Eliminating the cross-derivatives \( O_{sx} = O_{xs} \) yields the above expression for the gamma \( \Gamma = O_{xx} \).
Table 1

June '95 put options on S&P 500 futures
Settlement prices: May 17, 1995

<table>
<thead>
<tr>
<th>Strike Price</th>
<th>Put Price</th>
<th>Volatility (σ)</th>
<th>Delta (Δ)</th>
<th>Gamma (Γ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>505</td>
<td>1.35</td>
<td>14.03%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>510</td>
<td>1.80</td>
<td>13.06%</td>
<td>-.103</td>
<td>.0075</td>
</tr>
<tr>
<td>515</td>
<td>2.45</td>
<td>12.11%</td>
<td>-.156</td>
<td>.0133</td>
</tr>
<tr>
<td>520</td>
<td>3.45</td>
<td>11.29%</td>
<td>.240</td>
<td>.0194</td>
</tr>
<tr>
<td>525</td>
<td>4.95</td>
<td>10.59%</td>
<td>-.359</td>
<td>.0277</td>
</tr>
<tr>
<td>530</td>
<td>7.15</td>
<td>10.11%</td>
<td>-.494</td>
<td>.0262</td>
</tr>
<tr>
<td>535</td>
<td>10.00</td>
<td>9.59%</td>
<td>-.645</td>
<td>.0349</td>
</tr>
<tr>
<td>540</td>
<td>13.70</td>
<td>9.47%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The June '95 futures settlement price was 527.80. Implicit volatilities were computed using the Barone-Adesi and Whaley (1987) American option pricing formula for 29-day options with a 5.96% interest rate.

More generally, one may be trying to assess appropriate deltas and gammas using badly synchronized intradaily or closing price data. Since intradaily option prices fluctuate considerably with the underlying asset price, but intradaily implicit volatilities are more stable, it is convenient to express option derivatives in terms of the slope and convexity of the volatility function across different strike prices. Define $O_{BS}$ as the European or American option pricing variant of the Black-Scholes model used in computing implicit volatilities $\hat{\sigma}$: $O = O_{BS}(\hat{\sigma})$. By the chain rule,

$$O_x = O_{xBS} + O_{\hat{\sigma}BS} \frac{\partial \hat{\sigma}}{\partial X}$$  \hspace{1cm} (4)

$$O_{xx} = O_{xxBS} + 2O_{x\hat{\sigma}BS} \frac{\partial \hat{\sigma}}{\partial X} + O_{\hat{\sigma}^2BS} \left( \frac{\partial ^2 \hat{\sigma}}{\partial X^2} \right)^2 + O_{\hat{\sigma}BS} \frac{\partial ^2 \hat{\sigma}}{\partial X^2}$$  \hspace{1cm} (5)
Plugging these into (1) and (2) above and exploiting the fact that (1) and (2) also hold for the "Black-Scholes" option prices $O^{BS}$ yields the following expressions for delta and gamma:

\[
\Delta = \Delta^{BS} - \sigma^{BS} \frac{X}{S} \frac{\partial \hat{\sigma}}{\partial X}
\]

\[
\Gamma = \Gamma^{BS} + \left( \frac{X}{S} \right)^2 \left[ 2 \sigma^{BS} x \frac{\partial \hat{\sigma}}{\partial X} + \sigma^{BS} \left( \frac{\partial \hat{\sigma}}{\partial X} \right)^2 \right] + \sigma^{BS} \frac{\partial^2 \hat{\sigma}}{\partial X^2}
\]

where $\Delta^{BS}$ and $\Gamma^{BS}$ are the "Black-Scholes" delta and gamma computed at that option's implicit volatility.

Evaluating (6) and (7) requires computing the various partial derivatives of the "Black-Scholes" formula, and computing the slope and convexity of the implicit volatility function $\hat{\sigma}(X)$. The former can be done analytically if a European option pricing formula was used when computing implicit volatilities, and numerically if an American option pricing formula was used. It actually does not matter which formula is used when computing derivatives, provided that it is the same as used for computing implicit volatilities. Black's (1976) European futures option pricing model is perfectly acceptable, and is of course more tractable. Any errors in $\Delta^{BS}$ and $\Gamma^{BS}$ from a failure to take an early-exercise premium into account are corrected by the adjustment for the slope and convexity of the implicit volatility function. The applications in this article use the Barone-Adesi and Whaley (1987) American futures option pricing model primarily to emphasize that the observed volatility smirk is not attributable to the early-exercise premium of American puts on S&P 500 futures.
The slope and convexity of the implicit volatility function \( \hat{\sigma}(X) \) can be evaluated numerically for the above settlement price data. More typically, however, it will be necessary to estimate the volatility function from noisy intraday implicit volatilities. Many methods are viable; the simplest is probably the regression-based approach of Shimko (1993). The implicit volatilities are regressed on the strike price and strike price squared,

\[
\hat{\sigma}(X) = A_0 + A_1 X + A_2 X^2
\]

and the desired first and second derivatives can be estimated using the estimated coefficients:

\[
\frac{\partial \hat{\sigma}}{\partial X} = \hat{A}_1 + 2\hat{A}_2 X
\]

\[
\frac{\partial^2 \hat{\sigma}}{\partial X^2} = 2\hat{A}_2.
\]

For example, the regression-based estimate of the volatility function using implicit volatilities from the above settlement data is \( \hat{\sigma} = 8.0143 - 0.0289256 X + 2.640476 \times 10^{-5} X^2 \), and is graphed in Figure 1. Alternatively, the volatility function can be estimated in terms of the moneyness variable \( y = X/S \) rather than \( X \), and used in conjunction with appropriately modified versions of (6) and (7). \(^2\) This latter approach is somewhat more robust to intraday variation in the underlying asset price.

Expressions (6) and (7) indicate that the Black-Scholes delta and gamma badly misrepresent the true values if there is substantial slope to the volatility function across different strike prices --

\(^2\)The appropriate substitutions in (6) and (7) are \( \frac{\partial \hat{\sigma}}{\partial X} = \frac{\partial \hat{\sigma}}{\partial y} \frac{1}{S} \) and \( \frac{\partial^2 \hat{\sigma}}{\partial X^2} = \frac{\partial^2 \hat{\sigma}}{\partial y^2} \frac{1}{S^2} \) for this modification. For instance, the appropriate delta becomes \( \Delta = \Delta^{BS} - y \frac{O_y^{BS}}{S} \frac{\partial \hat{\sigma}}{\partial y} \), where \( O_y^{BS}/S \) depends only on \( y \) given the homogeneity of \( O_y^{BS} \) in \( S \) and \( X \).
which is, of course, the case for stock index options. Since the volatility function is downward sloping and the Black-Scholes "vega" $O^\text{BS}_\sigma$ is positive, call and put deltas computed using Black-Scholes implicit volatilities and hedge ratios understate the true deltas. Figure 2 compares the Black-Scholes deltas with those computed using the regression-based estimate of the volatility function, while Table 2 gives the values of the deltas and gammas.
Figure 2. Implicit deltas. Put options on S&P 500 futures: May 17, 1995.

Table 2

June '95 put options on S&P 500 futures
Settlement prices: May 17, 1995

<table>
<thead>
<tr>
<th>Strike Price</th>
<th>Smoothed Volatility $\hat{\sigma}(X)$</th>
<th>Delta</th>
<th>Gamma</th>
</tr>
</thead>
<tbody>
<tr>
<td>505</td>
<td>14.08%</td>
<td>$\Delta_{BS}$</td>
<td>$\Delta$</td>
</tr>
<tr>
<td>510</td>
<td>13.01%</td>
<td>-1.28</td>
<td>-0.61</td>
</tr>
<tr>
<td>515</td>
<td>12.08%</td>
<td>-1.169</td>
<td>-0.97</td>
</tr>
<tr>
<td>520</td>
<td>11.29%</td>
<td>-1.229</td>
<td>-1.53</td>
</tr>
<tr>
<td>525</td>
<td>10.62%</td>
<td>-1.313</td>
<td>-2.37</td>
</tr>
<tr>
<td>530</td>
<td>10.08%</td>
<td>-1.422</td>
<td>-3.53</td>
</tr>
<tr>
<td>535</td>
<td>9.68%</td>
<td>-1.551</td>
<td>-4.95</td>
</tr>
<tr>
<td>540</td>
<td>9.41%</td>
<td>-1.683</td>
<td>-6.47</td>
</tr>
</tbody>
</table>


Two caveats are in order regarding this method of computing deltas and gammas. First, the method is heavily dependent upon the assumption of homogeneity. And although this property appears desirable, and is satisfied by many popular American and European option pricing models, there do exist models that do not possess this property. Examples include the constant elasticity of variance model of Cox and Rubinstein (1985, pp.361-4), and implied binomial trees models such as Dupire (1994), Derman and Kani (1994), and Rubinstein (1994). Alternate methods of computing option derivatives are necessary for such models. However, nonhomogeneous models also imply that at-the-money implicit volatilities are nonstationary, contrary to the mean reversion evident in plots of implicit volatilities over time.3

Second, while the proposed methodology may be able to infer the deltas and gammas perceived by the market, that does not mean the market is correct. If options are mispriced, it is probable that the implicit deltas and gammas are also erroneous. Identifying mispriced options does of course require an assessment of what are the correct prices -- i.e., a proprietary model. Even in this case, however, the proposed method of computing implicit deltas and gammas may serve as an informative diagnostic for comparison with those estimated using a proprietary model.

3Nonhomogeneous models imply that the at-the-money option/asset price ratio $O^{ATM}/S$ depends upon the (nonstationary) underlying asset price $S$. Consequently, the implicit volatility computed from this ratio under such models must also be nonstationary.
References


