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The Skewness Premium:
Option Pricing Under Asymmetric Processes

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Abstract

This paper develops distribution-specific theoretical constraints on relative prices of out-of-the-money European call and put options that are also valid for American options on futures. The constraints can be used to identify which distributional hypotheses can and cannot explain observed moneyness biases. An application to S&P 500 futures options over 1983-1993 illustrates the inability of standard distributional hypotheses to explain observed moneyness biases -- especially those observed following the stock market crash of 1987. Some distribution-specific extensions to exotic option pricing are also discussed.

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Option pricing models such as Black and Scholes (1973) premised upon the underlying asset price following geometric Brownian motion have been found to exhibit serious specification error when fitted to market data. A "moneyness" or "striking price" bias is ubiquitous: model prices as a (monotone) function of the strike price are typically "tilted" relative to observed options prices, so that the residuals for low strike prices tend to be opposite in sign from the residuals for high strike prices. Such biases have been found in American call and put options on stocks traded on the Chicago Board Options Exchange;[1] in American calls and puts on S&P 500 futures traded on the Chicago Mercantile Exchange;[2] in American foreign currency call options traded on the Philadelphia Stock Exchange;[3] and in European Swiss franc-denominated call options on the dollar traded in Geneva.[4] The biases are not in the same direction for all markets, nor are they constant over time. The biases have been especially egregious in stock and stock index options since the stock market crash of 1987.

Observed moneyness biases and considerable time series evidence against the hypothesis that log-differenced asset prices are homoskedastic and normally distributed have spurred the development of option pricing models for alternative stochastic processes. The *constant elasticity of variance* (CEV) option pricing model of Cox and Ross (1976) and Cox and Rubinstein (1985) relaxed the constant volatility assumption and allowed the instantaneous conditional volatility of asset returns to depend deterministically upon the level of the asset price. Special cases include both arithmetic and geometric Brownian motion. The recent “implied binomial tree” models of Dupire (1994), Derman and Kani (1994), and Rubinstein (1994) can be viewed as flexible generalizations of the CEV model. The *stochastic volatility* option pricing models of Hull and White (1987), Scott (1987), and Wiggins (1987) permitted more general patterns of evolution for conditional volatility, parallelling the development of time series models such as ARCH and GARCH that were concerned with the same issue. The recognition that asset returns are leptokurtic, particularly for short holding periods, motivated Merton's (1976) model for pricing options on *jump-diffusion*

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[1]See Black (1975), MacBeth and Merville (1980), and Rubinstein (1985) for calls on CBOE stocks; see also the criticisms by Geske and Roll (1984) and Whaley (1982). See Blomeyer and Johnson (1988) for puts on CBOE stocks.

[2]See Whaley (1986) and Bates (1991).

[3]See Bodurtha and Courtadon (1987).

[4]See Chesney and Scott (1989).

processes.[5] Finally, the recognition that options involve future payoffs has spurred examination of the impact of *stochastic discount rates*.

Which of these alternative factors and associated option pricing models can generate the moneyness biases observed empirically? This question has motivated "horse race" papers such as Rubinstein (1985) and Shastri and Wethyavivorn (1987), and is a relevant criterion not just of option pricing models but of time series models as well. The central thesis of this paper is that most postulated processes can be ruled out *a priori* as inconsistent with observed moneyness biases. Those biases constitute *prima facie* evidence that the distributions implicit in option prices are markedly skewed relative to the benchmark lognormal distribution implied by geometric Brownian motion.[6] By contrast, the above option pricing models under the parameter values typically used generate implicit distributions that are essentially symmetric. Parameter values that are potentially consistent with observed moneyness biases are seldom studied, and in fact are often ruled out to increase tractability. For instance, although Merton's (1976) jump-diffusion model allowed jumps to have non-zero mean, empirical tests by Ball and Torous (1983, 1985) assumed mean-zero jumps -- precisely the parameterization incapable of eliminating observed moneyness biases. Stochastic volatility models typically focus on the special case in which volatility evolves independently of the asset price -- a case equally incapable of explaining moneyness biases.

The theoretical foundations for this central thesis rest on a method of measuring the moneyness bias that offers valuable insights into which option pricing models can and cannot explain observed biases. The method is based upon comparing the prices of out-of-the-money (OTM) calls and puts. Intuitively, since OTM calls pay out only if the asset price rises above the call's exercise price, while OTM puts pay off only if the asset price falls below the put's exercise price, call and put prices directly reflect the characteristics of the upper and lower tails of the distribution. Comparing the two for OTM options is consequently a direct gauge of the relative thicknesses of the tails, and consequently a

[5] McCulloch (1985,1987), motivated by the same concern, derives an option pricing model under the assumption that log-differenced asset prices are drawn from a stable Paretian distribution.

[6] This point is also made in Jarrow and Rudd (1982).

measure of implicit skewness. More precisely, relative prices of OTM options will reflect the skewness of the "risk-neutral" distribution generated from any given distributional hypothesis, and can therefore be used as a direct and easy diagnostic of that underlying hypothesis.

Section I of the paper quantifies the relationship between distributional hypotheses and the relative prices of European out-of-the-money calls and puts for the major families of option pricing models listed above. For the constant elasticity of variance processes, OTM call and put prices are roughly equal under arithmetic Brownian motion and diverge increasingly as one moves through intermediate CEV processes towards geometric Brownian motion. For the last process, an "*x*% rule" holds: call options *x*% out-of-the-money are priced <u>exactly</u> *x*% higher than the corresponding OTM puts. This "*x*% rule" also holds for the benchmark models in the other classes of option pricing models: stochastic volatility models in which volatility evolves independently of the asset price, jump-diffusion models with mean-zero jumps, and stochastic interest rates/dividend yields that follow Ornstein-Uhlenbeck processes. Equivalently, the implicit volatility patterns are perfectly symmetric functions of the log of the strike price under these benchmark models.

Section II extends the results to American options. Although the above results for European options will hold only approximately for American options in general, most of the results will hold exactly in the case of American options on futures. Furthermore, distribution-specific theoretical relationships between the immediate-exercise critical values for American calls and puts on futures can be derived. These results follow from the key insight that the put valuation problem is a distribution-specific transformation of variables of the call valuation problem for American options on futures. Based on these results, the "skewness premium" metric is defined as the percentage deviation of OTM call prices from the prices of puts correspondingly out-of-the-money. Section III applies the metric to American options on S&P 500 futures options traded on the Chicago Mercantile Exchange over 1983-1993, illustrating the substantial and persistent negative implicit skewness present since the crash of 1987.

Many of the results in the first two sections have been reported previously,[7] without proof. This article provides the proofs, and some extensions. These proofs are of interest not only for confirming the validity of previous claims, but also for the insights they provide, and for potential generalizations to other contingent claims. For instance, the key insight that the put valuation problem is a distribution-specific transformation of variables of the call valuation problem for European options in general and for American options on futures is largely unprecedented in the option pricing literature. Some applications to exotic option pricing are given in the concluding Section IV.

[7]Bates (1991) applies the skewness premium to S&P 500 futures options over 1985-87, while Bates (1996) applies it to foreign currency futures options over 1984-92.

I. Asymmetry and European Option Pricing

A. Assumptions and Notation

Throughout this paper, the following standard assumptions will typically be maintained:

A1) Markets are frictionless: there are no transactions costs or differential taxes, trading can take place continuously, there are no restrictions on borrowing or selling short.

A2) There exists a "risk-neutral" probability measure under which options can be priced at their expected discounted future payoff.

A3) The instantaneous risk-free interest rate r is known and constant.

A4) The "cost of carry" to maintaining a position in the underlying asset is a constant proportion b of the asset price.

Standard specifications of the "risk-neutral" probability measure will be examined below. Specific extensions of the results to stochastic interest rates and cost of carry will also be examined.

The following notation will be used:

S_t (or S) = the current price on the underlying asset;

$F_{t,t+T}$ (or F) = the forward price on the asset for delivery T periods from now;

T = the time to maturity of the option;

X = the strike price of the option;

$c(\Omega_t)$ [$C(\Omega_t)$] = European [American] call option price conditional upon current information Ω_t;

$p(\Omega_t)$ [$P(\Omega_t)$] = European [American] put option price conditional upon current information Ω_t.

When the cost of carry is constant, the forward and spot prices are related by

$$F_{t,t+T} = S_t e^{bT} \quad . \tag{1}$$

Without loss of generality, option prices will be specified in terms of the *forward* price rather than the spot price. Call options will be referred to as *out-of-the-money* (OTM) if the strike price X exceeds the forward price ($X > F$), and *in-the-money* (ITM) if the reverse is true. OTM put options will be those for which $X < F$, while ITM puts are those for which $X > F$.

In general, only a subset of the information set Ω_t will be highlighted: the current underlying forward asset price F and other state variables, the time to maturity T, and the strike price X. For instance, American call options will have prices written as $C_t = C(\Omega_t) = C(F, T; X)$. Model-specific parameters will be included in the list of arguments when option pricing formulas depend in interesting fashions upon those parameters.

I.B Symmetric and log-symmetric processes

When interest rates are constant, European call and put prices are given by

$$\begin{aligned} c(\Omega_t) &= e^{-rT} E^*[\max(S_{t+T} - X, 0) \mid \Omega_t] \\ &= e^{-rT} E^*[S_{t+T} - X \mid S_{t+T} > X; \Omega_t] \, Prob^*[S_{t+T} > X \mid \Omega_t] \end{aligned} \tag{2}$$

$$\begin{aligned} p(\Omega_t) &= e^{-rT} E^*[\max(X - S_{t+T}, 0) \mid \Omega_t] \\ &= e^{-rT} E^*[X - S_{t+T} \mid S_{t+T} < X; \Omega_t] \, Prob^*[S_{t+T} < X \mid \Omega_t] \, , \end{aligned} \tag{3}$$

where E^* is the expectation under the risk-neutral probability measure. The risk-neutral expected future spot price under nonstochastic interest rates is the forward price on the asset: $E^*(S_{t+T} | \Omega_t) = F_{t,t+T} = S_t e^{bT}$.

A key insight from (2) and (3) is that calls and puts *as functions of the exercise price X* are symmetric contingent claims. Call prices reflect conditions in the upper tail of the risk-neutral distribution, whereas put prices reflect conditions in the lower tail. If the strike prices of the put and call are spaced symmetrically around the mean $E^*(S_{t+T} | \Omega_t) = F_{t,t+T}$, symmetries or asymmetries in the risk-neutral distribution will be directly reflected in the relative prices of these out-of-the-money (OTM) calls and puts. Since different distributional hypotheses typically imply different risk-neutral distributions of S_{t+T}, options data can be used to distinguish amongst those hypotheses. The major theoretical contribution of this paper is to prove theoretical propositions that *quantify* the relationship between OTM calls and puts for different distributional hypotheses, thereby providing a metric for easily judging which hypotheses are consistent with observed option prices.

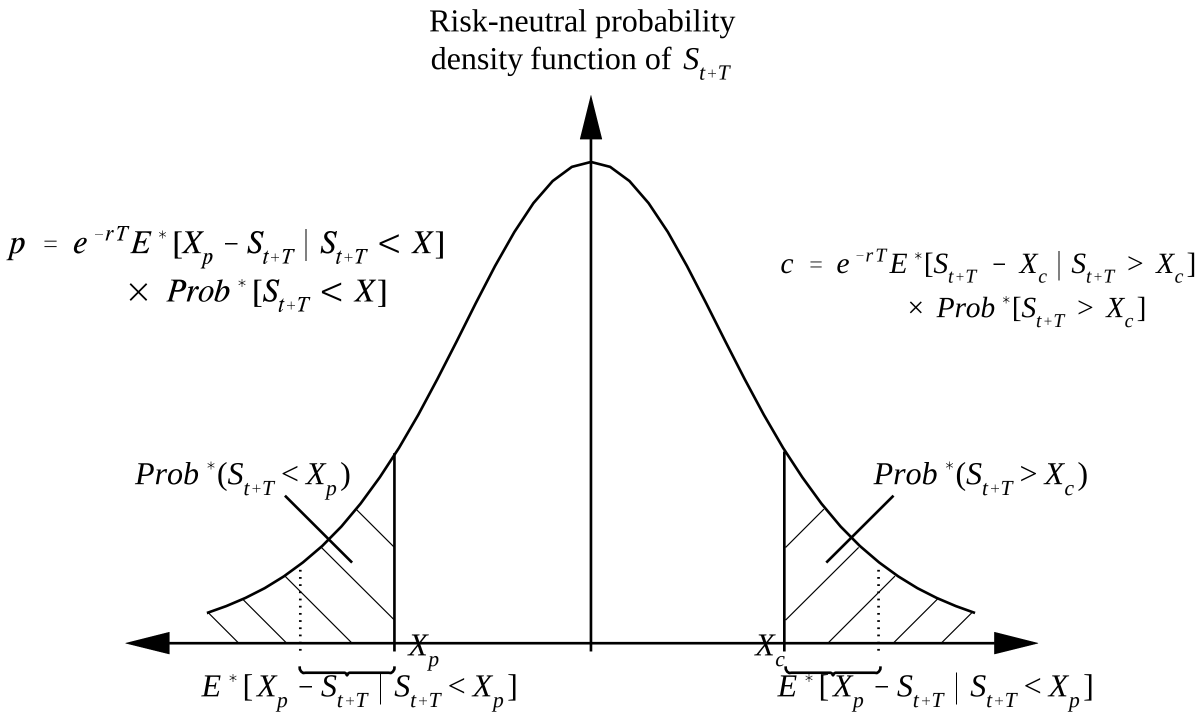

Figure 1. Determination of European call and put prices

The simplest relationship between out-of-the-money calls and puts is for those distributions that generate symmetric risk-neutral distributions for S_{t+T}.

Proposition 1 (symmetric processes): *If the terminal risk-neutral distribution of S_{t+T} is symmetric around its mean $F = S_t e^{bT}$, and the strike prices of the call and put are spaced symmetrically around F, then European call and put prices as functions of the spot price, time to maturity, and strike price are related by*

$$c(F,\ T;\ F + x) \quad = \quad p(F,\ T;\ F - x) \quad \text{for} \ |x| < F. \tag{4}$$

Proof: Obvious from Figure 1. ■

Proposition 1 says that a call option with strike price 5% above the forward price ($x = 0.05F$) will be priced identically to a put with strike price 5% below the forward price, if the distribution is symmetric. One historically important example of a symmetric distribution is Bachelier's (1900) Brownian motion process with zero drift and no absorbing barrier, generalized by Cox and Ross (1976) to processes of the form

$$dS = \mu S\, dt + \sigma\, dZ \quad . \tag{5}$$

While admittedly invalid given limited liability,[8] Bachelier option prices are indistinguishable when bankruptcy is unlikely from those generated by the Cox and Ross (1976) model with $\mu = 0$ and absorbing barrier at $S=0$. Consequently, Proposition 1 holds virtually exactly for processes of the form (5) when bankruptcy risk is negligible, and serves as a useful bound in other cases.

It is important to realize that (4) is unrelated to the European put-call relationship

$$c(F,\ T;\ X) = p(F,\ T;\ X) + e^{-rT}(F - X) \quad , \tag{6}$$

which is an arbitrage-based relationship between puts and call with *identical* strike prices, and holds regardless of distribution. Proposition 1 is a statement about calls and puts with different strike prices, and holds *only* for symmetric terminal risk-neutral distributions. If the distribution is asymmetric, prices of calls and puts symmetrically out of the money relative to the forward price will diverge.[9] For the major distributions studied hitherto, that divergence takes systematic forms. The positively skewed log-normal distribution used in the Black-Scholes model has a positive divergence between out-of-the-money (OTM) calls and puts:

$$c(F,\ T;\ F+x) > p(F,\ T;\ F-x) \quad for\ 0 < x < F. \tag{7}$$

The degree of divergence can be quantified for geometrically spaced strike prices.

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[8]Smith (1976) and Cox and Ross (1976) discuss this issue.

[9]Using put-call parity, one can interpret Proposition 1 as a statement about the relative prices of out-of-the-money and in-the-money calls when the distribution is symmetric:

$$c(S,\ T;\ F+x) = c(S,\ T;\ F\text{-}x) + e^{-rT}(F\text{-}x\text{-}F) = c(S,\ T;\ F\text{-}x) - x\,e^{-rT}\ .$$

Proposition 2 (Black-Scholes): *If the underlying asset price follows geometric Brownian motion, then European call and put prices are related by*

$$c(F,\ T;\ Fk)\ =\ k\,p(F,\ T;\ F/k)\quad \textit{for any } k > 0. \tag{8}$$

Proof: The Black-Scholes formulas are

$$c^{BS}(F,\ T;\ X)\ =\ e^{-rT}\,[FN(d_1)\ -\ XN(d_2)] \tag{9}$$

$$p^{BS}(F,\ T;\ X)\ =\ e^{-rT}\,[XN(-d_2)\ -\ FN(-d_1)] \tag{10}$$

where

$$d_1\ =\ \frac{\ln(F/X)\ +\ \tfrac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\ ,\quad d_2\ =\ d_1\ -\ \sigma\sqrt{T}\ , \tag{11}$$

and $N(\cdot)$ is the cumulative normal. Plugging $X_c = Fk$ into (9) and $X_p = F/k$ into (10) yields (8). ■

Proposition 2 states that given geometric Brownian motion, if the call and put options have strike prices 5% out-of-the-money relative to the forward price (and geometrically symmetric), then the call should be priced 5% higher than the put. If the asset price followed arithmetic Brownian motion, then from Proposition 1 the call and put options would have about the same price.[10]

Proposition 2 also holds for the benchmark stochastic volatility and jump-diffusion models, which can be written as mean-preserving randomizations of the Black-Scholes formula.[11]

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[10]Assuming bankruptcy risk is not substantial, and using the approximation $1/k\ =\ 1/(1+x)\ \approx\ 1\text{-}x$.

[11]I am indebted to an anonymous referee for this simplification of the original proof.

Corollary #1 (stochastic volatility) *If the risk-neutral representation of the underlying asset price is of the form*

$$
\begin{aligned}
dS/S &= b\,dt + \sigma_t\,dZ \\
d\sigma_t &= \mu(\sigma_t,\ t)\,dt + v(\sigma_t,\ t)\,dZ_\sigma \\
Cov(dS/S,\ d\sigma) &= \rho\,dt
\end{aligned}
\tag{12}
$$

then (8) *holds if* $\rho_{s\sigma} = 0$.

Proof: Under the independent-volatility assumptions of Hull and White (1987), option prices are

$$
\begin{aligned}
c^{HW}(F,\ T;\ Fk) &= \int_0^\infty c^{BS}(F,\ T;\ Fk,\ \overline{V})\ f(\overline{V})\ d\overline{V} \\
&= k\int_0^\infty p^{BS}(F,\ T;\ F/k,\overline{V})\ f(\overline{V})\,d\overline{V} \\
&= k\ p^{HW}(F,\ T;\ F/k)\quad ,
\end{aligned}
\tag{13}
$$

where $\overline{V}$ is the realized average variance per unit time and $f(\overline{V})$ is its risk-neutral density. ■

The corollary does not hold if the correlation $\rho_{s\sigma}$ between volatility shocks and price shocks is non-zero. As noted by Hull and White (1987), a positive $\rho_{s\sigma}$ biases OTM call option prices upward and ITM call options downward relative to Black-Scholes prices, whereas $\rho_{s\sigma} < 0$ induces the reverse biases. The implication in this framework is that $c(F,\ T;\ Fk) \gtrless k\,p(F,\ T;\ F/k)$ depending on whether $\rho_{s\sigma} \gtrless 0$. Figure 2 illustrates the impact of varying $\rho_{s\sigma}$ for the square root variance process used *inter alia* by Hull and White (1988) and Heston (1993).

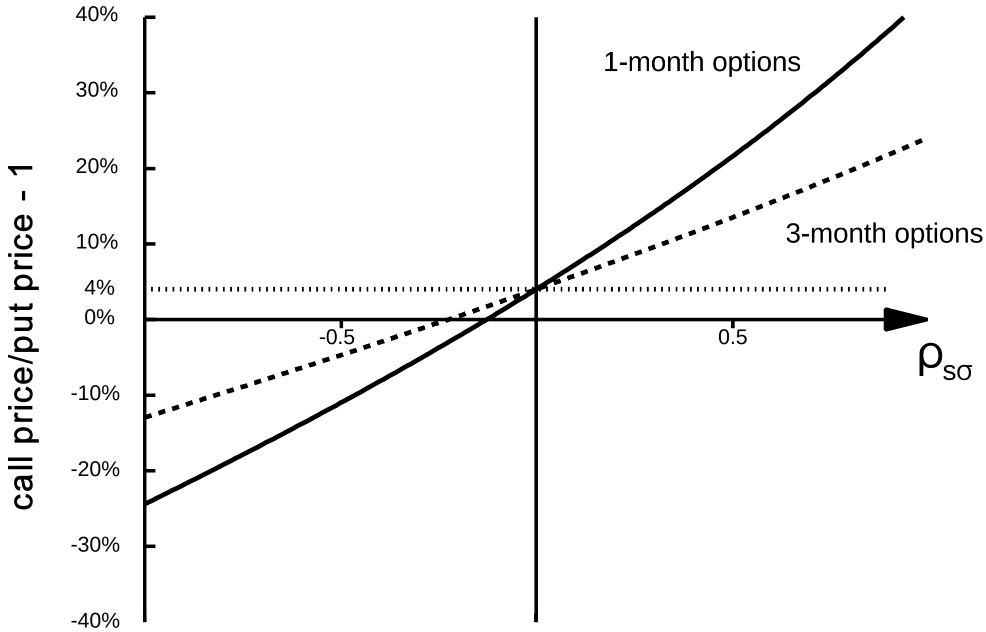

Figure 2. Relative prices of 4% OTM European call and put options on a stochastic volatility process, as a function of the correlation ρ_{sp} between price and volatility shocks. Risk-neutral process:

$$\begin{cases} dS/S = \sigma_t dZ \\ d\sigma_t^2 = 6\ln 2(.15^2 - \sigma_t^2)dt + .15\sigma_t dZ_v \end{cases}$$

(half-life to volatility shocks of 2 months); $\sigma_0 = .15$; $X_{call}/F = 1.04 = F/X_{put}$.

Corollary #2 (jump-diffusions) *If the risk-neutral representation of the underlying asset price is a jump-diffusion as in Merton* (1976),

$$dS/S = (b - \lambda^* \bar{k}^*)\,dt + \sigma\,dZ + k^*\,dq^* \tag{14}$$

where

λ^* *is the risk-adjusted jump frequency,*

q^* *is a Poisson counter with intensity* λ^**, and*

$\ln(1+k^*) \sim N\left[\ln(1+\bar{k}^*) - \tfrac{1}{2}\delta^2,\ \delta^2\right],$

then Proposition 2 holds if $\bar{k}^* = 0$ *and* λ^* *is constant.*

Proof: The mean-zero jump assumption implies call option prices of the form

$$c^{Merton}(F,\ T;\ X) \ = \ \sum_0^\infty \ w_n \ c^{BS}\left(F e^{n\ln(1+\bar{k}^*) - \lambda \bar{k}^* T},\ T;\ X,\ \sigma^2 + \frac{n\delta^2}{T} \right)$$

$$= \ \sum_0^\infty \ w_n \ c^{BS}\left(F,\ T;\ X,\ \sigma^2 + \frac{n\delta^2}{T} \right) \quad , \quad w_n \ \equiv \ \frac{e^{-\lambda^* T}(\lambda^* T)^n}{n!} \qquad \textbf{(15)}$$

with a similar expression for puts. Since (8) constrains Black-Scholes call and put prices conditional on the number of jumps n, it also holds for the weighted sum across all n. ■

Positive (negative) $\bar{k}^*$ raises (lowers) the skewness of the distribution relative to the lognormal, increasing (decreasing) OTM call option prices relative to OTM put option prices. Figure 3 illustrates the impact of varying $\bar{k}^*$

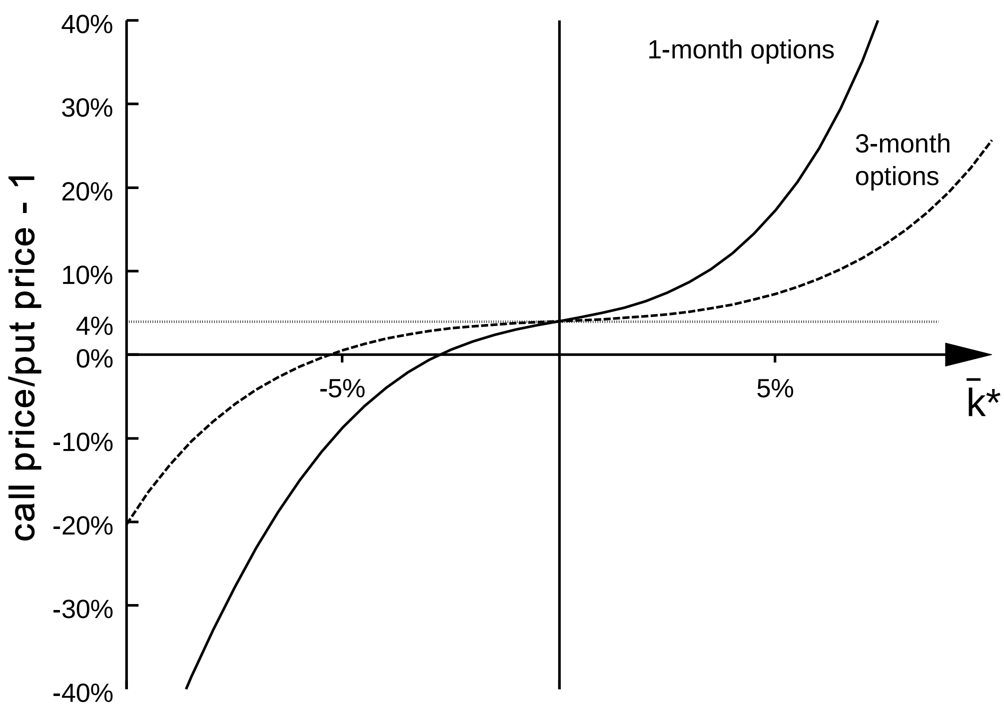

Figure 3. Relative prices of 4% OTM European call and put options on a jump-diffusion process, as a function of the mean jump size $\bar{k}^*$ under constant overall volality. Parameter values (equation (14)): $b = 0$, $\lambda^* = 1$ jump/year, $\delta = .02$; overall variance $\sigma^2 + \lambda^*\{[\ln(1+\bar{k}^*) - ½\delta^2]^2 + \delta^2\} = .15^2$ per year.

upon the deviation of calls from puts when the average variance $\sigma^2 + \lambda^*\{[\ln(1+\bar{k}^*) - ½\delta^2]^2 + \delta^2\}$ per unit time is held constant.[12]

Corollary #3 (Ornstein-Uhlenbeck interest rates/dividend yields) *If the underlying asset price has a risk-neutral representation of the form*

$$
\begin{aligned}
dS/S &= (r - r^*)dt + \sigma(t)dZ \\
dr &= [\alpha(t) - \beta(t)r]dt + \sigma_1(t)dZ_1 \\
dr^* &= [\alpha^*(t) - \beta^*(t)r]dt + \sigma_2(t)dZ_2
\end{aligned}
\qquad \textbf{(16)}
$$

where r is the foreign interest rate* (*for foreign currency options*) *or stock dividend yield* (*for stock options*), *then Proposition 2 holds* <u>*regardless*</u> *of the correlations between innovations in asset prices, interest rates, and dividend yields.*

Proof: Ornstein-Uhlenbeck interest rate processes with time-dependent parameters imply modified Black-Scholes foreign currency option prices of the form[13]

$$c(F, T; X) = B(r, T)[FN(d_1) - XN(d_2)] \qquad \textbf{(17)}$$

where

$B(r, T)$ is the price of a discount bond with time T to maturity;

$B^*(r^*, T)$ is the price of a foreign discount bond with time T to maturity;

$F = SB^*/B$ is the forward price on the asset;

$d_1 = [\ln(F/X) + ½V_{avg}T] / \sqrt{V_{avg}T}$;

$d_2 = d_1 - \sqrt{V_{avg}T}$; and

[12]It should be noted that $\bar{k}^*$ is the mean jump size under the risk-neutral distribution, and will typically differ from the *actual* mean jump size $\bar{k}$. A calibration of the bias is in Bates (1991).

[13]Examples of option pricing models directly or closely related to this specification include Merton (1973), Grabbe (1983), Rabinovitch (1989), Hilliard, Madura and Tucker (1991), and Amin and Jarrow (1991).

$$V_{avg} T \;=\; \int_0^T Var(dF/F),$$

where $Var(dF/F)$ is a deterministic function of time under processes of the form (16). Consequently, Proposition 2 applies directly. ■

Corollary #4 (volatility smile) *The Black-Scholes implicit volatility patterns inferred from observed European option prices will be symmetric in moneyness* $\ln(X/F)$ *for distributional hypotheses satisfying equation* (8); e.g., *Black-Scholes, stochastic volatility models with zero correlation between asset and volatility shocks, jump-diffusions with mean-zero jumps, and Ornstein-Uhlenbeck stochastic interest rates.*

Proof: Since Black-Scholes option prices satisfy (8), any other model satisfying (8) must have identical implicit volatilities for calls and puts with strike prices such that $\ln(X_c/F) \;=\; \ln k \;=\; -\ln(X_p/F)$ for k>0. By put-call parity, European calls and puts of identical strike prices and maturities must have identical implicit volatilities. ■

The Black-Scholes and Ornstein-Uhlenbeck stochastic interest rate models imply identical implicit volatilities for all strike prices of a given maturity. The benchmark stochastic volatility and jump-diffusion models are leptokurtic, and imply U-shaped implicit volatility patterns or “volatility smiles” that are perfectly symmetric in $\ln(X/F)$. Conversely, implicit volatility patterns will be a “tilted” U-shape for non-benchmark stochastic volatility and jump-diffusion models. Implicit volatilities from ITM calls/OTM puts will be higher than those from correspondingly OTM calls/ITM puts for stochastic volatility models with $\rho_{s\sigma} \;<\; 0$ and jump-diffusions with $\bar{k}* < 0$, and lower when the relevant parameters are positive.

I.C. CEV processes

Both arithmetic and geometric Brownian motions are special cases of the broader class of *constant elasticity of variance* (CEV) diffusions, with risk-neutral representation of the form

$$dS \;=\; b\,S\,dt \;+\; \sigma S^{\rho}\, dZ \tag{18}$$

where $2(\rho - 1)$ is the elasticity of the instantaneous conditional variance of asset returns with respect to the asset price. Setting $\rho=0$ yields arithmetic Brownian motion, while $\rho=1$ yields geometric Brownian motion. This generalization of simpler models by Cox and Ross (1976) and Cox and Rubinstein (1985) was motivated by Black's (1976) observation that volatility appeared to be inversely related to stock prices -- a phenomenon Black suggested could be partly but not entirely attributed to financial and/or operating leverage. Financial leverage effects discussed in Christie (1982) imply ρ is between 0 and 1, depending upon the debt/equity ratio. For such intermediate values of ρ, there is a relationship between OTM calls and puts intermediate between those of Propositions 1 and 2.

Lemma: *A European call with CEV parameter* $\rho<1$ *corresponds to a European put with CEV parameter* $\rho^* = 2 - \rho > 1$, *cost of carry* $b^* = -b$, *and a modified volatility parameter*:

$$c(F,\ T;\ Fk,\ \sigma,\ \rho,\ b) \;=\; k\ p(F,\ T;\ F/k,\ \sigma F^{2(1-\rho^*)},\ \rho^*,\ b^*)\,, \tag{19}$$

where the arguments are the forward price, the time to maturity, the strike price, the volatility parameter, the CEV parameter, and the cost of carry, respectively.

Proof: See appendix.

Proposition 3 (CEV processes): *If the risk-neutral asset price follows the constant elasticity of variance process* (18)*, then call and put prices are related by*

$$c(F,\ T;\ Fk,\ \sigma k^{1-\rho},\ \rho,\ b) \;=\; k\ p(F,\ T;\ F/k,\ \sigma,\ \rho,\ b)\ \text{for}\ k>0\ \text{and any}\ \rho, \tag{20}$$

and satisfy the following inequalities:

$$c(F,\ T;\ F+x,\ \sigma,\ \rho,\ b) \;<\; p(F,\ T;\ F-x,\ \sigma,\ \rho,\ b) \quad \text{for}\ 0<x<F\ \text{only if}\ \rho<0 \tag{21}$$

$$c(F,\ T;\ F+x,\ \sigma,\ \rho,\ b) \;>\; p(F,\ T;\ F-x,\ \sigma,\ \rho,\ b) \quad \text{for}\ 0<x<F,\ \rho>0 \tag{22}$$

$$c(F,\ T;\ Fk,\ \sigma,\ \rho,\ b) \ < \ k\ p(F,\ T;\ F/k,\ \sigma,\ \rho,\ b) \quad for\ k>1,\ \rho<1 \tag{23}$$

$$c(F,\ T;\ Fk,\ \sigma,\ \rho,\ b) \ > \ k\ p(F,\ T;\ F/k,\ \sigma,\ \rho,\ b) \quad for\ k>1,\ \rho>1, \tag{24}$$

with inequalities reversed for $x<0$ *and* $k < 1$ (*in-the-money options*).

Proof: Expression (20) is proved in Appendix I for $\rho<1$ via manipulation of the CEV option pricing formula. That it also holds for $\rho>1$ follows from the lemma. (23) and (24) follow from (20), since option prices are increasing functions of the volatility parameter σ. (22) also follows from (20) for $\rho>1$ (since $F/(1+x) > F(1\text{-}x)$ and $p(\cdot)$ is increasing in the strike price), and has been confirmed numerically for $0<\rho<1$. Computation also indicates that $\rho<0$ is a necessary but not a sufficient condition for OTM calls to be cheaper than OTM puts. An additional condition that ensures the inequality in (21) is that the probability of hitting the absorbing barrier at $S=0$ be small.[14] Figure 4 illustrates the impact on relative option prices of varying ρ. ■

Other more explicit models of financial leverage include Geske's (1979) *compound option* model and Rubinstein's (1983) *displaced diffusion* model. Geske's model recognizes that debt implies that the underlying stock is itself a call option on the value of a firm. If the firm's value follows geometric Brownian motion, as in Geske (1979), then equity options are similar to a CEV model with $0<\rho<1$ and satisfy (22) and (23). Rubinstein (1983) models the firm as a portfolio of geometric Brownian motion investments and riskless assets, where the latter may be negative. Rubinstein's model is similar to a CEV model with $0<\rho<1$ when the firm is a net debtor, and also satisfies (22) and (23). When the firm is a net holder of riskless assets, the displaced diffusion option model is similar to a CEV model with $\rho>1$, and satisfies (24).[15]

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[14]For $\rho<0$ the volatility of asset price increments rises as the asset price falls, which tends to induce negative skewness. This can, however, be more than offset by the absorbing barrier at $S=0$, which truncates the lower tail of the distribution.

[15]The above assertions for the compound and displaced diffusion option pricing models are based upon computation of the respective option pricing formulas.

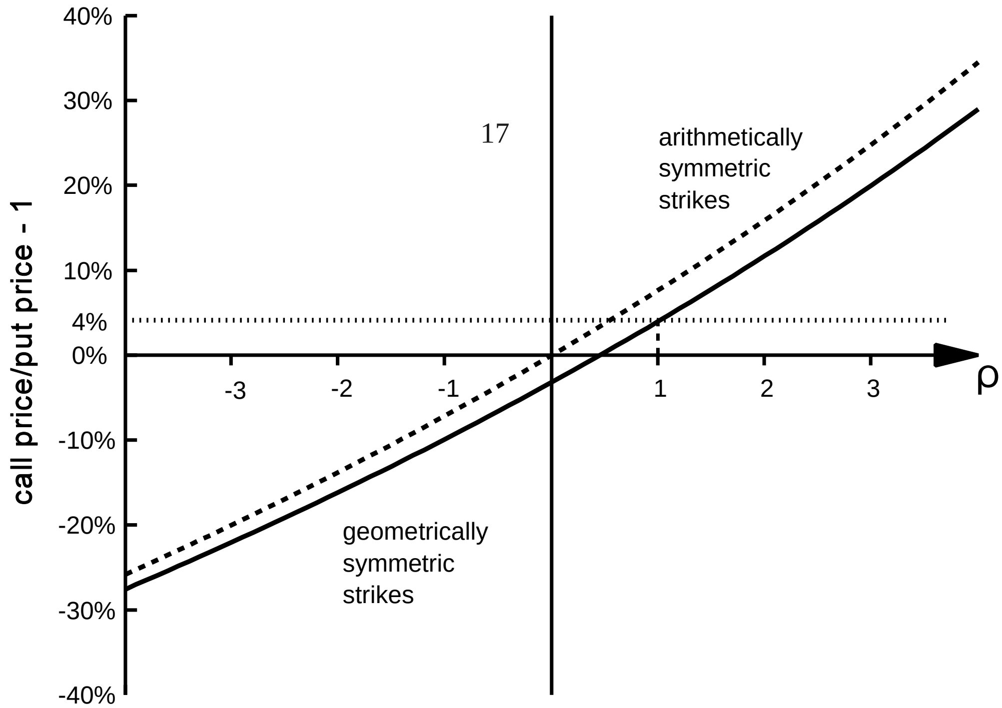

Figure 4. Relative prices of 4% OTM three-month European call and put options, as a function of the CEV parameter ρ. Cost of carry $b = 0$. Geometrically symmetric strikes: $X_{call}/F = 1.04 = F/X_{put}$; arithmetically symmetric: $X_{call}/F = 1.04$, $X_{put}/F = .96$.

Equations (23) and (24) can of course be translated into the corresponding implicit volatility patterns across different strike prices. An elasticity parameter $\rho > 1$ induces more positive skewness than the lognormal, and implies higher implicit volatilities for OTM calls/ITM puts with $\ln(X/F) > 0$ than for ITM calls/OTM puts with $\ln(X/F) < 0$. An elasticity parameter $\rho < 1$ has the opposite effect.

Equations (21) and (22) have no simple parallels in implicit volatility patterns, and are important diagnostics of whether leverage models can or cannot explain empirically observed moneyness biases. Thus, Rubinstein's (1985) and Sheikh's (1991) claim that volatility patterns in stock and stock index options are at times consistent with leverage models is based essentially upon option prices satisfying (23). Whether they also satisfy (22) was not tested. Daily estimates of implicit CEV parameters by MacBeth and Merville (1980) and Emmanuel and MacBeth (1982) often lay well outside the [0, 1] range required by standard leverage models.

II. Asymmetry and American Option Pricing

The relationships derived above for European out-of-the-money relative call and put prices will not hold for American OTM call and put prices in general. The problem is that the "early-exercise premium" markup of

American over European option prices is affected primarily by the cash flows of the underlying asset, and only secondarily by the skewness of the risk-neutral distribution. For example, an American option on a non-dividend paying stock will never be exercised early regardless of what the underlying distribution is, whereas it will be optimal to exercise an American put early if the asset price falls low enough. Since in this case American call prices equal European call prices while American put prices exceed European put prices, any distribution-specific relationship between prices of European calls and puts on non-dividend paying stocks cannot also hold exactly for the corresponding American prices. The relationships may hold approximately, however, for out-of-the-money American options with negligible early exercise premiums.

The influence of the cash flows of the underlying asset upon the markup of American over European prices is captured in the cost of carry parameter b. Significantly positive values of b (e.g., the $b = r$ case of a non-dividend paying stock) imply *ceteris paribus* a large markup for puts and small markup for calls, while significant negative values imply the reverse. An illustration of this is in Shastri and Tandon (1986), who examine the difference between American and European call and put prices under geometric Brownian motion for foreign currency options, for which the cost of carry $b = r - r^*$ can be positive or negative.

In the case of options on futures, the fact that the cost of carry equals zero creates a knife-edge case in which the markup for American over European call options parallels the markup for corresponding put prices. With zero cost of carry, the risk-neutral process used in pricing options becomes a martingale, and the early-exercise decision for calls and for puts is as symmetric or log-symmetric as the terminal (European) valuation problem. Consequently, Propositions 1-3 for European options with arbitrary cost of carry will also hold for American options on futures. Furthermore, relationships between the optimal early-exercise policies of American calls and puts can be derived under the various distributional hypotheses.

Proposition 4: *For American options on futures when interest rates are nonstochastic*:

1) *If the risk-neutral futures price follows arithmetic Brownian motion with no absorbing barrier then*

$$C(F,\ T;\ F+x) \quad = \quad P(F,\ T;\ F-x) \quad \textit{for}\ |x| < F\ \textit{and any}\ T, \tag{25}$$

and the critical, maturity-dependent futures price/exercise price <u>*differentials*</u> *above (below) which the call (put) will be exercised immediately are symmetric*:

$$(F - X)^*_c \quad = \quad -(F - X)^*_p\ . \tag{26}$$

2) *If the risk-neutral distribution of the futures price can be represented by*

a) *geometric Brownian motion*,
b) *a stochastic volatility process with independent evolution of volatility*, ***or***
c) *a "log-symmetric" jump-diffusion with* $\bar{k}^* = 0$,

then

$$c(F,\ T;\ Fk) \quad = \quad k\, p(F,\ T;\ F/k) \quad \textit{for}\ k > 0\ \textit{and any}\ T, \tag{27}$$

and the critical, maturity- and state-dependent futures price/exercise price <u>*ratios*</u> *above* (*below*) *which the call (put) will be exercised immediately are geometrically symmetric*:

$$(F/X)^*_c \quad = \quad 1/(F/X)^*_p\ . \tag{28}$$

Furthermore, implicit volatility patterns inferred using an American futures option pricing variant of Black-Scholes will be perfectly symmetric functions of moneyness $\ln(X/F)$.

3) *If the risk-neutral futures price follows a CEV process, then*

$$C(F,\ T;\ Fk,\ \sigma k^{1-\rho},\ \rho,\ b) \quad = \quad k\ P(F,\ T;\ F/k,\ \sigma,\ \rho,\ b)\ \textit{for}\ k > 0\ \textit{and}\ 0 \le \rho < 2 \tag{29}$$

and the following inequalities are satisfied:

$$C(F,\ T;\ F+x,\ \sigma,\ \rho,\ b) \quad < \quad P(F,\ T;\ F-x,\ \sigma,\ \rho,\ b) \quad \textit{for}\ 0 < x < F\ \textit{only if}\ \rho < 0 \tag{30}$$

$$C(F,\ T;\ F+x,\ \sigma,\ \rho,\ b) \quad > \quad P(F,\ T;\ F-x,\ \sigma,\ \rho,\ b) \quad \textit{for}\ 0 < x < F,\ 0 < \rho < 2 \tag{31}$$

$$C(F,\ T;\ Fk,\ \sigma,\ \rho,\ b) \ < \ k\ P(F,\ T;\ F/k,\ \sigma,\ \rho,\ b) \quad \textit{for}\ k>1,\ 0<\rho<1 \tag{32}$$

$$c(F,\ T;\ Fk,\ \sigma,\ \rho,\ b) \ > \ k\ p(F,\ T;\ F/k,\ \sigma,\ \rho,\ b) \quad \textit{for}\ k>1,\ 1<\rho<2, \tag{33}$$

with inequalities reversed for $x<0$ and $k < 1$ (in-the-money options).

Proof: The key insight underlying the proofs for arithmetic and geometric Brownian motion, stochastic volatility, and jump-diffusions is that the American put valuation problem for these distributions is the same under a distribution-specific transformation of variables as the American call valuation problem; see the appendix for details. That the results for European CEV options also apply to American options on futures has been confirmed numerically over the parameter range $0 \leq \rho < 2$ and in regions of $\rho < 0$ where bankruptcy risk is negligible.[16] The appendix also shows that the CEV lemma (19) also holds for American options on futures:

$$C(F,\ T;\ Fk,\ \sigma,\ \rho,\ b) \ = \ k\ P(F,\ T;\ F/k,\ \sigma F^{2(1-\rho^*)},\ \rho^*,\ b^*), \tag{34}$$

where $\rho^* = 2 - \rho > 1$ and $b^* = -b = 0$. ■

One corollary of Proposition 4 is that for these distributions, at-the-money American calls and puts on futures should be priced identically. This is an interesting theoretical result given that there are no arbitrage-based restrictions equating prices of at-the-money American calls and puts similar to those for European options -- only rather weak inequality constraints discussed in Stoll and Whaley (1986) limiting how far they can deviate. The results do depend upon the assumption of nonstochastic interest rates, however. It does not appear that the results

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[16]American CEV option prices were calculated using a variant of the Cox-Rubinstein (1985) binomial option pricing methodology. Option prices were expressed in terms of the transformed state variable

$$Z \equiv (F^{1-\rho}-1)/(1-\rho),$$

which follows a process with state-dependent drift but with <u>constant</u> conditional volatility, thereby permitting recursive option evaluation using a grid of evenly spaced Z-values. Option prices could be calculated only for ρ between 0 and 2. Boundary conditions precluded option evaluation outside this range, except when a low probability of hitting the boundary permitted truncation of the Z domain away from the boundary. Hull and White (1990) and Nelson and Ramaswamy (1990) discuss this transformation.

for European option prices under Ornstein-Uhlenbeck interest rate processes can be generalized to American futures options, except under excessively stringent conditions.

The above propositions can be used to construct a measure of asymmetry which, by analogy with the term premium, I call the "skewness premium."

Definition: The $x\%$ *skewness premium* is defined as the percentage deviation of $x\%$ out-of-the-money calls from $x\%$ out-of-the-money puts:

$$SK(x) \equiv c(F, T; X_c) / p(F, T; X_p) - 1 \quad \text{for European options in general}$$

$$SK(x) \equiv C(F, T; X_c) / P(F, T; X_p) - 1 \quad \text{for American futures options.}$$

where F is the forward price on the underlying asset. when options are European, or the underlying futures price for American futures options.[17] Two variants of the skewness premium are defined, depending on whether the strikes are symmetrically or geometrically symmetrically spaced around the forward price:

$$SK1(x): \quad X_p = F(1 - x) < F < X_c = F(1+x), \quad x>0.$$

$$SK2(x): \quad X_p = F/(1+x) < F < X_c = F(1+x), \quad x>0.$$

The skewness premiums are functions most importantly of the moneyness parameter x. From Propositions 1-4, the skewness premiums have the following properties for $x > 0$ regardless of the maturity of the options:

1) $SK1(x) < 0$ for CEV processes only if $\rho < 0$;

2) $SK1(x) = 0$ for arithmetic Brownian motion processes with no absorbing barrier;

3) $0 \leq SK1(x)$ for CEV processes with $0 \leq \rho$;

[17] Although there are relatively few markets in which European options are traded, there are circumstances in which European option pricing formulas are useful. For example, American call options on non-dividend paying stocks are priced identically to European calls. For such options, one could measure the skewness of the underlying distribution by comparing OTM call prices with the OTM European put prices imputed via put-call parity from the in-the-money call prices.

4) $SK2(x) < x$ for CEV processes with $\rho < 1$;

5) $SK2(x) = x$ for

a) geometric Brownian motion processes,
b) stochastic volatility processes with independent evolution of volatility ($\rho_{s\sigma}=0$)
c) jump-diffusions with a "log-symmetric" risk-neutral representation ($\bar{k}^* = 0$)

6) $SK2(x) > x$ for CEV process with $\rho > 1$.

For the non-benchmark jump-diffusion processes, numerical methods[18] indicate the following:

7) $SK2(x) \gtrless x$ for jump-diffusions with log-normal jumps depending on whether $\bar{k}^* \gtrless 0$.

Based upon results for European options, it is conjectured that a similar result holds for American futures options on stochastic volatility processes when volatility does not evolve independently of the asset price:

8) $SK2(x) \gtrless x$ for stochastic volatility processes depending on whether $\rho_{s\sigma} \gtrless 0$.

Results 1 through 8 are summarized in Figure 5.

$SK2$ is strictly less than $SK1$ for nonzero x given the higher put strike price $X_p = F/(1+x) = F(1-x)/(1-x^2)$; see Figure 4. However, the two skewness premium measures will be approximately equal for small x, implying results 1-8 will apply approximately to both measures. Consequently, for European options with arbitrary cost of carry and for American options on futures, calls $x\%$ out-of-the-money should be priced roughly between 0% and $x\%$ higher than puts $x\%$ out-of-the-money for the standard distributional hypotheses (categories 2 to 5) <u>regardless</u> of the maturity of the options. For in-the-money options ($x<0$), the propositions are reversed. Calls $x\%$ in-the-money should cost 0% - $x\%$ <u>less</u> than puts $x\%$ in-the-money under standard distributional hypotheses.

[18]Methods for pricing American options on jump-diffusion processes are discussed in Bates (1988, 1991).

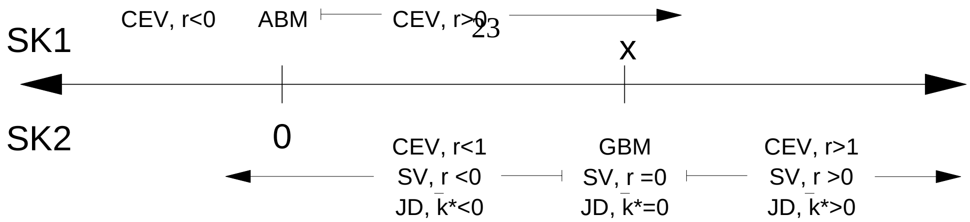

Figure 5. Skewness Premiums for Alternate Distributions and Parameter Values.

III. Options on S&P 500 futures, 1983-1993

To illustrate observed skewness premiums and the magnitude of post-1987 moneyness biases, transactions data for call and put options on S&P 500 futures and for the underlying futures contracts were obtained from the Chicago Mercantile Exchange from the options' inception on January 28, 1983 through December 31, 1993. A subset of the data was used, based on the criteria of Bates (1991,1996).[19]

Since options exist only for specific exercise prices, the skewness premium measure of skewness cannot be implemented directly. For each OTM call with exercise price $x\%$ above the futures price, there will not necessarily exist a corresponding OTM put with exercise price exactly $x\%$ below the futures price. Alternatively, there may be many puts roughly $x\%$ out-of-the-money. However, theoretical distributions and no-arbitrage conditions imply that options prices are continuous, monotone, and convex functions of the exercise price. Options prices for desired exercise prices were therefore interpolated from a constrained cubic spline fitted through the ratio of options prices to futures prices, as a function of the exercise price/futures price ratio X/F. Kuhn-Tucker constraints on the cubic

[19]Only those maturities meeting the following criteria were selected:

1) March, June, September, and December delivery dates;
3) 1-4 month (28-118 day) maturities;
4) At least 4 call strikes and 4 put strikes traded per maturity;
5) At least 20 call transactions and 20 put transactions per maturity.

Only actual transactions (no bids or asks) with relatively recent underlying futures transactions were used.

spline coefficients ensured that each spline was convex, was monotone with slope (in absolute value) between 0 and 1, and exceeded the immediate-exercise values.[20]

Skewness premiums from January 31, 1983 to December 31, 1993 were generated as the percentage difference between interpolated call and put prices using options 2% in-the-money, at-the-money and 2%, 4% and 6% out-of-the-money:

$$\hat{SK}(x) = [\hat{C}(x) - \hat{P}(x)] / \hat{P}(x)$$

with estimated standard error $\sigma_{SK} = (\hat{C}/\hat{P})[(\hat{\sigma}_C/\hat{C})^2 + (\hat{\sigma}_P/\hat{P})^2]^{½}$ computed from the spline estimations under the assumption that the interpolation error for calls is uncorrelated with the error for puts. Deep in-the-money skewness premiums are not reported. The major trading was in out-of-the-money options, and deep ITM calls and puts were rarely trading concurrently.[21]

For testing the skewness premium relative to benchmark hypotheses, two test statistics were constructed:

1. $T_1 = \hat{C}(x) - \hat{P}(x)$ for $X_{call}/F = 1+x$, $X_{put}/F = 1-x$ (test of symmetry)
2. $T_2 = \hat{C}(x) - (1+x)\hat{P}(x)$ for $X_{call}/F = 1+x = 1/(X_{put}/F)$ (test of x% rule).

with associated standard errors $[\hat{\sigma}^2_C(x) + \hat{\sigma}^2_P(x)]^{½}$ and $[\hat{\sigma}^2_C(x) + (1+x)^2\hat{\sigma}^2_P(x)]^{½}$ respectively. The former tests for positive or negative skewness; the latter tests for more or less skewness than those distributions (such as the lognormal) satisfying the x% rule.

[20]Details of the estimation of constrained cubic splines and interpolation error are available from the author on request.

[21]There were only 576 days out of 2258 in which 4% ITM calls and puts were traded simultaneously.

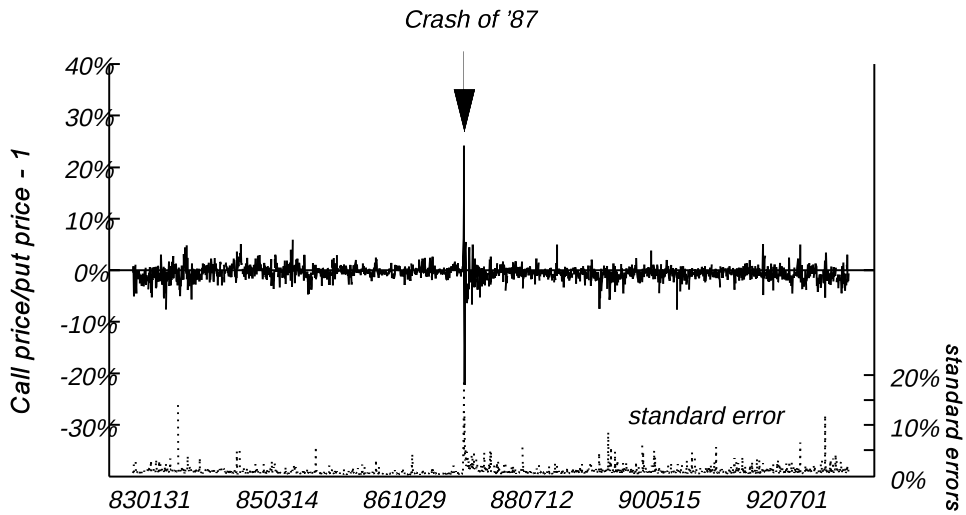

Figure 6. Percentage Deviation of ATM Call from ATM Put Prices, 1983-93.

The resulting skewness premiums and associated standard errors for options at-the-money and 4% out-of-the-money are shown in Figures 6 and 7, while Table I summarizes the overall results. As noted above, a corollary of Proposition 4 is that at-the money American futures options should be priced identically for all distributions considered, yielding a skewness premium of 0%. Figure 6 and Table I confirm that deviations from this theoretical value are small, and are well within the rather large no-arbitrage bounds of Stoll and Whaley (1986).[22] However, the deviations from 0% are statistically significant at the 2% level for 15% of the 2,198 days for which the 0% skewness premium could be measured.

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[22]For at-the-money options, Stoll and Whaley's bound can be written as $|C - P| \leq F(1 - e^{-rT})$, or

$$|SK| = |C/F - P/F| \; / \; (P/F) \quad \leq \quad (1 - e^{-rT}) / (P/F).$$

Using the American $\approx$ European option price approximation ($P \approx p$), the at-the-money Black option approximation $p/F \approx e^{-rT}\sigma\sqrt{T/2\pi}$, and the approximation $e^{rT} - 1 \approx rT$ yields an approximate bound of $(r/\sigma)\sqrt{2\pi T}$, where σ is the implicit volatility for an at-the-money option. For $r = 5\%$ and $\sigma = 10\%$, the arbitrage-based bounds on the at-the-money skewness premium for one-month options are roughly $\pm 24\%$.

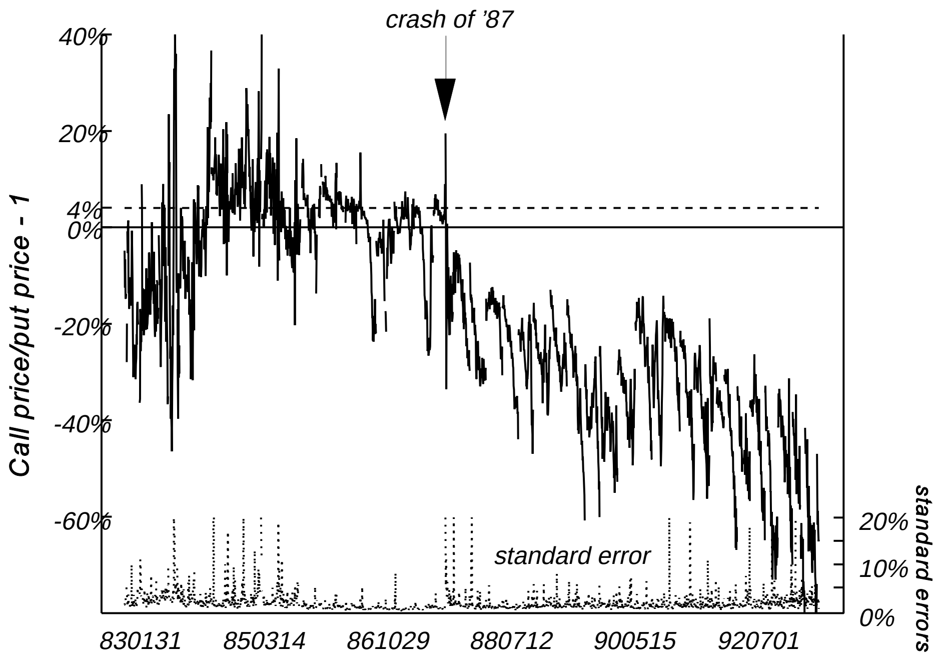

Figure 7. Percentage Deviation of 4% OTM Call from 4% OTM Put Prices, 1983-93.

The 4% skewness premiums shown in Figure 7 indicates that the 1983-1993 period can be essentially divided into four separate regimes. Over 1983-1984 and after the crash of 1987, OTM puts have been substantially more expensive than OTM calls, leading to substantially negative skewness premiums. Over 1985, by contrast, skewness premiums were predominantly positive, while the behavior over 1986-1987 was more mixed and has been discussed in Bates (1991). Interestingly, the post-crash moneyness biases were initially on the same order of magnitude as the those observed in the first two years the options were traded. The 1991-1993 decline in implicit volatilities shown in Figure 8 was accompanied by increasingly negative skewness premiums.

The behavior of skewness premiums for 2% and 6% OTM options was essentially identical and are consequently not shown. As a rule of thumb, the 2% (6%) OTM premiums are typically about ½ (1½) times the 4% OTM premiums. As one would expect, ITM premiums behave in reverse fashion to the OTM premiums. It should be clear from Figure 7 and Table I that observed skewness premiums for S&P 500 futures options typically

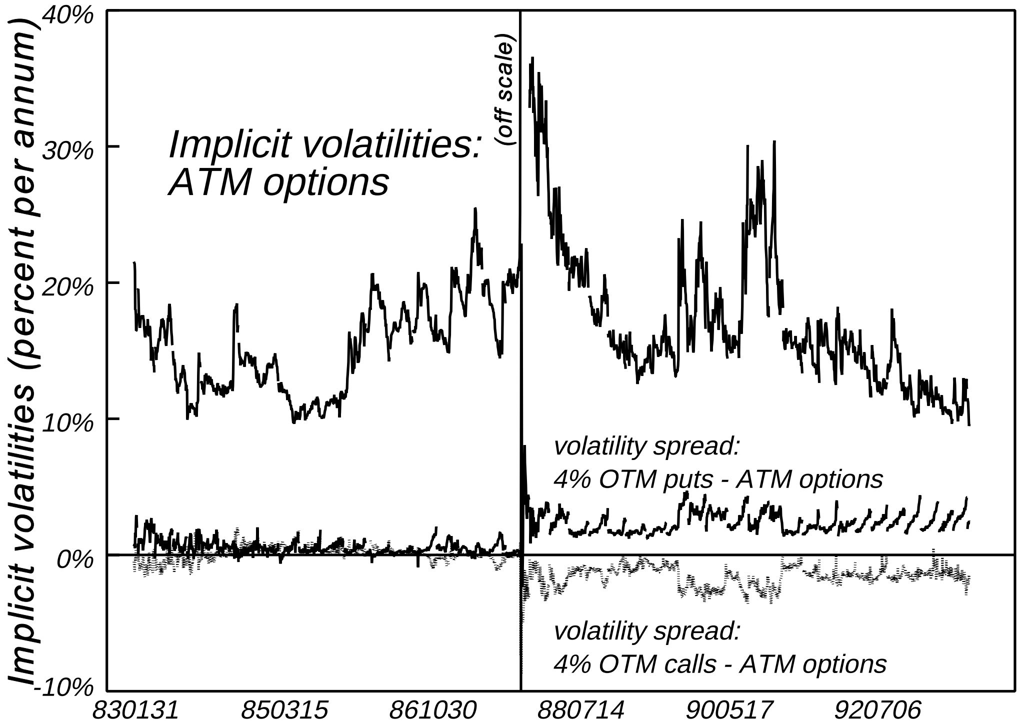

Figure 8. Implicit volatilities for ATM options, and the difference in implicit volatilities between 4% OTM and ATM options. Volatilities were inferred from interpolated option prices of 1-4 month maturities using Barone-Adesi and Whaley's (1987) American option pricing formula. ATM implicit volatilities are the average from ATM calls and puts.

fall well outside the narrow $[0\%, x\%]$ range implied by benchmark distributional hypotheses -- and that these deviations are too large to be attributed to interpolation error.

IV. Conclusions and Extensions

This paper has presented a "skewness premium" metric for readily identifying moneyness biases in options prices, and for judging which distributional hypotheses are and are not consistent with those biases. It was shown that the major distributional hypotheses considered hitherto -- geometric Brownian motion, standard CEV processes and other leverage models, standard symmetric stochastic volatility and jump-diffusion processes -- imply that call options $x\%$ out-of-the-money should be priced only 0% - $x\%$ more than put options $x\%$ out-of-the-money. An application to American options on S&P 500 futures revealed the biases to be substantially, significantly, and persistently outside (and typically below) this $[0\%, x\%]$ range, especially during the early trading years (1983-1984) and in the post-crash period (1988-1993). Negatively skewed distributions that are potentially consistent with the magnitude of the post-crash biases include stochastic volatility processes with a large negative correlation between volatility and market shocks, and jump-diffusion processes with negative-mean jumps. The skewness premium cannot identify which process and which parameter values would best fit observed option prices, but does provide an easily implementable preliminary diagnostic for narrowing the field.

The focus in this paper has been on using American and European option prices to infer the correct distribution. The key observation that the put valuation problem is a distribution-specific transformation of the call valuation problem is, however, more general than these particular contracts. This transformation of variables property is intrinsic to the partial differential equations used in pricing *all* derivatives, including exotics. Any other pairs of derivative contracts possessing a comparable call/put-type isomorphic mapping between boundary conditions will satisfy the same distribution-specific pricing relationships derived above for calls and puts. Knowing these relationships can be useful in pricing exotic options.

An obvious example is Bermudan call and put options on futures that can be exercised only on specific dates, and which satisfy the same pricing relationships as their European and American counterparts. A less obvious pair is cash-or-nothing binary put options that pay off X if and only if the asset price finishes below X, and asset-or-nothing binary calls that deliver the underlying asset if and only if the asset price finishes above X. An $x\%$ rule pricing relationship holds between the two under benchmark distributional hypotheses, yielding an asset-or-nothing binary call option pricing formula[23]

$$c^{AoN}(F,\ T,\ X) \ = \ e^{-rT} F \ Prob^{*}(S_T < F^2/X) \tag{35}$$

[23]It is straightforward to show that the terminal boundary conditions for the above asset-or-nothing calls and cash-or-nothing puts satisfy the the same mapping $c^{AoN}(y,\ 0;\ 1) = y\,p^{CoN}(y^{-1},\ 0;\ 1)$ as standard European calls and puts. Consequently, $c^{AoN}(F,\ T;\ Fk) = k\,p^{CoN}(F,\ T;\ F/k) = k\,e^{-rT}(F/k)\ Prob^{*}[S_T < F/k]$ under "log-symmetric" distri-butions, where $Prob^{*}(\cdot)$ is the risk-neutral distribution function. Using $k \equiv X/F$ yields (35). The standard geometric Brownian motion formula $c^{AoN} = e^{-rT}F\,N(d_1)$ for $d_1 = [\ln(F/X) + \tfrac{1}{2}\sigma^2 T] \,/\, \sigma\sqrt{T}$ is a special case.

that is valid under the "log-symmetric" distributions: geometric Brownian motion, stochastic volatility processes with uncorrelated asset and volatility shocks, and jump-diffusions with mean-zero jumps. A third example is Carr, Ellis and Gupta's (1996) ingenious use of Proposition 2 to show that down-and-in barrier calls on futures triggered by hitting barrier $H < X$ are priced proportionately to ordinary European puts on futures under geometric Brownian motion:[24]

$$c^{D\&I}(F,\ T;\ X,\ H) \ = \ \frac{X}{H} p(F,\ T;\ H^2/X). \tag{36}$$

The relationship also holds for Hull and White (1987)-type stochastic volatility processes, but not apparently for jump-diffusions.

[24]Under geometric Brownian motion and using Proposition 2, a down-and-in call on futures with strike $X > H$ is worth $c(F_t = H,\ T-t;\ X) = X/H\, p[H,\ T-t;\ H/(X/H)]$ if the futures price hits the triggering barrier at H at time t. Buying X/H puts with strike price H^2/X and switching to calls upon hitting the barrier is therefore a self-financing replicating strategy. The puts expire out-of-the-money if the barrier is never hit, as does the down-and-in call.

Appendix: Properties of Homogeneous Option Prices

Option prices under geometric Brownian motion, standard stochastic volatility specifications, and Merton (1976)-style jump diffusions are homogeneous of degree 1 in the futures price F and the strike price X:

$$\begin{aligned} O(F,\ T;\ X) &= X\,O\left(\frac{F}{X},\ T;\ 1\right) \\ &\equiv X\,O(y,\ T;\ 1) \end{aligned} \qquad \textbf{(A.1)}$$

where y is the standardized state variable F/X and $O(y,\ T;\ 1) = O(F,\ T;\ X)\ /\ X$ is the standardized option price. Similarly, CEV option prices are homogeneous of degree one in F, X, and $\sigma^{\frac{1}{1-\rho}}$ for $\rho \neq 1$:

$$\begin{aligned} O(F,\ T;\ X,\ \sigma,\ \rho,\ b) &= X\,O\left(\frac{F}{X},\ T;\ 1,\ \frac{\sigma}{X^{1-\rho}},\ \rho,\ b\right) \qquad \textbf{(A.2)} \\ &\equiv X\,O(y,\ T;\ 1,\ \sigma',\ \rho,\ b) \end{aligned}$$

where $\sigma' = \sigma / X^{1-\rho}$.

Standardized European call and put prices satisfy the terminal boundary conditions

$$c(y,\ 0;\ 1) = \max(y - 1,\ 0) \qquad \textbf{(A.3)}$$

$$p(y,\ 0;\ 1) = \max(1 - y,\ 0)\ , \qquad \textbf{(A.4)}$$

since the futures price equals the spot price at maturity. Standardized American call and put prices on *futures* must also satisfy the early-exercise boundary conditions

$$\begin{cases} C(y_c^*(T),\ T;\ 1) = y_c^*(T) - 1 > 0 \\ C_y(y_c^*(T),\ T;\ 1) = 1 \end{cases} \qquad \textbf{(A.5)}$$

$$\begin{cases} P(y_p^*(T),\ T;\ 1) = 1 - y_p^*(T) > 0 \\ P_y(y_p^*(T),\ T;\ 1) = -1 \end{cases} \qquad \textbf{(A.6)}$$

where y_c^* (y_p^*) is the critical maturity-dependent early-exercise ratio F/X above (below) which the call (put) is exercised immediately. For stochastic volatility models, there are also smooth-pasting conditions relative to the additional state variable, and the critical early-exercise ratios depend also upon the level of volatility.

I. Properties of CEV option prices

1. $\rho < 1$ ***versus*** $\rho > 1$ **(**expressions (19) and (34)):

Define the CEV differential operator

$$\boldsymbol{D}^{\sigma', \rho, b} \equiv -\frac{\partial}{\partial T} + \tfrac{1}{2}\left[\sigma' e^{bT(1-\rho)}\right]^2 y^{2\rho} \frac{\partial^2}{\partial y^2} . \qquad \textbf{(A.7)}$$

Standardized CEV option prices satisfy the partial differential equation $\boldsymbol{D}^{\sigma', \rho, b}\, O(y, T; 1, \sigma', \rho, b) = rO$.

Define $h(y, T; 1, \sigma', \rho^*, b^*) \equiv y\, c(y^{-1}, T; 1, \sigma', \rho, b)$ where $\rho^* \equiv 2\text{-}\rho$ and $b^* \equiv -b$. Since $h_T = yc_T$ and $h_{yy} = y^{-3}c_{yy}$, substitution confirms that $\boldsymbol{D}^{\sigma', \rho^*, b^*} h = rh$ given $\boldsymbol{D}^{\sigma', \rho, b} c(y, T; 1) = rc$. Furthermore, h satisfies (A.4) given $c(y, T; 1)$ satisfies (A.3). Therefore, h is the standardized put price for parameters ρ^* and b^*:

$$p(y, T; 1, \sigma', \rho^*, b^*) = h(y, T; 1, \sigma', \rho^*, b^*) = y\, c(y^{-1}, T; 1, \sigma', \rho, b) \qquad \textbf{(A.8)}$$

Exploiting the homogeneity of p and c, (A.8) can be rewritten as

$$y\, p(1, T; y^{-1}, \sigma'(y^{-1})^{1-\rho^*}, \rho^*, b^*) = c(1, T; y, \sigma' y^{1-\rho}, \rho, b) \qquad \textbf{(A.9)}$$

or

$$y\, p(1, T; y^{-1}, \sigma'', \rho^*, b^*) = c(1, T; y, \sigma'', \rho, b) \qquad \textbf{(A.10)}$$

where $\sigma'' \equiv \sigma' y^{1-\rho} = \sigma' y^{-(1-\rho^*)}$. Multiplying both sides by F yields

$$y\, p(F, T; F/y, \sigma'' F^{1-\rho^*}, \rho^*, b^*) = c(F, T; Fy, \sigma'' F^{1-\rho}, \rho, b) \qquad \textbf{(A.11)}$$

and redefining $\sigma \equiv \sigma'' F^{1-\rho}$ yields (19). ■

A similar relationship and proof applies to American CEV options on futures ($b = b^* = 0$). It is straightforward to confirm that $H(y, T; 1, \sigma', \rho^*, 0) \equiv y\, C(y^{-1}, T; 1, \sigma', \rho, 0)$ and $y_p^* \equiv 1/y_c^*$ satisfy conditions (A.4) and (A.6) for standardized American put options on futures if $C(y, T; 1)$ and y_c^* satisfy conditions (A.3) and (A.5) for standardized American calls on futures. ■

2. Relationships between OTM calls/ITM puts

Proposition:

$$c(y,\ T;\ 1,\ \sigma,\ \rho,\ b) \ = \ p(1,\ T;\ y,\ \sigma,\ \rho,\ b) \quad \textit{for}\ \rho < 1. \qquad \textbf{(A.12)}$$

Proof: Cox and Rubinstein's (1985) European CEV call option pricing formula for $\rho < 1$ is

$$c(F,\ T;\ X) \ = \ e^{-rT}\left[F \sum_{n=1}^{\infty} g(n,\ f)\, G(n+\lambda,\ x) \ - \ X \sum_{n=1}^{\infty} g(n+\lambda,\ f)\, G(n,\ x)\right] \qquad \textbf{(A.13)}$$

where $\lambda \equiv 1/[2(1-\rho)]$, $\theta \equiv 2\lambda b/[\sigma^2(e^{bT/\lambda}-1)]$, $f \equiv \theta F^{1/\lambda}$, $x \equiv \theta X^{1/\lambda}$, b is the cost of carry, and $\Gamma(\cdot)$, $g(\cdot)$, and $G(\cdot)$ are the gamma function, gamma density function, and complementary gamma distribution function, respectively:

$$g(n,\ z) \ \equiv \ \frac{e^{-z} z^{n-1}}{\Gamma(n)}, \quad G(n,\ z) \ \equiv \ \int_z^{\infty} g(n,\ x)\, dx\,. \qquad \textbf{(A.14)}$$

Using put-call parity,

$$\begin{aligned} c(y,\ T;\ 1) - p(1,\ T;\ y) \ &= \ c(y,\ T;\ 1) - c(1,\ T;\ y) - e^{-rT}(y-1) \\ &= \ e^{-rT}\Big[y \sum_{n=1}^{\infty} g(n,\ \theta y^{1/\lambda})\, G(n+\lambda,\ \theta) - \sum_{n=1}^{\infty} g(n+\lambda,\ \theta y^{1/\lambda})\, G(n,\ \theta) \\ &\quad - \sum_{n=1}^{\infty} g(n,\ \theta)\, G(n+\lambda,\ \theta y^{1/\lambda}) - y \sum_{n=1}^{\infty} g(n+\lambda,\ \theta)\, G(n,\ \theta y^{1/\lambda}) \ - \ (y-1)\Big] \end{aligned} \qquad \textbf{(A.15)}$$

Grouping terms pre-multiplied by y, and using the equalities $G(n, x) = \sum_{j=1}^{n} g(j,\ x)$ and $G(n\text{-}1+\lambda,\ x) + \sum_{j=n}^{\infty} g(j+\lambda, x) = 1$ yields after some manipulation the equalities

$$c(y,\ T;\ 1)\ -\ p(1,\ T;\ y) \ = \ e^{-rT} \sum_{n=1}^{\infty} \left[y\, g(n,\ \theta y^{1/\lambda})\, g(n+\lambda,\ \theta)\ -\ g(n,\ \theta)\, g(n+\lambda,\ \theta y^{1/\lambda}) \right]$$

$$= \ e^{-rT} e^{-\theta - y^{1/\lambda}} \sum_{n=1}^{\infty} \frac{\theta^{2n+\lambda-2}\, y^{(n-1)/\lambda} \left[y - (y^{1/\lambda})^{\lambda} \right]}{\Gamma(n)\, \Gamma(n+\lambda)} \qquad \textbf{(A.16)}$$

$$= \ 0. \qquad \blacksquare$$

Corollary #1 (Proposition 3 for ρ < 1):

$$c(F,\ T;\ Fk,\ \sigma k^{1-\rho},\ \rho,\ b) \ = \ k\ p(F,\ T;\ F/k,\ \sigma,\ \rho,\ b) \quad \textit{for}\ \ \rho < 1\ \ \textit{and any}\ \ k > 0. \qquad \textbf{(A.17)}$$

Proof: From (A.12),

$$Fk\ c(k^{-1},\ T;\ 1,\ \sigma F^{\rho-1},\ \rho,\ b) \ = \ Fk\ p(1,\ T;\ k^{-1},\ \sigma F^{\rho-1}\ \ \rho,\ b) \qquad \textbf{(A.18)}$$

which by homogeneity is equivalent to (A.17). ■

Corollary #2 (Proposition 3 for ρ > 1):

$$c(F,\ T;\ Fk,\ \sigma k^{1-\rho},\ \rho,\ b) \ = \ k\ p(F,\ T;\ F/k,\ \sigma,\ \rho,\ b) \quad \textit{for}\ \ \rho > 1\ \ \textit{and any}\ \ k > 0. \qquad \textbf{(A.19)}$$

Proof: For $\rho > 1$, define $\rho^* \equiv 2\text{-}\rho < 1$ and $b^* \equiv -b$. Then,

$$\begin{aligned} c(F,\ T;\ Fk,\ k^{1-\rho}\sigma,\ \rho,\ b) &= k\,p(F,\ T;\ F/k,\ (k^{1-\rho}\sigma)F^{2(1-\rho^*)},\ \rho^*,\ b^*) && \text{from (19)} \\ &= c(F,\ T;\ Fk,\ k^{1-\rho^*}[k^{1-\rho}\sigma F^{2(1-\rho^*)}],\ \rho^*,\ b^*) && \text{from (20)} \\ &= c(F,\ T;\ Fk,\ \sigma F^{2(1-\rho^*)},\ \rho^*,\ b^*) \\ &= k\,p(F,\ T;\ F/k,\ [\sigma F^{2(1-\rho^*)}]F^{2(1-\rho)},\ \rho,\ b) && \text{from (19)} \\ &= k\,p(F,\ T;\ F/k,\ \sigma,\ \rho,\ b). \end{aligned}$$

■

II. American option prices under geometric Brownian motion

Define the differential operator for standardized options under geometric Brownian motion,

$$\boldsymbol{D} \ \equiv \ -\frac{\partial}{\partial T} \ + \ \tfrac{1}{2}\sigma^2 y^2 \frac{\partial^2}{\partial y^2}\ . \qquad \textbf{(A.20)}$$

Define $H(y, T; 1) \equiv y\, C(y^{-1}, T; 1)$ Since $H_T = yC_T$, and $H_{yy} = y^{-3}C_{yy}$, direct substitution confirms that $\boldsymbol{D}h = rh$ given $\boldsymbol{D}C(y, T; 1) = rC$. Furthermore, H and $y_p^* \equiv 1/y_c^*$ satisfy (A.4) and (A.6) given $C(y, T; 1)$ and y^* satisfy (A.3) and (A.5). Therefore, H is the standardized put price:

$$P(y, T; 1) = H(y, T; 1) = y\, C(y^{-1}, T; 1). \tag{A.21}$$

Multiplying both sides by Fy^{-1} and exploiting homogeneity yields Proposition 4.2a. ■

III. Stochastic volatility with independent evolution of volatility

Same proof as above using the stochastic volatility differential generator

$$\boldsymbol{D} \equiv -\frac{\partial}{\partial T} + \frac{E^*(d\sigma)}{dt}\frac{\partial}{\partial \sigma} + \tfrac{1}{2}\sigma^2 y^2 \frac{\partial^2}{\partial y^2} + \tfrac{1}{2}\frac{Var(d\sigma)}{dt}\frac{\partial^2}{\partial \sigma^2}. \tag{A.22}$$

and the transformation of variables $H(y, \sigma, T) \equiv y\, C(y^{-1}, \sigma, T; 1)$ and $y_p^*(\sigma, T) \equiv 1/y_c^*(\sigma, T)$. An additional "smooth-pasting" condition $H_\sigma(y_p^*(\sigma, T), \sigma, T) = 0$ is met by H and $y_p^*(\sigma, T)$ given $C_\sigma(y_c^*(\sigma, T), \sigma, T) = 0$.

IV. Jump-diffusions with mean-zero jumps

In the jump-diffusion models given above, $C(y, T; 1)$ solves the general jump-diffusion equation

$$-V_T + \tfrac{1}{2}\sigma^2 y^2 V_{yy} + \lambda^* E^*[V(ye^J, T; 1) - V] = rV \tag{A.23}$$

for $V = C$ subject to call-specific boundary conditions (A.3) and (A.5). J is a random normal variable distributed $N(\gamma^* - \tfrac{1}{2}\delta^2, \delta^2)$ under the risk-neutral distribution. Mean-zero jumps implies $\gamma^* = 0$.

Define $H(y, T) \equiv y\, C(y^{-1}, T; 1)$. $H_T = yC_T$ and $H_{yy} = y^{-3}C_{yy}$, so

$$-H_T + \tfrac{1}{2}\sigma^2 y^2 H_{yy} + \lambda^* E^*[H(ye^J, T; 1) - H]$$

$$= -yC_T + \tfrac{1}{2}\sigma^2 y^{-1} C_{yy} + \lambda^* E^*[ye^J C(y^{-1}e^{-J}, T; 1) - yC] \qquad \textbf{(A.24)}$$

$$= y\left\{-C_T + \tfrac{1}{2}\sigma^2 y^{-2} C_{yy} + \lambda^* E^*[C(y^{-1}e^{+J}, T; 1) - C]\right\}$$

$$= y\, rC(y^{-1}, T; 1) = rH,$$

where the second equality follows from the property $E^*[e^J g(-J)] = E^* g(J)$ for $J \sim N(-\tfrac{1}{2}\delta^2, \delta^2)$ and arbitrary $g(\cdot)$. Since $H(y, T)$ solves (A.23), and H and $y_p^* \equiv 1/y_c^*$ solve put-specific boundary conditions given $C(y, T)$ and y_c^* satisfy the call-specific boundary conditions, $P(y, T; 1) = H(y, T)$. Prop. 4.2c then follows. ■

V. American option prices under Brownian motion with no absorbing barrier (Proposition 4.1)

It is straightforward to verify that under simple Brownian motion with no absorbing barrier at F=0, prices of American options on futures depend on the *difference* between the futures price and the exercise price:

$$O(F, T; X) = O(F - X, T; 0), \qquad \textbf{(A.25)}$$

where $x = F - X$ and O equals C or P. The relevant partial differential equation is

$$-O_T + \tfrac{1}{2}\sigma^2 O_{xx} = rO \qquad \textbf{(A.26)}$$

which for calls ($O = C$) is solved subject to the terminal condition

$$C(x, 0; 0) = \max(x, 0) \qquad \textbf{(A.27)}$$

and early-exercise smooth-pasting conditions

$$\begin{cases} C(x_c^*,\ T,\ 0) & = \ x_c^* > 0 \\ C_x(x_c^*,\ T,\ 0) & = \ 1 \end{cases} \tag{A.28}$$

where x_c^* is the critical futures price/strike price <u>differential</u> above which the call is always exercised early.

Define $H(x, T) \equiv C(-x, T; 0)$, which satisfies (A.26). $H(x, T)$ and $x_p^* \equiv -x_c^*$ satisfy the terminal and early-exercise boundary conditions for American puts, so $P(x, T; 0) = H(x, T)$. Proposition 4.1 then follows. ■

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Table I

Skewness premium SK ≡ $(C\text{-}P)/P$ for options symmetrically and geometrically symmetrically in-, at-, and out-of-the-money -- summary statistics and hypothesis tests. Skewness premiums interpolated using symmetric strike prices are used in testing for positive/negative implicit skewness. Skewness premiums interpolated using geometrically symmetric strike prices are used in testing for greater/less implicit skewness than those distributions (such as the lognormal) that satisfy the $x\%$ rule. Period: January 31, 1983 - December 31, 1993 (2,258 days).

Symmetric strikes: $X_{call}/F = 1 + x$, $X_{put}/F = 1 - x$	**2% ITM options**[a]	**ATM options**	**2% OTM options**	**4% OTM options**	**6% OTM options**
Number of observations	1,503	2,198	2,256	2,208	1,822
Mean	2.9%	-0.6%	-7.4%	-16.4%	-25.2%
Standard deviation	4.5%	1.6%	10.0%	22.5%	34.3%
Distribution of $(C\text{-}P)/P$					
Less than 0	35%	70%	68%	68%	70%
Greater than 0	65%	30%	32%	32%	30%
Frequency of rejection at 1% significance of H_0: $C = P$					
versus H_1: $C < P$	17%	12%	61%	63%	63%
versus H_1: $C > P$	49%	3%	22%	25%	22%

Geometrically symmetric strikes ($X_{call}/F = 1 + x = F/X_{put}$)	**2% ITM options**[a]	**ATM options**	**2% OTM options**	**4% OTM options**	**6% OTM options**
Number of observations	1,495	2,198	2,256	2,211	1,834
Mean	2.3%	-0.6%	-7.5%	-18.8%	-30.7%
Standard deviation	4.5%	1.5%	9.1%	20.9%	30.7%
Distribution of $(C\text{-}P)/P$					
less than x%	19%[a]	70%	83%	85%	87%
greater than x%	81%[a]	30%	17%	15%	13%
Frequency of rejection at 1% significance of H_0: $(C\text{-}P)/P = x\%$					
versus H_1: $(C\text{-}P)/P < x\%$	6%[a]	12%	71%	75%	76%
versus H_1: $(C\text{-}P)/P > x\%$	59%[a]	3%	6%	7%	6%

[a]For in-the-money options, $x = -2\%$.