The Market for Crash Risk

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Issues:
1) Deviations of “risk-neutral” stock index distributions implicit in options from conditional and unconditional estimates appear too pronounced to be attributable to risk premia.

2) Industrial organization of options markets doesn’t appear especially compatible with the risk-sharing assumptions underlying “representative agents.”
   - Stocks protected by exchange-traded index options are at most 2.6% of stock market.
   - Individual investors can buy options, but face barriers at broker level to writing options.
   - Options mostly written by investment banks.

Objective:
To explicitly model the financial intermediation of crash risk through the options markets.

Outline
1) Stylized facts of objective versus risk-neutral distributions
2) Equilibrium model with crash risk and homogeneous crash-averse preferences
3) Heterogeneous crash aversion
Implicit versus realized volatility

**Unconditional bias**
ISD 2% higher than realized volatility (annualized) over 1988-98, on average.

**Conditional bias**

\[
\sqrt{15.6} \sum_{t=1}^{T} (\Delta \ln F_t)^2 = .0160 + .756 ISD_t + \varepsilon_{t,T}, \quad R^2 = .45 (.0142) (.102)
\]

- May be attributable to risk premia on volatility and/or jump risk biasing upward risk-neutral variance relative to realized variance. E.g., Pan (2002)

- But, post-'87 profits from trading ATM options appear large by standard asset pricing criteria!

**Sharpe Ratio, 1988-98**
- Writing ATM straddles daily: \(1.7 \times\) that of S&P 500
- ISD-dependent straddle positions: \(2.7 \times\) that of S&P 500

Implicit versus realized skewness

- **Implicit** distributions have been strongly negatively skewed (volatility smirk) ever since the stock market crash of 1987.

This is specific to stock index options. Much less pronounced for other options (near-lognormal for FX options).
Implicit versus realized skewness

- Implicit distributions are substantially skewed and leptokurtic.
- Stock index returns are near-normal at weekly holding periods or longer.

<table>
<thead>
<tr>
<th>Skewness</th>
<th>Xkurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time series (daily,..., monthly)</td>
<td>-0.6 to -0.3</td>
</tr>
<tr>
<td>Options (monthly, over time)</td>
<td>-1 to -3</td>
</tr>
</tbody>
</table>

Options predict far more extreme events than have occurred over the post-crash period. Implicit jump parameters (1988-98):
- 0.75 jumps/year on average
- Mean jump size: -6.6%
- Jump std. deviation: 11.0%
- 8.3 jumps predicted over 1988-98.
- 3.7 daily moves predicted in excess of 10% in magnitude. None observed over 1988-98!

Implication: Even greater profit opportunities from writing OTM puts! (Jackwerth, 2000)

Furthermore, the *dynamic evolution* of implicit jump risk is puzzling.

V1: implicit variance and higher cumulants at all maturities; jump risk.
V2: implicit variance only. Little impact on higher cumulants

Affine models: $\lambda_s \propto \lambda_r$
Implies jump risk is highest following crashes

Alternate explanation:
“Price cycles” in crash insurance market, similar to those seen in earthquake and hurricane insurance.
A simple economy with traded crash risk

- Consumption at time $T$ of a terminal payoff $\tilde{D}_T$
- Information Structure:

$$d \ln D = \mu_d \, dt + \sigma_d \, dZ + \gamma_d \, dN$$

(2)

where $Z$ is a standard Wiener process, 
$N$ is a Poisson counter with constant intensity $\lambda$, 
$\gamma_d < 0$ is a deterministic announcement effect.

Discrete-time approximation: *trinomial tree*.

Time: $0 \quad T$

\[ \ln D_0 \]

\[ +\sigma \sqrt{\Delta t} \]

\[ -\sigma \sqrt{\Delta t} \]

\[ \gamma_d \]

Asset Markets

All assets are claims on terminal consumption.

Assume there exist three *primitive assets*:

<table>
<thead>
<tr>
<th>Net supply</th>
<th>Payoff at $T$</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>“Bond”</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(numeraire)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Equity</td>
<td>1</td>
<td>$D_T$</td>
</tr>
<tr>
<td>Jump</td>
<td>0</td>
<td>$1_{dN = 1}$</td>
</tr>
</tbody>
</table>

Insurance

Total terminal payoff of 1 insurance contract: $N_T - \int_{0}^{T} \lambda \cdot dt$

- Markets are *dynamically complete*. Equilibrium prices of these assets determines prices of all contingent claims.
- Jump insurance contract *embedded* in the stock index options actually traded. Easier to price options indirectly, via equilibrium price for crash insurance.
Preferences
State-dependent, “crash-averse” preferences:
\[ U_W(W_t, N_t, t) = E_t \left[ e^{\gamma N_T} \tilde{W}_T^{R} \right] \text{ for } R > 0, \ Y > 0 \]

Justifications
1) Revealed-preference: crash aversion \( Y > 0 \) increases the observed crash insurance premium \( \lambda^*/\lambda \).
- It is not possible with state-independent preferences to simultaneously match the equity and crash risk premia.
- \( U_W \) is the entropy-minimizing pricing kernel that matches both risk premia.

2) \( \lambda e^Y \) can roughly be interpreted as subjective beliefs regarding jump frequencies. Can vary across individuals.

3) Liu, Pan & Wang ‘02: uncertainty aversion

Properties
- Nests standard power & log utility and has similar properties:
  - homogeneity
  - myopic investment strategies (for \( R = 1 \))
- Under homogeneous agents, generates i.i.d. returns under actual and risk-neutral distributions (Ho et al, 1996)
- Jumps have two impacts on marginal utility \( U_W \):
  1) a direct wealth effect
  2) an additional crash aversion effect (size-invariant)
- Size-sensitive crash aversion also feasible, but tricker

General asset market equilibrium
\[
\eta_t = E_t \eta_T \\
S_t = \frac{E_t \eta_T D_t}{\eta_t} \\
\lambda^*_t = \lambda \left[ \frac{\eta_t d|dN=1}{\eta_t} \right]
\]

Homogeneous-agent equilibrium
\[
\eta_T = U_W(W_T, N_T) \bigg|_{W_T = D_T} \\
= D_T^{-R} e^{YN_T} \\
\lambda^*_t = \lambda e^{Y - R \gamma_t} \\
\eta_t = D_t^{-R} e^{YN_t} e^{(T-t)(-R \mu_d + \frac{1}{2} \sigma_d^2 + \lambda^*(e^Y - 1))} \\
\frac{S_t}{D_t} = \exp \left( (T - t) \left[ (\mu_d + \frac{1}{2} \sigma_d^2 - R \sigma_d^2) + \lambda^*(e^Y - 1) \right] \right)
\]
Features:

- $\lambda^*$ is constant, and can be much greater than $\lambda$
- Equity is a time-dependent multiple of the payoff signal, and therefore follows a comparable geometric jump-diffusion with constant coefficients:

$$dS/S = \mu dt + \sigma dZ + k(dN - \lambda dt) \quad (8)$$

$$\mu = R\sigma^2 + (\lambda - \lambda^*)k$$

$$\approx R(\sigma^2 + \lambda\gamma_d^2) - (\lambda\gamma_d)Y \quad (9)$$

- Equity jumps are equal to the jump in fundamentals:

$$\ln(1+k) = \gamma_d$$

Sample calibration

$\sigma = .15$

$\lambda = .25, \gamma_d = -.10$

$$\mu \approx R(\sigma^2 + \lambda\gamma_d^2) - \lambda\gamma_d Y$$

$$\approx .025R + .025Y$$

$$\ln(\lambda^*/\lambda) = -R\gamma_d + Y$$

$$= .10R + Y$$

For $R = 1, Y = 1$:

- $\mu = 5\%/\text{year}, \quad \lambda^*/\lambda = 3$.
- $Y$ roughly as important as $R$ for equity premium
- $Y$ substantially more important for crash premium.

Empirical anomalies

$\lambda^* \gg \lambda$ implies

- More jumps predicted than observed
- Implicit variances $\sigma^2 + \lambda\gamma_d^2$ overpredicts objective variance $\sigma^2 + \lambda\gamma_d^2$
- High Sharpe ratios from strategies that involve writing crash insurance:

$$\frac{\lambda^* dt - E_t[1_{dN = 1}]}{\sqrt{Var_t[1_{dN = 1}]} \cdot \lambda dt} = \frac{(\lambda^* - \lambda)dt}{\lambda dt(1 - \lambda dt)} = \frac{\lambda^*}{\lambda} - 1 \quad (20)$$

E.g., writing delta-neutral straddles (Fleming, 1998) or put options (Jackwerth, 2000)

Since objective and risk-neutral distributions i.i.d:

- Cannot explain volatility bias in regression
- Cannot explain time-varying $\lambda^*$
- ISD smirk should flatten out at longer maturities, contrary to fact.

Jackwerth (2002) implicit pricing kernel anomaly:

- Can match the substantial large risk aversion to large negative returns.
- Cannot match the local risk-loving implicit behavior found by Jackwerth and by Rosenberg & Engle.
Heterogeneous agents: \( U_Y(W_{YT}, N_T) = \frac{1}{1-R} W_{YT}^{1-R} e^{YN_T} \)

with common \( R \) and heterogeneous \( Y \).

**Market equilibrium**

- Market dynamically complete, so the solution to a central planner’s problem

\[
U = \max E_0 \sum_Y \omega_Y \tilde{W}_{YT}^{1-R} f_Y(N_T)
\]

\[
= E_0 U(W_T, N_T; \omega) \text{ for}
\]

\[
W_T = \sum_Y W_{YT}, \ W_{YT} \geq 0 \ \forall Y
\]

is the competitive equilibrium.

**Approach**

1) Work out the equivalent central planning problem/representative agent, and price assets off the associated pricing kernel.

2) Work out the dynamic trading strategies in primitive assets (equity, jump insurance) that replicate the competitive equilibrium.

3) Work out trading strategies in equities and options that replicate the trades in primitive assets.
Figure 3: Impact of relative wealth share $w_1 = W_1/W_{total}$ upon initial equilibrium quantities.

Two agents, with $Y = 0, 1$, respectively. Calibration: $\sigma = .20, \lambda = .25, \gamma_d = -.03; \; R = 1, \; T = 50, \; t = 0$.

Implications for portfolio choice

Investment opportunity set shifts greatly over time:

Equity: $\frac{dS}{S} = \mu(N_t, t)dt + \sigma d\tilde{Z} + k(N_t, t)[d\tilde{N} - \lambda dt]$

Crash insurance: $\lambda^*(N_t, t)$

Quantitative importance of hedging against such shifts?

Conflicting results in partial-equilibrium diffusion setting.

Important: Campbell and Viceira (1999), Campbell, Rodrigues & Viceira (2001)

Not very important: Chacko and Viceira (1999)
Here, myopic investment strategies can be compared with the optimal strategies that dynamically achieve the complete-markets equilibrium.

At equilibrium \((\mu_t, k_t, \lambda^*_t)\):
- Myopic and optimal equity positions are both 100% in equity
- Crash insurance positions differ

\[
q^*_Y(t) = (1 + k_t) \left[ \frac{w_Y(N_t + 1, t)}{w_Y(N_t, t)} - 1 \right]
\]

\[
q_Y^{\text{myopic}} = \left( \frac{\lambda e^Y}{\lambda^*} \right)^{\frac{1}{k}} - (1 + k)
\]

for \(w_Y = \text{the fraction of total wealth held by investors of type } Y\).

**Empirical anomalies**

\(\lambda^*_t \gg \lambda\) again consistent with unconditional anomalies:
- ISD’s too high relative to realized variance
- Too many jumps predicted
- High Sharpe ratios from strategies that write crash insurance

Heterogeneity helps explain the conditional puzzles.

1) **Volatility regressions:**

\[
\text{Var}_t[\ln S] = [\sigma^2 + \lambda_1^2] \, dt
\]

\[
\text{Var}_t^*[\ln S] = [\sigma^2 + \lambda^* (N_t, t) \gamma^2_t] \, dt
\]  

(50)

I.e., implicit variance informative, but overpredicts.
2) Dynamic evolution of option ISD’s and implicit jump risk:

Figure 5. Simulated instantaneous risk-neutral variance $R\sigma^2 + \lambda^2 t^2$, conditional upon jump timing matching that observed over 1988-98. Calibration: $\gamma_Y = 0.1$

Implicit factor estimates, 1988-98

Options

- Option prices: $c(S_t, N_t, \tau; X)$ are an $(N_t, t)$-dependent mixture of Black-Scholes prices.
- Negative jumps imply negative implicit skewness, and a “volatility smirk.”
- Maturity profile of implicit skewness sensitive to wealth distribution, but doesn’t especially match that observed in stock index options.

Replication of crash insurance through options markets

Options are instantaneous bundles of equity and crash insurance:

$c_S$ units of equity risk

$[\Delta c - c_S \Delta S]_{\Delta N} = 1$ units of crash risk

Crash-averse investors buy options

Crash-tolerant investors write options, and delta-hedge each to maintain an overall delta of 1.
Summary
- Paper has developed a complete-markets methodology for computing equilibria under heterogeneity and jumps. Options dynamically span the additional risks.
- Paper has proposed “crash-averse” preferences.

Relative to empirical “stylized facts:”
- Crash aversion helps explain static pricing puzzles.
- Heterogeneity helps explain some of the dynamic puzzles.

Other findings:
- Heterogeneity magnifies the impact of small adverse news in fundamentals.

Further issues
- Heterogeneity alone probably cannot explain the magnitude of post-’87 shocks to implicit distributions.
- Future research will focus on the situation when only market makers can write options/crash insurance for heterogeneous crash-averse investors.
  - Position constraints on crash insurance are equivalent to most of the existing constraints on “negative-gamma” option positions
    - Writing straddles
    - Writing naked OTM puts