Supplement to "Memoirs of an indifferent trader: Estimating forecast distributions from prediction markets": Appendix

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In this appendix, we do six things: (i) provide proofs of the propositions, (ii) discuss the sensitivity of the results to the hyperparameters, (iii) describe some unique characteristics of IEM contracts that help in constructing bounds, (iv) describe a general principle for deriving the bounds on the distributions implemented in Section 5 of the paper, (v) develop a common means of designating contracts so that contract designations describe the aspects of the distribution of vote shares that are tied to the contracts, and (vi) describe in detail the contracts and bounds for each of the elections studied in this paper. Prospectuses for all of the markets except the 1992 market can be found on the IEM website: www.tippie.uiowa.edu/iem. In 1992, the prospectus was similar to those in later years, but the 1992 market ran as a one-time market and the prospectus was part of the trader's manual. During each election, the specific contracts traded changed as the nature of the elections changed and the IEM directors became interested in studying different aspects of prediction markets.

A.1. Proofs of propositions

Proof of Proposition 1. First suppose $v \leq 1/2$. Then the upper bound on the probability of winning is given by mass probability m at $1/2$ in a distribution with mass points at 0 and $1/2$. Since $v = m/2$ in this case, the upper bound is $2v$.

Now suppose $v \geq 1/2$. Then the lower bound on the probability of winning is given by the mass probability m at 1 in a distribution with mass points at $1/2$ and 1. Since $v = (1-m)/2 + m$ in this case, the lower bound is $2v - 1$. □

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PROOF OF PROPOSITION 2. First suppose $v < 1/2$. Since the distribution of f is symmetric, $w = 0$ if $v < 1/4$. For $v \in (1/4, 1/2)$, the distribution with mass 1/2 at $v - 1/4$ and mass 1/2 at $v + 1/4$ implies $w = 1/2$, and because the distribution of f is symmetric, no greater w is possible. The distribution with mass 1 at v produces $w = 0$. The argument for $v > 1/2$ is entirely symmetric. □

PROOF OF PROPOSITION 3. From Proposition 2, $w > 0$ only if $1/4 < v < 3/4$. Suppose $1/4 < v < 1/2$. The relevant limiting case is a distribution uniform on $(0, 2v)$, for which $w = (2v)^{-1} \cdot (2v - 1/2) = 1 - (4v)^{-1}$. The argument for $v > 1/2$ is entirely symmetric. □

PROOF OF PROPOSITION 4. Suppose $f \sim \text{Beta}(a, b)$, $a \in (0, \infty)$, and $b \in (0, \infty)$. Let $c = a + b$ and $v = a/c = E(f)$; $c \in (0, \infty)$ and $v \in (0, 1)$. Then $p(f|c, v) \propto f^{vc-1}(1 - f)^{(1-v)c-1}$. Suppose $v < 1/2$.

For any two points $z < 1/2$ and $1 - z > 1/2$,

$$\log[p(z|c, v)/p(1 - z|c, v)] = (1 - 2v)c \log[(1 - z)/z]$$

is monotone increasing in c. Therefore, $P(f > 1/2|c, v)$ is monotone decreasing in c. The limits provide the bound on w.

Since $\text{var}(f|c, v) = v(1 - v)/(c + 1)$ (Zellner (1971, p. 374)),

$$\lim_{c\to\infty} \text{var}(f|c, v) = 0 \quad \Rightarrow \quad \lim_{c\to\infty} P(f > 0.5|c, v) = 0.$$

At the other extreme, $\lim_{c\to 0} \text{var}(f|c, v) = v(1 - v)$. This is the largest variance possible for a random variable with support on [0, 1] and mean v. The limiting distribution is that of a Bernoulli random variable with $P(f = 1) = v$. Therefore, $\lim_{c\to 0} P(f > 1/2|c, v) = v$. Since $w = P(f > 1/2)$ is a continuous function of c, all values $w \in (0, v)$ are possible.

A parallel argument provides the bounds for $v > 1/2$. If $v = 1/2$, the beta distribution is symmetric about 1/2, implying $w = 0.5$ for all c. □

PROOF OF PROPOSITION 5. Denote $f = h(x)$, where $h(x) = \exp(x)/[1 + \exp(x)]$ and x has probability density function (p.d.f.) $p(x) = \sigma^{-1}\phi[(x - \mu)/\sigma]$.

Consider any fixed $w = P(f > 1/2) \in (0, 1/2)$. Because $h(x)$ is monotone increasing and $h(0) = 1/2$, $P(f > 1/2) = P(x > 0) = \Phi(\mu/\sigma)$. Hence given $w < 1/2$, then $\mu < 0$ and $\sigma = a\mu$, where $a = 1/\Phi^{-1}(w)$.

For any $\varepsilon > 0$, $\lim_{\mu\to 0} P[x \in (-\varepsilon, \varepsilon)] = 1$; because $h(x)$ is continuous at $x = 0$,

$$\lim_{\mu\to 0} P(1/2 - \varepsilon < f < 1/2 + \varepsilon) = 1.$$

Therefore, as $\mu \to 0$, $v \to 1/2$.

For any $c > 0$, $\lim_{\mu\to-\infty} P[x \in (-c, c)] = 0$; because $\lim_{x\to-\infty} h(x) = 0$ and $\lim_{x\to\infty} h(x) = 1$, $\lim_{\mu\to-\infty} P(c < f < 1 - c) = 0$. Therefore, as $\mu \to -\infty$, $v \to \lim_{\mu\to-\infty} P(f = 1) = w$.

The moment v is a continuous function of μ and, therefore, any $v \in (w, 1/2)$ is attained for some μ.

It remains to show that $v > 1/2$ and $v < w$ are not possible. Exploiting the fact that $h(x) + h(-x) = 1$, we can write

$$\begin{aligned} v = E[h(x)] &= \int_{-\infty}^{\infty} h(x)\sigma^{-1}\phi\left(\frac{x-\mu}{\sigma}\right)dx \\ &= \frac{1}{2} + \sigma^{-1}\int_0^{\infty}\left[h(x) - \frac{1}{2}\right]\phi\left(\frac{x-\mu}{\sigma}\right)dx \\ &\quad + \sigma^{-1}\int_0^{\infty}\left[h(-x) - \frac{1}{2}\right]\phi\left(\frac{-x-\mu}{\sigma}\right)dx \\ &= \frac{1}{2} + \sigma^{-1}\int_0^{\infty}\left[h(x) - \frac{1}{2}\right]\cdot\left[\phi\left(\frac{x-\mu}{\sigma}\right) - \phi\left(\frac{-x-\mu}{\sigma}\right)\right]dx. \end{aligned} \tag{A.1}$$

Because $h(x)$ is monotone increasing and $h(0) = 1/2$, then $h(x) - 1/2 > 0$ for all $x > 0$. For all $x > 0$, $\phi'(x) = -\phi'(-x) > 0$, and $|x - \mu| > |-x - \mu|$ because $\mu < 0$. Therefore,

$$\phi\left(\frac{x-\mu}{\sigma}\right) - \phi\left(\frac{-x-\mu}{\sigma}\right) < 0 \quad \text{for all } x > 0. \tag{A.2}$$

From (A.1), $E(f) = E[h(x)] < 1/2$.

Define $q(x) = h(x) - I_{(0,\infty)}(x)$. Note that $E[q(x)] = v - w$, and $q(-x) = -q(x)$ because $h(x) + h(-x) = 1$:

$$E[q(x)] = \int_0^{\infty} q(x)\sigma^{-1}\left[\phi\left(\frac{x-\mu}{\sigma}\right) - \phi\left(\frac{-x-\mu}{\sigma}\right)\right]dx.$$

In view of (A.2) and $q(x) = h(x) - 1$ for all $x > 0$, $E[q(x)] = v - w > 0$, establishing $v > w$.

The argument for $w \in (1/2, 1)$ is symmetric. The point $(v = 1/2, w = 1/2)$ corresponds to $\mu = 0$. □

The proof of Proposition 5 in fact establishes a more general result. The p.d.f. of x can be of the form $g[(x - \mu)/\sigma]$ with $g'(y) = -g'(-y) < 0$ for all $y > 0$. The fraction f can be of the form $f = h(x)$, with $\lim_{x\to-\infty} h(x) = 0$, $\lim_{x\to\infty}(x) = 1$, $h(x) + h(-x) = 1$, and $h(x)$ continuous at $x = 0$. For any parametric family of distributions for f with this construction, the set of possible (v, w) will be S_*.

A.2. Sensitivity of the results to the selection of the hyperparameters

How does the choice of hyperparameters affect the results? Over a reasonable range, not much. Decreasing τ or c makes p_{ti} smoother as a function of i, and the smoothness tends to push more weight into the tails of the distribution. Increasing τ or c sufficiently moves the solution eventually to a discrete distribution on a small number of points corresponding to the endpoints of INT contracts. These effects are illustrated in Figure A.1 for a typical day in our data, October 28, 1992. Each panel in this figure shows the function $\mathbf{p}_t$ that solves the quadratic programming problem as dots connected by lines. For comparison, they also show as a light green line the logistic normal distribution cen-

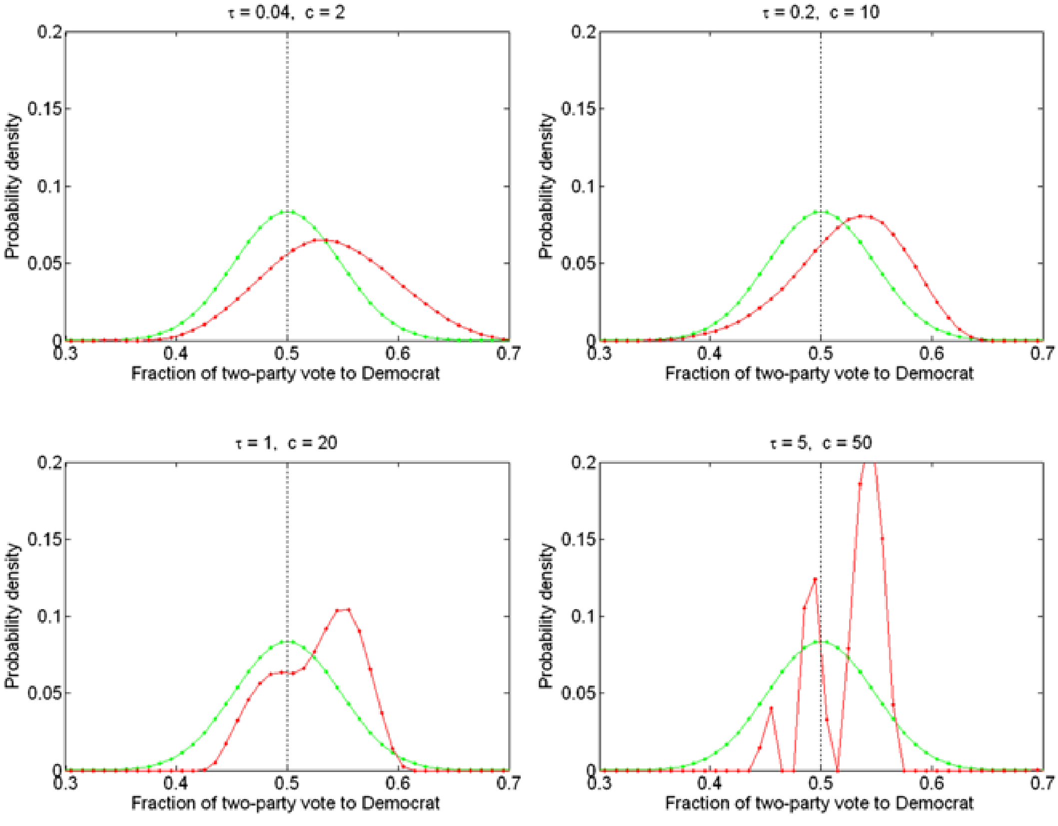

FIGURE A.1. Estimated probability distribution for October 28, 1992, corresponding to alternative values of the prior hyperparameters τ and c (heavy lines). Light lines show naive distribution based on historical data.

tered at 0.5 with variance equal to the sample variance taken from the popular vote in U.S. presidential elections in 1868–2004; this distribution might reasonably be taken as an estimate of $\mathbf{p}_t$ uninformed by any information about the current election. We will use this as our benchmark throughout the rest of this appendix. In interpreting Figure A.1, recall that there are contracts with payoffs that depend on whether the fraction f of the Democratic vote is greater or less than 0.46, 0.50, and 0.54. These points account for the modes in the distribution in the lower right panel.

On the whole, results are only moderately sensitive to halving or doubling the values of the hyperparameters. Decreasing both hyperparameters by a scale factor of more than one-half leads to distributions that are more diffuse than a prior distribution based on the historical record of elections since 1868 as shown in the upper left panel of Figure A.1. In some cases, sufficiently small hyperparameters lead to modes in the tails that we do not find reasonable. Increasing both hyperparameters by a scale factor of more than 2 leads to substantial multimodality for the 1992 and 2004 elections in which there were five or more contracts, as exemplified by the lower right panel of Figure A.1. Distributions with multiple modes and little probability mass between them also seem unreasonable.

A.3. Common contract designations

As the IEM ran markets over the four elections studied in this paper, the contract names and specifications of contract payoffs changed. We create a common designation that allows the reader to infer the aspects of the vote-share distribution tied to each contract. We use the following symbols: C denotes the generic designation of a candidate or party, D denotes Democratic party or the nominee of the Democratic party, R denotes the Republican party or the nominee of the Republican party, F denotes the Reform party or the nominee of the Reform party, O denotes other parties and/or nominees, *VS* denotes a contract with a payoff proportional to the popular vote shares received by various candidates, W denotes a contract based on winning the majority or plurality of the popular vote, and (a, b) denotes a contract that pays off if a popular vote share falls in the range $(a\%, b\%)$.

These symbols provide a consistent designation of contracts across election years. For example, *VS.D|DR* denotes a contract with a liquidation value equal to the popular vote share of the Democratic party given the votes received by the Democratic and Republican parties (i.e., the two-party vote share of the Democratic party). Similarly, *W.D|DRF* denotes a contract with a liquidation value of $1 if the Democratic party wins given the popular votes received by the Democratic, Republican, and Reform parties (i.e., if the Democratic party takes the plurality of the three-party popular vote).

These symbols also allow us to designate liquidation values in a convenient and consistent manner. For example, the liquidation value of the *VS.D|DR* contract is $D/(D+R)$, where D and R are the popular votes received by the Democratic and Republican parties. Similarly, the liquidation value of the *W.D|DRF* contract is $\text{IF}(D > \max(R, F))$, where D, R, and F are the popular votes received by the Democratic, Republican, and Reform parties, and the IF function takes on the value of 1 if the argument is true and 0 otherwise.

Finally, we designate the best outstanding bids and asks by the contract name followed by b or a, respectively. We use the outstanding bids and asks to form the bounds on the distribution.

A.4. Some special IEM characteristics

A.4.1 *Unit portfolios*

Every IEM contract trades as part of a unit portfolio. This means that there are always two ways to effectively buy a contract and two ways to effectively sell. A trader can buy a contract by purchasing it at the ask at the price Ca (the current best ask for candidate C). Alternatively, a trader can effectively buy the contract by purchasing the unit portfolio at $1 and selling the other contracts in the portfolio at their best outstanding bids. This leaves the trader with one more C contract at an effective price of $1 - \sum_{O \neq C} Ob$, where O indexes the other contracts in the unit portfolio. Similarly, there are two effective prices to sell a contract: Cb or $1 - \sum_{O \neq C} Oa$.

A.4.2 *No-arbitrage restrictions*

Arbitrage violations occur if a trader can buy and resell some portfolio of contracts for a certain profit. Regardless of his expectations, any trader would trade against arbitrage violations. There are two basic arbitrage restrictions on the IEM:

Definition A.1 (Individual Contract Arbitrage Restriction). $Cb \leq Ca$ for all contracts.

This restriction is generally enforced by the IEM exchange itself. Unless there are computer errors, whenever a bid or ask is entered such that $Cb \geq Ca$, the exchange automatically executes trades until $Cb < Ca$.

Definition A.2 (Portfolio Arbitrage Restrictions). $\sum_{C \in P} Cb \leq 1$ and $\sum_{C \in P} Ca \geq 1$ for every unit portfolio, where P represents the set of contracts in the unit portfolio.

If either restriction is violated, the trader can make a profit without risk and without changing his net portfolio. If $\sum_{C \in P} Cb > 1$, the trader can (a) buy the unit portfolio from the exchange for \$1 and (b) sell the portfolio at individual contract bids for a net profit of $\sum_{C \in P} Cb - 1 > 0$. If $\sum_{C \in P} Ca < 1$, the trader can (a) buy the unit portfolio at individual contract asks and (b) sell the portfolio to the exchange for \$1 for a net profit of $1 - \sum_{C \in P} Ca > 0$. The trader can execute all the transactions to buy or sell the unit portfolio at bids or asks simultaneously using a portfolio purchase or sale option available through the IEM.

Oliven and Rietz (2004) showed that portfolio arbitrage restrictions are sometimes violated. However, they also show that they are quickly exploited. Next, we show that portfolio arbitrage violations lead to inconsistencies in how bids and asks bound aspects of the implied distribution of the indifferent trader. Then we discuss how to deal with any inconsistencies that may arise.

A.5. Construction of bounds

Generally, the vote-share markets are based on the share of the two-party vote and so are winner-takes-all contracts. In these cases, we work with the Democratic share of the two-party vote. However, this is not always the case. Sometimes one or both include contracts based on other outcomes. In 2000, for example, both the vote-share and winner-takes-all market were based on the shares of the three-party vote. In this case, we work with the Democratic share of the three-party vote. More difficult problems arise when there are contracts trading in the market that represent other actual or potential candidates. Some of these may be Democratic, some Republican, and some third party. To create consistency across election years, we develop a standard way of to deal with such situations.

Make the following definitions:

Definition A.3 (Democratic Party). D represents all of the candidates and/or contracts issued that are associated with the Democratic party. Define

$$Db := \sum_{\substack{C \in \text{Democratic} \\ \text{party}}} Cb \quad \text{and} \quad Da := \sum_{\substack{C \in \text{Democratic} \\ \text{party}}} Ca.$$

DEFINITION A.4 (Republican Party). R represents all of the candidates and/or contracts issued that are associated with the Republican party. Define

$$Rb := \sum_{\substack{C \in \text{Republican} \\ \text{party}}} Cb \quad \text{and} \quad Ra := \sum_{\substack{C \in \text{Republican} \\ \text{party}}} Ca.$$

DEFINITION A.5 (Third Party). O represents all of the candidates and/or contracts issued that are associated with parties or candidates other than the Democratic and Republican parties. Define

$$Ob := \sum_{\substack{C \in \text{Other} \\ \text{parties}}} Cb \quad \text{and} \quad Oa := \sum_{\substack{C \in \text{Other} \\ \text{parties}}} Ca.$$

In general, bounds are constructed as follows. For two-contract markets based on the two-party vote, we can bound aspects of the distribution of the Democratic share of the two-party vote easily.

DEFINITION A.6. Bounds on f as a fraction of the two-party vote with two contracts are

$$\textit{lower bound} := \max(Db, 1 - Ra),$$
$$\textit{upper bound} := \min(Da, 1 - Rb).$$

The easiest way to see that these are the correct bounds is to consider a winner-takes-all market. If the trader thinks that $P(D > R) < Db$, he would sell the Democratic contract at Db. Alternatively, if the trader thinks that $P(D > R) < 1 - Ra \rightarrow Ra < 1 - P(D > R) = P(R > D)$, he would buy the Republican contract at Ra. Similarly, if the trader thinks that $P(D > R) > Da$, he would buy at Da, and if $P(D > R) > 1 - Rb$, he would sell at Rb. Similar arguments hold in a vote-share market.

It is also relatively simple to bound aspects of the distribution for the Democratic share of the three-way vote if there are Democratic, Republican, and Other contracts trading in the markets.

DEFINITION A.7. Bounds on f as a fraction of the multiparty vote with third-party contracts are

$$\textit{lower bound} := \max(Db, 1 - Ra - Oa),$$
$$\textit{upper bound} := \min(Da, 1 - Rb - Ob).$$

Again, the easiest way to see that these are the correct bounds is to consider a winner-takes-all market. If the trader thinks that $P(D > R, O) < Db$, he would sell the Democratic contract(s) at Db. Alternatively, if the trader thinks that $P(D > R, O) < 1 - Ra - Oa \rightarrow Ra + Oa < 1 - P(D > R, O) = P(R, O > D)$, he would buy the Republican and third-party contracts at $Ra + Oa$. Similarly, if the trader thinks that $P(D > R, O) > Da$, he would buy at Da, and if $P(D > R, O) > 1 - Rb - Ob$, he would sell at $Rb + Ob$. Similar arguments hold in a vote-share market.

The most difficult case is when we are trying to bound aspects of the distribution for the two-party vote when there are third-party contracts being traded. We will assume that the trader believes that the two-party vote share and the probability of winning the two-party vote is independent of the third-party vote share or probability of winning. As a result, we can divide the bounds in the preceding definitions by the trader's expected value of the third-party contracts. But, we do not know the exact expectations for the third-party candidates. We can only bound them by looking at the third-party contract bids and asks. We will take the conservative approach, dividing by the upper bound of the third-party vote when determining the lower bounds and vice versa.

Definition A.8. Bounds on f as a fraction of the two-party vote with third-party contracts are

$$\textit{lower bound} := \frac{\max(Db, 1 - Ra - Oa)}{\min(Da + Ra, 1 - Ob)},$$
$$\textit{upper bound} := \frac{\min(Da, 1 - Rb - Ob)}{\max(Db + Rb, 1 - Oa)}.$$

The logic for the numerator follows the previous definition. The denominator follows similar logic that bounds the distribution of the third-party votes represented by O.

A.6. Consistency of bounds

Generally, the bounds we construct are consistent with each other. That is, the lower bound falls below the upper bound. However, from time to time, inconsistencies arise. Here we show that inconsistencies only arise when there are pure arbitrage violations. As a result, any trader would be willing to trade at outstanding bids and asks. He cannot be indifferent to trading per se. However, exploiting arbitrage opportunities does not result in any reallocation of his portfolio.

Proposition A.1. *Defined in the above manner, the lower bound never exceeds the upper bound unless there is a pure arbitrage opportunity in the market.*

Proof. The proof simply considers all cases, assuming the no-arbitrage restrictions hold leads to a contradiction in all cases. Consider the most general Definition A.8, where there are 16 possible cases that violate consistency. The following case results in inconsistent bounds:

Case 1 ($\frac{Db}{Da+Ra} > \frac{Da}{Db+Rb}$). No arbitrage $\rightarrow Db \leq Da$ and $Da + Ra \geq Db + Rb \rightarrow \frac{Db}{Da+Ra} \leq \frac{Da}{Db+Rb}$ a contradiction.

Similarly, in all 15 other cases, no-arbitrage restrictions imply that (a) the lower bound numerator is less than or equal the upper bound numerator, and (b) the lower bound denominator is greater than or equal to the upper bound denominator, creating a contradiction. Similar relationships hold for the other bounds definitions. □

Portfolio arbitrage violations sometimes arise in the data and, once, a bid which exceeded an ask had not yet cleared when the exchange recorded midnight bids and asks. We could deal with these inconsistencies in several ways:

(i) We could simply exclude these observations from the data. However, we find this solution unsatisfactory.

(ii) We could construct weaker bounds by using the lowest lower bound and the highest upper bound. Bounds constructed in this manner never result in inconsistencies in the data. This imposes minimal restriction on the estimated distribution. To the degree we are interested in showing that the implied distributions have information content, this is a conservative approach. However, we know that traders quickly exploit arbitrage opportunities.

(iii) We could assume that the indifferent trader will trade to exploit arbitrage opportunities, but not beyond. To implement this, we look for the arbitrage violation whenever a set of bounds is inconsistent. We assume that the trader trades against these violations, moving the prices until they disappear, but trades no further. When a bid exceeds an ask, we proxy this by assuming that the bid–ask midpoint is unaffected by trading and, hence, use the bid–ask midpoint as both the upper and lower bound. When a sum of bids exceeds $1, we proxy trading until the violation is eliminated by dividing each bid by their sum. This drives each bid down proportionately to the point that there is no longer a violation. Similarly, when a sum of asks is less than $1, we divide each ask by their sum. This increases them each proportionately until there is no longer a violation. Since exploiting arbitrage opportunities always involves purchasing and reselling unit portfolios, the process does not result in a net reallocation of the indifferent trader's portfolio.

For the analysis in the paper, we follow the third approach, using the strictest arbitrage-free bounds. Following the second approach leads to somewhat smoother distributions, sometimes with fewer modes, but leaves the rest of the results largely unaffected.

A.7. The 1992 IEM presidential markets

In 1992, the IEM traded three sets of contracts on the presidential election: (i) a vote-share market with contracts on the share of the two-party popular vote received by George Bush as the Republican nominee (contract name: R.BU) versus Bill Clinton as the Democratic nominee (contract name: D.CL), (ii) a winner-takes-all market with contracts based on which party took the plurality of the three-party popular vote between Bush, Clinton, and Ross Perot (contract names: P.CL, P.BU, and P.PE), and (iii) several "derivatives" markets with winner-takes-all contracts based on various popular vote splits between Bush and Clinton (example pair: B42d, which paid $1 if Bush took less than 42% of the two-party popular vote, and B42u, which paid $1 if Bush took more than 42%). Table A.1 lists the contracts traded, our common designation for them and their liquidation values.

In addition to the typical bounds constructed from bids and asks within a market as discussed above, there was an additional set of intermarket bounds in 1992. This arose

TABLE A.1. 1992 IEM markets and contracts.

Original Contract Name	Contract Designation	Liquidation Value
D.CL	$VS.D\|DR$	$D/(D+R)$
R.BU	$VS.R\|DR$	$R/(D+R)$
P.CL	$W.D\|DRF$	$\text{IF}(D > \max(R, F))$
P.BU	$W.R\|DRF$	$\text{IF}(R > \max(D, F))$
P.PE	$W.F\|DRF$	$\text{IF}(F > \max(D, R))$
B42d*	$(0, 42).R\|DR$	$\text{IF}(R/(R+D) < 0.42)$
B42u*	$(42, 100).R\|DR$	$\text{IF}(R/(R+D) \geq 0.42)$
B46d	$(0, 46).R\|DR$	$\text{IF}(R/(R+D) < 0.46)$
B46u	$(46, 100).R\|DR$	$\text{IF}(R/(R+D) \geq 0.46)$
B50d	$(0, 50).R\|DR$	$\text{IF}(R/(R+D) < 0.50)$
B50u	$(50, 100).R\|DR$	$\text{IF}(R/(R+D) \geq 0.50)$
B54d	$(0, 54).R\|DR$	$\text{IF}(R/(R+D) < 0.54)$
B54u	$(54, 100).R\|DR$	$\text{IF}(R/(R+D) \geq 0.54)$

Note: The asterisk indicates that trading began 21 days before the election.

because, effectively, there were two separate sets of contracts trading in whether the Democrats won the popular vote. There was a typical winner-takes-all market based on whether the Democratic nominee, the Republican nominee, or the Reform nominee would win the plurality of the vote. We derive bounds on f in this market according to the standard definitions discussed above. However, there was also a derivative contract that paid \$1 dependent on whether the Democratic nominee took more or less than 50% of the two-party vote, which also bounds f. This gives the following overall restrictions.

DEFINITION A.9. Additional bounds in 1992 were

$$\textit{lower bound on } P(f > 0.5) := \max\left\{(0, 50).R|DRb, 1 - (50, 100).R|DRa, \frac{\max(W.D|DRFb, 1 - W.R|DRFa - W.F|DRFa)}{\min(W.D|DRFa + W.R|DRFa, 1 - W.F|DRFb)}\right\},$$

$$\textit{upper bound on } P(f > 0.5) := \min\left\{(0, 50).R|DRa, 1 - (50, 100).R|DRb, \frac{\min(W.D|DRFa, 1 - W.R|DRFb - W.F|DRFb)}{\max(W.D|DRFb + W.R|DRFb, 1 - W.F|DRFa)}\right\}.$$

These bounds create intermarket restrictions that are often violated. We could deal with these inconsistencies in four ways:

(i) We could simply exclude these observations from the data. However, we find this solution unsatisfactory.

(ii) We could exclude one market or the other from the analysis. Since the derivative market was much more thinly traded, this is the logical market to eliminate. However, this is still throwing away information.

TABLE A.2. Construction of lower bounds on the probability distribution for 1992: $f = D/(D+R)$.

Characteristic of the Distribution	Lower Bounds
$E(f)$	$\max\{VS.D\|DRb, 1 - VS.R\|DRa\}$
$P(f > 0.50)$	$\max\{(0,50).R\|DRb, 1-(50,100).R\|DRa, \frac{\max(W.D\|DRFb, 1-W.R\|DRFa-W.F\|DRFa)}{\min(W.D\|DRFa+W.R\|DRFa, 1-W.F\|DRFb)}\}$
$P(f > 0.46)$	$\max\{(0,54).R\|DRb, 1-(54,100).R\|DRa\}$
$P(f > 0.54)$	$\max\{(0,46).R\|DRb, 1-(46,100).R\|DRa\}$
$P(f > 0.58)$	$\max\{(0,42).R\|DRb, 1-(42,100).R\|DRa\}$

TABLE A.3. Construction of upper bounds on the probability distribution for 1992: $f = D/(D+R)$.

Characteristic of the Distribution	Upper Bounds
$E(f)$	$\min\{VS.D\|DRa, 1 - VS.R\|DRb\}$
$P(f > 0.50)$	$\min\{(0,50).R\|DRa, 1-(50,100).R\|DRb, \frac{\min(W.D\|DRFa, 1-W.R\|DRFb-W.F\|DRFb)}{\max(W.D\|DRFb+W.R\|DRFb, 1-W.F\|DRFa)}\}$
$P(f > 0.46)$	$\min\{(0,54).R\|DRa, 1-(54,100).R\|DRb\}$
$P(f > 0.54)$	$\min\{(0,46).R\|DRa, 1-(46,100).R\|DRb\}$
$P(f > 0.58)$	$\min\{(0,42).R\|DRa, 1-(42,100).R\|DRb\}$

(iii) We could construct weaker bounds by using the lowest lower bound and the highest upper bound. Bounds constructed in this manner never result in inconsistencies in the data. This imposes minimal restriction on the estimated distribution. To the degree we are interested in showing that the implied distributions have information content, this is a conservative approach.

(iv) We could assume that the indifferent trader will trade to exploit arbitrage opportunities, but not beyond. To handle this, we look for the arbitrage violation whenever a set of bounds is inconsistent. We assume that the trader trades against these violations, moving the prices until they disappear, but trades no further. Because the derivative market had much thinner queues, we move prices only in this market until we achieve no intermarket arbitrage violations.[1]

For the analysis in the paper, we follow the fourth approach, using the strictest arbitrage-free bounds. Following the third approach leads to somewhat smoother distributions, sometimes with fewer modes, but leaves the rest of the results largely unaffected. Tables A.2 and A.3 outline the construction of the bounds we use.

[1] This effectively collapses the interval to a point equal to (a) the WTA lower bound if the INT upper bound falls below the WTA lower bound or (b) the WTA upper bound if the INT lower bound falls above the WTA upper bound. In both cases, the INT contract price is assumed to move just enough to eliminate the arbitrage opportunity.

A.8. The 1996 IEM presidential markets

In 1996, the IEM traded two sets of contracts on the presidential election: (i) a vote-share market with contracts based on the share of the two-party popular vote taken by Bill Clinton as the Democratic nominee against several potential Republican nominees, including Robert Dole, the eventual nominee, and (ii) a winner-takes-all market with contracts based on whether Bill Clinton as the Democratic nominee, another Democratic nominee, the Republican nominee, or another candidate took the plurality of the popular vote. The contract names in the Clinton–Dole two-party popular vote race were V.CLIN and V.DOLE. While Dole was the de facto nominee, contracts in other potential Republican nominees traded in pairs until Dole was actually nominated at the Republican convention, 20 days into our sample period. As a result, other contracts traded for the first 19 days. These included pairs for potential races between Clinton and Lamar Alexander (contract names: CL|ALEX and V.ALEX), between Clinton and Steve Forbes (CL|FORB and V.FORB), Phil Gramm (CL|GRAM and V.GRAM), and any other Republican nominee (CL|OREP and V.OREP). Each contract pair would have liquidated at the two-party vote shares received if the associated candidate had become the nominee. For example, if Lamar Alexander had been the nominee, the contract pair associated with Alexander and the corresponding Clinton contract would have liquidated at $1 times the popular vote share taken by the associated candidate. Because Alexander was not nominated, both contracts in this pair liquidated at $0. After the Republican Convention, all contracts associated with nonnominees had a value of zero and were delisted. Because Robert Dole actually became the nominee, the contract pair associated with Dole continued to trade and liquidated at the two-party vote shares after the election. The delisting of other contracts occurred on August 16, 1996, 19 days into our sample period. Before this date, we used all contract pairs to determine our bounds. After that date, we used only the Dole associated contracts. Table A.4 lists the contracts traded,

Table A.4. 1996 IEM markets and contracts.

Original Contract Name	Contract Designation	Liquidation Value
V.CLIN	$VS.D\|DR\&Dole$	$D/(D+R)\|Dole$
V.DOLE	$VS.R\|DR\&Dole$	$R/(D+R)\|Dole$
CL|ALEX*	$VS.D\|DR\&Alex$	$D/(D+R)\|Alex$
V.ALEX*	$VS.R\|DR\&Alex$	$R/(D+R)\|Alex$
CL|FORB*	$VS.D\|DR\&Forb$	$D/(D+R)\|Forb$
V.FORB*	$VS.R\|DR\&Forb$	$R/(D+R)\|Forb$
CL|GRAM*	$VS.D\|DR\&Gram$	$D/(D+R)\|Gram$
V.GRAM*	$VS.R\|DR\&Gram$	$R/(D+R)\|Gram$
CL|OREP*	$VS.D\|DR\&ORep$	$D/(D+R)\|ORep$
V.OREP*	$VS.R\|DR\&ORep$	$R/(D+R)\|ORep$
CLIN	$W.D(C)\|DRO$	$\text{IF}((D=WC)>\max(R,O))$
OTDEM	$W.D(O)\|DRO$	$\text{IF}((D\neq WC)>\max(R,O))$
REP	$W.R\|DRO$	$\text{IF}(R>\max(D,O))$
ROF96	$W.O\|DRO$	$\text{IF}(O>\max(R,D))$

Note: The asterisk indicates that trading stopped 81 days before the election.

TABLE A.5. Construction of lower bounds on the probability distribution for 1996: $f = D/(D+R)$.

Characteristic of the Distribution	Lower Bounds
$E(f)$	$\max\{VS.D\vert DRb^{*}, 1 - VS.R\vert DRa^{*}\}$
$P(f > 0.50)$	$\frac{\max(W.D\vert DROb, 1-W.R\vert DROa-W.O\vert DROa)}{\min(W.D\vert DROa+W.R\vert DROa, 1-W.ODROa)}$

Note: The asterisk indicates that during the first 19 days, the bids (or asks) in the vote-share market are sums, across the Republican candidates, of bids (or asks) on all contracts associated with the party. The Democratic bid (or ask) in the winner-takes-all market is the sum of bids (or asks) across the Clinton and other Democratic contracts.

TABLE A.6. Construction of upper bounds on the probability distribution for 1996: $f = D/(D+R)$.

Characteristic of the Distribution	Upper Bounds
$E(f)$	$\min\{VS.D\vert DRa^{*}, 1 - VS.R\vert DRb^{*}\}$
$P(f > 0.50)$	$\frac{\min(W.D\vert DROa, 1-W.R\vert DROb-W.O\vert DROb)}{\max(W.D\vert DROb+W.R\vert DROb, 1-W.O\vert DROa)}$

Note: The asterisk indicates that during the first 19 days, the bids (or asks) in the vote-share market are sums, across the Republican candidates, of bids (or asks) on all contracts associated with the party. The Democratic bid (or ask) in the winner-takes-all market is the sum of bids (or asks) across the Clinton and other Democratic contracts.

our common designation for them, and their liquidation values, where *&C* represents the particular associated Republican candidate becoming the nominee:

Tables A.5 and A.6 outline the construction of the bounds we use. They show the multiple lower bounds from which we use the maximum and the multiple upper bounds from which we take the minimum.

Two issues deserve special note. First, during the first 19 days of our data, there were multiple Clinton contracts trading in the vote-share market. There were also multiple Republican contracts trading. Each pair represented the vote share Clinton and the Republican nominee would get conditional on the actual Republican nominee. To deal with this, we simply sum the bids (or asks) of all the contracts associated with Clinton and sum all the bids (or asks) of all the contracts associated with the Republicans. Second, in the winner-takes-all market, the IEM ran contracts associated with Clinton as the Democratic nominee, any other Democratic nominee, the Republican nominee, and any other candidate. To deal with the two Democratic contracts, we add together the bids (or asks) of the two contracts.

A.9. THE 2000 IEM PRESIDENTIAL MARKETS

In 2000, the IEM traded two sets of contracts on the presidential election: (i) a vote-share market with contracts based on the share of the three-party popular vote taken by the Democratic, Reform, and Republican nominees, and (ii) a winner-takes-all market with contracts based on which party took the plurality of the three-party popular

TABLE A.7. 2000 IEM markets and contracts.

Original Contract Name	Contract Designation	Liquidation Value
DemVS	$VS.D\|DRF$	$D/(D+R+F)$
RepVS	$VS.R\|DRF$	$R/(D+R+F)$
ReformVS	$VS.F\|DRF$	$F/(D+R+F)$
Dem	$W.D\|DRF$	$\text{IF}(D > \max(R, F))$
Rep	$W.R\|DRF$	$\text{IF}(R > \max(D, F))$
Ref	$W.F\|DRF$	$\text{IF}(F > \max(D, R))$

TABLE A.8. Construction of lower bounds on the probability distribution for 2000: $f = D/(D+R+F)$.

Characteristic of the Distribution	Lower Bounds
$E(f)$	$\max\{VS.D\|DRFb, 1 - VS.R\|DRFa - VS.F\|DRFa\}$
$P(f > 0.50)$	$\max\{W.D\|DRFb, 1 - W.R\|DRFa - W.F\|DRFa\}$

TABLE A.9. Construction of upper bounds on the probability distribution for 2000: $f = D/(D+R+F)$.

Characteristic of the Distribution	Upper Bounds
$E(f)$	$\min\{VS.D\|DRFa, 1 - VS.R\|DRFb - VS.F\|DRFb\}$
$P(f > 0.50)$	$\min\{W.D\|DRFa, 1 - W.R\|DRFb - W.F\|DRFb\}$

vote. Table A.7 lists the contracts traded, our common designation for them, and their liquidation values.

Tables A.8 and A.9 outline the construction of the bounds we use. They show the multiple lower bounds from which we use the maximum and the multiple upper bounds from which we take the minimum.

Because both markets are based on the three-party vote, we simply construct the bounds and distribution as the Democratic percentage of the three-party vote.

A.10. THE 2004 IEM PRESIDENTIAL MARKETS

In 2004, the IEM traded two sets of contracts on the presidential election: (i) a vote-share market with contracts based on the share of the two-party popular vote taken by the Democratic and Republican nominees, and (ii) a winner-takes-all market with contracts based on which party took the majority of the two-party popular vote. During our sample period, the winner-takes-all contracts were split into interval contracts that liquidated at $1, depending on whether the popular vote split was in particular ranges.

TABLE A.10. 2004 IEM markets and contracts.

Original Contract Name	Contract Designation	Liquidation Value
KERR	$VS.D\|DR$	$D/(D+R)$
BU|KERR	$VS.R\|DR$	$R/(D+R)$
DEM04	$W.D\|DR^*$	IF$(D > R)$
REP04	$W.R\|DR^*$	IF$(R > D)$
DEM04_G52	$(52, 100).D\|DR^*$	IF$(D/(R+D) \geq 0.52)$
DEM04_L52	$(50, 52).D\|DR^*$	IF$(0.50 < D/(R+D) < 0.52)$
REP04_G52	$(52, 100).R\|DR^*$	IF$(R/(R+D) \geq 0.52)$
REP04_L52	$(50, 52).R\|DR^*$	IF$(0.50 < R/(R+D) < 0.52)$

Note: The asterisk indicates that the WTA contracts split into interval contracts on September 21, 2004. Individual contracts DEM04 and REP04 ceased trading and were replaced by the four interval contracts.

TABLE A.11. Construction of lower bounds on the probability distribution for 2004: $f = D/(D+R)$.

Characteristic of the Distribution	Lower Bounds
$E(f)$	$\max\{VS.D\|DRb, 1 - VS.R\|DRa\}$
$P(f > 0.50)$	$\max\{W.D\|DRb, 1 - W.R\|DRa\}$
$P(f < 0.48)$	$\max\{(52, 100).R\|DRb, 1 - (50, 52).R\|DRa - (50, 52).D\|DRa - (52, 100).D\|DRa\}$
$P(0.48 < f < 0.50)$	$\max\{(50, 52).R\|DRb, 1 - (52, 100).R\|DRa - (50, 52).D\|DRa - (52, 100).D\|DRa\}$
$P(0.50 < f < 0.52)$	$\max\{(50, 52).D\|DRb, 1 - (52, 100).R\|DRa - (50, 52).R\|DRa - (52, 100).D\|DRa\}$
$P(0.52 < f)$	$\max\{(52, 100).D\|DRb, 1 - (52, 100).R\|DRa - (50, 52).R\|DRa - (50, 52).D\|DRa\}$

TABLE A.12. Construction of bounds on the probability distribution for 2004: $f = D/(D+R)$.

Characteristic of the Distribution	Upper Bounds
$E(f)$	$\min\{VS.D\|DRa, 1 - VS.R\|DRb\}$
$P(f > 0.50)$	$\min\{W.D\|DRa, 1 - W.R\|DRb\}$
$P(f < 0.48)$	$\min\{(52, 100).R\|DRa, 1 - (50, 52).R\|DRb - (50, 52).D\|DRb - (52, 100).D\|DRb\}$
$P(0.48 < f < 0.50)$	$\min\{(50, 52).R\|DRa, 1 - (52, 100).R\|DRb - (50, 52).D\|DRb - (52, 100).D\|DRb\}$
$P(0.50 < f < 0.52)$	$\min\{(50, 52).D\|DRa, 1 - (52, 100).R\|DRb - (50, 52).R\|DRb - (52, 100).D\|DRb\}$
$P(0.52 < f)$	$\min\{(52, 100).D\|DRa, 1 - (52, 100).R\|DRb - (50, 52).R\|DRb - (50, 52).D\|DRb\}$

Table A.10 lists the contracts traded, our common designation for them and their liquidation values.

Tables A.11 and A.12 outline the construction of the bounds we use. They show the multiple lower bounds from which we use the maximum and the multiple upper bounds from which we take the minimum.

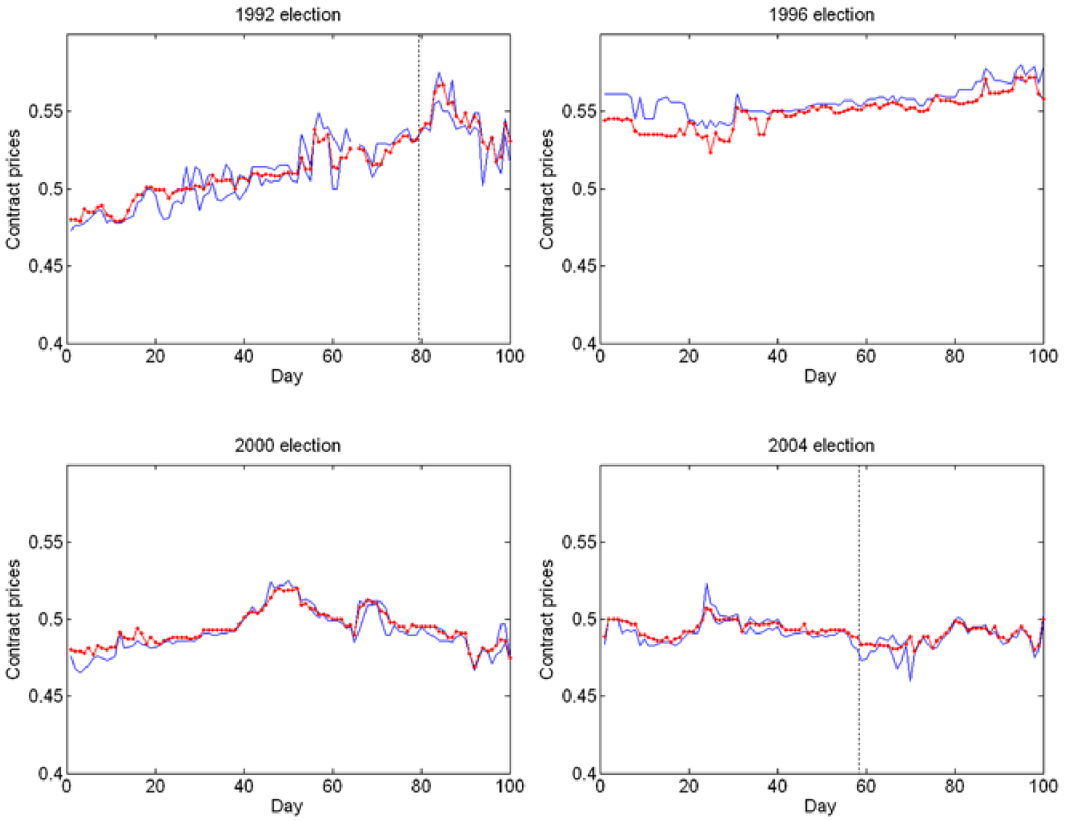

Figure A.2. The (blue) light lines show upper and lower bounds for vote-share contracts as detailed in paper and this appendix. The (red) heavy lines connecting dots show the estimated values of $E(f)$.

A.11. Summary of bounds

Figure A.2 shows the bounds of the VS contracts, along with our estimates of $E(f)$, for all the days running up to each election. Figure A.3 does the same for the WTA contracts and $P(f > 0.5)$. In each panel, the lighter (blue) lower line is the time series of lower bounds and the lighter (blue) upper line is the time series of upper bounds, while the heavier (red) line with dots is the time series of the $E(f)$ in Figure A.2, and $P(f > 0.5)$ in Figure A.3.

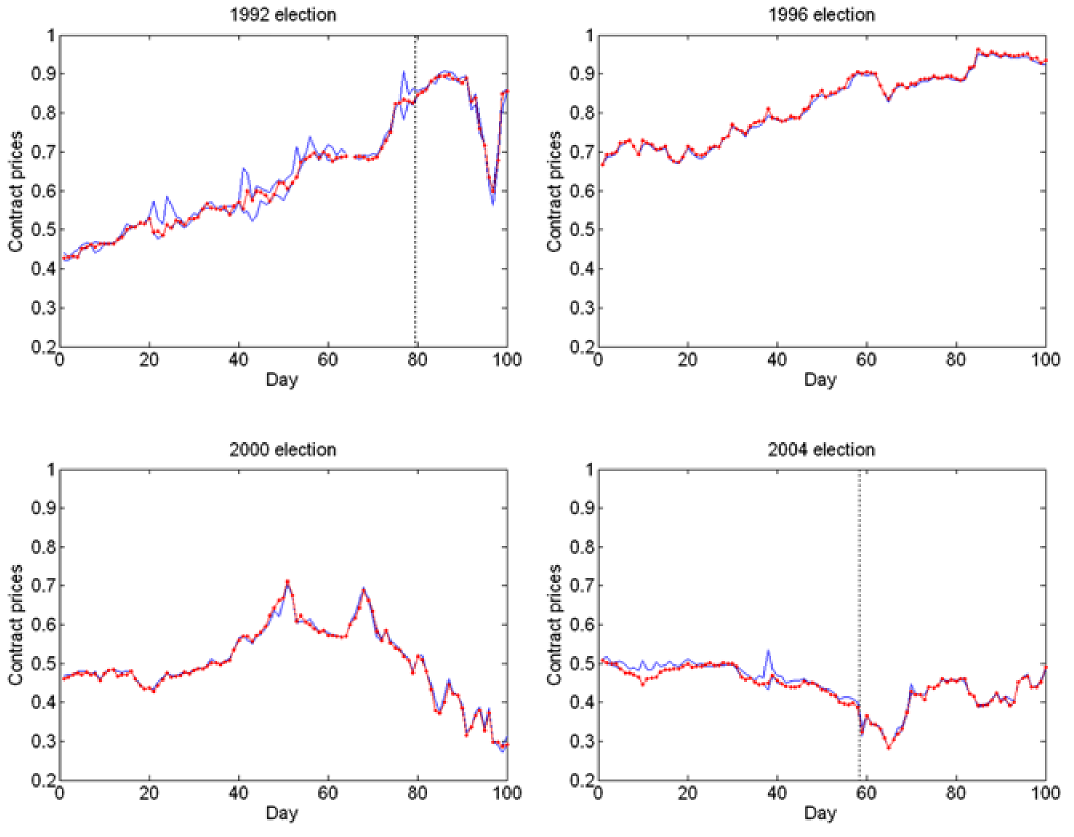

FIGURE A.3. The (blue) light lines show upper and lower bounds for winner-takes-all contracts as detailed in the paper and this appendix. The (red) heavy lines connecting dots show the estimated values of $P(f > 0.5)$.