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# On a Market Equilibrium Theorem with an Infinite Number of Commodities

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A direct proof is given of the market equilibrium theorem of Gale, Nikaido and Debreu for an infinite-dimensional commodity space. The theorem is closely related to a recent result of Aliprantis and Brown, but allows for excess demand correspondences rather than excess demand functions. 

## 1. INTRODUCTION

The purpose of this paper is to extend the well-known market equilibrium theorem of Gale, Nikaido and Debreu (see [5, p. 82]) from a Euclidean commodity space to a Hausdorff locally convex linear topological space. Our extension of the above result is established by combining a selection theorem for correspondences having open lower sections with the Tychonoff fixed point theorem. Thus, our result includes an alternative proof of the Gale–Nikaido–Debreu theorem for a finite-dimensional commodity space.

A different proof of the Gale–Nikaido–Debreu result for a Hausdorff locally convex linear topological space was given in Florenzano [6]. In particular, she considers finite-dimensional subspaces and establishes the existence of an equilibrium in each subspace by appealing to the finite-dimensional Gale–Nikaido–Debreu theorem. Then, by a limiting argument she shows the existence of equilibrium for the full space. Thus, contrary to our method of proof the finite-dimensional version of the Gale–Nikaido–Debreu theorem becomes indispensable to her argument. A similar type of argument to that of Florenzano [6] is used by Khan [8], who proves a very general existence of a competitive equilibrium result for an economy where agents' preferences need not be transitive or complete.

A different proof of the Gale–Nikaido–Debreu result for a Riesz space of commodities has been given in Aliprantis and Brown [1], but they consider excess demand functions satisfying a boundary condition rather than



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excess demand correspondences. Hence, our result could be also seen as an extension of the Aliprantis–Brown [1] approach to excess demand correspondences.

The paper proceeds as follows. Section 2 contains some notation, definitions and a selection theorem, which is of fundamental importance for the proof of our equilibrium theorem. Section 3 contains the main result of the paper, i.e., a generalization of the Gale–Nikaido–Debreu market equilibrium theorem to an infinite-dimensional commodity space. Finally, Section 4 compares our main result with that of Aliprantis and Brown [1].

## 2. A Preliminary Result

### 2.1. *Notation*

$2^A$ denotes the set of all subsets of $A$.

$\mathbb{R}^l$ denotes $l$-fold Cartesian product of the set of real numbers $\mathbb{R}$.

int $A$ denotes the interior of the set $A$.

$\Omega$ denotes the positive orthant of $\mathbb{R}^l$.

If $X$ is a linear topological space, its dual is the space $X^*$ of all continuous linear functionals on $X$.

If $q \in X^*$ and $y \in X$ the value of $q$ at $y$ is denoted by $q \cdot y$.

### 2.2. *Definitions*

Let $X$ be a topological space, and let $Y$ be a linear topological space. A correspondence $\phi: X \to 2^Y$ is said to be *upper-demicontinuous* (u.d.c.) if the set $\{x \in X: \phi(x) \subset V\}$ is open in $X$ for every open half space $V$ of $Y$. An *open cover* of a topological space $X$ is a collection of open subsets of $X$ whose union is $X$. A *refinement* of an open cover $U$ is an open cover $V$ such that every $v \in V$ is a subset of some $u \in U$. An open cover $U$ is *locally finite* if every $x \in X$ has a neighborhood intersecting only finitely many $u \in U$. A Hausdorff space $X$ is called *paracompact* if every open cover of $X$ has a locally finite refinement.

### 2.3. *A Selection Theorem*

The following selection theorem will be of fundamental importance to prove our main result. For other applications of this theorem see [11].

THEOREM 2.1. *Let $X$ be a paracompact Hausdorff space and $Y$ be a linear topological space. Suppose $\phi: X \to 2^Y$ is a correspondence such that:*

(i) *$\phi(x)$ is nonempty, for each $x \in X$,*

(ii) $\phi(x)$ *is convex, for each* $x \in X$,

(iii) $\phi^{-1}(y) = \{x \in X\colon y \in \phi(x)\}$ *is open in* $X$, *for each* $y \in Y$.

*Then there exists a continuous function* $f\colon X \to Y$ *such that* $f(x) \in \phi(x)$ *for all* $x \in X$.

*Proof.* For each $y \in Y$ the set $\phi^{-1}(y)$ is open in $X$, and by (i), for each $x \in X$ there is a $y \in Y$ such that $x \in \phi^{-1}(y)$. Hence the collection $C = \{\phi^{-1}(y)\colon y \in Y\}$ is an open cover of $X$. Since $X$ is paracompact, there is a locally finite refinement $F = \{U_a\colon a \in A\}$ of $C$. ($A$ is an index set and $U_a$ is an open set in $X$.) By Theorem 4.2 in [4, p. 170] we can find a family of continuous functions $\{g_a\colon a \in A\}$ such that $g_a\colon X \to [0, 1]$, $g_a(x) = 0$ for $x \notin U_a$ and $\sum_{a \in A} g_a(x) = 1$ for all $x \in X$. For each $a \in A$ choose $y_a \in Y$ such that $U_a \subset \phi^{-1}(y_a)$. This can be done since $F$ is a refinement of $C$. Define $f\colon X \to Y$ by $f(x) = \sum_{a \in A} g_a(x)\, y_a$, for all $x \in X$. By the local finiteness of $F$, each $x \in X$ has a neighborhood $N_x$ which intersects only finitely many $U_a$'s. Hence, $f(x)$ is a finite sum of continuous functions on $N_x$ and is therefore continuous on $N_x$. So $f$ is a continuous function from $X$ to $Y$. Further, for any $a \in A$ such that $g_a(x) \neq 0$, $x \in U_a \subset \phi^{-1}(y_a)$ and so $y_a \in \phi(x)$. Thus, $f(x)$ is a convex combination of elements $y_a$ in $\phi(x)$ and therefore $f(x) \in \phi(x)$ for all $x \in X$. Q.E.D.

## 3. The Excess Demand Theorem

Let $\Delta^{l-1} = \{p \in \Omega\colon \sum_{i=1}^{l} p_i = 1\}$ be the price simplex in $\mathbb{R}^l$. An economy will be described by an excess demand correspondence $\zeta\colon \Delta^{l-1} \to 2^{\mathbb{R}^l}$ satisfying the weak Walras law, i.e., for every $p \in \Delta^{l-1}$ there exists some $x \in \zeta(p)$ with $p \cdot x \leqslant 0$. For such an economy a price vector $\mathbf{p} \in \Delta^{l-1}$ is a *free disposal equilibrium* if $\zeta(\mathbf{p}) \cap (-\Omega) \neq \emptyset$. Gale, Nikaido and Debreu (see Debreu [5] and its references) have proved the existence of a free disposal equilibrium for a commodity space which is $\mathbb{R}^l$. We will now prove a more general result which allows for the dimensionality of the commodity space to be infinite. We must note that our commodity space is general enough to include the space of essentially bounded measurable functions $L_\infty$ used in Bewley [2] and Magill [9], and certain Riesz spaces used in Aliprantis and Brown [1] and Khan [8].

Theorem 3.1. *Let* $X$ *be a Hausdorff locally convex linear topological space,* $C$ *a closed, convex cone of* $X$ *having an interior point* $e$, $C^* = \{p \in X^*\colon p \cdot x \leqslant 0$ *for all* $x \in C\} \neq \{0\}$ *the dual cone of* $C$ *and* $\Delta = \{p \in C^*\colon p \cdot e = -1\}$. *Let* $\zeta\colon \Delta \to 2^X$ *be a correspondence such that*:

(i) $\zeta$ *is u.d.c. in the weak* topology* (*i.e.,* $\zeta\colon (\Delta, w^*) \to 2^X$ *is u.d.c.*),

(ii) *$\zeta(p)$ is convex nonempty and compact for all $p \in \Delta$,*

(iii) *for all $p \in \Delta$ there exists $x \in \zeta(p)$ such that $p \cdot x \leqslant 0$.*

*Then there exists $\mathbf{p} \in \Delta$ such that $\zeta(\mathbf{p}) \cap C \neq \varnothing$.*

*Proof.* Suppose not; i.e., for all $p \in \Delta$ we have that $\zeta(p) \cap C = \varnothing$. Fix $p$ in $\Delta$, then by Theorem 3.2 [10, p. 65] there exists a continuous linear functional which strictly separates the convex compact set $\zeta(p)$ from the closed convex set $C$, i.e., for all $p \in \Delta$ there exist $r \in X^*$, $r \neq 0$ and $b \in \mathbb{R}$ such that $\sup_{y \in C} r \cdot y < b < \inf_{x \in \zeta(p)} r \cdot x$.

Notice that $b > 0$ and $r \in C^*$. Without loss of generality we can take $r$ to be in $\Delta$. Indeed, if $r \notin \Delta$, then $e \in \operatorname{int} C$ implies that $r \cdot e < 0$,[1] and we can replace $r$ by $r/(-r \cdot e)$.

Define $F\colon \Delta \to 2^{\Delta}$ by $F(p) = (q \in \Delta\colon q \cdot x > 0$ for all $x \in \zeta(p)\}$. In view of $r \in F(p)$, each $F(p)$ is nonempty for all $p \in \Delta$ and it can be easily checked that $F(p)$ is also convex for all $p \in \Delta$. Clearly, for any $q \in \Delta$, $F^{-1}(q) = \{p \in \Delta\colon q \in F(p)\} = \{p \in \Delta\colon q \cdot x > 0$ for all $x \in \zeta(p)\}$. Let $V_q = \{x \in X\colon q \cdot x > 0\}$ be an open half space in $X$. Since $\zeta$ is u.d.c. in the weak* topology, the set $W = \{p \in \Delta\colon \zeta(p) \subset V_q\}$ is weak* open in $\Delta$. It is easy to check that $W = F^{-1}(q)$. Consequently, for all $q \in \Delta$ the set $F^{-1}(q)$ is weak* open in $\Delta$. By Theorem 2.1 there exists a weak* continuous function $f\colon \Delta \to \Delta$ such that $f(p) \in F(p)$ for all $p \in \Delta$. Further, by Alaoglu's theorem [7, Theorem 3.8, p. 123], $\Delta$ is weak* compact. Since $f\colon \Delta \to \Delta$ is a weak* continuous function from the weak* compact, convex set $\Delta$ into itself, by Tychonoff's fixed point theorem [4, Theorem 2.2, p. 414] there exists $\mathbf{p} \in \Delta$ such that $\mathbf{p} = f(\mathbf{p}) \in F(\mathbf{p})$, i.e., $\mathbf{p} \cdot x > 0$ for all $x \in \zeta(\mathbf{p})$, a contradiction to assumption (iii). Hence there exists $\mathbf{p} \in \Delta$ such that $\zeta(\mathbf{p}) \cap C \neq \varnothing$. Q.E.D.

*Remark* 3.1. When $X = \mathbb{R}^l$, the above result gives as a corollary Debreu's [5, p. 82] excess demand theorem. An alternative proof of Debreu's [5] excess demand theorem for $X = \mathbb{R}^l$ has been also given in Border [3].[2] His proof is in the spirit of Aliprantis and Brown [1].

## 4. Relationship with the Work of Aliprantis and Brown

We now indicate how our market equilibrium theorem is related to the work of Aliprantis and Brown [1]. In the Aliprantis–Brown [1] paper, the important examples were $AM$-spaces with units. Consider an $AM$-space $L$

[1] To see this, assume by way of contradiction that $r \cdot e = 0$. Pick a symmetric neighborhood $V$ of zero with $e + V \subseteq C$. Now if $x \in X$, then for some $\lambda > 0$ we have that $\pm\lambda x \in V$ and so $\pm\lambda r \cdot x = r \cdot (e \pm \lambda x) \leqslant 0$. Therefore, $r \cdot x = 0$ for each $x \in X$, i.e., $r = 0$, which is a contradiction.

[2] I owe this reference to a referee.

with unit $e$ and positive cone $L_+$, and let $\Delta = \{p \in L^*_+ : p \cdot e = 1\}$. In [1] a function $E: \Delta \to L$ is said to be an *excess demand function* whenever

(a) there exists a consistent locally convex topology $t$ on $L$ such that $E: (\Delta, w^*) \to (L, t)$ is continuous, and

(b) $E$ satisfies the Walras law, i.e., $p \cdot E(p) = 0$ for all $p \in \Delta$.

Under these conditions, the main result of Aliprantis and Brown [1] guarantees the existence of some $p \in \Delta$ with $E(p) \leqslant 0$. This conclusion can be also obtained from our Theorem 3.1 as follows:

Put $C = -L_+$, and note that $-e$ is an interior point of $C$. Thus $C^* = L^*_+ = \{p \in L^*: p \cdot x \geqslant 0 \text{ for all } x \in L_+\}$, and our simplex $\Delta$ agrees with that of Aliprantis and Brown [1]. Now define $\zeta: \Delta \to 2^L$ by $\zeta(p) = \{E(p)\}$, and note that $\zeta$ satisfies the hypothesis of Theorem 3.1. Consequently, for some $p \in \Delta$ we have $\zeta(p) \cap C \neq \emptyset$, i.e., $E(p) \leqslant 0$ holds for some $p \in \Delta$.

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