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Integration of Banach-Valued Correspondences

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Abstract. We study the basic properties of the integral of a Banach-valued correspondence. In particular, we examine the convergence, compactness and convexity properties of the Bochner and Gel'fand integrals of a set-valued function. The above properties are applied to prove the existence of an equilibrium for an abstract economy with a continuum of agents.

1. Introduction

The classical model of exchange under perfect competition known as the "Arrow-Debreu-McKenzie model" was formulated in terms of a finite set of agents taking prices as given and engaging in sale and purchase of commodities. Aumann (1966) argued that the classical model is clearly at odds with itself as the finitude of agents means that each individual is able to exercise some influence and therefore the assumption of price taking behavior is not sensible. In a path breaking paper, Aumann (1966) resolves this problem by assuming that the set of agents is an atomless measure space and consequently the influence of each agent in the economy as a whole is "negligible." Hence, the "Aumann economy," that is, an economy with an atomless measure space of agents, captures precisely the meaning of perfect competition.

In order to prove the existence of a competitive equilibrium in a perfectly competitive economy, Aumann (1966) faced the following fundamental problem. What is the definition of the aggregate demand set if the set of agents is an atomless measure space? For instance, if we denote the set of agents by T and denote by $D(t,p)$ the demand set of agent $t \in T$, at prices p, we know that if T is finite, the aggregate demand set is given by the summation of the individual demand sets, i.e., $\sum_{t\in T} D(t,p)$. However, if T is an atomless measure space, then we have to integrate the set $D(t,p)$. But what does it mean to integrate a set-valued function? In a seminal paper Aumann (1965) introduced the notion of the integral of a set-valued function (or correspondence) and proved some basic results needed to tackle the problem of the existence of a competitive equilibrium in an economy with an atomless measure space of agents, and with a finite dimensional commodity space.

However, if one wishes to allow for perfectly competitive economies with an infinite dimensional commodity space, then an extension of the work of Aumann (1965) is required. In particular, from the integration of finite dimensional-valued correspondences we must now pass to integration of Banach-valued correspondences.

The main purpose of this paper is to study the integral of a Banach-valued correspondence and prove some basic theorems needed in general equilibrium and game theory. Results due to Debreu (1967), Datko (1972), Diestel (1977), Hiai-Umegaki (1977), Khan (1982, 1984, 1985), Papageorgiou (1985), Khan-Majumdar (1986), Balder (1988), Yannelis (1988, 1989, 1990), Rustichini (1989), and Castaing (1988) have drastically influenced the present paper which in a way may be considered as a synthesis of the work of the above authors.

2. Preliminaries

2.1 Notation.

$\mathbb{R}^n$ denotes the n-fold Cartesian product of the set of real numbers $\mathbb{R}$.

con A denotes the convex hull of the set A.

$\overline{\text{con}}A$ denotes the closed convex hull of the set A.

2^A denotes the set of all nonempty subsets of the set A.

$\emptyset$ denotes the empty set.

$/$ denotes the set of theoretic subtraction.

dist denotes the distance.

proj denotes the projection.

If $A \subset X$, where X is a Banach space, cl A denotes the norm closure of A.

If X is a linear topological space, its dual is the space X^* of all continuous linear functionals on X, and if $p \in X^*$ and $x \in X$ the value of p at x is denoted either by $\langle p, x\rangle$ or $p \cdot x$.

If $\{F_n : n = 1, 2, \ldots\}$ is a sequence of nonempty subsets of a Banach space X, we will denote by s-Ls F_n and s-Li F_n the set of its (strong) limit superior and (strong) limit inferior points respectively, i.e.,

$$\text{s-Ls}\, F_n = \{x \in X : x = s - \lim_{k\to\infty} x_{n_k},\ x_{n_k} \in F_{n_k},\ k = 1,2,\ldots\}$$
$$\text{s-Li}\, F_n = \{x \in X : x = s - \lim_{n\to\infty} x_n,\ x_n \in F_n,\ n = 1,2,\ldots\}.$$

A w in front of $\text{Ls}\, F_n$ ($\text{Li}\, F_n$) will mean limit superior (limit inferior) with respect to the weak topology $\sigma(X, X^*)$.

2.2 Definitions. Let X and Y be sets. The *graph* of the set-valued function (or correspondence), $\phi : X \to 2^Y$ is denoted by $G_\phi = \{(x,y) \in X \times Y : y \in \phi(x)\}$. Let (T, τ, μ) be a complete, finite measure space, and X be a separable Banach space. The correspondence $\phi : T \to 2^X$ is said to have a *measurable graph* if $G_\phi \in \tau \otimes \beta(X)$, where $\beta(X)$ denotes the Borel σ-algebra on X and $\otimes$ denotes product σ-algebra. The correspondence $\phi : T \to 2^X$ is said to be *lower measurable* if for every open subset V of X, the set $\{t \in T : \phi(t) \cap V \neq \emptyset\}$ is an element of τ. Recall [see for instance Debreu (1966), p. 359 or Yannelis (1990a), Lemma 3] that if $\phi : T \to 2^X$ has a measurable graph, then ϕ is lower measurable. Furthermore, if $\phi(\cdot)$ is closed valued and lower measurable then $\phi : T \to 2^X$ has a measurable graph. A well-known result of Aumann (1967) which will be of fundamental importance in this paper says that if (T, τ, μ) is a complete, finite measure space, X is a separable metric space and $\phi : T \to 2^X$ is a nonempty valued correspondence having a measurable graph, then $\phi(\cdot)$ admits a *measurable selection*, i.e., there exists a measurable function $f : T \to X$ such that $f(t) \in \phi(t)$ μ-a.e.

We now define the notion of a Bochner integrable function. We will follow closely Diestel-Uhl (1977). Let (T, τ, μ) be a finite measure space and X be a Banach space. A function $f : T \to X$ is called *simple* if there exist $x_1, x_2, \ldots, x_n$ in X and $\alpha_1, \alpha_2, \ldots, \alpha_n$ in τ such that $f = \sum_{i=1}^n x_i \chi_{\alpha_i}$, where $\chi_{\alpha_i}(t) = 1$ if $t \in \alpha_i$ and $\chi_{\alpha_i}(t) = 0$ if $t \notin \alpha_i$. A function $f : T \to X$ is said to be *μ-measurable* if there exists a sequence of simple functions $f_n : T \to X$ such that $\lim_{n\to\infty} \|f_n(t) - f(t)\| = 0$ for almost all $t \in T$. A μ-measurable function $f : T \to X$ is said to be *Bochner integrable* if there exists a sequence of simple functions $\{f_n : n = 1,2,\ldots\}$ such that

$$\lim_{n\to\infty} \int_T \|f_n(t) - f(t)\| d\mu(t) = 0.$$

In this case we define for each $E \in \tau$ the integral to be $\int_E f(t)d\mu(t) = \lim_{n\to\infty} \int_E f_n(t)d\mu(t)$. It can be shown [see Diestel-Uhl (1977), Theorem 2, p. 45] that, if $\phi : T \to X$ is a μ-measurable function then f is Bochner integrable if and only if $\int_T \|f(t)\|d\mu(t) < \infty$. It is important to note that the *Dominated Convergence Theorem* holds for Bochner integrable functions, in particular, if $f_n : T \to X$ $(n = 1, 2, \dots)$ is a sequence of Bochner integrable functions such that $\lim_{n\to\infty} f_n(t) = f(t)$ μ-a.e., and $\|f_n(t)\| \le g(t)$ μ-a.e., where $g \in L_1(\mu, \mathbb{R})$, then f is Bochner integrable and $\lim_{n\to\infty} \int_T \|f_n(t) - f(t)\|d\mu(t) = 0$.

For $1 \le p < \infty$, we denote by $L_p(\mu, X)$ the space of equivalence classes of X-valued Bochner integrable functions $x : T \to X$ normed by

$$\|x\|_p = \left(\int_T \|x(t)\|_p d\mu(t)\right)^{1/p}.$$

It is a standard result that normed by the functional $\|\cdot\|_p$ above, $L_1(\mu, X)$ becomes a Banach space [see Diestel-Uhl (1977), p. 50]. We denote by S^p_ϕ the *set of all selections from* $\phi : T \to 2^X$ that belong to the space $L_p(\mu, X)$, i.e.,

$$S^p_\phi = \{x \in L_p(\mu, X) : x(t) \in \phi(t) \ \mu\text{-a.e.}\}.$$

We will also consider the set $S^1_\phi = \{x \in L_1(\mu, X) : x(t) \in \phi(t)\ \mu\text{-a.e.}\}$, i.e., S^1_ϕ is the *set of all Bochner integrable selections from* $\phi(\cdot)$. Using the above set and following Aumann (1965) we can define the *integral of the correspondence* $\phi : T \to 2^X$ as follows:

$$\int_T \phi(t)d\mu(t) = \left\{\int_T x(t)d\mu(t) : x \in S^1_\phi\right\}.$$

In the sequel we will denote the above integral by $\int \phi$. Recall that the correspondence $\phi : T \to 2^X$ is said to be *integrally bounded* if there exists a map $h \in L_1(\mu, \mathbb{R})$ such that $\sup\{\|x\| : x \in \phi(t)\} \le h(t)$ μ-a.e. Moreover, note that if T is a complete measure space, X is a separable Banach space and $\phi : T \to 2^X$ is an integrably bounded, nonempty valued correspondence having a measurable graph, then by the Aumann measurable selection theorem we can conclude that S^1_ϕ is nonempty and therefore $\int_T \phi(t)d\mu(t)$ is nonempty as well. It should be noted that the measurability of ϕ is a sufficient condition for the nonemptiness of $\int \phi$, but it is not necessary. In fact, $\int \phi$ may be nonempty even if ϕ does not have a measurable graph [see Schechter (1989) for an example to that effect].

A Banach space X has the *Radon-Nikodym Property with respect to the measure space* (T, τ, μ) if for each μ-continuous vector measure $G : \tau \to X$ of bounded variation there exists $g \in L_1(\mu, X)$ such that $G(E) = \int_E g(t)d\mu(t)$ for all $E \in \tau$. A Banach space X has the *Radon-Nikodym Property* (RNP) if X has the RNP with respect to every finite measure space. Recall now [see Diestel-Uhl (1977, Theorem 1, p. 98)] that if (T, τ, μ) is a finite measure space $1 \leq p < \infty$, and X is a Banach space, then X^* has the RNP if and only if $(L_p(\mu, X))^* = L_q(\mu, X^*)$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Let A_n $(n = 1, 2, \ldots)$ be a sequence of nonempty subsets of a Banach space. Following Kuratowski (1966, p. 339) we say that A_n *converges* in A (written as $A_n \to A$) if and only if $\text{s-Li}\, A_n = \text{s-Ls}\, A_n = A$. Also, we say that A_n *converges in the Kuratowski-Mosco sense to* A (written as $A_n \xrightarrow{\text{K-M}} A$) if and only if $\text{s-Li}\, A_n = \text{w-Ls}\, A_n = A$. It may be useful to remind the reader that $\text{Li}\, A_n$ and $\text{Ls}\, A_n$ are both closed sets and that $\text{s-Li}\, A_n \subset \text{s-Ls}\, A_n$ [see Kuratowski (1966), pp. 336–338].

Let X be a metric space and Y be a Banach space. The correspondence $\phi : X \to 2^Y$ is said to be *upper semicontinuous* (u.s.c.) at $x_0 \in X$, if for any neighborhood $N(\phi(x_0))$ of $\phi(x_0)$, there exists a neighborhood $N(x_0)$ of x_0 such that for all $x \in N(x_0)$, $\phi(x) \subset N(\phi(x_0))$. We say that ϕ is u.s.c. if ϕ is u.s.c. at every point $x \in X$. Recall that this definition is equivalent to the fact that the set $\{x \in X : \phi(x) \subset V\}$ is open in X for every open subset V of Y [see for instance Kuratowski (1966), Theorem 3, p. 176].

Let ν be a small positive number and let B be the open unit ball in Y. The correspondence $\phi : X \to 2^Y$ is said to be *quasi upper-semicontinuous* (q.u.s.c.) at $x \in X$, if whenever the sequence x_n $(n = 1, 2, \ldots)$ in X converges to x, then for some n_0, $\phi(x_n) \subset \phi(x) + \nu B$ for all $n \geq n_0$. We say that ϕ is q.u.s.c. if ϕ is q.u.s.c. at every point $x \in X$. It can be easily checked that if ϕ is compact valued, quasi upper-semicontinuity implies upper-semicontinuity and vice-versa.

Let now P and X be metric spaces. The correspondence $F : P \to 2^X$

is said to be *lower semicontinuous* (l.s.c.) if the sequence p_n $(n = 1, 2, \ldots)$ in P converges to $p \in P$, then $F(p) \subset \operatorname{Li} F(p_n)$. Finally recall that the correspondence $F : P \to 2^X$ is said to be *continuous*, if and only if it is u.s.c. and l.s.c.

3. Weak Compactness in $L_p(\mu, X)$

The result below has found several applications in general equilibrium and game theory [see for example Khan (1986), Yannelis (1987, 1990b) and Yannelis-Rustichini (1990, 1991)] and it is known in the literature of economic theory as Diestel's theorem on weak compactness in $L_1(\mu, X)$.

Theorem 3.1. *Let (T, τ, μ) be a complete finite measure space, X be a separable Banach space and $\phi : T \to 2^X$ be an integrally bounded, convex, weakly compact and nonempty valued correspondence. Then S^1_ϕ is weakly compact in $L_1(\mu, X)$.*

Proof. First note that $(L_1(\mu, X))^* = L_\infty(\mu, X^*_{w^*})$ [see for instance Tulcea-Tulcea (1969)]. Pick an arbitrary $x \in L_\infty(\mu, X^*_{w^*})$. If we show that x attains its supremum on S^1_ϕ the result will follow from James's theorem [James (1964)]. To this end, let

$$\sup_{f \in S^1_\phi} f \cdot x = \sup_{f \in S^1_\phi} \int_{t \in T} (f(t) \cdot x(t)) d\mu(t).$$

By Lemma 1 in Debreu-Schmeidler (1972) or Theorem 2.2 in Hiai-Umegaki (1977) we have that

$$\sup_{f \in S^1_\phi} \int_{t \in T} (f(t) \cdot x(t)) d\mu(t) = \int_{t \in T} \sup_{g \in \phi(t)} (g \cdot x(t)) d\mu(t).$$

Define the correspondence $\theta : T \to 2^X$ by

$$\theta(t) = \{y \in \phi(t) : y \cdot x = \sup_{g \in \phi(t)} g \cdot x\}.$$

Since the correspondence $\phi : T \to 2^X$ is weakly compact valued we have that $\theta(t) \neq \emptyset$ for all $t \in T$. Define the function $F : T \times X \to [-\infty, \infty]$ by $F(t, y) = y \cdot x - \sup_{g \in \phi(t)} g \cdot x$. Note that for each fixed $t \in T$, $F(t, \cdot)$

is continuous and for each fixed $y \in X$, $F(\cdot, y)$ is measurable. Hence by a standard result [see for instance Yannelis (1990a, Proposition 3.1)], $F(\cdot,\cdot)$ is jointly measurable and consequently the set

$$F^{-1}(0) = \{(t,y) \in T \times X : F(t,y) = 0\} \quad \text{belongs to } \tau \otimes \beta(X).$$

Since $\phi(\cdot)$ has a measurable graph, the set $G_\phi = \{(t,y) \in T \times X : y \in \phi(t)\}$ is an element of $\tau \otimes \beta(X)$. Observe that

$$G_\theta = F^{-1}(0) \cap G_\phi.$$

Since $F^{-1}(0)$ and G_ϕ belong to $\tau \otimes \beta(X)$ so does G_θ, i.e., $\theta(\cdot)$ has a measurable graph. By the Aumann measurable selection theorem, there exists a measurable function $z : T \to X$ such that $z(t) \in \theta(t)$ μ-a.e. Hence $z \in S^1_\phi$ and

$$\sup_{g \in S^1_\phi} g \cdot x = \int_{t \in T} (z(t) \cdot x(t)) d\mu(t) = z \cdot x.$$

Since $x \in L_\infty(\mu, X^*_{w^*})$ was arbitrary, we can conclude that every element of $(L_1(\mu, X))^* = L_\infty(\mu, X^*_{w^*})$ attains its supremum on S^1_ϕ. This completes the proof of the Theorem.

Remark 3.1. Note that if (T, τ, μ) is a finite measure space, and X is a Banach space then $(L_p(\mu, X))^* = L_q(\mu, X^*_{w^*})$ where $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ [see Tulcea-Tulcea (1969)]. Hence, in Theorem 3.1 we can replace the fact that S^1_ϕ is weakly in $L_1(\mu, X)$ with the statement that S^p_ϕ, $(1 \leq p < \infty)$ is weakly compact in $L_p(\mu, X)$.

Bibliographical notes. Theorem 4.1 was proved by Diestel (1977) in a less general form [see also Byrne (1978)]. However, it should be noted that Castaing had earlier proved a related result to that of Diestel's. Also, Datko (1973) proved a version of Diestel's theorem for a reflexive separable Banach space. The proof of Theorem 4.1 is based on the celebrated theorem of James (1964) and it is patterned after that of Khan (1982, 1987) and Papageorgiou (1985). Recently, Balder (1990) has given an alternative proof of Diestel's theorem using a.e. convergence of arithmetic averages.

4. Weak Sequential Convergence in $L_p(\mu, X)$

We begin by proving the following result:

Theorem 4.1. *Let (T, τ, μ) be a finite measure space and X be a separable Banach space. Let $\{f_\lambda : \lambda \in \Lambda\}$ (Λ is a directed set), be a net in $L_p(\mu, X)$, $1 \leq p < \infty$ such that f_λ converges weakly to $f \in L_p(\mu, X)$. Suppose that for all $\lambda \in \Lambda$, $f_\lambda(t) \in F(t)$ μ-a.e., where $F : T \to 2^X$ is a weakly compact, integrably bounded, convex, nonempty valued correspondence. Then we can extract a sequence $\{f_{\lambda_n} : n = 1, 2, \ldots\}$ from the net $\{f_\lambda : \lambda \in \Lambda\}$ such that:*

(i) f_{λ_n} converges weakly to f, and
(ii) $f(t) \in \overline{\text{con}}\, \text{w-Ls}\{f_{\lambda_n}(t)\}$ μ-a.e.

Proof. We begin the proof of Theorem 4.1 by stating the following result of Artstein (1979, Proposition C, p. 280).

Proposition 4.1. *Let (T, τ, μ) be a finite measure space and let $f_n : T \to R^\ell$ $(n = 1, 2, \ldots)$ be a uniformly integrable sequence of functions converging weakly to f. Then,*

$$f(t) \in \text{con}\, \text{w-Ls}\{f_n(t)\} \ \mu\text{-}a.e.$$

Using Artstein's result we can prove the following proposition.

Proposition 4.2. *Let (T, τ, μ) be a finite measure space and X be a separable Banach space whose dual X^* has the RNP. Let $\{f_n : n = 1, 2, \ldots\}$ be a sequence in $L_p(\mu, X), 1 \leq p < \infty$ such that f_n converges weakly to $f \in L_p(\mu, X)$. Suppose that for all n $(n = 1, 2, \ldots)$, $f_n(t) \in F(t)$ μ-a.e. where $F : T \to 2^X$ is a weakly compact, nonempty valued correspondence. Then*

$$f(t) \in \overline{\text{con}}\, \text{w-Ls}\{f_n(t)\} \ \mu\text{-}a.e.$$

Proof. Since f_n converges weakly to f and X^* has the RNP, for any $\psi \in (L_p(\mu, X))^* = L_q(\mu, X^*)$ (where $\frac{1}{p} + \frac{1}{q} = 1$), we have that $\langle \psi, f_n \rangle = \int_T \langle \psi(t), f_n(t) \rangle d\mu(t)$ converges to $\langle \psi, f \rangle = \int_T \langle \psi(t), f(t) \rangle d\mu(t)$. Define the functions $h_n : T \to R$ and $h : T \to R$ by $h_n(t) = \langle \psi(t), f_n(t) \rangle$ and $h(t) = \langle (t), f(t) \rangle$ respectively. Since for each n, $f_n(t) \in F(t)$ μ-a.e. and $F(\cdot)$ is weakly compact, h_n is bounded and uniformly integrable. Also, it is easy to check that h_n converges weakly to h. In fact, let $g \in L_\infty(\mu, \mathbb{R})$ and let $M = \|g\|_\infty$, then

$$\left| \int_T g(t)(h_n(t) - h(t)) d\mu(t) \right| = \left| \int_T g(t)(\langle \psi(t), f_n(t) \rangle - \langle \psi(t), f(t) \rangle) d\mu(t) \right| \leq M |\langle \psi, f_n \rangle - \langle \psi, f \rangle| \tag{4.1}$$

and (4.1) can become arbitrarily small since as it was noted above $\langle \psi, f_n \rangle$ converges to $\langle \psi, f \rangle$.

By Proposition 4.1, we have that $h(t) \in \text{con}\, \text{w-Ls}\{h_n(t)\} \subset \overline{\text{con}}\, \text{w-Ls}\{h_n(t)\}$ μ-a.e., i.e., $\langle \psi(t), f(t) \rangle \in \overline{\text{con}}\, \text{w-Ls}\{\langle \psi, f_n(t) \rangle\} = \langle \psi(t), \overline{\text{con}}\, \text{w-Ls}\{f_n(t)\} \rangle$ μ-a.e. and consequently,

$$\int_T \langle \psi(t), f(t) \rangle d\mu(t) \in \int_T \langle \psi(t), x(t) \rangle d\mu(t), \tag{4.2}$$

where $x(\cdot)$ is a selection from $\overline{\text{con}}\, \text{w-Ls}\{f_n(\cdot)\}$. It follows from (4.2) that:

$$f \in S^p_{\overline{\text{con}}\, \text{w-Ls}\{f_n\}}. \tag{4.3}$$

To see this, suppose by way of contradiction that $f \notin S^p_{\overline{\text{con}}\, \text{w-Ls}\{f_n\}}$, then by the separating hyperplane theorem,[1] there exists $\psi \in (L_p(\mu, X))^* = L_q(\mu, X^*)$, $\psi \neq 0$ such that $\langle \psi, f \rangle > \sup\{\langle \psi, x \rangle : x \in S^p_{\overline{\text{con}}\, \text{w-Ls}\{f_n\}}\}$, i.e., $\int_T \langle \psi(t), f(t) \rangle d\mu(t) > \int_T \langle \psi(t), x(t) \rangle d\mu(t)$, where $x(\cdot)$ is a selection from $\overline{\text{con}}\, \text{w-Ls}\{f_n(\cdot)\}$, a contradiction to (4.2). Hence, (4.3) holds and we can conclude that $f(t) \in \overline{\text{con}}\, \text{w-Ls}\{f_n(t)\}$ μ-a.e. This completes the proof of Proposition 4.2.

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[1] Note that the set $S^p_{\overline{\text{con}}\, \text{w-Ls}\{f_n\}}$ is nonempty. In fact, since $\text{w-Ls}\{f_n\}$ is lower measurable and nonempty valued, so is $\overline{\text{con}}\, \text{w-Ls}\{f_n\}$. So, $\overline{\text{con}}\, \text{w-Ls}\{f_n\}$ admits a measurable selection (recall the Kuratowski and Ryll-Nardzewski measurable selection theorem). Obviously the measurable selection is also integrable since $\overline{\text{con}}\, \text{w-Ls}\{f_n\}$ lies in a weakly compact subset of X. Therefore, we can conclude that $S^p_{\overline{\text{con}}\, \text{w-Ls}\{f_n\}}$ is nonempty.

Remark 4.1. Proposition 4.2 remains true without the assumption that X^* has the RNP. The proof proceeds as follows: Since f_n converges weakly to f we have that $\langle \phi, f_n \rangle$ converges to $\langle \phi, f \rangle$ for all $\phi \in (L_p(\mu, X))^*$. It follows from a standard result [see for instance Dinculeanu (1973, p. 112)] that ϕ can be represented by a function $\psi : T \to X^*$ such that $\langle \psi, x \rangle$ is measurable for every $x \in X$ and $\|\psi\| \in L_q(\mu, \mathbb{R})$. Hence, $\langle \psi, f_n \rangle = \int_T \langle \psi(t), f_n(t) \rangle d\mu(t)$ and $\langle \psi, f \rangle = \int_T \langle \psi(t), f(t) \rangle d\mu(t)$. Define the functions $h_n : T \to R$ and $h : T \to R$ by $h_n(t) = \langle \psi(t), f_n(t) \rangle$ and $h(t) = \langle \psi(t), f(t) \rangle$ respectively. One can now proceed as in the proof of Proposition 4.2 to complete the argument.

We are now ready to complete the proof of Theorem 4.1. Denote the net $\{f_\lambda : \lambda \in \Lambda\}$ by B. Since by assumption for all $\lambda \in \Lambda$, $f_\lambda(t) \in F(t)$ μ-a.e. where $F : T \to 2^X$ is an integrably bounded, weakly compact, convex, nonempty valued correspondence we can conclude that for all $\lambda \in \Lambda$, f_λ lies in the weakly compact set S_F^p (recall Diestel's theorem on weak compactness, Theorem 3.1). Hence, the weak closure of B, i.e., $w - \mathrm{cl}\, B$, is weakly compact. By the Eberlein-Smulian Theorem [see Dunford-Schwartz (1958, p. 430)], $w-\mathrm{cl}\, B$ is weakly sequentially compact. Obviously the weak limit of f_λ, i.e., f, belongs to $w-\mathrm{cl}\, B$. From Whitley's theorem[2] [Aliprantis-Burkinshaw (1985, Lemma 10.12, p. 155)], we know that if $f \in w - \mathrm{cl}\, B$, then there exists a sequence $\{f_{\lambda_n} : n = 1, 2, \ldots\}$ in B such that f_{λ_n} converges weakly to f. Since the sequence $\{f_{\lambda_n} : n = 1, 2, \ldots\}$ satisfies all the assumptions of Proposition 4.2 and Remark 4.1 we can conclude that $f(t) \in \overline{\mathrm{con}}\, \text{w-Ls}\{f_{\lambda_n}(t)\}$ μ-a.e. This completes the proof of Theorem 4.1.

An immediate conclusion of Theorem 4.1 is the following useful corollary.

Corollary 4.1. *Let (T, τ, μ) be a finite measure space and X be a separable Banach space. Let $\{f_n : n = 1, 2, \ldots\}$ be a sequence of functions in $L_p(\mu, X)$, $1 \leq p < \infty$ such that f_n converges weakly to $f \in L_p(\mu, X)$. Suppose that for all n $(n = 1, 2, \ldots)$, $f_n(t) \in F(t)$ μ-a.e., where $F : T \to 2^X$ is a weakly compact, integrably bounded, nonempty valued correspondence. Then*

$$f(t) \in \overline{\mathrm{con}}\, \text{w-Ls}\{f_n(t)\} \ \mu\text{-a.e.}$$

[2] See also Kelley-Namioka (1963, exercise L, p. 165).

Bibliographical Notes. Theorem 4.1 and its proof are due to Yannelis (1989). Corollary 4.1 generalizes previous results of Artstein (1979) and Khan-Majumdar (1986). A related result to Corollary 4.1 has also been obtained by Balder (1988) and Castaing (1988). Ostroy-Zame (1988) have used Corollary 4.1 in order to prove the existence of an equilibrium in economies with a continuum of agents and commodities.

5. Properties of the Set of Integrable Selections from a Correspondence

We begin by proving s-Li and w-Ls versions of Fatou's Lemma for the set of integrable selections.

Theorem 5.1. *Let (T, τ, μ) be a complete, finite measure space and let X be a separable Banach space. If $\phi_n : T \to 2^X$ $(n = 1, 2, \dots)$ is a sequence of integrably bounded correspondences having a measurable graph, i.e., $G_{\phi_n} \in \tau \otimes \beta(X)$, then*

$$S^1_{\text{s-Li}\,\phi_n} \subset \text{s-Li}\, S^1_{\phi_n}.$$

Proof. Let $x \in S^1_{\text{s-Li}\,\phi_n}$, i.e., $x(t) \in \text{s-Li}\,\phi_n(t)$ μ-a.e., we must show that $x \in \text{s-Li}\, S^1_{\phi_n}$. First note that $x(t) \in \text{s-Li}\,\phi_n(t)$ μ-a.e. implies that there exists a sequence $\{x_n : n = 1, 2, \dots\}$ such that $s - \lim_{n\to\infty} x_n(t) = x(t)$ μ-a.e. and $x_n(t) \in \phi_n(t)) = 0$ μ-a.e. For each n $(n = 1, 2, \dots)$, define the correspondence $A_n : T \to 2^X$ by

$$A_n(t) = \left\{ y \in \phi_n(t) : \|y - x(t)\| \le \text{dist}(x(t), \phi_n(t)) + \frac{1}{n} \right\}.$$

Clearly for all n $(n = 1, 2, \dots)$ and for all $t \in T$, $A_n(t) \ne \emptyset$. Moreover, $A_n(\cdot)$ has measurable graph. Indeed, the function $g : T \times X \to [-\infty, \infty]$ defined by $g(t, y) = \|y - x(t)\| - \text{dist}(x(t), \phi_n(t))$ is measurable in t and continuous in y and therefore by a standard result [see Yannelis (1990a, Proposition 3.1)], $g(\cdot\,, \cdot)$ is jointly measurable with respect to the product σ-algebra $\tau \otimes \beta(X)$. It is easy to see that:

$$\begin{aligned} G_{A_n} &= \left\{ (t, y) \in T \times X : g(t, y) \le \frac{1}{n} \right\} \cap G_{\phi_n} \\ &= g^{-1}\left(\left[-\infty, \frac{1}{n} \right] \right) \cap G_{\phi_n}. \end{aligned}$$

Since $\phi_n(\cdot)$ has a measurable graph and $g(\cdot,\cdot)$ is jointly measurable, we can conclude that G_{A_n} belongs to $\tau \otimes \beta(X)$, i.e., $A_n(\cdot)$ has a measurable graph. By the Aumann measurable selection theorem there exists a measurable function $f_n : T \to X$ such that $f_n(t) \in A_n(t)$ μ-a.e. Since $x(t) \in \text{s-Li}\,\phi_n(t)$ μ-a.e., $\lim_{n\to\infty} \text{dist}(x(t), \phi_n(t)) = 0$ μ-a.e. which implies that $\lim_{n\to\infty} \|f_n(t) - x(t)\| = 0$ μ-a.e. Since $f_n(t) \in \phi_n(t)$ μ-a.e. and $\phi_n(\cdot)$ is integrably bounded, by the dominated convergence theorem [see Diestel-Uhl (1977, p. 45)], $f_n(\cdot)$ is Bochner integrable, i.e., $f_n \in L_1(\mu, X)$. Hence, $x \in \text{s-Li}\, S^1_{\phi_n}$ and this completes the proof of Theorem 5.1.

Theorem 5.2. *Let (T, τ, μ) be a finite measure space, X be a separable Banach space and let $\phi_n : T \to 2^X$ $(n = 1, 2, \ldots)$ be a sequence of nonempty, closed valued correspondences such that:*

(i) For all n $(n = 1, 2, \ldots)$, $\phi_n(t) \subset F(t)$ μ-a.e., where $F : T \to 2^X$ is an integrably bounded, weakly compact, convex, nonempty-valued correspondence.

Then

$$\text{w-Ls}\, S^1_{\phi_n} \subset S^1_{\overline{\text{con}}\,\text{w-Ls}\,\phi_n}.$$

Moreover, assume that w-Ls $\phi_n(\cdot)$ *is closed and convex valued.*

Then,

$$\text{w-Ls}\, S^1_{\phi_n} \subset S^1_{\text{w-Ls}\,\phi_n}.$$

Proof. Let $x \in \text{w-Ls}\, S^1_{\phi_n}$, i.e., there exists $x_k \in S^1_{\phi_{n_k}}$ $(k = 1, 2, \ldots)$ such that x_k converges weakly to x. We wish to know that $x \in S^1_{\overline{\text{con}}\,\text{w-Ls}\,\phi_n}$. Since x_k converges weakly to x and x_k lies in a weakly compact set, it follows from Proposition 4.2 that $x(t) \in \overline{\text{con}}\,\text{w-Ls}\{x_k(t)\}$ μ-a.e. and therefore $x(t) \in \overline{\text{con}}\,\text{w-Ls}\,\phi_n(t)$ μ-a.e. Since by assumption for each n, $\phi_n(\cdot)$ lies in the integrably bounded, convex set $F(\cdot)$, we can conclude that $x \in S^1_{\overline{\text{con}}\,\text{w-Ls}\,\phi_n}$. This completes the proof of the fact that:

$$\text{w-Ls}\, S^1_{\phi_n} \subset S^1_{\overline{\text{con}}\,\text{w-Ls}\,\phi_n}. \tag{5.1}$$

Since w-Ls $\phi_n(\cdot)$ is closed and convex (hence weakly closed), we have that $\text{w-Ls}\,\phi_n(\cdot) = \overline{\text{con}}\,\text{w-Ls}\,\phi_n(\cdot)$ and therefore,

$$S^1_{\text{w-Ls}\,\phi_n} = S^1_{\overline{\text{con}}\,\text{w-Ls}\,\phi_n}. \tag{5.2}$$

Combining now (5.1) and (5.2) we can conclude that $\text{w-Ls}\, S^1_{\phi_n} \subset S^1_{\text{w-Ls}\,\phi_n}$. This completes the proof of the theorem.

Combining Theorems 5.1 and 5.2 we can obtain the following dominated convergence result for the set of integrable selections from a correspondence.

Corollary 5.1. *Let (T, τ, μ) be a complete finite measure space and X be a separable Banach space. Let $\phi_n : T \to 2^X$ $(n = 1, 2, \ldots)$ be a sequence of closed valued and lower measurable correspondences such that:*

(i) For each n $(n = 1, 2, \ldots)$, $\phi_n(t) \subset F(t)$ μ-a.e., where $F : T \to 2^X$ is an integrably bounded, weakly compact, convex, nonempty valued correspondence,

(ii) $\phi_n(t) \xrightarrow{\text{K-M}} \phi(t)$ μ-a.e., and

(iii) $\phi(\cdot)$ is convex valued.

Then

$$S^1_{\phi_n} \xrightarrow{\text{K-M}} S^1_{\phi}.$$

Proof. First note that since for each n $(n = 1, 2, \ldots)$, $\phi_n(\cdot)$ is closed valued and lower measurable, $G_{\phi_n} \in \tau \otimes \beta(X)$, i.e., $\phi_n(\cdot)$ has a measurable graph and so does $\text{s-Li}\,\phi_n(\cdot)$. Now if $\phi(t) = \text{s-Li}\,\phi_n(t) = \text{w-Ls}\,\phi_n(t)$ μ-a.e., it follows from Theorems 5.1 and 5.2 that:

$$S^1_{\phi} = S^1_{\text{s-Li}\,\phi_n} \subset \text{s-Li}\,S^1_{\phi_n} \subset \text{w-Ls}\,S^1_{\phi_n} \subset S^1_{\text{w-Ls}\,\phi_n} = S^1_{\phi}.$$

Therefore

$$S^1_{\phi} = \text{s-Li}\,S^1_{\phi_n} = \text{w-Ls}\,S^1_{\phi_n},$$

and we can conclude that $S^1_{\phi_n} \xrightarrow{\text{K-M}} S^1_{\phi}$. This completes the proof of the Corollary.

The lemma below will be used to prove Theorem 5.3.

Lemma 5.1. *Let (T, τ, μ) be a complete finite measure space, X be a separable Banach space and $F : T \to 2^X$ be a nonempty closed valued and lower measurable correspondence. Let $\{f_i : i = 1, 2, \ldots\}$ be a sequence in S^p_F, $(1 \le p < \infty)$ such that $F(t) = \text{cl}\{f_i(t) : i = 1, 2, \ldots\}$ μ-a.e. Then, for each $f \in S^p_F$ and $\delta > 0$, there exists a finite measurable partition $\{A_1, A_2, \ldots, A_m\}$ of (T, τ) such that*

$$\left\| f - \sum_{i=1}^{m} \chi_{A_i} f_i \right\|_p < \delta.$$

Proof. Consider a strictly positive $v \in L_1(\mu, \mathbb{R})$ such that $\int_{t\in T} v(t)d\mu(t) < \frac{\delta^p}{3}$. We can find a countable measurable partition $\{B_i\}$ of (T,τ) such that

$$\|f(t) - f_i(t)\|_p < v(t), \quad \text{for almost all } t \in B_i,\ i \geq 1.$$

Pick an integer m so that

$$\sum_{i=m+1}^{\infty} \int_{B_i} \|f(t)\|_p d\mu(t) < \frac{(\delta/2)^2}{3},$$
$$\sum_{i=m+1}^{\infty} \int_{B_i} \|f_1(t)\|_p d\mu(t) < \frac{(\delta/2)^2}{3},$$

and define a measurable partition $\{A_1, \ldots, A_m\}$ as follows:

$$A_1 = B_1 \cup \left(\bigcup_{i=m+1}^{\infty} B_i \right), \quad A_j = B_j \text{ for } j = 2, \ldots, m.$$

Then it can be easily seen that:

$$\begin{aligned} \left\| f - \sum_{i=1}^{m} \chi_{A_i} f_i \right\|_p &= \sum_{i=1}^{m} \int_{B_i} \|f(t) - f_i(t)\|_p d\mu(t) \\ &\quad + \sum_{i=m+1}^{\infty} \int_{B_i} \|f(t) - f_1(t)\|_p d\mu(t) \\ &= \int_T v(t)d\mu(t) + \sum_{i=m+1}^{\infty} 2^p \int_{B_i} (\|f(t)\|_p \\ &\quad + \|f_1(t)\|_p) d\mu(t) < \delta. \end{aligned}$$

Theorem 5.3. *Let (T,τ,μ) be a complete finite measure space, X be a separable Banach space and $F : T \to 2^X$ be a closed, nonempty valued and lower measurable correspondence. Suppose that S_F^p, $(1 \leq p < \infty)$ is nonempty. Then*

$$S^p_{\overline{\text{con}}F} = \overline{\text{con}} S^p_F.$$

Proof. Define the correspondence $\tilde{F} : T \to 2^X$ by $\tilde{F}(t) = \overline{\text{con}}F(t)$. It can be easily checked that $\tilde{F}(\cdot)$ is lower measurable and obviously closed and convex valued. Moreover, $S^p_{\tilde{F}}$ is closed and convex. Clearly, $\overline{\text{con}}S^p_F \subset S^p_{\overline{\text{con}}F}$ since $S^p_F \subset S^p_{\overline{\text{con}}F}$. To prove that $S^p_{\overline{\text{con}}F} \subset \overline{\text{con}}S^p_F$, consider the sequence $\{f_i : i = 1,2,\dots\}$ in S^p_F where $\mathrm{cl}\{f_i(t) : i = 1,2,\dots\} = F(t)$ μ-a.e. Define the set

$$U = \left\{ g : \sum_{i=1}^{n} \lambda_i f_i,\ \lambda_i \geq 0,\ \text{rational},\ \sum_{i=1}^{n} \lambda_i = 1,\ n \geq 1 \right\}.$$

Observe that U is a countable subset of $S^p_{\overline{\mathrm{con}}F}$ and $\overline{\mathrm{con}}F(t) = \mathrm{cl}\{g(t) : g \in U\}$ μ-a.e. It follows from Lemma 5.1 that for each $f \in S^p_{\overline{\mathrm{con}}F}$ and for each $\delta > 0$ we can find a finite measurable partition $\{A_1, A_2, \dots, A_m\}$ of (T, τ) and functions $g_1, g_2, \dots, g_m$ in U such that:

$$\left\| f - \sum_{k=1}^{m} \chi_{A_k} g_k \right\|_p < \lambda.$$

We can now find an integer n so that, for

$$1 \leq k \leq m,\ g_k = \sum_{i=1}^{n} \lambda_{ki} f_i \text{ where } \lambda_{ki} \geq 0,\ \sum_{i=1}^{n} \lambda_{ki} = 1.$$

Observe that:

$$\begin{aligned} \sum_{k=1}^{m} \chi_{A_k} g_k &= \sum_{k=1}^{m} \chi_{A_k} \left(\sum_{i=1}^{n} \lambda_{ki} f_i \right) \\ &= \sum_{(i_1,\dots,i_m)} (\lambda_{1i_1}, \dots, \lambda_{mi_m}) \left(\sum_{k=1}^{m} \chi_{A_k} f_{i_k} \right), \end{aligned}$$

where $(i_1, \dots, i_m)$ is taken for $1 \leq i_k \leq n$, $k = 1, 2, \dots, m$. Therefore, $\sum_{k=1}^{m} \chi_{A_k} g_k$ is a convex combination of functions in S^p_F and we can conclude that $f \in \overline{\mathrm{con}} S^p_F$. This completes the proof of Theorem 5.3.

Below we consider correspondences of two variables and assume that they are measurable in the one variable and u.s.c. or l.s.c. in the other. We then ask the question as to whether the set of all integrable selections of the correspondence is either u.s.c. or l.s.c.

Theorem 5.4. *Let (T, τ, μ) be a complete, finite measure space, P be a metric space and X be a separable Banach space. Let $\psi : T \times P \to 2^X$ be a nonempty valued, integrably bounded correspondence, such that for* *each fixed $t \in T$, $\psi(t,\cdot)$ is q.u.s.c. and for each fixed $p \in P$, $\psi(\cdot\,,p)$ has a measurable graph. Then*

$$S^1_\psi(\cdot) \text{ is q.u.s.c.}$$

Proof. Let $\tilde{B}$ be the open unit ball in $L_1(p,X)$ and ν be a small positive number. We must show that if $\{p_n : n = 1,2,\ldots\}$ is a sequence in p converging to $p \in P$, then for a suitable n_0, $S^1_\psi(p_n) \subset S^1_\psi(p) + \nu\tilde{B}$ for all $n \geq n_0$.

We begin by finding the suitable n_0. Since for each fixed $t \in T$, $\psi(t,\cdot)$ is q.u.s.c. we can find a minimal M_t such that

$$\psi(t,p_n) \subset \psi(t,p) + \delta B \text{ for all } n \geq M_t, \tag{5.3}$$

where $\delta = \frac{\nu}{3\mu(T)}$ (and B is the open unit ball in X).

We now show that M_t is a measurable function of t. However, first we make a few observations. By assumption for each fixed p and n, $G_{\psi(\cdot\,,p_n)+\delta B} \in \tau\otimes\beta(X)$ and so does $(G_{\psi(\cdot\,,p_n)+\delta B})^c$, (where S^c denotes the complement of the set S). It is easy to see that $G_{\psi(\cdot\,,p)} \cap (G_{\psi(\cdot\,,p_n)+\delta B})^c \in \tau \otimes \beta(X)$. Therefore, the set

$$U = \{(t,x) \in T \times X : (t,x) \in G_{\psi(\cdot\,,p)} \cap (G_{\psi(\cdot\,,p_n)+\delta B})^c\}$$

belongs to $\tau \otimes \beta(X)$.

It follows from the projection theorem [see for instance Yannelis (1990a)] that

$$\text{proj}_T(U) \in \tau.$$

Notice that,

$$\begin{aligned} \text{proj}_T(U) &= \{t \in T : \psi(t,p) \not\subset \psi(t,p_n) + \delta B\} \\ &= \{t \in T : \psi(t,p)/(\psi(t,p_n) + \delta B) \neq \emptyset\}. \end{aligned}$$

By virtue of the measurability of the above set we can now conclude that M_t is a measurable function of t. In particular, simply notice that,

$$\begin{aligned} \{t \in T : M_t = m\} = &\bigcap_{n \geq m} \{t \in T : \psi(t,p_n) \subset \psi(t,p) + \delta B\} \\ &\cap \{t \in T : \psi(t,p_{m-1}) \not\subset \psi(t,p) + \delta B\}. \end{aligned}$$

We are now in a position to choose the desired n_0. Since $\psi(\cdot,\cdot)$ is integrably bounded there exists $h \in L_1(\mu, \mathbb{R})$ such that for almost all $t \in T$, $\sup\{\|x\| : x \in \psi(t,p\} \leq h(t)$ for each $p \in P$.

Choose δ_1 such that if $\mu(S) < \delta_1$, $(S \subset T)$, then $\int_S h(t)d\mu(t) < \frac{\nu}{3}$. Since M_t is a measurable function of t, we can choose n_0 such that $\mu(\{t \in T : M_t \geq n_0\}) < \delta_1$. This is the desired n_0. Let $n \geq n_0$ and $y \in S^1_\psi(p_n)$. We must show that $y \in S^1_\psi(p) + \nu\tilde{B}$.

By assumption, for each fixed $p \in P$, $\psi(\cdot,p)$ has a measurable graph and $\psi(\cdot,\cdot)$ is nonempty valued. Hence, by the Aumann measurable selection theorem there exists a measurable function $f_1 : T \to X$ such that $f_1(t) \in \psi(t,p)$ μ-a.e. Define the correspondence $\phi : T \to 2^X$ by $\theta(t) = (\{y(t)\} + \delta B) \cap \psi(t,p)$. It follows from (5.3) that for all $t \in T_0 = \{t : M_t \leq n_0\}$, $\theta(t) \neq \emptyset$. Moreover, $\theta(\cdot)$ has a measurable graph. Another application of the Aumann measurable selection theorem allows us to guarantee the existence of a measurable function $f_2 : T \to X$ such that $f_2(t) \in \theta(t)$ μ-a.e. Define $f : T \to X$ by

$$f(t) = \begin{cases} f_1(t) & \text{for } t \notin T_0 \\ f_2(t) & \text{for } t \in T_0. \end{cases}$$

Then $f(t) \in \psi(t,p)$ μ-a.e. and since $\psi(\cdot,\cdot)$ is integrably bounded we can conclude that $f \in S^1_\psi(p)$. If we show that $\|f-y\| < \nu$ then $y \in S^1_\psi(p)+\nu\tilde{B}$ and we will be done. But this is easy to see. We have

$$\begin{aligned}\|f-y\| &= \int_{T/T_0} \|f_1(t) - y(t)\| d\mu(t) + \int_{T_0} \|f_2(t) - y(t)\| d\mu(t) \\ &< 2\int_{T/T_0} h(t)d\mu(t) + \int_{T_0} \delta d\mu(t) \\ &< \frac{2\nu}{3} + \delta\mu(T) = \frac{2\nu}{3} + \frac{\nu}{3\mu(T)} \cdot \mu(T) = \nu.\end{aligned}$$

This completes the proof of the theorem.

Remark 5.1. If in addition to the assumptions of Theorem 5.4, it is assumed that $S^1_\psi(\cdot)$ is compact valued, then we can conclude that $S^1_\psi(\cdot)$ is u.s.c. Moreover, by adding in Theorem 5.3 the assumption that $\psi(\cdot,\cdot)$ is convex valued and that for all $(t,p) \in T \times P$, $\psi(t,p) \subset K$ where K is a weakly compact, convex, nonempty subset of X, then it follows from Theorem 3.1 that $S^1_\psi(\cdot)$ is weakly compact valued and we can conclude that $S^1_\psi(\cdot)$ is weakly u.s.c., i.e., the set $\{p \in P : S^1_\psi \subset V\}$ is open in P for every weakly open subset V of X.

Theorem 5.5. *Let (T, τ, μ) be a complete, finite separable measure space, P be a metric space and X be a separable Banach space. Let $\psi : T \times P \to 2^X$ be a nonempty, closed, convex valued correspondence such that:*

(i) for each fixed $t \in T$, $\psi(t, \cdot)$ is weakly u.s.c.
(ii) for all $(t,p) \in T \times P$, $\psi(t,p) \subset K(t)$ where $K : T \to 2^X$ is an integrably bounded, weakly compact and nonempty valued correspondence.

Then

$$S^1_\psi(\cdot) \quad \textit{is weakly u.s.c.}$$

Proof. First note that by Theorem 3.1 S^1_K is weakly compact in $L_1(\mu, X)$. Since for each $p \in P$, $S^1_\psi(p)$ is a weakly closed subset of S^1_K, it is weakly compact. Since the measure space (T, τ, μ) is separable and X is a separable Banach space, $L_1(\mu, X)$ is separable. Hence, S^1_K is metrizable as it is a weakly compact subset of $L_1(\mu, X)$ [Dunford-Schwartz (1958, Theorem V.6.3, p. 434)]. Consequently, in order to show that $S^1_\psi(\cdot)$ is weakly u.s.c., it suffices that to show that $S^1_\psi(\cdot)$ has a weakly closed graph, i.e., if $\{p_n : n = 1, 2, \ldots\}$ is a sequence in P converging to $p \in P$, then

$$\text{w-Ls}\, S^1_\psi(p_n) \subset S^1_\psi(p).x$$

To this end let $x \in \text{w-Ls}\, S^1_\psi(p_n)$, i.e., there exists x_k $(k = 1, 2, \ldots)$ in $L_1(\mu, X)$ such that x_k converges weakly to $x \in L_1(\mu, X)$ and $x_k(t) \in \psi(t, p_{n_k})$ μ-a.e. We must show that $x \in S^1_\psi(p)$. It follows Theorem 4.1 that $x(t) \in \overline{\text{con}}\, \text{w-Ls}\{x_k(t)\}$ μ-a.e. and therefore,

$$x(t) \in \overline{\text{con}}\, \text{w-Ls}\, \psi(t, p_n) \ \mu\text{-a.e.} \tag{5.4}$$

Since for each fixed $t \in T$, $\psi(t, \cdot)$ has a weakly closed graph we have that:

$$\text{w-Ls}\, \psi(t, p_n) \subset \psi(t, p) \ \mu\text{-a.e.} \tag{5.5}$$

Combining (5.2) and (5.3) and taking into account the fact that ψ is convex valued we have that $x(t) \in \psi(t,p)$ μ-a.e. Since ψ is integrably bounded, we can conclude that $x \in S^1_\psi(p)$. This completes the proof of Theorem 5.5.

Alternatively, Theorem 5.5 can be proved by means of the Mazur lemma. As noted above, it suffices to show that $S^1_\psi(\cdot)$ has a weakly closed graph. To this end let $(p_n, y_n) \in G_{S^1_\psi}$ be a sequence such that p_n converges (in the metric topology) to p and y_n converges weakly to y. We must show that $y \in S^1_\psi(p)$. Since $y_n \in S^1_\psi(p_n)$, we have that $y_n(t) \in \psi(t, p_n)$ μ-a.e. By Mazur's lemma there exists $z_n(\cdot) \in \text{con} \bigcup_{n_0 \geq n} y_{n_0}(\cdot)$ such that $z_n(\cdot)$ converges in norm to $y(\cdot)$. Without loss of generality we may assume (otherwise pass to a subsequence) that $z_n(t)$ converges in norm to $y(t)$ for all $t \in T/S$, where S is a set of measure zero. Fix $t \in T/S$. Since by assumption $\psi(t, \cdot)$ is weakly u.s.c. for every small positive number δ, there exists n such that for all $n_0 \geq n$, $\psi(t, p_{n_0}) \subset \psi(t,p) + \delta B$, where B is the open unit ball in X. But then $\text{con} \bigcup_{n_0 \geq n} \psi(t, p_{n_0}) \subset \psi(t,p) + \delta B$ which implies that $z(t) \in \psi(t,p) + \delta B$ and consequently, $y(t) \in \psi(t,p) + \delta B$. Hence, $y(t) \in \psi(t,p)$ by letting δ converge to zero. Since t was arbitrary, $y(t) \in \psi(t,p)$ μ-a.e. Finally, since ψ is integrably bounded, we can conclude that $y \in S^1_\psi(p)$. This completes the proof.

Theorem 5.6. *Let (T, τ, μ) be a complete, finite measure space, X be a separable Banach space and P be a metric space. Let $\phi : T \times P \to 2^X$ be an integrably bounded correspondence such that for each fixed $t \in T$, $\phi(t, \cdot)$ is l.s.c. and for each fixed $p \in P$, $\phi(\cdot, p)$ has a measurable graph. Then*

$$S^1_\phi(\cdot) \quad \textit{is l.s.c.}$$

Proof. Let $\{p_n : n = 1, 2, \dots\}$ be a sequence in P converging to $p \in P$. We must show that $S^1_\phi(p) \subset \text{Li}\, S^1_\phi(p_n)$. Since by assumption for each fixed $t \in T$, $\phi(t, \cdot)$ is l.s.c. we have that $\phi(t,p) \subset \text{Li}\, \phi(t, p_n)$ for all $t \in T$, and therefore

$$S^1_\phi(p) \subset S^1_{\text{Li}\,\phi}(p_n). \tag{5.6}$$

It follows now from Theorem 5.1 that (5.6) can be written as:

$$S^1_\phi(p) \subset S^1_{\text{Li}\,\phi}(p_n) \subset \text{Li}\, S^1_\phi(p_n).$$

Hence,

$$S^1_\phi(\cdot) \quad \text{is l.s.c.}$$

The Corollary below follows directly from Theorems 5.4 and 5.6 and Remark 5.1.

Corollary 5.6. *Let (T, τ, μ) be a complete, finite measure space, P be a metric space and X be a separable Banach space. Let $\psi : T \times P \to 2^X$ be an integrably bounded, nonempty valued correspondence such that for each fixed $p \in P$, $\psi(\cdot, p)$ has a measurable graph and for each fixed $t \in T$, $\psi(t, \cdot)$ in continuous. Moreover, suppose that $S^1_\psi(\cdot)$ is compact valued. Then*

$$S^1_\psi(\cdot) \quad \textit{is continuous.}$$

Bibliographical Notes. Theorems 5.1, 5.2 and Corollary 5.1 are taken from Yannelis (1989). Theorem 5.3 and its proof is due to Hiai-Umegaki (1977). Theorems 5.4 and 5.6 are variations of some results given in Yannelis (1990). The proof of Theorem 5.5 is taken from Yannelis (1990). The alternative proof of Theorem 5.5 is due to Khan-Papageorgiou (1988).

6. Properties of the Integral of a Correspondence

In this section we present an infinite-dimensional generalization of the work of Aumann (1965).

Theorem 6.1. *Let (T, τ, μ) be a finite measure space and X be a separable Banach space. Let $\phi : T \to 2^X$ be a correspondence satisfying the following condition:*

(i) $\phi(t) \subset K(t)$ μ-a.e., where $K : T \to 2^X$ is an integrably bounded, weakly compact, convex, nonempty valued correspondence.

Then $\int \overline{\text{con}}\phi$ is weakly compact.

Proof. Note that since $\overline{\text{con}}\phi(\cdot)$ is (norm) closed and convex so is $S^1_{\overline{\text{con}}\phi}$. It is a consequence of the Separation Theorem that the weak and norm topologies coincide on closed convex sets. Thus, $S^1_{\overline{\text{con}}\phi}$ is weakly closed. Since $S^1_{\overline{\text{con}}\phi}$ is a subset of the set S^1_K and the latter set is weakly compact in $L_1(\mu, X)$ (recall Theorem 3.1), we can conclude that $S^1_{\overline{\text{con}}\phi}$ is weakly compact. Define the mapping $\psi : L_1(\mu, X) \to X$ by $\psi(x) = \int_{t \in T} x(t) d\mu(t)$. Certainly ψ is linear and norm continuous. By Theorem 15 in Dunford-Schwartz (1958, p. 422), ψ is also weakly continuous. Hence, $\psi(S^1_{\overline{\text{con}}\phi}) = \{(\psi(x) : x \in S^1_{\overline{\text{con}}\phi}\} = \int \overline{\text{con}}\phi$ is weakly compact. This completes the proof of the Theorem.

Theorem 6.2. *Let (T, τ, μ) be a finite atomless measure space, X be a Banach space and $\phi : T \to 2^X$ be a correspondence. Then* $\mathrm{cl} \int \phi$ *is convex.*

Proof. Let x, y be elements of the set $\mathrm{cl} \int \phi$, we must show that for any $\delta > 0$ and $\lambda \in (0,1)$ there exists $z \in \mathrm{cl} \int \phi$ such that $\|z-(\lambda x+(1-\lambda)y\| < \delta$. Fix $\delta > 0$ and choose x_δ, y_δ in $\int \phi$, such that $\|x - x_\delta\| < \frac{\delta}{2}$ and $\|y - y_\delta\| < \frac{\delta}{2}$. By the definition of the integral of the set-valued function ϕ, we have that there exist h, g in S^1_ϕ such that

$$\left(\int h, \int g\right) = (x_\delta, y_\delta).$$

Define the vector measure $V : \tau \to X \times X$ by

$$V(S) = \left(\int_S h, \int_S g\right).$$

Since the measure space (T, τ, μ) is atomless it follows from Uhl's theorem [see for instance Uhl (1969) or Diestel-Uhl (1977, p. 266)][3] that the norm closure of V is convex. Hence, we can find $\Omega \in \tau$ such that

$$\|V(\Omega) - \lambda V(T)\| < \frac{\delta}{2}.$$

Define the function $z : T \to X$ by

$$z(t) = \begin{cases} h(t) & \text{if } t \in \Omega \\ g(t) & \text{if } t \notin \Omega. \end{cases}$$

Then $z = \int z(t) d\mu(t) \in \int \phi$ and it can be easily checked that

$$\begin{aligned} \|z - (\lambda x + (1-\lambda)y)\| &\le \|z - (\lambda x_\delta + (1-\lambda)y_\delta)\| + \lambda\|x_\delta - x\| \\ &\quad + (1-\lambda)\|y_\delta - y\| \\ &< \delta. \end{aligned}$$

This completes the proof of Theorem 6.2.

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[3] Note that the assumption X has the RNP is not needed for proving that the norm closure of the vector measure V is convex.

Define the mapping $\pi : T \to X$ by $\pi(x) = \int_{t\in T} x(t)d\mu(t)$. Note that the integral of the correspondence $\phi : T \to 2^X$ is $\pi(S^1_\phi) = \{\pi(x) : x \in S^1_\phi\}$. With this observation in mind the reader can easily see that the result below is an immediate conclusion of Theorems 5.3, 6.1 and 6.2.

Theorem 6.3. *Let (T, τ, μ) be a finite, atomless measure space and X be a separable Banach space. Suppose that the correspondence $\phi : T \to 2^X$ satisfies assumption (i) of Theorem 6.1. Then*

$$\overline{\text{con}} \int \phi = \int \overline{\text{con}} \phi = \text{cl} \int \phi.$$

The results below are w-Ls and s-Li versions of the Fatou lemma and follow directly from Theorems 5.1 and 5.2 respectively.

Theorem 6.4. *Let (T, τ, μ) be a complete, finite measure space and X be a separable Banach space. If $\phi_n : T \to 2^X$ $(n = 1, 2, \dots)$ is a sequence of integrably bounded correspondences having a measurable graph, i.e., $G_{\phi_n} \in \tau \otimes \beta(X)$, then*

$$\int \text{s-Li}\, \phi_n \subset \text{s-Li} \int \phi_n.$$

Theorem 6.5. *Let (T, τ, μ) be a finite measure space, and X be a separable Banach space. Let $\phi_n : T \to 2^X$ $(n = 1, 2, \dots)$ be a sequence of nonempty, closed valued correspondences such that*

(i) For all n $(n = 1, 2, \dots)$, $\phi_n(t) \subset K(t)$ μ-a.e., where $K : T \to 2^X$ is an integrably bounded, weakly compact, convex, nonempty-valued correspondence.

Then

$$\text{w-Ls} \int \phi_n \subset \text{cl} \int \text{w-Ls}\, \phi_n.$$

Furthermore, if w-Ls $\phi_n(\cdot)$ *is closed and convex valued then*

$$\text{w-Ls} \int \phi_n \subset \int \text{w-Ls}\, \phi_n.$$

As a corollary of Theorems 6.4 and 6.5 (or alternatively from Corollary 5.1), we obtain a Lebesgue-Aumann-type dominated convergence result for the integral of a correspondence.

Corollary 6.1. *Let $\phi_n : T \to 2^X$ $(n = 1, 2, \ldots)$ be a sequence of correspondences satisfying all the assumptions of Theorems 6.4 and 6.5. Suppose that*

(i) $\phi_n(t) \xrightarrow{\text{K-M}} \phi(t)$ *μ-a.e.*

Then,

$$\int \phi_n \xrightarrow{\text{K-M}} \text{cl} \int \phi.$$

Moreover, if $\phi(\cdot)$ is convex valued, then

$$\int \phi_n \xrightarrow{\text{K-M}} \int \phi.$$

It should be noted that Theorems 6.1, 6.3 and 6.5 have been established using stronger assumptions than those adopted by Aumann (1965). However, the following example below will show that Aumann's results are false in infinite-dimensional spaces. In particular, without assumption (i) of Theorems 6.1, 6.3 and 6.5, all these results above become false.

Example 6.1. Let X in Theorem 6.1 be equal to ℓ_2, i.e., the space of real sequences (a_n) for which the norm $\|a_n\| = (\sum |a_n|^2)^{1/2}$ is finite, and let $T = [0, 2\pi]$, τ the Borel sets in $[0, 2\pi]$ and μ the Lebesgue measure on (T, τ). Let $K = \{x \in \ell_2 : \|x\| \leq 4\pi\}$. Since the space $X = \ell_2$ is reflexive the weak and weak* topologies coincide and thus by the Alaoglu theorem we can conclude that K is weakly compact. Choose a complete orthogonal system $\{w_n : n = 0, 1, \ldots\}$ in $L_2(\mu)$ such that each w_n assumes only the valued ± 1, $w_0 = \chi_{[0,2\pi]}$ and $\int_{t \in [0,2\pi]} w_n(t) d\mu(t) = 0$ for $n = 1, 2, \ldots$. For each n and each $E \in \tau$ let

$$\lambda_n(E) = 2^{-n} \int_{t \in E} \left(\frac{1 + w_n(t)}{2} \right) d\mu(t).$$

Define the vector measure $V : \tau \to \ell_2$ by

$$V(E) = (\lambda_0(E), \lambda_1(E), \ldots).$$

Then $\|V(E)\| < 2\mu(E)$ for each $E \in \tau$. Therefore, the vector measure V is countably additive, V is of bounded variation and it is obviously atomless. Clearly, 0 and $V(T)$ are in $V(\tau) = \{x \in \ell_2 : x = V(E),\ E \in \tau\}$ and note that $\frac{1}{2}VT$ is the convex hull of $V(\tau)$. The argument now of

Lyapunov adopted by Diestel-Uhl (1977, p. 262) can be used here to prove that there is no $E \in \tau$ such that $V(E) = \frac{1}{2}V(T)$, i.e., *the ℓ_2-valued atomless vector measure V of bounded variation is nonconvex.*

Observe now that ℓ_2 has the RNP. Hence, there exists a function $g \in L_1(\mu, \ell_2)$ such that for each $E \in \tau$, $V(E) = \int_{t\in T} \chi_E(t)g(t)d\mu(t)$. Since the norm closure of the range of V is convex [Theorem 10, p. 266 in Diestel-Uhl (1977)] we can conclude that $\frac{1}{2}V(T)$ is in the closure. Consequently, there exists a sequence $\{E_n : n = 1,2,\ldots\}$ in τ such that $\lim_{n\to\infty} V(E_n) = \frac{1}{2}V(T)$. For each n, define $\phi_n : T \to \ell_2$ by $\phi_n(t) = \chi_{E_n}(t)g(t)$. It can be easily checked that w-Ls$\,\phi_n$ is measurable [see for instance Yannelis (1990a, Lemma 3.12 and Remark 3.1)]. We now show that the inclusion w-Ls$\int \phi_n \subset \int$ w-Ls$\,\phi_n$ does not hold. In particular, since s-Ls$\int \phi_n \subset$ w-Ls$\int \phi_n$ we will prove a slightly stronger result, i.e., the inclusion s-Ls$\int \phi_n \subset$ w-Ls$\int \phi_n$ does not hold. Note that for each n, $\phi_n(t) \in \{0, g(t)\}$ μ -a.e. and so w-Ls$\,\phi_n \subset \{0, g(t), \{0, g(t)\}, \emptyset\}$. For any $\phi \in S^1_{\text{w-Ls}\,\phi_n}$ we have that $\phi(t) = \chi_E(t)g(t)$ μ-a.e., for $E \in \tau$. In order now for the inclusion s-Li$\int \phi_n \subset \int$ w-Ls$\,\phi_n$ to hold, we must have that $\frac{1}{2}V(T) \in \int$ w-Ls$\,\phi_n$, i.e., $\frac{1}{2}V(T) = \int_{t\in E} g(t)d\mu(t) = V(E)$. But as it was remarked above no such $E \in \tau$ exists (since the vector measure V is not convex). hence, the w-Ls version of *the Fatou Lemma fails in infinite-dimensional spaces.* Note that the above example also showed that the integral of the closed valued correspondences $F : T \to 2^{\ell_2}$ defined by $F(t) = \{0, g(t)\}$ *is not compact* (in fact it is not even closed!).[4] Finally, note that $\frac{1}{2}V(T) = \frac{1}{2}\int_{t\in T} g(t)d\mu(t) \in \text{con} \int F$ and $\frac{1}{2}V(T) \notin \int F$, i.e., the integral of the correspondence $F : T \to 2^{\ell_2}$ *is not convex.*[5]

The results below follow directly from Theorems 5.4, 5.6 and Corollary 5.3.

Theorem 6.6. *Let (T, τ, μ) be a complete, finite measure space, P be a metric space and X be a separable Banach space. Let $\psi : T \times P \to 2^X$ be a nonempty valued, integrably bounded correspondence, such that for each fixed $t \in T$, $\psi(t, \cdot)$ is q.u.s.c. and for each fixed $p \in P$, $\psi(\cdot, p)$ has a*

[4] Recall that Aumann (1965) demonstrated that if X is finite dimensional and $F : T \to 2^X$ is integrably bounded and closed valued, then $\int F$ is compact.

[5] Note that when X is finite dimensional the well-known result of Richter (1963) assures that $\int F$ is convex.

measurable graph. Then

$$\int \psi(t,\cdot) \quad \text{is q.u.s.c.}$$

Theorem 6.7. *Let (T,τ,μ) be a complete, finite measure space, X be a separable Banach space and P be a metric space. Let $\phi : T \times P \to 2^X$ be an integrably bounded correspondence such that for each fixed $t \in T$, $\phi(t,\cdot)$ is l.s.c. and for each fixed $p \in P$, $\phi(\cdot\,,p)$ has a measurable graph. Then*

$$\int \phi(t,\cdot) \quad \text{is l.s.c.}$$

Remark 6.1. If in addition to the assumptions of Theorem 6.7, it is assumed that $\int \psi(t,\cdot)$ is compact valued, then we can conclude that $\int \psi(t,\cdot)$ is u.s.c.

Corollary 6.2. *Let (T,τ,μ) be a complete, finite measure space, P be a metric space and X be a separable Banach space. Let $\psi : T \times P \to 2^X$ be a nonempty valued, integrably bounded correspondence, such that for each fixed $p \in P$, $\psi(\cdot\,,p)$ has a measurable graph and for each fixed $t \in T$, $\psi(t,\cdot)$ is continuous. Moreover, suppose that $\int_T \psi(t,\cdot)d\mu(t)$ is compact valued. Then*

$$\int_T \psi(t,\cdot)d\mu(t) \quad \text{is continuous.}$$

Below we prove a s-Ls version of the Fatou Lemma in infinite dimensions.

Theorem 6.8. *Let (T,τ,μ) be a complete, finite measure space and X be a separable Banach space. Let $\phi_n : T \to 2^X$ $(n = 1,2,\dots)$ be a sequence of nonempty valued, graph measurable correspondences, taking values in a compact, nonempty subset of X, Then*

$$\text{s-Ls} \int_T \phi_n(t)d\mu(t) \subset \text{cl} \int_T \text{s-Ls}\, \phi_n(t)d\mu(t).$$

Moreover, if $\text{Ls}\,\phi_n(\cdot)$ is convex valued, then

$$\text{s-Ls} \int_T \phi_n(t)d\mu(t) \subset \int_T \text{s-Ls}\, \phi_n d\mu(t).$$

Proof. Denote by P the interval $[0,1)$. Define the correspondence $\psi : T \times P \to 2^X$ by

$$\psi(t,p) = \begin{cases} \phi_n(t) & \text{if } \frac{1}{n+1} < p < \frac{1}{n} \\ \phi_n(t) \cup \phi_{n+1}(t) & \text{if } p = \frac{1}{n+1} \\ \mathrm{Ls}\, \phi_n(t) & \text{if } p = 0. \end{cases}$$

It can be easily checked that for each fixed $t \in T$, $\psi(t,\cdot)$ is u.s.c. and that for each fixed $p \in P$, $\psi(\cdot\,,p)$ has a measurable graph. Moreover, ψ is integrably bounded. Hence, ψ satisfies all the assumptions of Theorem 6.6 and thus, $\int_T \psi(t,\cdot)d\mu(t)$ is q.u.s.c. Let now $x \in \mathrm{Ls} \int_T \phi_n(t)d\mu(t)$, i.e., there exists x_{n_k} such that $\lim_{n\to\infty} x_{n_k} = x$, $x_{n_k} \in \int_T \phi_{n_k}(t)d\mu(t)$ $(k = 1,2,\ldots)$. We wish to show that $x \in \mathrm{cl} \int \phi_n(t)d\mu(t)$.

Since $\int_T \psi(t,\cdot)d\mu(t)$ is q.u.s.c. (see Section 2 for a definition) it follows that if p_{n_k} converges to 0 then $\int_T \psi(t,p_{n_k})d\mu(t) \subset \int_T(t,0)d\mu(t) + \nu B$ for all sufficiently large k (where ν is a small positive number and B denotes the open unit ball in X). Consequently, $x_{n_k} \in \int_T \psi(t,0)d\mu(t) + \nu B$ for all sufficiently large k and therefore, $x \in \mathrm{cl} \int_T \psi(t,0)d\mu(t) \equiv \mathrm{cl} \int_T \text{s-Ls}\, \phi_n(t)d\mu(t)$ as was to be shown. If now $\mathrm{Ls}\, \phi_n$ is convex valued (recall that s-Ls $\phi_n(\cdot)$ is closed valued as well), it follows from Theorem 6.1 and the first conclusion of Theorem 6.8 that

$$\text{s-Ls} \int_T \phi_n d\mu(t) \subset \mathrm{cl} \int_T \text{s-Ls}\, \phi_n(t)d\mu(t) = \int_T \text{s-Ls}\, \phi_n(t)d\mu(t).$$

This completes the proof of the Theorem.

We close this section by obtaining the following dominated converge result:

Theorem 6.9. *Let (T,τ,μ) be a complete, finite measure space and X be a separable Banach space. Let $\phi_n : T \to 2^X$ $(n = 1,2,\ldots)$ be a sequence of integrably bounded, nonempty valued correspondence having a measurable graph, such that*

(i) For all n $(n = 1,2,\ldots)$, $\phi_n(t) \subset K$ μ-a.e., where K is a compact, nonempty subset of X, and

(ii) $\phi_n(t) \to \phi(t)$ μ-a.e.

Then

$$\int_T \phi_n(t)d\mu(t) \to \mathrm{cl} \int_T \phi(t)d\mu(t).$$

Moreover, if $\phi(\cdot)$ is convex valued then

$$\int_T \phi_n(t)d\mu(t) \to \int_T \phi(t)d\mu(t).$$

Proof. Since by assumption $\phi_n(t) \to \phi(t)$ μ-a.e., i.e., $\phi(t) = \text{s-Li}\,\phi_n(t)$ $= \text{s-Ls}\,\phi_n(t)$ μ-a.e., it follows from Theorems 6.4 and 6.8 that:

$$\int \phi = \int \text{s-Li}\,\phi_n \subset \text{s-Li} \int \phi_n \subset \text{s-Ls} \int \phi_n \subset \text{cl} \int \text{s-Ls}\,\phi_n = \text{cl} \int \phi.$$

Therefore,

$$\text{cl} \int_T \phi(t)d\mu(t) = \text{s-Li} \int_T \phi_n(t)d\mu(t) = \text{s-Ls} \int_T \phi_n(t)d\mu(t),$$

i.e.,

$$\int_T \phi_n(t)d\mu(t) \to \text{cl} \int_T \phi(t)d\mu(t).$$

If now $\phi(\cdot)$ is convex valued, we can conclude (recall the second conclusion of Theorem 6.8) that:

$$\int \phi = \int \text{s-Li}\,\phi_n \subset \text{s-Li} \int \phi_n \subset \text{s-Ls} \int \phi_n \subset \int \text{s-Ls}\,\phi_n = \int \phi.$$

Thus,

$$\int_T \phi(t)d\mu(t) = \text{s-Li} \int_T \phi_n(t)d\mu(t) = \text{s-Ls} \int_T \phi_n(t)d\mu(t),$$

i.e.,

$$\int_T \phi_n(t)d\mu(t) \to \int_T \phi(t)d\mu(t),$$

and this completes the proof of Theorem 6.9.

Bibliographic Notes. A version of Theorem 6.1 is proved by Yannelis (1988). Theorem 6.2 is due to Datko (1973). The proof given here is taken from Khan (1985). It has been useful in general equilibrium theory [see for instance Rustichini-Yannelis (1989, 1991) and Ostroy-Zame (1988)]. It should be noted that Theorem 6.2 is the infinite dimensional version of Theorem 2 of Aumann (1965) [see also Debreu (1967)] and it was first proved by Datko (1973) for X being a reflexive separable

Banach space. The reflexivity assumption was subsequently relaxed by Khan (1985). Rustichini-Yannelis (1990) showed that if the dimensionality of the measure space is bigger than the dimensionality of the space X, then the conclusion of Theorem 6.3 can be strengthened to $\int \overline{\text{con}}\phi = \int \phi$.

The Example 6.1 is due to Lyapunov [see also Diestel-Uhl (1977, p. 262)]. The argument used to prove that several of the properties of the Aumann integral fail in an infinite dimensional setting is due to Rustichini (1989). Theorems 6.6–6.9 are due to Yannelis (1990). Related results to Theorems 6.6–6.9 were obtained by Debreu (1967).

7. The Gel'fand Integral

Let (T, τ, μ) be a finite measure space and X be a Banach space. Let $f : T \to X^*$ be a function such that $\langle f, x\rangle \in L_1(\mu)$ for all $x \in X$, then for each $A \in \tau$ the element x_A^* in X^* is called the *Gel'fand integral of f* over A, where

$$x_A^*(x) = \int_A \langle f(t), x\rangle d\mu(t) \quad \text{for all } x \in X.$$

We denote by $(S_\phi^1)^*$ the *set of all Gel'fand integrable selections from the correspondence* $\phi : T \to 2^{X^*}$, i.e.,

$$\begin{aligned}(S_\phi^1)^* &= \{x \in (L_1(\mu, X))^* : x(t) \in \phi(t)\ \mu\text{-a.e.}\}\\ &= \{x \in L_\infty(\mu, X^*) : x(t) \in \phi(t)\ \mu\text{-a.e.}\}.\end{aligned}$$

The *Gel'fand integral of the correspondence* $\phi : T \to 2^{X^*}$ is defined as follows:

$$\int \phi(t) d\mu(t) = \left\{ \int \langle f(t), x\rangle d\mu(t) : f \in (S_\phi^1)^* \text{ for all } x \in X \right\}.$$

Note that the above integral may be empty unless ϕ admits weak* measurable selections. A very useful result due to Khan (1985) which has found several applications in game theory and general equilibrium is the fact that the weak* closure of the Gel'fand integral of a correspondence is convex. This result can be proved adopting a similar argument used to prove Theorem 6.2 except that instead of using Uhl's Theorem one can now appeal to a result of Kluvalek [Kluvalek (1973, p. 46, Lemma 5)]. We state below a very useful result for the Gel'fand integral of a correspondence.

Theorem 7.1. *Let (T, τ, μ) be a complete, finite measure space, X^* be the dual of a separable Banach space and $\phi : T \to 2^{X^*}$ be a correspondence with a weak* measurable graph (i.e., $G_\phi \in \tau \otimes \beta_{w^*}(X^*)$, where $\beta_{w^*}(X^*)$ are the Borel subsets of X^* in the weak* topology of X^*) such that $\phi(t)$ is weak* closed and bounded for almost all t in T. Then for all $A \in \tau$,*

$$w^* - \text{cl} \int_A \phi = \int_A w^* - \overline{\text{con}}\phi.$$

Moreover, $\int_A w^* - \text{con}\,\phi$ is weak compact and convex.*

Bibliographical Notes. Theorem 7.1 is due to Khan (1985) and it has found important applications in general equilibrium theory [Rustichini-Yannelis (1989, 1991), Ostroy-Zame (1988)], game theory [Cotter (1990), Khan (1986)] and demand theory [Border (1989), Kim (1990)].

8. An Application

In this section we will indicate how some of the results in Yannelis (1990a) as well as theorems of this paper can be used to prove the existence of an equilibrium for an abstract economy with a measure space of agents.

An *abstract economy* Γ is a quadruple $[(T, \tau, \mu), X, P, A]$, where

(1) (T, τ, μ) is the *measure space of agents*,
(2) $X : T \to 2^Y$ is the *strategy correspondence* (where Y is a linear topological space),
(3) $P : T \times S^1_X \to 2^Y$ is a *preferences correspondence* such that $P(t, x) \subset X(t)$ for all $(t, x) \in T \times S^1_X$, and
(4) $A : T \times S^1_X \to 2^Y$ is a *constraint correspondence* such that $A(t, x) \subset X(t)$ for all $(t, x) \in T \times S^1_X$.

The interpretation of the preference correspondence $P : T \times S^1_X \to 2^Y$ is as follows: We read $y \in P(t, x)$ as "agent t strictly prefers y to $x(t)$ if the given strategies of other agents are fixed." Throughout this section we set $Y = \mathbb{R}^n$ and endow S^1_X with the weak topology.

An *equilibrium* for Γ is an $x^* \in S^1_X$ such that for almost all t in T the following conditions hold:

(i) $x^*(t) \in A(t, x^*)$ and

(ii) $P(t,x^*) \cap A(t,x^*) = \emptyset$.

Below we state the assumptions needed for the proof of our equilibrium existence theorem.

(8.1) (T, τ, μ) is a complete finite separable measure space.

(8.2) $X : T \to 2^Y$ is a correspondence such that:

- (a) It is integrably bounded and for all $t \in T$, $X(t)$ is a closed convex nonempty subset of Y;
- (b) $X(\cdot)$ is lower measurable.

(8.3) $A : T \times S^1_X \to 2^Y$ is a correspondence such that:

- (a) for each fixed $t \in T$, $A(t, \cdot)$ is continuous;
- (b) $A(\cdot\,,\cdot)$ is closed, convex and nonempty valued;
- (c) for each fixed $x \in S^1_X$, $A(\cdot\,, x)$ is lower measurable.

(8.4) $P : T \times S^1_X \to 2^Y$ is a correspondence such that:

- (a) for each fixed $t \in T$, $P(t, \cdot)$ has an open graph in $S^1_X \times Y$;
- (b) $x(t) \notin \operatorname{con} P(t,x)$ for all $x \in S^1_X$, μ-a.e.;
- (c) for every open subset V of Y, the set $\{(t,x) \in T \times S^1_X : A(t,x) \cap \operatorname{con} P(t,x) \cap V \neq \emptyset\}$ belongs to $\tau \otimes \beta_w(S^1_X)$, where $\beta_w(S^1_X)$ denotes the Borel σ-algebra for the weak topology on S^1_X.

We are now ready to state the following result:

Theorem 8.1. *Let $\Gamma = [(T, \tau, \mu), X, P, A]$ be an abstract economy satisfying (8.1)–(8.4). Then an equilibrium in Γ exists.*

Proof. Define the set-valued function $\psi : T \times S^1_X \to 2^Y$ by $\psi(t,x) = \operatorname{con} P(t,x)$. It can be easily checked that for each fixed $t \in T$, $\psi(t, \cdot)$ has an open graph in $S^1_X \times Y$ [see for instance Lemma 4.1 in Yannelis (1987)]. Define the set-valued function $\phi : T \times S^1_X \to 2^Y$ by $\phi(t,x) = \psi(t,x) \cap A(t,x)$. It follows from Lemma 4.2 in Yannelis (1987) that for each fixed $t \in T$, $\phi(t, \cdot)$ is weakly u.s.c., i.e., for every open subset V of Y, the set $\{x \in S^1_X : \phi(t,x) \cap V \neq \emptyset\}$ is weakly open in S^1_X. By assumption (8.4)(c), $\phi(\cdot\,,\cdot)$ is lower measurable. Let $U = \{(t,x) \in T \times S^1_X : \phi(t,x) \neq \emptyset\}$. By Theorem 4.2 in Yannelis (1990a) we can guarantee the existence of a Carathéodory-type selection, i.e., there exists a function $f : U \to Y$ such that $f(t,x) \in \phi(t,x)$ for all $(t,x) \in U$ and for each $t \in T$, $f(t, \cdot)$ is continuous on $U_t = \{x \in S^1_X : \phi(t,x) \neq \emptyset\}$ and for each $x \in S^1_X$, $f(\cdot\,, x)$ is measurable on $U_x = \{t \in T : \phi(t,x) \neq \emptyset\}$. Moreover, $f(\cdot\,,\cdot)$ is jointly measurable. Define the set-valued function $F : T \times S^1_X \to 2^Y$ by

$$F(t,x) = \begin{cases} \{f(t,x)\} & \text{if } (t,x) \in U \\ A(t,x) & \text{if } (t,x) \notin U. \end{cases}$$

It follows at once from the l.s.c. of $\phi(t,\cdot)$ that for each $t \in T$ the set $U_t = \{x \in S^1_X : \phi(t,x) \neq \emptyset\}$ is weakly open in S^1_X. Thus, by Lemma 6.1 in Yannelis-Prabhakar (1983) for each fixed $t \in T$, $F(t,\cdot)$ is weakly u.s.c. in the sense that the set $\{x \in S^1_X : F(t,x) \subset V\}$ is weakly open in S^1_X for every open subset V of Y. As in Yannelis (1987) one can easily check that for each $x \in S^1_X$, $F(\cdot, x)$ has a measurable graph. Also, $F(\cdot,\cdot)$ is closed, convex and nonempty valued. Define the set-valued function $\theta : S^1_X \to 2^{S^1_X}$ by

$$\theta(x) = \{y \in S^1_X : y(t) \in F(t,x) \ \mu\text{-a.e.}\}.$$

Note that by Theorem 3.1, S^1_X is weakly compact in $L_1(\mu, Y)$. Since the measure space (T, τ, μ) is separable, $L_1(\mu, Y)$ is a separable Banach space. Since, weakly compact subsets of a separable Banach space are metrizable, we can conclude that S^1_X is metrizable. Hence, it follows from Theorem 5.5 that $\theta(\cdot)$ is weakly u.s.c., i.e., for every weakly open subset V of S^1_X the set $\{x \in S^1_X : \theta(x) \subset V\}$ is weakly open in S^1_X. Appealing to the Aumann measurable selection theorem, we can conclude that $\theta(\cdot)$ is nonempty valued. Similarly, the set S^1_X is nonempty. Obviously $\theta(\cdot)$ is convex valued and so is the set S^1_X. It follows from the Fan-Glicksberg fixed point theorem that there exists $x^* \in S^1_X$ such that $x^* \in F(x^*)$. It can be easily now checked that the fixed point is by construction an equilibrium for the abstract economy Γ.

Bibliographical Notes. This section is based on Yannelis (1987) where we refer the reader for related results. A version of Theorem 8.1 was first proved by Khan-Vohra (1984). It should be noted that the notion of an equilibrium for an abstract economy is due to Debreu (1952) which in turn generalizes the notion of a noncooperative equilibrium for a game in normal form introduced by Nash (1951). For more applications of Carathéodory-type selections theorems as well as recent results on integration of set-valued function we recommend, the papers of Kim-Prikry-Yannelis (1989), Yannelis-Rustichini (1990, 1991), Balder-Yannelis (1990), and Yannelis (1990b). Finally a paper by Debreu (1967a) uses measure theory and measurable selections extensively.

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