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Equilibrium theory with unbounded consumption sets and non-ordered preferences Part I. Non-satiation☆

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ABSTRACT

A new condition is introduced for the existence of equilibrium for an economy where preferences need not be transitive or complete and the consumption set of each agent need not be bounded from below. The new condition allows us to extend the literature in two ways. First, the result of the paper can cover the case where the utility set for individually rational allocations may not be compact. As illustrated in Page et al. [Page Jr., F.H., Wooders, M.H., Monteiro, P.K., 2000. Inconsequential arbitrage. Journal of Mathematical Economics 34, 439–469], the no arbitrage conditions do not apply to an economy with a non-compact utility set. Second, we generalize the arbitrage-based equilibrium theory to the case of non-transitive preferences.

1. Introduction

The consumption set need not be bounded from below in an asset market economy where unlimited short sales are allowed. The existence of a Walrasian equilibrium with unbounded-from-below choice sets was initially addressed in Hart (1974) who introduced a condition on preferences eventually known as a no arbitrage condition. To generalize the condition of Hart (1974) on preferences, different arbitrage notions have been introduced in the literature (see for example, Hammond, 1983; Page, 1987; Werner, 1987; Chichilnisky, 1995; Page et al., 2000; Dana et al., 1999; Allouch, 2002, among others).

The arbitrage conditions are not only sufficient but also necessary for the existence of a Walrasian equilibrium in certain cases. The notion of arbitrage, however, has some limitations as a conceptual framework for explaining equilibrium beyond either the transitivity of preferences or the compactness of the utility set for individually rational allocations. First of all, equilibrium theory with unbounded consumption sets makes use of the assumption of transitivity of preferences. The no

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arbitrage conditions of the literature are based upon a utility representation of preferences.[1] One may be tempted to consider a naive extension of the notions of arbitrage to the case with non-transitive preferences simply by dropping the transitivity of preferences in their definition. We provide an example where the naive extension of the no arbitrage conditions fails to explain the existence of equilibrium with non-transitive preferences. Specifically, the set of Pareto optimal allocations is compact but the naively extended conditions fail in the example. This example illustrates the difficulty of finding sufficient conditions for the existence of equilibrium with non-transitive preferences as an extension of the no arbitrage conditions. To the best of our knowledge, there is no literature which attempts to extend the notion of arbitrage to the case of non-transitive preferences in the framework of Hart (1974).

Another limitation of the arbitrage-based approach lies in its capability to explain the existence of equilibrium even in the case with transitive preferences. As illustrated in Page et al. (2000), the no arbitrage conditions do not apply to the case where the utility set for individually rational allocations is not compact. Specifically, the no arbitrage conditions are no longer necessary for the existence of equilibrium in economies with a noncompact utility set.

The purpose of this paper is to prove the existence of equilibrium with unbounded consumption sets and non-transitive preferences. To this end we introduce a new condition which subsumes as a special case all the arbitrage conditions found in the literature in two respects. First, it covers the case where the utility set for individually rational allocations is compact and moreover, it explains the existence of equilibrium in the counterexample of Page et al. (2000) to the no arbitrage conditions. Second, we generalize the arbitrage-based equilibrium theory to the case of non-transitive preferences. In particular, the result of the paper applies to the aforementioned economy with non-transitive preferences in which the set of Pareto optimal allocations is compact, an equilibrium exists, but any known conditions are violated. In other words, the economy of the example has all the desired properties except for the transitivity of preferences, however, the existence of equilibrium cannot be explained by any conditions found in the existing literature. Furthermore, our analysis covers interdependent preferences. Thus, the equilibrium existence result of the paper includes as a special case not only all the equilibrium existence results with unbounded consumption sets but also gives as a corollary the standard equilibrium existence results without transitivity or completeness of preferences.

One example of the economies with non-ordered preferences on unbounded choice sets is a recent development of the capital asset pricing model (CAPM). Traditional CAPMs assume that agents' preferences are represented by a mean-variance utility function. Recently, Boyle and Ma (2006) show that the return–risk relationship of the CAPM can hold in the case where agents are risk averse with respect to mean-preserving spread (MPS).[2] The MPS-risk-averse preferences need not be transitive and thus, admit no utility representation in general. It is also widely recognized in the literature that transitive preferences fail to explain important anomalous phenomena such as preference reversal.[3] For example, experimental methods, Grether and Plott (1979) and Loomes and Sugden (1991) among others document the violation of transitivity of preferences by detecting the preference reversal phenomenon.

There have been also controversies over the presence of money pump which is frequently mentioned as a refutable evidence against non-transitive preferences. These controversies, however, are quite misleading from the viewpoint of equilibrium theory. Non-transitive preferences may allow a triad of objects $\{A, B, C\}$ which contributes to money pumps. Suppose that there exists an agent who strictly prefers A to B, B to C, and C to A, and he is currently endowed with A. Then an arbitrageur can pump money out of his pocket by offering repeatedly to exchange A for C, then C for B, and then B for A at a fee small enough to induce him to accept each offer. The literature of decision theory argues that money pumps are hard to find in the real world and therefore, the non-transitivity of preferences is not convincing. It will be illustrated later (Example 3.1.2) that the naive arguments are irrelevant to confirming the non-transitivity of preferences because the money pump never works and therefore, is not observable in equilibrium.

The paper is organized as follows: In Section 2 we present an auxiliary theorem which generalizes the standard equilibrium existence results. In particular, the auxiliary theorem is proved via an extension of the abstract equilibrium theorem of Shafer and Sonnenschein (1975), and Borglin and Keiding (1976), and it is the main mathematical tool used to prove our main theorem which is the focus of Section 3. Appendix contains all the proofs of the results of the paper.

2. A first extension of the classical equilibrium existence theorem

2.1. Notation

For a set A in a finite-dimensional Euclidean space $\mathbb{R}^\ell$, the following notation will be used.

- 2^A denotes the set of *all subsets* of the set A.

[1] Allouch (2002) is an exception because preferences are allowed to be incomplete. However, Allouch (2002) assumes that preferences are transitive.

[2] An agent is said to exhibit *MPS-risk-aversion* if for any random payoffs X and $Y = X + \epsilon$, he prefers X to Y whenever $E(\epsilon) = 0$ and $\text{Cov}(X, \epsilon) = 0$. For more rigorous treatment of MPS-risk-aversion, see Boyle and Ma (2006).

[3] Experimental work of Berg et al. (1985) documents that arbitrage profits can be extracted from the optimal choices which reveal the preference reversal phenomenon. Preference reversals arise when an agent prefers lottery A to lottery B but sets a higher selling price on B than on A. Such behavior violates transitivity of preferences.

- con$A$ denotes the *cone generated by the set A*, i.e., con $A = \cup_{\lambda \geq 0} \lambda A$.
- clcon$A$ denotes the *the closure of the cone generated by the set A.*
- co$A$ denotes the *convex hull* of the set A.
- cl$A$ denotes the *closure* of the set A.
- int$A$ denotes the *interior* of the set A.
- ∂A denotes the *boundary* of the set A.
- rint$A$ denotes the *relative interior* of the convex set A.[4]
- $\partial^r A$ denotes the *relative boundary* of the convex set A, i.e., $\partial^r A = (\text{cl}A) \setminus (\text{rint}A)$.

We adopt the following additional notation.

- $\|x\| = \sqrt{\sum_{j=1}^{\ell} x_j^2}$ denotes the *Euclidean norm* of the vector $x = (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell$.
- $x^n \to x$ denotes the *convergence* of the sequence $\{x^n : n = 1, 2, \ldots\}$ to the point x.
- $C^\circ(x, r)$ denotes the *open ball in* $\mathbb{R}^\ell$ centered at the point x with radius $r > 0$.
- $C(x, r)$ denotes the *closed ball in* $\mathbb{R}^\ell$ centered at the point x with radius $r > 0$.

2.2. Definitions

For two nonempty subsets Z and Y in $\mathbb{R}^\ell$, consider a correspondence $\varphi : Z \to 2^Y$. Let clφ, intφ and coφ denote the correspondence from Z to 2^Y which has the value cl$\varphi(z)$, int$\varphi(z)$ and co$\varphi(z)$ for all $z \in Z$, respectively. The correspondence φ is said to have an *open graph* if $G_\varphi \equiv \{(z, y) \in Z \times Y : y \in \varphi(z)\}$ is open in $Z \times Y$. The correspondence φ is said to have *open lower sections* if the set $\varphi^{-1}(y) = \{z \in Z : y \in \varphi(z)\}$ is open in Z for every $y \in Y$ and φ is said to have *open upper sections* (or *open-valued*) if $\varphi(z)$ is open in Y for every $z \in Z$. The correspondence φ is said to be *lower semi-continuous* if for every open set V of Y, $\{z \in Z : \varphi(z) \cap V \neq \emptyset\}$ is open in Z and φ is said to be *upper semi-continuous* if for every open set V of Y, $\{x \in Z : \varphi(x) \subset V\}$ is open in Z.

2.3. Auxiliary theorem: a generalization of the classical Walrasian equilibrium existence theorems

We consider the exchange economy which is populated by finitely many *agents* in I. We let I denote both the number and the set of agents and ℓ the finite number of commodities. For each $i \in I$, let $e_i \in \mathbb{R}^\ell$ denote the *initial endowment* and $X_i \subset \mathbb{R}^\ell$ denote the *choice set* of agent $i \in I$. We denote the *exchange economy* by $E = \{(X_i, e_i, P_i) : i \in I\}$ where $P_i : X \to 2^{X_i}$ is a *preference correspondence* where $X = \prod_{i \in I} X_i$. For points $x \in X$ and $y_i \in X_i$, we read $y_i \in P_i(x)$ as "agent i strictly prefers y_i to x_i provided that the other agents choose x_j for all $j \neq i$." For example, given a binary relation $\succ_i \subset X \times X$, we can define P_i as follows:

$$\forall x = (x_1, \ldots, x_I) \in X, \quad P_i(x) = \{y_i \in X_i : (x_1, \ldots, x_{i-1}, y_i, x_{i+1} \ldots, x_I) \succ_i x\}.$$

The preference ordering $\succ_i$ is so general that it allows interdependence among agents and need not be either transitive or complete. For each $p \in \mathbb{R}^\ell \setminus \{0\}$ and each $i \in I$, we define the sets $\beta_i(p) = \{x_i \in X_i : px_i < pe_i\}$ and $B_i(p) = \{x_i \in X_i : px_i \leq pe_i\}$. Notice that cl$\beta_i(p) = B_i(p)$ whenever $\beta_i(p) \neq \emptyset$ and X_i is convex.

Definition 2.3.1. An *equilibrium* for the exchange economy E is a pair $(p, x) \in (\mathbb{R}^\ell \setminus \{0\}) \times X$ such that

(i) $x_i \in B_i(p)$ for all $i \in I$,
(ii) $P_i(x) \cap B_i(p) = \emptyset$ for all $i \in I$, and
(iii) $\sum_{i \in I}(x_i - e_i) = 0$.

The pair (x, p) is a *quasi-equilibrium* for the economy E if it satisfies (i), (iii), and the following condition.

(ii$'$) $P_i(x) \cap \beta_i(p) = \emptyset$ for all $i \in I$.

Let F denote the set of *feasible allocations* $\{x \in X : \sum_{i \in I}(x_i - e_i) = 0\}$. For each $x \in X$ and each $i \in I$, let $R_i(x_i) = \{z \in X : x_i \notin \text{co}P_i(z)\}$. We set $R(e) = \{x \in X : e_i \notin \text{co}P_i(x) \quad \text{for all } i \in I\}$. An allocation $x \in R(e)$ is called *individually rational*. Such individual rationality is appropriate in the sense that if x is an equilibrium allocation, then $e_i \notin \text{co}P_i(x)$ for all $i \in I$.[5] We set $H = F \cap R(e)$. Let cl$H$ denote the closure of H in X. A point $x \in H$ is an allocation which is feasible and individually rational. In the special case that the preference ordering of agent i is defined on X_i and representable by a quasiconcave utility function u_i for all $i \in I$, $R(e)$ is equal to the set $\{x \in X : u_i(e_i) \leq u_i(x_i) \text{ for all } i \in I\}$ and therefore, it is convex. We assume that E satisfies the following conditions for all $i \in I$.

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4 The relative interior of A is the interior of A in the smallest affine subspace of $\mathbb{R}^\ell$ which contains A.

5 For all i, let $\succ_i$ denote the strict ordering on X_i induced by a reflexive ordering $\succsim_i$ on X_i, i.e., for all z_i, x_i in X_i, $z_i \succ_i x_i$ if $z_i \succsim_i x_i$ but not $x_i \succsim_i z_i$. In this case, one might be tempted to define the set of individually rational allocations as $\prod_{i \in I}\{x_i \in X_i : x_i \succsim_i e_i\}$. This is fine as far as $\succ_i$ is transitive for all $i \in I$. It is not the case, however, with non-transitive preferences because $x_i \succ_i e_i$ does not imply $P_i(x_i) \subset P_i(e_i)$, as illustrated in Example 3.1.2.

B1. X_i is a closed, nonempty and convex set in $\mathbb{R}^\ell$.
B2. e_i is in the interior of X_i.
B3. P_i is lower semi-continuous.
B4. For all $x \in X$, $x_i \notin \text{co}P_i(x)$.
B5. For all $x \in H$, $P_i(x) \neq \emptyset$.
B6. Let x be a point in H. Then for each $z_i \in \text{co}P_i(x)$ and $v_i \in X_i$, there exists $\lambda \in (0, 1)$ such that $\lambda z_i + (1 - \lambda)v_i \in \text{co}P_i(x)$.

Assumption B5 states that no satiation occurs on the set of feasible and individually rational allocations. Assumption B6 holds true when $P_i(x)$ is relatively open in X_i for all $x \in H$.[6] As shown later, this condition is not required for the existence of quasi-equilibrium. It is used only in verifying that a quasi-equilibrium is an equilibrium. Define $\phi_i : X \to 2^{X_i}$ by $\phi_i(x) = \text{co}P_i(x)$. For $e_i \in X_i$, notice that

$$\begin{aligned}\phi_i^{-1}(e_i) &= \{x \in X : e_i \in \phi_i(x)\}\\ &= \{x \in X : e_i \in \text{co}P_i(x)\}.\end{aligned}$$

Observe that

$$\begin{aligned}X \setminus \phi_i^{-1}(e_i) &= \{x \in X : e_i \notin \phi_i(x)\}\\ &= \{x \in X : e_i \notin \text{co}P_i(x)\} = R_i(e_i).\end{aligned}$$

Since P_i is lower semi-continuous under the condition B3, $\phi_i^{-1}(e_i)$ need not be open and therefore, $R_i(e_i)$ need not be closed. Consequently, H need not be closed.[7] By B4, $e_i \notin \text{co}P_i(e)$ for all $i \in I$ and therefore, $e \in R(e)$. Since $e \in F$, it follows that $e \in \text{cl}H$. In particular, cl$H$ is not empty.

Below we prove a very general Walrasian equilibrium existence theorem, which will be used in the proof of our main result.

Auxiliary Theorem 2.3.1. *Suppose that E satisfies the assumptions B1–B6. Then there exists an equilibrium for the economy E if H is bounded.*

Proof. See Appendix B. □

The auxiliary theorem is based on the assumption B5 and the condition that H is bounded. Thus, this result subsumes as a special case the classical existence theorems of Shafer (1976) and Gale and Mas-Colell (1975), among others, which are built upon the condition that F is bounded. As shown in Appendix, the proof of Auxiliary Theorem is distinct from the standard proof of the classical theorems because it involves delicate arguments in verifying that candidate equilibrium allocations lie in H.

3. Main results

We briefly review the notion of arbitrage used in the literature and provide generalizations of the no arbitrage conditions to the case with non-transitive preferences. Two examples are given to motivate the need for the new conditions for the existence of equilibrium. The first example borrowed from Page et al. (2000) illustrates that the no arbitrage conditions are not useful to explain equilibrium when the utility set for allocations in H is not compact. Specifically, the no arbitrage conditions are no longer necessary for the existence of equilibrium in economies with the noncompact utility set. Another limitation of the arbitrage conditions is that they explicitly or implicitly assume the transitivity of preferences. One may be tempted to consider a naive extension of the notions of arbitrage to the case with non-transitive preferences simply by replacing 'transitive preferences' by 'non-transitive preferences' in their definition. The second example shows that such a naive extension of the no arbitrage conditions fails to explain the existence of equilibrium with non-transitive preferences. Moreover, the set of Pareto optimal allocations in H is compact but the naively extended conditions fail in the example.[8] The second example illustrates the difficulty of finding sufficient conditions for the existence of equilibrium with non-transitive preferences as an extension of the no arbitrage conditions. This example is also used to demonstrate the irrelevance of the money pump arguments against non-transitive preferences. In the second subsection, we introduce a new condition which can cover the cases not only of non-transitive preferences, but also of noncompact utility sets. In particular, this condition subsumes all the previous no arbitrage conditions as a special case. Moreover, it covers the counterexample of Page et al. (2000). Finally, the main theorem of the paper is provided under the condition which further generalizes the condition of the second subsection.

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6 Suppose that $P_i(x)$ is relatively open in X_i. Let z_i be a point in co$P_i(x)$. Then there exist $z_i' \in P_i(x)$, $z_i'' \in P_i(x)$ and $\alpha \in [0, 1]$ such that $z_i = \alpha z_i' + (1 - \alpha)z_i''$. Let v_i be a point in X_i. Since $P_i(x)$ is relatively open in X_i, there exists $\lambda \in (0, 1)$ such that $\lambda z_i' + (1 - \lambda)v_i \in P_i(x)$ and $\lambda z_i'' + (1 - \lambda)v_i \in P_i(x)$. It follows that $\lambda z_i + (1 - \lambda)v_i = \alpha[\lambda z_i' + (1 - \lambda)v_i] + (1 - \alpha)[\lambda z_i'' + (1 - \lambda)v_i] \in \text{co}P_i(x)$.

7 If each P_i has open lower sections and is convex-valued, then $R(e)$ is closed. But this is not warranted by B3. For example, the budget correspondence is lower semi-continuous but does not have open lower sections. See Yannelis and Prabhakar (1983) for an example of a lower semi-continuous correspondence which does not have open lower sections.

8 A feasible allocation $x \in X$ is said to be *Pareto optimal* if there is no feasible allocation $y \in X$ such that $y_i \in P_i(x_i)$ for every $i \in I$.

3.1. Arbitrage conditions and counterexamples

For illustrative purposes, we consider the case where preferences for agent i are represented by a reflexive ordering $\succsim_i$ on X_i for all $i \in I$. Let $\succ_i$ denote the strict ordering on X_i induced by $\succsim_i$; for all z_i, x_i in X_i, $z_i \succ_i x_i$ if $z_i \succsim_i x_i$ and not $x_i \succsim_i z_i$. Define the correspondence $P^i : X_i \to 2^{X_i}$ by $P^i(x_i) = \{z_i \in X_i : z_i \succ_i x_i\}$. In this section, we assume that for all $x \in X$ and all $i \in I$, $P_i(x) = P^i(x_i)$, and P_i is convex-valued.[9] For each $x_i \in X_i$, we set $R^i(x_i) = \{z_i \in X_i : \text{not}(x_i \succ_i z_i)\}$.[10] Then it is easy to see that $R(e) = \prod_{i \in I} R^i(e_i)$.

Dana et al. (1999) and Page et al. (2000) extend the notions of arbitrage used in Werner (1987) and Page and Wooders (1996) to the case where H need not be bounded, and they compare various notions of arbitrage. Remarkably, Dana et al. (1999) provide a condition which is equivalent to the compactness of the utility set $\{\mu \in \mathbb{R}^I_+ : \exists x \in H \text{ s.t. } 0 \le \mu_i \le u_i(x_i) \text{ for all } i \in I\}$ when preferences of agent i are represented by a quasiconcave function $u_i : X_i \to \mathbb{R}$.[11] Allouch (2002) further generalizes the notion of arbitrage to the case where preferences are transitive but need not be complete.

CPP. (i) The preference ordering $\succsim_i$ on X_i is transitive and reflexive for each $i \in I$. (ii) For each sequence $\{x^n\}$ in H, there exist a subsequence $\{x^{n_k}\}$ and a sequence $\{y^{n_k}\}$ in X convergent to some point $y \in \text{cl}H$ which satisfies $y_i^{n_k} \succ_i x_i^{n_k}$ for all n_k and for all $i \in I$.

Allouch (2002) shows that the CPP condition is equivalent to the compactness of the utility set of allocations in H when preferences are numerically representable.

The aforementioned literature, however, does not explain the existence of equilibrium in the following two examples.

Example 3.1.1. Page et al. (2000), and Monteiro et al. (2000) provide an example where the economy has an equilibrium but the utility set is not compact. They show that the example does not satisfy any known conditions based on arbitrage.

It is assumed in the economy that $I = \{1, 2\}$, $X_1 = X_2 = \mathbb{R}^2$, $e_1 = e_2 = (0, 0)$. Agent 1's utility function is given by

$$u_1(v) = \begin{cases} v_1, & \text{if } v_1 \le 0, \text{ or } v_2 \ge -1 \\ -\dfrac{v_1}{v_2}, & \text{if } v_1 \ge 0, \text{ and } v_2 \le -1 \end{cases}$$

while preferences of agent 2 is given by

$$u_2(w) = w_1 + 2w_2.$$

The utility set is not compact.[12] It is easy to see that the economy has an equilibrium (p, v, w) where $p = (1, 2)$, and $v = (2, -1)$ and $w = (-2, 1)$ are an optimal choice for agent 1 and 2, respectively.

Example 3.1.2. We consider a two-agent, two-good economy where agents have non-transitive preferences in $\mathbb{R}^2$ with $e_1 = e_2 = (0, 0) \in \mathbb{R}^2$. For all $v = (v_1, v_2) \in \mathbb{R}^2$, we introduce the sets

$$W^1(v) = \begin{cases} \{v' = (v'_1, v'_2) \in \mathbb{R}^2 : v'_1 > 0 \,\&\, v'_2 \ge 0\} \cup \{(0, 0)\} & \text{if } v = (0, 0) \\ \{v' = (v'_1, v'_2) \in \mathbb{R}^2 : v'_2 \ge v_2\} & \text{if } v \ne (0, 0) \end{cases}$$

$$W^2(v) = \begin{cases} \{v' = (v'_1, v'_2) \in \mathbb{R}^2 : v'_1 \ge 0 \,\&\, v'_2 > 0\} \cup \{(0, 0)\} & \text{if } v = (0, 0) \\ \{v' = (v'_1, v'_2) \in \mathbb{R}^2 : v'_1 \ge v_1\} & \text{if } v \ne (0, 0). \end{cases}$$

For each $i = 1, 2$, we define $\succsim_i$ on $\mathbb{R}^2$ such that $v' \succsim_i v$ if $v' \in W^i(v)$. Then it follows that for all $v \in \mathbb{R}^2$,

$$P^1(v) = \begin{cases} R^2_{++} & \text{if } v = (0, 0) \\ \{v' = (v'_1, v'_2) \in \mathbb{R}^2 : v'_2 > v_2\} & \text{if } v \ne (0, 0) \end{cases}$$

$$P^2(v) = \begin{cases} R^2_{++} & \text{if } v = (0, 0) \\ \{v' = (v'_1, v'_2) \in \mathbb{R}^2 : v'_1 > v_1\} & \text{if } v \ne (0, 0). \end{cases}$$

Since $(-1, 2) \in P^1(1, 1)$ and $(1, 1) \in P^1(0, 0)$ but $(-1, 2) \notin P^1(0, 0)$, P^1 is not transitive. Preferences of agent 1 are incomplete because $(0, 0)$ and $(-1, 1)$ are not comparable. Similarly, we can show that P^2 is non-transitive and incomplete.

It is easy to see that the sets $R^1(e_1)$ and $R^2(e_2)$ is written as

$$R^1(e_1) = \{v = (v_1, v_2) \in \mathbb{R}^2 : v_2 \ge 0\}$$
$$R^2(e_2) = \{v = (v_1, v_2) \in \mathbb{R}^2 : v_1 \ge 0\}.$$

We show that the economy satisfies all the conditions imposed by B1–B6. Clearly, P^1 and P^2 are convex and open valued. We claim that they are lower semi-continuous. For a point $v \in \mathbb{R}^2$, let $z = (z_1, z_2)$ be a point in $P^1(v)$. Let $v^n \to v$ in $\mathbb{R}^2$. Suppose that $v = (0, 0)$. Then we see that $z_1 > 0$ and $z_2 > 0$. Since $v^n \to (0, 0)$, $z_2 > v_2^n$ for sufficiently large n and therefore, $z \in \mathbb{R}^2_{++} \cap \{v = (v_1, v_2) \in \mathbb{R}^2 : v_2 > v_2^n\}$. It implies that $z \in P^1(v^n)$.

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9 By Proposition A.1 of Appendix, the convexity assumption can replace B4 without loss of generality.

10 Notice that "not $(x_i \succ_i z_i)$" means either "not $(x_i \not\succsim_i z_i)$" or "$z_i \not\succsim_i x_i$".

11 Dana et al. (1999) and Page et al. (2000) also provide an excellent review of various notions of arbitrage and their economic implications.

12 It is easy to see that $u_1(n, -n) \to 1$ and $u_2(-n, n) \to \infty$ as $n \to \infty$.

Suppose that $v \neq (0, 0)$. Since $\mathbb{R}^2 \setminus \{(0, 0)\}$ is open, v^n is in $\mathbb{R}^2 \setminus \{(0, 0)\}$ for sufficiently large n. On the other hand, $z_2 > v_2^n$ so that we have $z \in P^1(v^n)$ for sufficiently large n. Therefore, we conclude that P^1 is lower semi-continuous. By the same argument we can show that P^2 is lower semi-continuous. Since P^1 and P^2 are convex and open valued, B6 trivially holds for the economy. Thus, we see that the economy satisfies B1–B6.

In Section 1, we argue that money pumps are not observable in equilibrium. Clearly, the triad $\{(0, 0), (1, 1), (-1, 3)\}$ constitutes a money pump against agent 1. For any equilibrium price $p \in \mathbb{R}^2_+$, $(1, 1)$ is not budget-feasible in equilibrium. Agent 1 cannot accept the offer to exchange $(0, 0)$ for $(1, 1)$ at any positive fee because of the budget constraint. Thus, the money pump does not work in equilibrium.

We consider an extension of the CPP condition to economies with non-transitive preferences simply by dropping (i) and keeping (ii) of the CPP condition. We show that the naive extension of the CPP condition is not satisfied with the example. For each n, set $x_1^n = (-a^n, b^n)$ and $x_2^n = (a^n, -b^n)$ for some a^n and b^n in $\mathbb{R}$. We assume that $x^n = (x_1^n, x_2^n)$ is in H for all n. Then we have $a^n \geq 0$ and $b^n \geq 0$. Suppose that a^n and b^n are strictly positive and increasing for all n, and $a^n \to \infty$ and $b^n \to \infty$. Then the distance of $P^i(x_i^n)$ from the origin goes to infinity as $n \to \infty$. Thus, there is no bounded sequence $\{y^n\}$ which satisfies $y_i^n \in P^i(x_i^n)$ for all n and all $i = 1, 2$, and therefore, (ii) of the CPP condition is violated.

Remark 3.1.1. The economy of Example 3.1.2 has the desired properties in terms of preferences except for the non-transitivity. Moreover, the initial allocation $\{(e_1, e_2)\}$ is the unique Pareto optimal allocation in H and thus, the set of Pareto optimal allocations in H is trivially compact. Nonetheless, there is no literature which covers Example 3.1.2. The classical works do not apply simply because consumption sets have no lower bound. The arbitrage-based literature following the seminal work of Hart (1974) is not applicable because preferences are not transitive. Moreover, the existence of equilibrium in Example 3.1.2 cannot be explained by the version of the CPP condition of Allouch (2002) where the transitivity of preferences is dropped.

In the subsequent sections, we will discuss new conditions which can cover Examples 3.1.1 and 3.1.2.

3.2. New conditions without transitivity

We provide a new sufficient condition for the existence of equilibrium without transitivity of preferences. This condition encompasses all the arbitrage-related conditions as a special case. Remarkably, it is illustrated that the utility set need not be compact under this condition. In particular, our new condition is satisfied in the economy of Example 3.1.1 which is given by Page et al. (2000).

For a point $x \in X$ and $i \in I$, we set

$$r_i(x) = \max\{\|x_j\|, j \neq i\}.$$

Note that the closed ball $C(0, r_i(x))$ contains x_j for all $j \neq i$. We make the following assumption.

B7a. There exists $h \in I$ such that for any sequence $\{x^n\}$ in H, there exist a subsequence $\{x^{n_k}\}$, and a sequence $\{y^{n_k}\}$ convergent to a point $y \in \text{cl}H$ such that for all n_k,

$$P_h(y^{n_k}) \subset \text{con}[P_h(x^{n_k}) - \{e_h\}] + \{e_h\} \tag{1}$$

and for all $i \neq h$,

$$P_i(y^{n_k}) \cap C(0, r_h(x^{n_k})) \subset \text{con}[P_i(x^{n_k}) \cap C(0, r_h(x^{n_k})) - \{e_i\}] + \{e_i\}. \tag{2}$$

It will be illustrated that B7a may hold in economies with a noncompact utility set and therefore, it is strictly more general than the no arbitrage conditions of the literature. The asymmetric treatment of agents in the condition B7a deserves a special remark.

Remark 3.2.1. The asymmetric treatment of agents in B7a plays a crucial role in extending the no arbitrage conditions beyond the class of economies which have the compact utility set for individually rational allocations. There are two conceivable conditions which give symmetric treatment to agents. The first one is the following alternative to B7a: for any sequence $\{x^n\}$ in H, there exist a subsequence $\{x^{n_k}\}$, and a sequence $\{y^{n_k}\}$ convergent to a point $y \in \text{cl}H$ such that for all n_k and all $i \in I$,

$$P_i(y^{n_k}) \cap C(0, r(x^{n_k})) \subset \text{con}[P_i(x^{n_k}) \cap C(0, r(x^{n_k})) - \{e_i\}] + \{e_i\}. \tag{3}$$

where for a point $x \in X$, $r(x) = \max\{\|x_i\|, i \in I\}$. This alternative condition is quite restrictive because, as illustrated below, it may fail to explain the existence of equilibrium in economies with a noncompact utility set.

The other conceivable symmetric treatment of agents is to apply the restriction on agent h represented by (1) to the other agents: for any sequence $\{x^n\}$ in H, there exist a subsequence $\{x^{n_k}\}$, and a sequence $\{y^{n_k}\}$ convergent to a point $y \in \text{cl}H$ such that for all n_k and all $i \in I$,

$$P_i(y^{n_k}) \subset \text{con}[P_i(x^{n_k}) - \{e_i\}] + \{e_i\} \tag{4}$$

This condition is weaker than B7a but is not required for the existence of equilibrium. For example, the condition is satisfied in the economy with two agents and two goods where preferences of the agents are represented by a utility function $u_1(v) = v_2$ and $u_2(v) = v_1$ on $\mathbb{R}^2$, but the economy has no equilibrium.

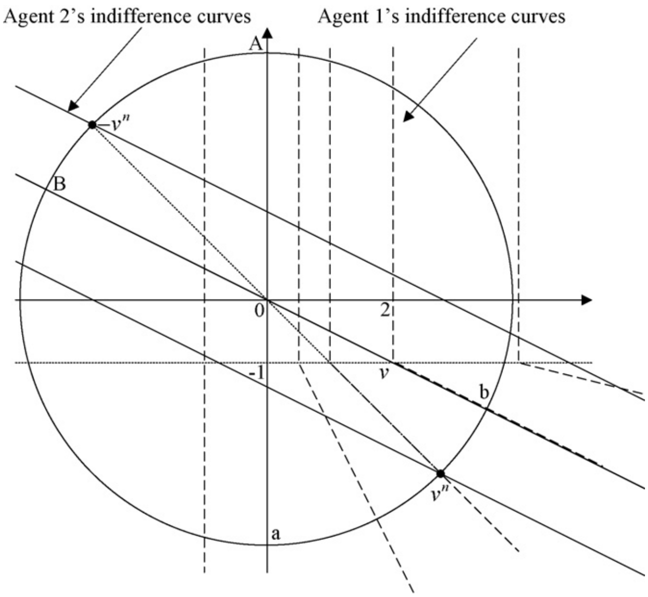

Fig. 1. A non-compact utility set.

The following condition is a simple extension of the CPP condition of Allouch (2002) to the case where preferences need not be transitive.

B7b. For any sequence $\{x^n\}$ in H, there exist a subsequence $\{x^{n_k}\}$, and a sequence $\{y^{n_k}\}$ convergent to a point $y \in \mathrm{cl}H$ such that for all n_k and all $i \in I$,

$$P_i(y^{n_k}) \subset P_i(x^{n_k}). \tag{5}$$

Condition B7b can be considered as a simple extension of the CPP condition of Allouch (2002) to the case where preferences need not be transitive. Clearly, B7a subsumes B7b and therefore, the CPP condition of Allouch (2002) becomes a special case.

As mentioned earlier, the CPP condition is equivalent to the compactness of the utility set of allocations in H when preferences are numerically representable. Moreover, the no arbitrage conditions used by Werner (1987); Page (1987) or Dana et al. (1999) imply the compactness of the utility set and therefore, they also imply B7a.[13] The following example shows that B7a may hold with the noncompact utility set for allocations in H.

Example 3.2.1. We continue with Example 3.1.1. The utility set for allocations in H is not compact. As illustrated below, however, the economy satisfies B7a. We recall that there exists an equilibrium (p, v, w) where $p = (1, 2)$, and $v = (2, -1)$ and $w = (-2, 1)$. Note that $w + v = 0$.

Let $\{(v^n, -v^n)\}$ be a sequence of allocations in H. By setting $x_1^n = v^n$ and $x_2^n = -v^n$ for all n, we can match the notation of B7a. If $\{(v^n, -v^n)\}$ is bounded, we set $y_1^n = v^n$ and $y_2^n = -v^n$ for each n. Then it is trivial to see that $\{x^n\}$ and $\{y^n\}$ satisfy (1) and (2).

From now on, we assume that $\|v^n\| \to \infty$. We set $y_1^n = v$ and $y_2^n = w$ for all n, and $y_1 = v$ and $y_2 = w$. Note that for each n, $P^2(y_2^n) = P^2(w)$ is the open half space above the line through b and B. Since $(v^n, -v^n) \in H$ and $\|v^n\| \to \infty$, without loss of generality, we can assume that v^n lies in the fourth quadrant of $\mathbb{R}^2$ and $\|v^n\| > \sqrt{5}$ for all n (see Fig. 1). Then $-v^n$ lies in the arc AB while v^n in the arc ab.

Noting that $e_2 = (0, 0)$, we see that for all n, $\mathrm{con}[P^2(-v^n) - \{e_2\}] - \{e_2\} = \mathrm{con}P^2(-v^n)$, and $\mathrm{con}P^2(-v^n)$ is the open half space above the line through b and B. Recalling that $w = -v$, for all n we have

$$P^2(w) = \mathrm{con}[P^2(-v^n) - \{e_2\}] + \{e_2\}.$$

Since $e_1 = (0, 0)$ and b is the best point for agent 1 in the arc ab, it follows that $P^1(b) = P^1(v) \subset P^1(v^n)$ for all n and therefore,

$$\begin{aligned} P^1(v) \cap C(0, \|v^n\|) &\subset P^1(v^n) \cap C(0, \|v^n\|) \\ &= [P^1(v^n) \cap C(0, \|v^n\|) - \{e_1\}] + \{e_1\} \\ &\subset \mathrm{con}[P^1(v^n) \cap C(0, \|v^n\|) - \{e_1\}] + \{e_1\}. \end{aligned}$$

Thus, we conclude that the economy satisfies B7a.

[13] A complete comparison of the no arbitrage conditions can be found in Dana et al. (1999), and Page et al. (2000).

In Remark 3.2.1, we have mentioned that the alternative condition to B7a where every agent is subject to the restriction represented by (3) may fail to explain the existence of equilibrium. This is true in this example. It is easy to see that for each n with $-v^n \in P^2(w)$,

$$P^2(w) \cap C(0, \|v^n\|) \not\subset P^2(-v^n) \cap C(0, \|v^n\|),$$

and therefore,

$$P^2(w) \cap C(0, \|v^n\|) \not\subset \mathrm{con}[P^2(-v^n) \cap C(0, \|v^n\|) - \{e_2\}] + \{e_2\}.$$

Thus, the restriction on each agent imposed by (3) does not hold for the economy.

3.3. Main existence theorems

In this section, we provide the main existence theorem of the paper based on a condition which generalizes B7a. As illustrated below, B7a calls for further generalization because it does not cover the economy of Example 3.1.2. The exemplary economy is standard except for the non-transitivity of preferences. Nevertheless, the existence of equilibrium of the economy cannot be explained by any known conditions including B7a. The main existence result of the paper is based on a generalization of B7a which covers the economy of Example 3.1.2.

The new condition to be discussed is motivated by the following distinct characterization of the equilibrium conditions. For a point $x \in X$, we define the set

$$G(x) = \sum_{i \in I} \mathrm{clcon}[P_i(x) - \{e_i\}].$$

It is shown below that the set $G(\cdot)$ summarizes the conditions for x to be a quasi-equilibrium for the economy E.

Lemma 3.3.1. *Suppose that for all $x \in H$ and all $i \in I$, $P_i(x)$ is convex and $x_i \in \mathrm{cl}P_i(x)$. Then for a point $x \in H$, $G(x) \neq \mathbb{R}^\ell$ if and only if there exists a nonzero vector $p \in \mathbb{R}^\ell$ such that $pz \geq 0$ for all $z \in G(x)$, i.e., $pe_i \leq pz_i$ for all $z_i \in \mathrm{cl}P_i(x)$ and all $i \in I$.*

Proof. See Appendix B. □

This lemma shows that for a point $x \in H$, $G(x) \neq \mathbb{R}^\ell$ is a necessary and sufficient condition for the existence of $p \in \mathbb{R}^\ell$ such that (p, x) is a quasi-equilibrium for the economy E. The condition that $G(x) \neq \mathbb{R}^\ell$ will be guaranteed in the proof of the main theorem of the paper by a fixed point theorem. In addition to the conditions of Lemma 3.3.1, suppose that B2 and B6 hold for the economy. Then by the same arguments made in Step 5 of the proof of Theorem 2.3.1, we can show that (p, x) is an equilibrium for the economy E. The following result is immediate from the assumptions B2 and B6 and the results of Lemma 3.3.1.

Proposition 3.3.1. *Suppose that B2 and B6 hold, and for all $x \in H$ and all $i \in I$, $P_i(x)$ is convex and $x_i \in \mathrm{cl}P_i(x)$. Then for any $x \in X$, $x \in H$ and $G(x) \neq \mathbb{R}^\ell$ if and only if there exists a nonzero $p \in \mathbb{R}^\ell$ such that (p, x) is an equilibrium for the economy E.*

Proposition 3.3.1 shows that for an allocation $x \in H$, $G(x) \neq \mathbb{R}^\ell$ is a necessary and sufficient condition for the existence of equilibrium. Thus for an allocation $x \in H$, the set $G(x)$ fully characterizes the conditions for x to be an equilibrium allocation.

For a point $x \in X$ and $i \in I$, we define the set

$$G_i(x) = \mathrm{clcon}[P_i(x) - \{e_i\}] + \sum_{j \neq i} \mathrm{clcon}[P_j(x) \cap C(0, r_i(x)) - \{e_j\}].$$

Let $E_i(x)$ denote the economy which is the same as E except for that agent i has the consumption set X_i and each $j \neq i$ has the consumption set $X_j \cap C(0, r_i(x))$ which is equal to the consumption set X_j truncated by the closed ball $C(0, r_i(x))$. The economy $E_i(x)$ will be used in the proof of the main theorem of the paper. The following corollary shows that the set $G_i(x)$ summarizes the conditions for x to be equilibrium of the economy $E_i(x)$.

Corollary 3.3.1. *Suppose that for all $x \in H$ and all $i \in I$, $x_i \in \mathrm{cl}P_i(x)$. Let $x \in H$ and $i \in I$ such that $P_j(x) \cap C(0, r_i(x)) \neq \emptyset$ for every $j \neq i$.*[14] *Then $G_i(x) \neq \mathbb{R}^\ell$ if and only if there exists a nonzero $p \in \mathbb{R}^\ell$ such that (p, x) is an equilibrium for the economy $E_i(x)$.*

Example 3.1.2 illustrates that arbitrage-related conditions may not be useful for the existence of equilibrium with non-transitive preferences. By taking advantage of the properties of the set $G_i(\cdot)$ for each i, we provide new conditions which subsume the no arbitrage conditions as a special case and are relevant to the case of non-transitive preferences.

B7. There exists $h \in I$ such that for any sequence $\{x^n\}$ in H with $\{r_h(x^n)\}$ increasing to infinity, there exist a subsequence $\{x^{n_k}\}$, and a sequence $\{y^{n_k}\}$ convergent to a point $y \in \mathrm{cl}H$ such that for all n_k,

$$P_h(y^{n_k}) - \{x_h^{n_k}\} \subset G_h(x^{n_k}) \tag{6}$$

[14] By the same arguments made in Step 2 of the proof of Theorem 3.1.1, we can show that $x_j \in \mathrm{cl}P_j(x)$ and $P_j(x) \cap C(0, r_i(x)) \neq \emptyset$ for every $j \neq i$ imply $x_j \in \mathrm{cl}[P_j(x) \cap C(0, r_i(x))]$.

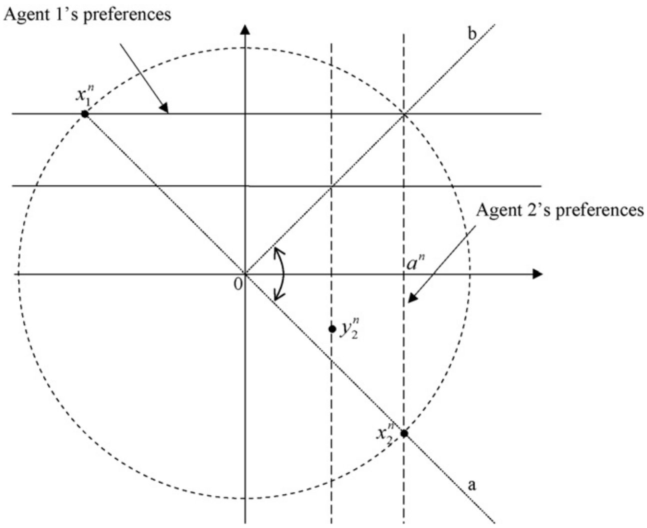

Fig. 2. A simple example without satisfying B7a.

and for all $j \neq h$,

$$[P_j(y^{n_k}) \cap C(0, r_h(x^{n_k}))] - \{x_j^{n_k}\} \subset G_h(x^{n_k}). \tag{7}$$

Remark 3.3.1. We have emphasized the importance of the asymmetric treatment of agents in Remark 3.2.1. Since B7 is a generalization of B7a as shown below, Remark 3.2.1 applies to B7 as well.

Lemma 3.3.2. *Suppose that $x_i \in \text{cl}P_i(x)$ for all $x \in X$ and for all $i \in I$.*[15] *Then B7a implies B7.*

Proof. See Appendix B. □

The following illustrates an economy which satisfies B7 but does not B7a. Thus, B7 is strictly more general than B7a.

Example 3.3.1. We show that the economy of Example 3.1.2 satisfies B7 but does not satisfy B7a. For each n, set $x_1^n = (-a^n, b^n)$ and $x_2^n = (a^n, -b^n)$ for some a^n and b^n in $\mathbb{R}$. We assume that $x^n = (x_1^n, x_2^n)$ is in H for all n and $a^{n_k} + b^{n_k} \to \infty$. The condition $x^n \in H$ implies that $a^n \geq 0$ and $b^n \geq 0$. Since $\|x_1^n\| = \|x_2^n\|$, we have $r_i(x^n) = \|x_i^n\|$ for all $i = 1, 2$ and all n.

For each n, we take x^n such that x_2^n is on the line a. Let $\{y_2^n\}$ be a bounded sequence in $R^2(e_2)$. Since $\|x_2^n\| \to \infty$, without loss of generality, we can assume that $\|y_2^n\| < a^n$. Then $y_2^n \in Q^2(e_2) \cap C(0, \|x_2^n\|)$. As shown in Fig. 2, the cone $\text{con}[P^2(x_2^n) - \{e_2\}] + \{e_2\}$ is an open set which has the lines $0a$ and $0b$ as its boundaries. Thus, we see that

$$P^2(y_2^n) \cap C(0, \|x_2^n\|) \not\subset \text{con}[P^2(x_2^n) - \{e_2\}] + \{e_2\}.$$

Let $\{y_1^n\}$ be a bounded sequence in $R^1(e_1)$. By the same argument, we can show that $P^1(y_1^n) \cap C(0, \|x_1^n\|) \not\subset \text{con}[P^1(x_1^n) - \{e_1\}] + \{e_1\}$. Thus, B7a does not hold for the economy.

It is trivial to see that (e_1, e_2) is a unique equilibrium allocation for each $E_1(x^n)$. That is, any nonzero allocation in F cannot be an equilibrium allocation of $E_1(x^n)$. Then by Corollary 3.3.1, $G_1(x^n) = \mathbb{R}^2$ and therefore, B7 holds trivially in this example.

With all these preliminary results out of the way, we can now turn to the main existence theorem of this paper.

Theorem 3.3.1. *Suppose that E satisfies the assumptions B1–B6. Then there exists an equilibrium for the economy E if the condition B7 is satisfied.*

Sketch of Proof. The rigorous proof of the theorem will be given in Appendix B. The following sketch will be useful to understand the idea of the proof.

When the consumption sets are not bounded, traditional approaches to the existence proof rely on the truncation method introduced by Debreu (1959). Moreover, truncations of the consumption sets need to be taken sequentially so that the whole consumption sets can be covered in the limit when F is bounded. In the case where F is unbounded, however, the truncation method alone does not work because no sequential truncations can contain F as a whole.

Assumption B7 allows us to circumvent this problem. The idea is to take advantage of information elicited from each truncated economy. (It is worth noting that the truncation method used here differs from the conventional truncation methods in that one agent is allowed to make choices in the whole consumption set in all the truncated economies.) By applying Auxiliary Theorem 2.3.1, we show that each economy with the truncated consumption sets has an equilibrium

[15] As mentioned earlier in this section, this condition does no harm to the existence result as far as B4 is satisfied.

under Assumption B7. Again by B7, there exists a bounded sequence of allocations which are asymptotically supported by the sequence of equilibrium prices for the truncated economies. The limit of those prices and bounded allocations turns out to be an equilibrium of the economy E. □

This theorem is a generalization of all the existence results in the literature in several respects. The condition B7 along with the other conditions B1–B6 are sufficient for the existence of equilibrium. As illustrated in Example 3.2.1, the utility set for allocations in H need not be compact under the condition B7. Preferences need not be transitive and moreover, they are interdependent and satisfy the condition B3, weak continuity assumptions on preferences. Most importantly, B7 is much weaker than the no arbitrage conditions used in the literature because all other arbitrage conditions cannot apply to the case where either the utility set is not compact or agents have non-transitive preferences.

Remark 3.3.2. One of the referees of the paper suggested an interesting alternative condition for B7 where agents are treated symmetrically. Its formulation is based upon the following definition.

Definition 3.3.1. A sequence $\{T_n\}$ of sets in $\mathbb{R}^\ell$ is called a *compactification procedure* of the economy E if it satisfies the following conditions

(i) for each n, T_n is non-empty, closed and satisfies $T_n \subset T_{n+1}$,
(ii) for each $x \in \mathbb{R}^\ell$ and each $\epsilon > 0$, there exists n such that $C(x, \epsilon) \subset T_n$,
(iii) for each n, the set $H \cap T_n^I$ is bounded, and
(iv) for each $i \in I$ and each $x \in R(e)$, it holds that $P_i(x) \cap T_1 \neq \emptyset$.

If each X_i is bounded from below, then there exists $z \in \mathbb{R}^\ell$ such that $X_i \subset \{x \in \mathbb{R}^\ell : x \geq z\}$ for all $i \in I$. In this case, a trivial compactification procedure $\{T_n\}$ is available where $T_n = \{x \in \mathbb{R}^\ell : x \geq z\}$ for all n. The following is the condition suggested by the referee as a possible substitute for B7.

B7s. There exists a compactification procedure $\{T_n\}$ such that for each $\{x^n\}$ with $x^n \in H \cap T_n$, there exist $\epsilon > 0$ and $y \in \mathrm{cl}H$ which satisfy[16]

$$P_i(y) \cap C(y_i, \epsilon) - \{e_i\} \subset \mathrm{Ls}\{\mathrm{con}[P_i(x^n) \cap T_n - \{e_i\}]\}.$$

Clearly, agents are symmetrically treated in B7s. As shown below, an advantage of B7s over B7 is it makes the existence proof much simpler. It is easy to check that B7s is weaker than B7b. We also see that the compactification procedure $\{T_n\}$ defined by

$$T_n = \left\{(v_1, v_2) \in \mathbb{R}^2 : v_2 \geq -\left(\frac{1}{2} + \frac{1}{n}\right)(v_1 + n) - n,\ v_2 \geq -\left(\frac{1}{2} - \frac{1}{n}\right)(v_1 + n) - n\right\}$$

satisfies B7s in the economy of Example 3.1.1. A disadvantage of B7s, however, is that no systematic theory is yet available which answers the existence of a compactification procedure $\{T_n\}$ which satisfies the conditions of B7s. Indeed, there would be no definite way of checking B7s in the simple economy mentioned in the end of Remark 3.2.1.[17] We now provide an existence theorem based on assumption B7s.

Theorem 3.3.1s. *Suppose that E satisfies the assumptions B1–B6. Then there exists an equilibrium for the economy E if the condition B7s is satisfied.*[18]

Proof. See Appendix B. □

4. Concluding remarks

We have shown the existence of equilibrium in an economy with non-ordered preferences and unbounded-from-below consumption sets. The consequences of the paper not only subsume the arbitrage-based equilibrium theory as a special case but also can cover the case with non-compact utility sets.

The main results of the paper stated in Theorem 3.3.1 depend critically on the assumption B5 which excludes the satiability of preferences. One possible extension is to examine the effect of satiation on equilibrium. Such an extension is attempted in Won and Yannelis (2002a,b) as a sequel to the current paper. Allouch and Le Van (2006); Allouch et al. (2006), and Martins-da-Rocha and Monteiro (2007) address the equilibrium existence problem with satiable preferences under the condition that satiation occurs both inside and outside the set of feasible and individually rational allocations. Won and Yannelis (2002a,b) differ from them in that it also covers the case where preferences are possibly satiated only inside the set of feasible and

[16] If $\{A_n\}$ is a sequence of subsets of $\mathbb{R}^\ell$, then $\mathrm{Ls}\{A_n\}$ is the set of limit points of $\{A_n\}$, i.e., $b \in \mathrm{Ls}\{A_n\}$ if and only if there exists a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \in A_{n_k}$ for each n_k and $a_{n_k} \to b$.

[17] Let us call the aforementioned economy 'economy Z'. Any compactification procedure could be picked up to be tested against the conditions of B7s in the economy Z. It would fail. But this does not say anything about whether B7s holds in the economy Z because there is still an inexhaustible set of compactification procedures to be tested. That is, there is no way of seeing whether B7s holds in the economy Z.

[18] The proof of Theorem 3.3.1s to be shown in Appendix B heavily rely on the suggestions of the referee.

individually rational allocations. As mentioned in Won and Yannelis (2002a,b), such a distinction is particularly important in the context of the capital asset pricing models. By specializing the framework of Won and Yannelis (2002a,b) in asset pricing models, Won et al. (2008) show the existence of equilibrium in the capital asset pricing model with heterogeneous expectations where mean-variance utility functions reach satiation due to the absence of risk-free assets. One challenging problem is to extend the outcomes of the paper to cover the case discussed in Won (2001) where contract curves are unbounded.

Appendix A. Equilibrium in abstract economies

As a preliminary step for the existence of equilibrium, we introduce an abstract economy. For each $i \in I$, let X_i be a nonempty set in $\mathbb{R}^\ell$. We set $X = \prod_{i \in I} X_i$. An *abstract economy* $\Gamma = \{(X_i, A_i, P_i) : i \in I\}$ is a set of ordered triples (X_i, A_i, P_i) where $A_i : X \to 2^{X_i}$ and $P_i : X \to 2^{X_i}$ are correspondences. The abstract economy provides a simple but powerful conceptual framework for studying an exchange economy in a general setting.

Definition A.1. A *quasi-equilibrium* for Γ is a point $x \in X$ such that for all $i \in I$,

(i) $x_i \in \text{cl}A_i(x)$
(ii) $P_i(x) \cap A_i(x) = \emptyset$.

The point $x \in X$ is an *equilibrium* for Γ if it satisfies (i) and the following condition

(ii$'$) $P_i(x) \cap \text{cl}A_i(x) = \emptyset$.

We are now ready to provide the following preliminary theorem which will be useful in proving the existence of quasi-equilibrium of an exchange economy.

Theorem A.1. *Let $\Gamma = \{(X_i, A_i, P_i) : i \in I\}$ be an abstract economy satisfying the following conditions for each $i \in I$*

A1. *Each X_i is convex, compact and nonempty in $\mathbb{R}^\ell$.*
A2. *Each P_i is lower semi-continuous.*
A3. *A_i is convex-valued, nonempty-valued and has an open graph.*
A4. *$\text{cl}A_i$ is upper semi-continuous.*
A5. *$x_i \notin \text{co}P_i(x)$ for all $x \in X$.*

Then Γ has a quasi-equilibrium, i.e., there exists $x^* \in X$ such that for all $i \in I$,

(i) $x_i^* \in \text{cl}A_i(x^*)$, and
(ii) $P_i(x^*) \cap A_i(x^*) = \emptyset$.

Proof. For each $i \in I$, define $\psi_i : X \to 2^{X_i}$ by $\psi_i(x) = [\text{co}P_i(x)] \cap A_i(x)$. Clearly, ψ_i is convex-valued. For each $i \in I$, let $U_i = \{x \in X : \psi_i(x) \neq \emptyset\}$. Since P_i is lower semi-continuous, by Proposition 2.6 in Michael (1956) $\text{co}P_i$ is lower semi-continuous. Hence, by Lemma 4.2 of Yannelis (1987), ψ_i is lower semi-continuous.[19] It follows from the lower semi-continuity of ψ_i that for each $i \in I$, U_i is open in X (recall that $U_i = \{x \in X : \psi_i(x) \cap X \neq \emptyset\}$).

There are two cases to be examined: either (a) $U_i = \emptyset$ for all $i \in I$ or (b) $U_i \neq \emptyset$ for some $i \in I$. It is easily seen in case (a) that for all i and for all $x \in X$, $\psi_i(x) = [\text{co}P_i(x)] \cap A_i(x) = \emptyset$ and therefore, $P_i(x) \cap A_i(x) = \emptyset$. Hence, condition (ii) of the theorem holds. To show that (i) is also fulfilled, we define the correspondence $A : X \to 2^X$ by $A(x) = \prod_{i \in I} \text{cl}A_i(x)$. Since each $\text{cl}A_i$ is upper semi-continuous, closed-valued, convex-valued and nonempty-valued, so is A. By the Kakutani fixed point theorem there exists $x^* \in X$ such that $x^* \in A(x^*)$, which implies that $x_i^* \in \text{cl}A_i(x^*)$ for all $i \in I$. Thus (i) also holds.

We turn to case (b). For each i with $U_i \neq \emptyset$, we denote by $\psi_i|_{U_i}$ the restriction of ψ_i to U_i, i.e., $\psi_i|_{U_i} : U_i \to 2^{X_i}$. Since U_i is open in X, it is also paracompact.[20] By applying Theorem 3.1$'''$ of Michael (1956, p. 368) to $\psi_i|_{U_i}$, there exists a continuous function $f_i : U_i \to X_i$ such that $f_i(x) \in \psi_i(x)$ for all $x \in U_i$. For each $i \in I$, define $g_i : X \to 2^{X_i}$ by

$$g_i(x) = \begin{cases} \{f_i(x)\}, & \text{if } x \in U_i, \\ \text{cl}A_i(x), & \text{if } x \notin U_i. \end{cases}$$

By Lemma 6.1 in Yannelis and Prabhakar (1983), g_i is upper semi-continuous and it is clearly convex-valued, nonempty-valued and closed-valued. Define $g : X \to 2^{X_i}$ by $g(x) = \prod_{i \in I} g_i(x)$. Then g is upper semi-continuous, convex-valued, nonempty-valued, and closed-valued. By the Kakutani fixed point theorem, there exists $x^* \in X$ such that $x^* \in g(x^*)$, i.e. $x_i^* \in g_i(x^*)$ for all $i \in I$. If $x^* \in U_i$ for some $i \in I$, then $x_i^* = f_i(x^*) \in \text{co}P_i(x^*) \cap A_i(x^*) \subset \text{co}P_i(x^*)$ which contradicts A5. Hence, for all $i \in I$, $x^* \notin U_i$, i.e.,

[19] Let T and Y be any topological spaces, and $\theta_1 : T \to 2^Y$ and $\theta_2 : T \to 2^Y$ be correspondences. Then Lemma 4.2 of Yannelis (1987) shows that if θ_1 has an open graph and θ_2 is lower semi-continuous, the correspondence $\theta : T \to 2^Y$ defined by $\theta(t) = \theta_1(t) \cap \theta_2(t)$ is lower semi-continuous.
[20] X is metrizable because it is a countable product of metric spaces. It is also well-known (Stone's Theorem) that metrizable spaces are paracompact.

$x_i^* \in \mathrm{cl}A_i(x^*)$ and $\mathrm{co}P_i(x^*) \cap A_i(x^*) = \emptyset$, which implies $P_i(x^*) \cap A_i(x^*) = \emptyset$ for each $i \in I$. Thus, we can conclude that $P_i(x^*) \cap A_i(x^*) = \emptyset$, i.e., x^* is a quasi-equilibrium for Γ. □

If each P_i has open upper sections, $P_i(x^*) \cap A_i(x^*) = \emptyset$ implies that $P_i(x^*) \cap \mathrm{cl}A_i(x^*) = \emptyset$. Thus, the following corollary is immediate from Theorem A.1.

Corollary A.1. *Suppose that Γ satisfies A1–A5 of Theorem A.1.If each P_i has open upper sections, then Γ has an equilibrium.*

Remark A.1. Corollary A.1 does not follow from Borglin and Keiding (1976); Shafer and Sonnenschein (1975) or Yannelis and Prabhakar (1983) because the assumptions on the correspondences P_i are weaker here than those papers. In particular, Yannelis and Prabhakar (1983) assume that P_i must have open lower sections which implies that P_i is lower semi-continuous but the reverse is not true. Shafer and Sonnenschein (1975) and Borglin and Keiding (1976) assume that preference correspondences must have an open graph which implies that both sections (upper and lower) must be open.

Remark A.2. The assumptions on the constraint correspondences in Theorem A.1, however, are slightly stronger than those of Shafer and Sonnenschein (1975). Nonetheless, they are automatically fulfilled by the standard exchange economy, and as we will see in the next section Theorem A.1 will enable us to provide a more general Walrasian equilibrium existence result than that of Shafer (1976).

The condition A5 can be replaced by a more tractable condition without losing any generality. Since P_i is lower semi-continuous, by Proposition 2.6 in Michael (1956), the convex hull correspondence $\mathrm{co}P_i$ is lower semi-continuous. Now we consider the economy $\tilde{\Gamma} = (X_i, A_i, \tilde{P}_i)_{i \in I}$ where $\tilde{P}_i$ is defined as follows: for all $x_i \in X_i$,

$$\tilde{P}_i(x) = \{(1-\alpha)x_i + \alpha x_i' : 0 < \alpha \le 1, x_i' \in \mathrm{co}P_i(x)\}.$$

For each $x \in X$, $\tilde{P}_i(x)$ is convex. Clearly, $P_i(x) \subset \mathrm{co}P_i(x) \subset \tilde{P}_i(x)$, and x_i is in the boundary of $\tilde{P}_i(x)$ for all $x \in X$ and all $i \in I$. By the lower semi-continuity of $\mathrm{co}P_i$, $\tilde{P}_i$ is lower semi-continuous (for details, see Gale and Mas-Colell (1975, 1979) or Allouch (2002)).

For each $i \in I$, we consider the following condition.

A5′. For all $x \in X$ with $P_i(x) \neq \emptyset$, $P_i(x)$ is convex, $x_i \notin P_i(x)$, and for each $y_i \in P_i(x)$, $(x_i, y_i]$ is in $P_i(x)$.[21]

This condition is stronger than A5 but the following result shows that instead of A5, A5′ can be used together with the other assumptions to prove Theorem A.1.

Proposition A.1. *If Γ satisfies the assumptions A1–A5, then $\tilde{\Gamma}$ satisfies the assumptions A1–A4 and A5′. Moreover, if $x \in X$ is an equilibrium (a quasi-equilibrium) of $\tilde{\Gamma}$, it is also an equilibrium (a quasi-equilibrium, resp.) of Γ.*

Proof. Suppose that Γ satisfies the assumptions A1–A5. As discussed above, $\tilde{P}_i$ is lower semi-continuous and convex-valued. Let x be a point in X with $P_i(x) \neq \emptyset$. By A5 we immediately see $x_i \notin \tilde{P}_i(x)$ and by construction, x_i is in the relative boundary of $\tilde{P}_i(x)$. Thus, $\tilde{\Gamma}$ satisfies A1–A4 and A5′.

Suppose that $x \in X$ is an equilibrium of $\tilde{\Gamma}$. Then $x \in \mathrm{cl}A_i(x)$ and $\tilde{P}_i(x) \cap \mathrm{cl}A_i(x) = \emptyset$ for all $i \in I$. Since $P_i(x) \subset \tilde{P}_i(x)$, we trivially see that $x \in \mathrm{cl}A_i(x)$ and $P_i(x) \cap \mathrm{cl}A_i(x) = \emptyset$ for each i. Thus x is also an equilibrium of Γ. Similarly, we can show that if $x \in X$ is a quasi-equilibrium of $\tilde{\Gamma}$, then it is a quasi-equilibrium of Γ. □

Lemma A.1. *Suppose that P_i satisfies B6. Let x be a point in H. Then for each $z_i \in \tilde{P}_i(x)$ and $v_i \in X_i$, there exists $\lambda \in (0, 1)$ such that $\lambda z_i + (1-\lambda)v_i \in \tilde{P}_i(x)$.*

Proof. Let z_i be a point in $\tilde{P}(x)$. Then there exist $\alpha \in (0, 1]$ and $x_i' \in \mathrm{co}P_i(x)$ such that $z_i = (1-\alpha)x_i + \alpha x_i'$. Let v_i be a point in X_i. Since $x_i' \in \mathrm{co}P_i(x)$, by B6 there exists $\lambda \in (0, 1)$ such that $\lambda x_i' + (1-\lambda)v_i \in \mathrm{co}P_i(x)$. We set $\tilde{\lambda} = \lambda/(\alpha - \alpha\lambda + \lambda)$. It follows that $\tilde{\lambda} \in (0, 1)$ and

$$\begin{aligned}
&\tilde{\lambda} z_i + (1-\tilde{\lambda})v_i \\
&\quad = \tilde{\lambda}[(1-\alpha)x_i + \alpha x_i'] + (1-\tilde{\lambda})v_i \\
&\quad = \tilde{\lambda}(1-\alpha)x_i + [1-\tilde{\lambda}(1-\alpha)]\left(\frac{\tilde{\lambda}\alpha}{1-\tilde{\lambda}(1-\alpha)}x_i' + \frac{1-\tilde{\lambda}}{1-\tilde{\lambda}(1-\alpha)}v_i\right) \\
&\quad = \tilde{\lambda}(1-\alpha)x_i + [1-\tilde{\lambda}(1-\alpha)][\lambda x_i' + (1-\lambda)v_i] \in \tilde{P}_i(x).
\end{aligned}$$
□

Appendix B. Proofs of the results of the main text

Proof of Auxiliary Theorem 2.3.1. By Proposition A.1 and Lemma A.1, without loss of generality we may assume that E satisfies B1–B3, B5 and instead of B4 and B6, the following conditions B4′and B6′.

[21] For two vectors x and y in $\mathbb{R}^\ell$, we denote by $(x, y]$ the set $\{z \in \mathbb{R}^\ell : z = (1-\lambda)x + \lambda y$ for some $\lambda \in (0, 1]\}$.

B4$'$. For all $x \in X$ with $P_i(x) \neq \emptyset$, $P_i(x)$ is convex, $x_i \notin P_i(x)$, and for each $y_i \in P_i(x)$, $(x_i, y_i]$ is in $P_i(x)$.
B6$'$. Let x be a point in H. Then for each $z_i \in P_i(x)$ and $v_i \in X_i$, there exists $\lambda \in (0, 1)$ such that $\lambda z_i + (1-\lambda)v_i \in P_i(x)$.

For each $i \in I$, let X_i^H denote the projection of cl$H$ onto X_i. Since H is bounded, so is cl$H$ and therefore, X_i^H is bounded for all $i \in I$. Hence, we can choose a closed and bounded ball K centered at the origin in $\mathbb{R}^\ell$ which contains X_i^H and e_i in its interior for all $i \in I$. We introduce the truncated economy $\hat{E} = (\hat{X}_i, e_i, \hat{P}_i)$ where for all $i \in I$,

$$\hat{X}_i = X_i \cap K, \hat{X} = \prod_{i \in I} \hat{X}_I \text{ and } \hat{P}_i(x) = P_i(x) \cap K \text{ for all } x \in \hat{X}.$$

We will break the proof into several steps.

Step 1. We introduce the sets Δ and Δ_1 in $\mathbb{R}^\ell$ defined by

$$\Delta = \{p \in \mathbb{R}^\ell : \|p\| \leq 1\}$$
$$\Delta_1 = \{p \in \mathbb{R}^\ell : \|p\| = 1\}.$$

To apply Theorem A.1, we need to convert $\hat{E}$ into our abstract economy $\Gamma = (\hat{X}_i, A_i, G_i)_{i \in I'}$ where $I' = I \cup \{0\}$ by adding the agent 0 as follows; if $i = 0$, we set $\hat{X}_0 = \Delta$ and define

$$G_0(p, x) = \left\{ q \in \Delta : q\left(\sum_{i \in I}(x_i - e_i)\right) > p\left(\sum_{i \in I}(x_i - e_i)\right)\right\},$$
$$A_0(p, x) = \Delta \text{ for all } (p, x) \in \Delta \times \hat{X},$$

and if $i \in I$, for all $(p, x) \in \Delta \times \hat{X}$ we set

$$G_i(p, x) = \hat{P}_i(x), \quad \text{and}$$
$$A_i(p, x) = \{x_i \in X_i : px_i < pe_i + 1 - \|p\|\} \cap K.$$

Since e_i is in the interior of both X_i and K, $A_i(p, x)$ is not empty for all $(p, x) \in \Delta \times \hat{X}$. On the other hand, cl$A_i$: $\Delta \times \hat{X} \to 2^K$ has a closed graph and K is compact. These imply that the correspondence clA_i is upper semi-continuous. Thus, Γ satisfies A1–A4 and A5$'$ (B4$'$) of Theorem A.1, and consequently, Γ has a quasi-equilibrium, i.e., there exists $(\hat{p}, \hat{x}) \in \Delta \times \hat{X}$ such that

(a) $\hat{p} \in \text{cl}A_0(\hat{p}, \hat{x}) = \Delta$ and $G_0(\hat{p}, \hat{x}) \cap \Delta = \emptyset$ and for all $i \in I$,
(b) $\hat{x}_i \in \text{cl}A_i(\hat{p}, \hat{x})$, i.e., $\hat{p}\hat{x}_i \leq \hat{p}e_i + 1 - \|\hat{p}\|$, and
(c) $G_i(\hat{p}, \hat{x}) \cap A_i(\hat{p}, \hat{x}) = \emptyset$, i.e., $\hat{P}_i(\hat{x}) \cap A_i(\hat{p}, \hat{x}) = \emptyset$.

We will show that $(\hat{p}, \hat{x})$ is an equilibrium for the original exchange economy E.

Step 2. We show that $\hat{x} \in F$, i.e., $\sum_{i \in I}(\hat{x}_i - e_i) = 0$. Since $\hat{p} \in \text{cl}A_0(\hat{p}, \hat{x})$ and $G_0(\hat{p}, \hat{x}) \cap \Delta = \emptyset$, we see that $\|\hat{p}\| \leq 1$ and for all $q \in \Delta$,

$$\hat{p}\left(\sum_{i \in I}(\hat{x}_i - e_i)\right) \geq q\left(\sum_{i \in I}(\hat{x}_i - e_i)\right).$$

Suppose that $\hat{x} \notin F$. We set $q' = (\sum_{i \in I}(\hat{x}_i - e_i))/\|\sum_{i \in I}(\hat{x}_i - e_i)\|$. It follows that $q' \in \Delta$ and therefore,

$$\hat{p}\left(\sum_{i \in I}(\hat{x}_i - e_i)\right) \geq q'\sum_{i \in I}(\hat{x}_i - e_i) = \|\sum_{i \in I}(\hat{x}_i - e_i)\| > 0.$$

In particular, this implies that $\|\hat{p}\| = 1$. On the other hand, $\hat{x}_i \in \text{cl}A_i(\hat{p}, \hat{x})$ for each $i \in I$ implies that $\hat{p}(\hat{x}_i - e_i) \leq 1 - \|\hat{p}\|$. Summing up over $i \in I$, we obtain $\hat{p}(\sum_{i \in I}(\hat{x}_i - e_i)) \leq I(1 - \|\hat{p}\|)$ and therefore, $\|\hat{p}\| < 1$, which is impossible. Therefore, $\hat{x}$ is in F.

Step 3. We claim that $\hat{x}\in R(e)$. Suppose otherwise, i.e., $e_i\in P_i(\hat{x})$ for some $i\in I$. Since $e_i\in \text{int}K$, e_i is in $\hat{P}_i(\hat{x})$. If $\|\hat{p}\|<1$, then e_i is in $A_i(\hat{p},\hat{x})$. This implies that $\hat{P}_i(\hat{x})\cap A_i(\hat{p},\hat{x})\neq\emptyset$, which contradicts (c). We turn to the case that $\|\hat{p}\|=1$. Since e_i is in the interior of $\hat{X}_i$ and $\|\hat{p}\|=1$, we can pick $x_i'\in\hat{X}_i$ such that $\hat{p}x_i'<\hat{p}e_i$. Recalling that $e_i\in P_i(\hat{x})$, by B6$'$ there exists $\lambda\in(0,1)$ such that $\lambda e_i+(1-\lambda)x_i'\in P_i(\hat{x})$ and therefore, $\lambda e_i+(1-\lambda)x_i'\in\hat{P}_i(\hat{x})$. Then (c) implies that $\hat{p}[\lambda e_i+(1-\lambda)x_i']\geq\hat{p}e_i$ or $\hat{p}x_i'\geq\hat{p}e_i$, which is impossible. Thus, we conclude that $\hat{x}\in R(e)$.

Step 4. The results of Steps 2 and 3 imply that $\hat{x}\in H$. We want to show that

$$\|\hat{p}\|=1 \text{ and } \hat{p}\hat{x}_i=\hat{p}e_i \text{ for all } i\in I.$$

By B5, we have $P_i(\hat{x})\neq\emptyset$ for all $i\in I$. Let y_i be a point in $P_i(\hat{x})$. Then by B4$'$, $(\hat{x}_i,y_i]$ is in $P_i(\hat{x})$. Since $\hat{x}_i$ is in the interior of K, we have $K\cap(\hat{x}_i,y_i]\neq\emptyset$ and thus, $\hat{P}_i(\hat{x})\neq\emptyset$. Then we can choose $t_i\in\hat{P}_i(\hat{x})$. By B4$'$, $\alpha t_i+(1-\alpha)\hat{x}_i$ is in $\hat{P}_i(\hat{x})$ for any $\alpha\in(0,1]$. By (c), we have $\hat{P}_i(\hat{x})\cap A_i(\hat{p},\hat{x})=\emptyset$, which implies that for all $\alpha\in(0,1]$,

$$\hat{p}(\alpha t_i+(1-\alpha)\hat{x}_i)\geq\hat{p}e_i+1-\|\hat{p}\|.$$

By letting $\alpha\to 0$, we have $\hat{p}\hat{x}_i\geq\hat{p}e_i+1-\|\hat{p}\|$. On the other hand, (b) gives $\hat{p}\hat{x}_i\leq\hat{p}e_i+1-\|\hat{p}\|$. Hence, for all $i\in I$,

$$\hat{p}\hat{x}_i=\hat{p}e_i+1-\|\hat{p}\|.$$

Summing it over I, we see that

$$\sum_{i\in I}\hat{p}(\hat{x}_i-e_i)=\sum_{i\in I}(1-\|\hat{p}\|).$$

Since $\sum_{i\in I}(\hat{x}_i-e_i)=0$, we obtain $\|\hat{p}\|=1$. Moreover, we can conclude that $\hat{p}\hat{x}_i=\hat{p}e_i$ for all $i\in I$.

Step 5. What is proved up to Step 4 is that $(\hat{p},\hat{x})$ is a quasi-equilibrium for the truncated economy $\hat{E}$. Now we show that $(\hat{p},\hat{x})$ is an equilibrium of E by verifying that $P_i(\hat{x})\cap B_i(\hat{p})=\emptyset$ for all $i\in I$. By Steps 3 and 4, $\hat{x}$ is in cl$H$. Since $X_i^H\subset\text{int}K$, it implies that $\hat{x}_i\in\text{int}K$ for all $i\in I$. Suppose that $P_i(\hat{x})\cap\beta_i(\hat{p})\neq\emptyset$ for some $i\in I$. Let z_i be a point in $P_i(\hat{x})\cap\beta_i(\hat{p})$. Then we can choose $\alpha'\in(0,1)$ such that $\alpha'\hat{x}_i+(1-\alpha')z_i\in K$. It follows from B4$'$ that $\alpha'\hat{x}_i+(1-\alpha')z_i\in\hat{P}_i(\hat{x})$. Recalling that $\|\hat{p}\|=1$, $\hat{p}\hat{x}_i=\hat{p}e_i$ and $\hat{p}z_i<\hat{p}e_i$, we have $\alpha'\hat{p}\hat{x}_i+(1-\alpha')\hat{p}z_i<\hat{p}e_i=\hat{p}e_i+1-\|\hat{p}\|$, and therefore, $\alpha'\hat{x}_i+(1-\alpha')z_i\in\hat{P}_i(\hat{x})\cap A_i(\hat{p},\hat{x})$, which contradicts (c). Thus, we have $P_i(\hat{x})\cap\beta_i(\hat{p})=\emptyset$ for each $i\in I$.

We claim that $\hat{p}x_i>\hat{p}e_i$ for all $x_i\in P_i(\hat{x})$. Suppose that there exists $x_i'\in P_i(\hat{x})$ such that $\hat{p}x_i'=\hat{p}e_i$. Since $e_i\in\text{int}X_i$, we can pick $v_i\in X_i$ such that $\hat{p}v_i<\hat{p}e_i$. By B6$'$, there exists $\lambda\in(0,1)$ such that $\lambda x_i'+(1-\lambda)v_i\in P_i(\hat{x})$. On the other hand, we have $\hat{p}[\lambda x_i'+(1-\lambda)v_i]<\hat{p}e_i$, and therefore, $\lambda x_i'+(1-\lambda)v_i\in\beta_i(\hat{p})$. This contradicts the fact that $P_i(\hat{x})\cap\beta_i(\hat{p})=\emptyset$. We conclude that $P_i(\hat{x})\cap B_i(\hat{p})=\emptyset$ for all $i\in I$ and therefore, $(\hat{p},\hat{x})\in\Delta_1\times X$ is an equilibrium for E. □

Proof of Lemma 3.3.1. For a point $x\in H$, suppose that $G(x)\neq\mathbb{R}^\ell$. Since $G(x)$ is a convex cone, by the separating hyperplane theorem there exists a nonzero $p\in\mathbb{R}^\ell$ such that for all $z\in G(x)$, $0\leq pz$. Recalling that $x_i\in\text{cl}P_i(x)$ for all $i\in I$, we have $x_i-e_i\in\text{cl}[P_i(x)-\{e_i\}]$ and therefore, $x_i-e_i\in\text{clcon}[P_i(x)-\{e_i\}]$ for all $i\in I$. Since $x\in H$, it follows that

$$\text{clcon}[P_i(x)-\{e_i\}]-\{x_i-e_i\}=\text{clcon}[P_i(x)-\{e_i\}]+\sum_{j\neq i}\left\{x_j-e_j\right\}\subset G(x).$$

This implies that for all $z_i'\in\text{clcon}[P_i(x)-\{e_i\}]$,

$$0\leq p[-(x_i-e_i)+z_i']. \tag{8}$$

In particular, for any $\lambda>0$ we have $-(x_i-e_i)+\lambda(x_i-e_i)\in G(x)$. This implies that for each $i\in I$, $0\leq p[-(x_i-e_i)+\lambda(x_i-e_i)]=(\lambda-1)p(x_i-e_i)$. If $\lambda>1$, then $0\leq p(x_i-e_i)$, and if $\lambda<1$, then $0\geq p(x_i-e_i)$. Thus, we have $px_i=pe_i$ for all $i\in I$.

Let $z_i\in\text{cl}P_i(x)$. Then there exist $\lambda\geq 0$ and $z_i'\in\text{clcon}[P_i(x)-\{e_i\}]$ such that $z_i'=\lambda(z_i-e_i)$. Since $px_i=pe_i$, it follows from (8) that $0\leq pz_i'=\lambda p(z_i-e_i)$ or $pe_i\leq pz_i$.

Suppose that $G(x)=\mathbb{R}^\ell$. Then it is easy to see that $0\leq pz$ for all $z\in G(x)$ implies $p=0$, which is impossible. □

Proof of Lemma 3.3.2. Suppose that B7a holds for agent h. Let $\{x^n\}$ be a sequence in H where $r_h(x^n)$ increases to infinity. Then there exist a subsequence $\{x^{n_k}\}$ and a sequence $\{y^{n_k}\}$ convergent to a point $y\in\text{cl}H$ which satisfy (1) and (2). Since $r_h(x^{n_k})\to\infty$

and $\{y^{n_k}\}$ is bounded, we have $P_i(y^{n_k}) \cap C(0, r_h(x^{n_k})) \neq \emptyset$ and by (2), $P_i(x^{n_k}) \cap C(0, r_h(x^{n_k})) \neq \emptyset$ for all $i \neq h$ and for sufficiently large n_k. Thus, we have $G_h(x^{n_k}) \neq \emptyset$ for sufficiently large n_k.

It follows from (1) and (2) that for sufficiently large n_k,

$$\begin{aligned} P_h(y^{n_k}) - \{x_h^{n_k}\} &\subset \text{con } [P_h(x^{n_k}) - \{e_h\}] - \{x_h^{n_k} - e_h\} \\ &= \text{con } [P_h(x^{n_k}) - \{e_h\}] + \sum_{i \neq h} \{x_i^{n_k} - e_i\} \\ &\subset G_h(x^{n_k}), \end{aligned}$$

and for each $j \neq h$,

$$\begin{aligned} P_j(y^{n_k}) \cap C(0, r_h(x^{n_k})) - \{x_j^{n_k}\} &\subset \text{con}[P_j(x^{n_k}) \cap C(0, r_h(x^{n_k})) - \{e_j\}] - \{x_j^{n_k} - e_j\} \\ &= \text{con}[P_j(x^{n_k}) \cap C(0, r_h(x^{n_k}))] + \sum_{i \neq j} \{x_i^{n_k} - e_i\} \\ &\subset G_h(x^{n_k}). \end{aligned}$$

Therefore, we conclude that B7a implies B7. □

Proof of Theorem 3.3.1. As mentioned earlier in this section, without loss of generality we can assume that E satisfies B4$'$ instead of B4. (For details, we refer the reader to the proof of the Auxiliary Theorem 2.3.1) Then P_i is convex-valued, and for all $x \in X$, $x_i \in \text{cl}P_i(x)$.

Let $\{K^n\}$ denote a sequence of increasing closed balls centered at the origin in $\mathbb{R}^\ell$ such that

$$\mathbb{R}^\ell \subset \cup_{n=1}^\infty K^n.$$

We can take K^1 to be a sufficiently large ball that e_i is contained in the interior of K^1 for all $i \in I$. Since K^n is increasing, all e_i's are contained in the interior of K^n for all n.

Let h be an agent which satisfies Assumption B7. Since the following arguments do not rely on the choice of h, without loss of generality, we can assume that $h = 1$.

For each n, we set

$$X_1^n = X_1, \quad X_j^n = X_j \cap K^n \quad \text{for all } j \neq 1, \quad \text{and} \quad X^n = \prod_{i \in I} X_i^n.$$

Let $x \in X^n$ and $p \in \Delta$. For all n and $i \in I$, we set

$$P_i^n(x) = P_i(x) \cap X_i^n,$$
$$\beta_i^n(p) = \{x_i \in X_i^n : px_i < pe_i\}.$$

Note that $P_1^n(x) = P_1(x)$ and $\beta_1^n(p) = \beta_1(p)$ for all n. For each n, let $E^n = \{(X_i^n, e_i, P_i^n) : i \in I\}$ denote the truncated economy of E.

Remarks B.1. For each n, the first agent's choices are not restricted in any E^n while the other agents' choices are restricted to K^n in E^n for all n. Since X_i^n's are bounded except for the first agent, it is easy to check that $H \cap X^n$ is bounded for all n.

Step 1. Since E^n satisfies B1–B4 and $H \cap X^n$ is bounded for all n, by the same arguments made in Steps 1–3 of the proof of the Auxiliary Theorem 2.3.1, there exists a pair $(p^n, x^n) \in \Delta \times X^n$ for each n such that $x^n \in H$, and for all $i \in I$,

(a) $p^n x_i^n \le p^n e_i + 1 - \|p^n\|$ and
(b) $P_i^n(x^n) \cap \{x_i \in X_i^n : p^n x_i < p^n e_i + 1 - \|p^n\|\} = \emptyset$.

For each $i \neq 1$, $x_i^n \in K^n$ and therefore, $C(0, r_1(x^n)) \subset K^n$. First, we consider the case that there exists n such that x_i^n is in the interior of K^n for each $i \neq 1$, or $C(0, r_1(x^n))$ is in the interior of K^n. Then by B4$'$, $P_i^n(x^n) = P_i(x^n) \cap K^n$ is not empty for each $i \neq 1$. Recalling that the choices of agent 1 are not restricted, we have $P_i^n(x^n) \neq \emptyset$ for all $i \in I$. Thus, by applying the arguments of Steps 1–4 of the proof of the Auxiliary Theorem 2.3.1 to the truncated economy E^n, we see that (p^n, x^n) is an equilibrium for E^n. Since $P_1(x^n) = P_1^n(x^n)$ and x_i^n is in the interior of K^n for each $i \neq 1$, it follows by applying the arguments of Step 5 of the proof of the Auxiliary Theorem 2.3.1 for each $i \neq 1$ that (p^n, x^n) is an equilibrium for E, and in this case, we are done.

Thus, we only need to examine the case that for each n there exists some $i_n \neq 1$ such that $x_{i_n}^n$ is in the boundary of K^n. In this case, we have $C(0, r_1(x^n)) = K^n$ for all n. This implies that $r_1(x^n)$ increases to infinity and therefore, $\{x^n\}$ has no bounded subsequences, and for all n and $i \neq 1$,

$$P_i^n(x^n) = P_i(x^n) \cap C(0, r_1(x^n)). \tag{9}$$

By Assumption B7, there exist a subsequence $\{x^{n_k}\}$ of $\{x^n\}$, and a sequence $\{y^{n_k}\}$ convergent to a point $y \in \text{cl}H$ such that for all n_k,

$$P_1^{n_k}(y^{n_k}) = P_1(y^{n_k}) \subset G_1(x^{n_k}) + \{x_h^{n_k}\} \tag{10}$$

and for all $i \neq 1$,

$$P_i^{n_k}(y^{n_k}) = P_i(y^{n_k}) \cap C(0, r_1(x^{n_k})) \subset G_1(x^{n_k}) + \{x_i^{n_k}\}. \tag{11}$$

In particular, (10) implies that $G_1(x^{n_k})$ is not empty for all n_k. Therefore, $P_i(x^{n_k}) \cap C(0, r_1(x^{n_k})) \neq \emptyset$, and by (9), $P_i^{n_k}(x^{n_k}) \neq \emptyset$ for all n_k and for all $i \neq 1$. Since $P_1^{n_k}(x^{n_k}) = P_1(x^{n_k})$, $P_1^{n_k}(x^{n_k})$ is also not empty for all n_k.

Step 2. We show that (p^{n_k}, x^{n_k}) is an equilibrium of the economy $E^{n_k}(x^{n_k})$. First, we claim that $x_i^{n_k}$ is in the relative boundary of $P_i^{n_k}(x^{n_k})$. By B4$'$, this trivially holds for $i = 1$. For each $i \neq 1$, we choose a point y_i in $P_i^{n_k}(x^{n_k})$. By B4$'$, $(x_i^{n_k}, y_i]$ is in $P_i(x^{n_k})$. Since $x_i^{n_k}$ and y_i are in K^{n_k}, we have $(x_i^{n_k}, y_i] \in P_i^{n_k}(x^{n_k})$ and therefore, $x_i^{n_k}$ is in the relative boundary of $P_i^{n_k}(x^{n_k})$.

For a given n_k, we choose $t_i \in P_i^{n_k}(x^{n_k})$ for each $i \in I$. By B4$'$, $\alpha t_i + (1-\alpha)x_i^{n_k}$ is in $P_i^{n_k}(x^{n_k})$ for any $\alpha \in (0, 1]$. It follows from (b) that for all $\alpha \in (0, 1]$,

$$p^{n_k}(\alpha t_i + (1-\alpha)x_i^{n_k}) \geq p^{n_k} e_i + 1 - \|p^{n_k}\|.$$

By letting $\alpha \to 0$, we have $p^{n_k} x_i^{n_k} \geq p^{n_k} e_i + 1 - \|p^{n_k}\|$. On the other hand, (a) gives $p^{n_k} x_i^{n_k} \leq p^{n_k} e_i + 1 - \|p^{n_k}\|$. Hence, for all $i \in I$,

$$p^{n_k} x_i^{n_k} = p^{n_k} e_i + 1 - \|p^{n_k}\|.$$

Summing it over I, we see that

$$\sum_{i \in I} p^{n_k}(x_i^{n_k} - e_i) = \sum_{i \in I} (1 - \|p^{n_k}\|).$$

Since $\sum_{i \in I}(x_i^{n_k} - e_i) = 0$, we obtain $\|p^{n_k}\| = 1$. Moreover, we can conclude that $p^{n_k} x_i^{n_k} = p^{n_k} e_i$ for all $i \in I$.

Thus, it follows from the conditions (a) and (b) that (p^{n_k}, x^{n_k}) is an equilibrium of the economy $E^{n_k}(x^{n_k})$, or

(A) $\|p^{n_k}\| = 1$, and $p^{n_k} x_i^{n_k} = p^{n_k} e_i$ for all $i \in I$,
(B) $p^{n_k} e_1 \leq p^{n_k} z_1$ for all $z_1 \in P_1(x^{n_k})$, and
(C) $p^{n_k} e_i \leq p^{n_k} z_i$ for all $z_i \in P_i(x^{n_k}) \cap C(0, r_1(x^{n_k}))$ and each $i \in I \setminus \{1\}$.

Since $G_1(x^{n_k})$ has the form

$$G_1(x^{n_k}) = \text{clcon}[P_1(x^{n_k}) - \{e_1\}] + \sum_{j \neq 1} \text{clcon}[P_j(x^{n_k}) \cap C(0, r_1(x^{n_k})) - \{e_j\}],$$

the last two outcomes (B) and (C) lead to the following relation.

$$p^{n_k} z \geq 0 \quad \text{for all } z \in G_1(x^{n_k}). \tag{12}$$

Step 3. It follows from (10) and (12) that for all $z_1 \in P_1(y^{n_k})$,

$$p(z_1 - x_1^{n_k}) \geq 0. \tag{13}$$

Since $y_i^{n_k}$ is in the interior of $C(0, r_1(x^{n_k}))$ for sufficiently large n_k and for all $i \neq 1$, $P_i^{n_k}(y^{n_k})$ is not empty for sufficiently large n_k and for all $i \neq 1$. Thus, it follows from (11) and (12) that for all $i \neq 1$ and for all $z_i \in P_i(y^{n_k}) \cap C(0, r_1(x^{n_k}))$,

$$p(z_i - x_i^{n_k}) \geq 0. \tag{14}$$

Since $y_i^{n_k}$ belongs to the relative boundary of $P_i(y^{n_k})$ for all $i \in I$ and n_k, (13) and (14) imply that for sufficiently large n_k and for all $i \in I$,

$$p^{n_k}(y_i^{n_k} - x_i^{n_k}) \geq 0.$$

Recalling that $p^{n_k}x_i^{n_k} = p^{n_k}e_i$, we see that $p^{n_k}y_i^{n_k} \geq p^{n_k}e_i$ for all n_k and all $i \in I$. By passing to the limit, this implies that $py_i \geq pe_i$ for all $i \in I$. By summing up the inequalities over $i \in I$, we obtain $p\sum_{i \in I} y_i \geq p\sum_{i \in I} e_i$. Since $y \in F$, this implies that for all $i \in I$,

$$py_i = pe_i.$$

Step 4. To complete the proof we must show that $P_i(y) \cap B_i(p) = \emptyset$ for all i. First, we claim that $P_i(y) \cap \beta_i(p) = \emptyset$. Suppose otherwise. Then we can pick $z_i \in P_i(y) \cap \beta_i(p)$. By Lemma 4.2 of Yannelis (1987), the correspondence $P_i \cap \beta_i$ defined by $P_i(x) \cap \beta_i(p)$ for all $(p, x) \in \Delta \times X$ is lower semi-continuous.[22] Thus, there exists $z_i^{n_k} \in P_i(y^{n_k}) \cap \beta_i(p^{n_k})$ for each n_k which converges to z_i. In particular, it implies that

$$p^{n_k} z_i^{n_k} < p^{n_k} e_i.$$

Since $z_1^{n_k} \in P_1(y^{n_k})$ and $z_i^{n_k} \in P_i(y^{n_k}) \cap C(0, r_1(x^{n_k}))$ for all $i \neq 1$ and sufficiently large n_k, by Step 3 we must have

$$p^{n_k} e_i = p^{n_k} x_i^{n_k} \leq p^{n_k} z_i^{n_k},$$

which leads to a contradiction. Thus, we have $P_i(y) \cap \beta_i(p) = \emptyset$ for all $i \in I$. By the same arguments made in Step 5 of the proof of Theorem 2.3.1, we see that $P_i(y) \cap B_i(p) = \emptyset$ for all $i \in I$.

Step 5. By the results of Steps 1–5, we see that $y \in F$, $\|p\| = 1$, and $y_i \in B_i(p)$ and $P_i(y) \cap B_i(p) = \emptyset$ for all i. Therefore we conclude that (p, y) is an equilibrium of E. □

Proof of Theorem 3.3.1s. Let $\{T_n\}$ denote the compactification procedure of the assumption B7s. For each n, we denote by $\mathcal{E}^n = \{(X_i^n, e_i, P_i^n) : i \in I\}$ the truncated economy defined such that

$$X_i^n := X_i \cap T_n \quad \text{and} \quad P_i^n(x) = P_i(x) \cap T_n, \quad \forall x \in X^n := \prod_{i \in I} X_i^n.$$

For a sufficiently large n, $\mathcal{E}^n$ satisfies B1–B4. The set $\text{cl}H \cap T_n^I$ contains the set of individually rational and feasible allocations for $\mathcal{E}^n$, and by (iii) of Definition 3.3.1, it is compact. By (iv) of Definition 3.3.1, $P_i^n(x) \neq \emptyset$ for all $x \in R(e)$ and thus, $\mathcal{E}^n$ satisfies B5. Then by Auxiliary Theorem 2.3.1, there exists an equilibrium (p^n, x^n) of $\mathcal{E}^n$ with $p^n \in \Delta_1$. In particular, $\{p^n\}$ has a subsequence convergent to a point $p \in \Delta_1$. By Assumption B7s, there exist $\epsilon > 0$ and $y \in \text{cl}H$ such that

$$P_i(y) \cap C(y_i, \epsilon) - \{e_i\} \subset \text{Ls}\left\{con\left[P_i(x^n) \cap T_n - \{e_i\}\right]\right\}. \tag{15}$$

We claim that (p, y) is an equilibrium of the economy E. All we have to show is that if $z_i \in P_i(y)$, then $pz_i > pe_i$ for all $i \in I$. For each $\alpha \in (0, 1)$, we set $z_i(\alpha) = \alpha z_i + (1 - \alpha)y_i$. Then for each α sufficiently close to 0, the point $z_i(\alpha)$ is in $P_i(y) \cap C(y_i, \epsilon)$. It follows from (15) that there exist $\{z_i^n(\alpha)\}$ and $\lambda^n \geq 0$ such that $\lambda^n(z_i^n(\alpha) - e_i) \to z_i(\alpha) - e_i$ and $z_i^n(\alpha) \in P_i(x^n) \cap T_n$ for each n. Since (p^n, x^n) is an equilibrium of $\mathcal{E}^n$, we have $p^n(z_i^n(\alpha) - e_i) > 0$ and by passing to the limit, $pz_i(\alpha) \geq pe_i$. By letting $\alpha \to 0$, we obtain $py_i \geq pe_i$. It follows from the market clearing condition that $py_i = pe_i$ for each $i \in I$. Since $z_i(\alpha) = \alpha z_i + (1 - \alpha)y_i$ and $pz_i(\alpha) \geq pe_i$, this implies that $pz_i \geq pe_i$ for each $i \in I$. By the same arguments made in Step 5 of the proof of Theorem 2.3.1, one can show that $z_i \in P_i(y)$ implies that $pz_i > pe_i$ for each $i \in I$. □

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[22] For details, see footnote 19.

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